

Feynman integral reduction and Landau singularities

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We propose that Feynman integral reduction is controlled by solutions of the Landau equations. We study integral relations with prescribed propagator powers using syzygy methods and discuss how syzygies can be expressed as a sum over components of the Landau singularity locus. This leads to a determinantal approach to solving the syzygy problem, giving rise to highly compact and physically transparent solutions. We demonstrate the method in applications to planar two-loop five-point integrals relevant for the $pp \rightarrow t\bar{t}H$ process. Our results suggest an efficient method of Feynman integral reduction and provide a novel physical perspective on the problem.

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Introduction. Perturbative calculations of scattering amplitudes are essential for precise understanding of fundamental physics at particle collider and gravitational-wave experiments. The defining difficulty of such calculations is both the large number and complexity of the Feynman integrals that must be computed. In modern approaches, this issue is tackled by exploiting the linear relations that the integrals satisfy [1]. This both reduces the computational burden to that of a smaller set of “master integrals” as well as facilitates the calculation of the master integrals themselves via the differential equations method [2–5]. While algorithms for integral reduction exist [6] and public implementations are available [7–16], the large demands for precision placed by experimental uncertainties have driven the development of novel reduction methods such as syzygy methods [17–19], finite-field reconstruction [20,21], and twisted cohomology approaches [22–27].

In this Letter we argue that Feynman integral reduction is controlled by the Landau equations [28]. The Landau equations have recently been under intense study [29–36] as singularities of scattering amplitudes are described by solutions that constrain external kinematics. In this work we discuss how solutions of the Landau equations that do not constrain external kinematics organize the set of relations between Feynman integrals. We uncover this physical correspondence by considering the set of integral relations with prescribed propagator powers. It is well known that these relations are controlled by a

so-called syzygy equation [17]. Despite intense study over the past 15 years [17–19,37,38–44] and impressive recent computational progress [15,16,45], both the physical meaning of the syzygy equation and general solutions have remained elusive. In this work, we tackle both questions by connecting the syzygy problem to the Landau equations.

While a link between syzygies and the Landau equations has been noticed before [37], we employ two advances to make this connection constructive. First, we exploit the recent observation [46] that distinguished “critical syzygies” are often sufficient for integral reduction purposes. Second, we use advanced technologies from commutative algebra [47,48]. Together, this allows us to write the set of critical syzygies as a sum over components of the solutions of the Landau equations. Importantly, this leads to an approach to construct a highly compact collection of analytic syzygies. We illustrate our approach by applying it to a cutting-edge example of a two-loop pentabox family for the frontier $pp \rightarrow t\bar{t}H$ process. We discuss a double box case in detail and present the remaining set of five-point syzygy solutions in the Supplemental Material [49]. Altogether, our results highlight how Feynman integral reduction is controlled by infrared singularities and provide an efficient method of syzygy construction for phenomenological application.

Singularities and integral relations. In this work, we study relations between Feynman integrals with prescribed propagator powers. It is well known [17] that these relations are easily constructed once a “syzygy equation” is solved. In the Baikov representation [50], this equation can be formulated as [18,46]

$$0 = a_0 B + \sum_{i \in \text{ISPs}(\Gamma)} a_i \partial_i B + \sum_{e \in \text{props}(\Gamma)} (\tilde{a}_e z_e B + \bar{a}_e z_e \partial_e B), \quad (1)$$

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where Γ is the graph associated to the Feynman integral, $\text{ISPs}(\Gamma)$ labels the irreducible scalar product variables, and $\text{props}(\Gamma)$ labels the propagator variables z_e . In Eq. (1), B is the Baikov polynomial which, in two-loop, five-point applications, can be normalized and written as

$$B = \mu_{11}\mu_{22} - \mu_{12}^2, \quad \mu_{ij} = \frac{G(\bar{\ell}_i p_1 p_2 p_3 p_4)}{G(p_1 p_2 p_3 p_4)}, \quad (2)$$

where $\bar{\ell}_i$ are D -dimensional loop momenta, p_k are external momenta, $G(V)$ is the determinant of the matrix of dot products of vectors in V and W and, $G(V) = G(V)$.

Despite being a linear equation, solving Eq. (1) is highly nontrivial since integral reduction demands solutions a_0 , a_i , $\bar{a}_e$, and $\bar{a}_e$ that are elements of R , the polynomial ring in Baikov variables. In Ref. [46], it was observed that integral relations induced by solutions to Eq. (1) are governed by the a_0 term. This led to the definition of the set of critical syzygy solutions, $\text{CSyz}(\Gamma)$, that control this feature at the level of Eq. (1). In the following, we develop a novel approach to finding critical syzygy solutions, by highlighting a connection to the Landau equations.

To begin, we consider the maximal cut Landau locus in Baikov parametrization [34,51]. This is the variety $U_{\text{Landau}}^\Gamma = V(J_{\text{Landau}}^\Gamma)$ associated to the polynomial ideal

$$J_{\text{Landau}}^\Gamma = \langle B \rangle + \sum_{i \in \text{ISPs}(\Gamma)} \langle \partial_i B \rangle + J_{\text{cut}}^\Gamma, \quad (3)$$

where $+$ denotes the sum of polynomial ideals and $J_{\text{cut}}^\Gamma = \sum_{e \in \text{props}(\Gamma)} \langle z_e \rangle$. We consider U_{Landau}^Γ for fixed external kinematics and hence as a variety in Baikov variables alone. Physically, U_{Landau}^Γ describes integration space regions that potentially give rise to divergences. For two-loop, five-point examples, we find that U_{Landau}^Γ admits a decomposition into subvarieties as

$$U_{\text{Landau}}^\Gamma = V(J_{4d} + J_{\text{cut}}^\Gamma) \cup \left[\bigcup_{i=1}^{n_{\text{IR}}} U_i^\Gamma \right], \quad (4)$$

where $J_{4d} = \langle \mu_{11}, \mu_{22}, \mu_{12} \rangle$ and each U_i^Γ is an irreducible component of U_{Landau}^Γ that is not contained in $U_0^\Gamma = V(J_{4d} + J_{\text{cut}}^\Gamma)$. We observe that, physically, the $U_{i>0}$ are associated to soft and collinear configurations.

Our main result is to write the critical syzygy module as a sum over the varieties in Eq. (4) as

$$\text{CSyz}(\Gamma) = \sum_{i=0}^{n_{\text{IR}}} \text{FittSyz}(U_i^\Gamma), \quad (5)$$

which we dub a “Landau decomposition” of $\text{CSyz}(\Gamma)$. In the following, we define $\text{FittSyz}(U_i^\Gamma)$ by giving an explicit procedure to construct a generating set. To this end, we

recall that the module of critical syzygies, $\text{CSyz}(\Gamma)$, is defined such that it is isomorphic to an ideal [46] as

$$\text{CSyz}(\Gamma) \simeq A_0^\Gamma / J_{\text{cut}}^\Gamma, \quad A_0^\Gamma = J_{\text{syzy}}^\Gamma : \langle B \rangle, \quad (6)$$

$$J_{\text{syzy}}^\Gamma = \sum_{i \in \text{ISPs}(\Gamma)} \langle \partial_i B \rangle + \sum_{e \in \text{props}(\Gamma)} \langle z_e B, z_e \partial_e B \rangle, \quad (7)$$

where $A_0^\Gamma / J_{\text{cut}}^\Gamma$ is the image of A_0^Γ in the quotient ring $R / J_{\text{cut}}^\Gamma$ and $J_{\text{syzy}}^\Gamma : \langle B \rangle$ is an ideal quotient. Geometrically, if this quotient is saturated, i.e., $J_{\text{syzy}}^\Gamma : \langle B \rangle = J_{\text{syzy}}^\Gamma : \langle B^2 \rangle$, then it can be understood as removing components where $B = 0$ from $V(J_{\text{syzy}}^\Gamma)$ [46]. We will make use of this “saturation hypothesis” throughout this work. Here, we interpret the ideal quotient as removing Landau singularities from $V(J_{\text{syzy}}^\Gamma)$ and Eq. (5) as an algebraic shadow of this geometric intuition. That is, one can remove Landau locus components one by one in a way that automatically yields critical syzygies.

To set up Eq. (5), we introduce some technology. The main observation is that we can decompose A_0^Γ as

$$A_0^\Gamma = \sum_{i=0}^{n_{\text{IR}}} \text{Fitt}_0(M_i) + J_{\text{cut}}^\Gamma, \quad (8)$$

$$M_0 = J_{4d} / J_{\text{syzy}}^\Gamma, \quad M_i = (J_{\text{syzy}}^{\Gamma[U_i^\Gamma]} \cap J_{4d}) / J_{\text{syzy}}^\Gamma, \quad (9)$$

where the M_i are viewed as R modules and the $i = 0$ term in both Eq. (8) and Eq. (5) can be dropped if $n_{\text{IR}} > 0$. This result, and thus Eq. (5), can be shown to hold under the saturation hypothesis and

$$V(J_{\text{syzy}}^\Gamma : \sqrt{J_{\text{syzy}}^\Gamma}) \cap U_{\text{Landau}}^\Gamma \subseteq \bigcup_{i=1}^{n_{\text{IR}}} U_i^\Gamma, \quad (10)$$

where $\sqrt{J}$ denotes the radical of J . $V(J : \sqrt{J})$ is known as the nonreduced locus of J and describes the locus where J is not a radical ideal. The proof is technical, and we leave it to the Supplemental Material [49].

There are two main pieces of mathematical technology in Eq. (8). The first is $\text{Fitt}_0(M_i)$, the zeroth Fitting ideal of the module M_i [48]. Denoting a generating set of the numerator ideal of M_i as $\{v_{i,1}, \dots, v_{i,n}\}$, $\text{Fitt}_0(M_i)$ is generated by the $n \times n$ minors of a presentation matrix $\mathcal{M}_i$ of M_i : a matrix whose rows generate the module of syzygies of a generating set of M_i . Geometrically, for two ideals J, K such that $J \subset K$, the zeroth Fitting ideal satisfies

$$V(\text{Fitt}_0[K/J]) = V(J:K), \quad (11)$$

and is thus intimately related to the ideal quotient. The second technology is $J^{[U]}$, the “prelocalization” of an ideal J at a variety U , which can be built from a primary decomposition of J . Denoting the primary components of J

as $\mathcal{Q}_j$, we define the prelocalization as those components whose associated variety intersects U , i.e.,

$$J^{[U]} = \bigcap_{j: V(\mathcal{Q}_j) \cap U \neq \emptyset} \mathcal{Q}_j. \quad (12)$$

Thus, each term in Eq. (8) excises $U_0^\Gamma \cup U_i^\Gamma$ from $V(J_{\text{syz}}^\Gamma)$.

In practice, we can view Eq. (8) as a determinantal construction of A_0^Γ . Moreover, via judicious, block construction of the $\mathcal{M}_i$, we can also use this to build $\text{CSyz}(\Gamma)$. To this end, writing the generators of J_{syz}^Γ in Eq. (7) as $\{b_1, \dots, b_m\}$, we construct a polynomial matrix $\mathcal{C}_i$ such that $b_j = (\mathcal{C}_i)_{jk} v_{i,k}$. Next, we construct a matrix $\mathcal{R}_i$ whose rows generate the module of syzygies of $\{v_{i,1}, \dots, v_{i,n}\}$. It can be shown that the augmented matrix $\mathcal{M}_i = \begin{pmatrix} \mathcal{C}_i \\ \mathcal{R}_i \end{pmatrix}$ is a presentation matrix of M_i . To use this to construct syzygies, we exploit that, under the saturation hypothesis, there exists a $\vec{c}_i$ such that $B = \vec{c}_i \cdot \vec{v}_i$. We can combine this vector and $\mathcal{M}_i$ into the matrix

$$\mathcal{F}_i = \begin{pmatrix} B & \vec{c}_i^T \\ \vec{b} & \mathcal{C}_i \\ \vec{0} & \mathcal{R}_i \end{pmatrix}, \quad (13)$$

which satisfies $\mathcal{F}_i \begin{pmatrix} -1 \\ \vec{v}_i \end{pmatrix} = 0$, and so maximal minors of $\mathcal{F}_i$ vanish. Therefore, Laplace expansion along the leftmost column yields solutions to Eq. (1). We define $\text{FittSyz}(U_i^\Gamma)$ as the module implicitly generated by Laplace expansions of the maximal minors of $\mathcal{F}_i$ that contain the first row. By construction, the coefficients of B generate $\text{Fitt}_0(M_i)$, and we conclude that the decomposition of Eq. (5) holds.

In summary, to construct the set of critical syzygies for a topology Γ , we first construct an irreducible decomposition of the maximal-cut Landau locus. Then, for each component, we construct the matrix $\mathcal{F}_i$ by computing the set of polynomial matrices c_i , $\mathcal{C}_i$, and $\mathcal{R}_i$. Laplace expansion of each $\mathcal{F}_i$ gives syzygy solutions which, when taken together, generate the syzygy module $\text{CSyz}(\Gamma)$.

Notation and variable choice. In the rest of this Letter, we demonstrate how Eq. (13) can be used to efficiently build syzygy solutions. For concreteness, we study applications to cases arising in the two-loop contributions to amplitudes for $pp \rightarrow t\bar{t}H$. Here, the external momenta satisfy

$$\sum_{i=1}^5 p_i = 0, \quad p_{1,3}^2 = m_t^2, \quad p_2^2 = m_H^2, \quad p_{4,5}^2 = 0. \quad (14)$$

We route the loop momenta as in Fig. 1(a) and, writing $q_k = \sum_{j=1}^k p_j$, $q_0 = 0$, we define Baikov variables as

$$z_{i,j} = (\vec{\ell}_i - q_j)^2 - m_{i,j}^2, \quad \tilde{z} = (\vec{\ell}_1 - \vec{\ell}_2)^2, \quad (15)$$

where $m_{i,j}$ is the mass of the propagator associated to $z_{i,j}$. We abbreviate $\partial_{a,j} = \frac{\partial}{\partial z_{a,j}}$ and $\partial_{\tilde{z}} = \frac{\partial}{\partial \tilde{z}}$.

For practical computations, as well as ease of physical understanding, we make the observation that, for fixed external kinematics, Eq. (15) defines an isomorphism between the polynomial ring in Baikov variables and a polynomial ring in momentum variables. Specifically,

$$R \cong \mathbb{C}[\ell_1^\mu, \ell_2^\mu, \mu_{11}, \mu_{22}, \mu_{12}], \quad (16)$$

where ℓ_i are the four-dimensional projections of the $\vec{\ell}_i$ and $\mu_{ij} = \vec{\ell}_i \cdot \vec{\ell}_j - \ell_i \cdot \ell_j$. The isomorphism in Eq. (16) follows as both the forward and the inverse map are polynomial, with the inverse map given by the well-known

$$\ell_i = \frac{1}{2} \sum_j (z_{i,0} - z_{i,j} + q_j^2 - m_{i,j}^2 + m_{i,0}^2) v_j, \quad (17)$$

where the v_i are defined by $v_i \cdot q_j = \delta_{ij}$, alongside Eq. (2).

The Baikov polynomial and its derivatives can easily be expressed in loop-momentum variables via Eq. (2) and

$$\partial_{\tilde{z}} B = \mu_{12}, \quad (18)$$

$$\begin{aligned} \partial_{a,j} B &= \delta_{j,0} (\mu_{\bar{a}\bar{a}} - \mu_{12}) + w_j \cdot \Lambda_a, \\ \Lambda_a^\nu &= \ell_a^\nu \mu_{\bar{a}\bar{a}} - \ell_{\bar{a}}^\nu \mu_{12}, \end{aligned} \quad (19)$$

where $\bar{a} = 3 - a$ and $w_i^\mu = -2\partial_{a,i} \ell_a^\mu$. Moreover, we make use of the Sudakov parametrization,

$$\ell_i = x_i p + \beta_i \eta + \ell_i^\perp, \quad (20)$$

where p is the relevant massless momentum, η is a massless reference vector, and $\ell_i^\perp$ is the projection of ℓ_i into the space transverse to p and η .

A five-point double box example. In this section, we apply Eq. (5) in the case of the double box of Fig. 1(a). All explicit computations are performed with SINGULAR [52]. To begin, we check the validity of the Landau decomposition, by computing the ingredients of Eq. (10). We first compute the irreducible decomposition of the maximal cut Landau locus, finding

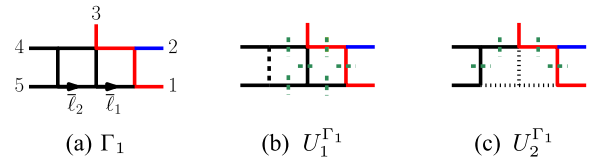


FIG. 1. Double box topology from $pp \rightarrow t\bar{t}H$ (a) and its leading Landau singularities (b), (c). Black lines are massless particles; red, the top quark; blue, the Higgs boson; dashed, soft particles; dotted, collinear particles.

$$U_{\text{Landau}}^{\Gamma_1} = U_0^{\Gamma_1} \cup U_1^{\Gamma_1} \cup U_2^{\Gamma_1}, \quad (21)$$

where $U_1^{\Gamma_1}$ and $U_2^{\Gamma_2}$ correspond to configurations of loop momenta that both satisfy the maximal cut, as well as some internal soft/collinear configuration. We depict them graphically in Figs. 1(b) and 1(c).

Next, to compute the nonreduced locus of $J_{\text{syz}}^{\Gamma_1}$, we calculate the radical of $J_{\text{syz}}^{\Gamma_1}$. Our approach is to express it as the intersection of the minimal associated primes [47] of $R/J_{\text{syz}}^{\Gamma_1}$, which we write as $\text{mAss}(J_{\text{syz}}^{\Gamma_1})$. As minimal primes correspond to irreducible components of $V(J_{\text{syz}}^{\Gamma_1})$, and generators of $J_{\text{syz}}^{\Gamma_1}$ factorize, it can be argued that

$$\text{mAss}(J_{\text{syz}}^{\Gamma_1}) = \text{mAss}(A_0^{\Gamma_1}) \cup \min_{\subseteq} \left[\bigcup_{\Gamma' \subseteq \Gamma_1} \text{mAss}(J_{\text{Landau}}^{\Gamma'}) \right], \quad (22)$$

where the minimum is with respect to inclusion. For the two-loop examples we have studied, Eq. (22) can be efficiently implemented for numerical external kinematics.

For the double box example, we find the result

$$\sqrt{J_{\text{syz}}^{\Gamma_1}} = A_0^{\Gamma_1} \cap J_{4d} \cap J_{\text{e-col}}^{\ell_2, p_5} \cap J_{\text{e-col}}^{\ell_2 - q_4, p_4} \cap J_{\text{soft}}^{\ell_1 - \ell_2} \cap J_{\text{soft}}^{\ell_1} \cap J_{\text{i-col}}^{\ell_1, \ell_2}. \quad (23)$$

The novel ideals in Eq. (23) correspond to uncut soft or collinear singularities, which we depict in Fig. 2. Two ideals describe configurations where an internal momentum is collinear to an external momentum, as in Fig. 2(a), two ideals describe a soft internal momentum, as in Fig. 2(b), and a final ideal describes three collinear internal momenta, as in Fig. 2(c). Explicitly, they are given by

$$J_{\text{e-col}}^{\ell_i - q_j, p_k} = \langle \mu_{12}, \mu_{ii}, (\ell_i - q_j)^{\perp \nu}, (\ell_i - q_j) \cdot p_k \rangle, \quad (24)$$

$$J_{\text{soft}}^{\ell_1} = \langle \ell_1^\nu, \mu_{11}, \mu_{12} \rangle, \quad (25)$$

$$J_{\text{soft}}^{\ell_1 - \ell_2} = \langle \ell_1^\nu - \ell_2^\nu, \mu_{11} - \mu_{12}, \mu_{22} - \mu_{12} \rangle, \quad (26)$$

$$J_{\text{i-col}}^{\ell_1, \ell_2} = \langle \ell_1^\mu \ell_2^\nu - \ell_1^\nu \ell_2^\mu, z_{1,0}, z_{2,0}, \tilde{z} \rangle, \quad (27)$$

where we employ the parametrization of Eq. (20) for $p = p_k$. With these data, we compute the nonreduced locus of $J_{\text{syz}}^{\Gamma_1}$ and find that its intersection with the maximal cut is empty, implying that Eq. (10) is satisfied.

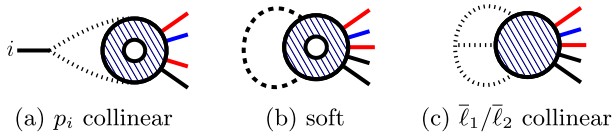


FIG. 2. Diagrams of Landau singularities arising in the computation of $\sqrt{J_{\text{syz}}^{\Gamma_1}}$ (a, b, c). Notation matches Fig. 1.

These ideals allow us to easily compute the necessary prelocalizations and their associated $\mathcal{C}$ matrices. As the nonreduced locus does not intersect the maximal cut, all prelocalizations are simply intersections of the ideals in Eq. (23). For $U_1^{\Gamma_1}$, it is physically clear that it intersects only the p_4/p_5 collinear loci and the 4d locus, giving

$$J_{\text{syz}}^{\Gamma_1, [U_1^{\Gamma_1}]} = J_{4d} \cap J_{\text{e-col}}^{\ell_2, p_5} \cap J_{\text{e-col}}^{\ell_2 - q_4, p_4} = \langle \mu_{12}, \mu_{22}, \mu_{11} \ell_2^{\perp \nu}, z_{2,0} z_{2,3} \mu_{11}, z_{2,4} \mu_{11} \rangle, \quad (28)$$

where we employ the parametrization of Eq. (20) with $p = p_5, \eta = p_4$. To construct $\mathcal{C}$, we note that $J_{\text{syz}}^{\Gamma_1, [U_1^{\Gamma_1}]}$ contains J_{4d} . We express the generators of $J_{\text{syz}}^{\Gamma_1}$ in terms of those of Eq. (28) by inspection, using Eq. (18) and

$$\partial_{1,j} B = (\partial_{1,j} \mu_{11}) \mu_{22} - 2(\partial_{1,j} \mu_{12}) \mu_{12}, \quad (29)$$

$$z_{2,4} \partial_{2,4} B = (\partial_{2,4} \mu_{22}) z_{2,4} \mu_{11} - 2z_{2,4} (\partial_{2,4} \mu_{12}) \mu_{12}, \quad (30)$$

$$\partial_{2,j} B = (w_j \cdot \ell_2^{\perp}) \mu_{11} - 2(\partial_{2,j} \mu_{12}) \mu_{12}, \quad (31)$$

$$z_{2,j} \partial_{2,j} B = z_{2,j} \left[\frac{z_{2,(3-j)} - z_{2,4}}{2p_4 \cdot p_5} + w_j \cdot \ell_2^{\perp} \right] \mu_{11} - z_{2,j} \partial_{2,j} \mu_{12}^2, \quad (32)$$

where Eq. (31) holds for $j = 1, 2$ and Eq. (32) for $j = 0, 3$.

The prelocalization at $U_2^{\Gamma_1}$ can be handled analogously. Physically, the maximal-cut ℓ_1/ℓ_2 collinear singularity intersects with the four-dimensional locus, as well as uncut soft and collinear loci, implying

$$J_{\text{syz}}^{\Gamma_1, [U_2^{\Gamma_1}]} = J_{4d} \cap J_{\text{soft}}^{\ell_1 - \ell_2} \cap J_{\text{i-col}}^{\ell_1, \ell_2} = \langle \Lambda_1^\nu, \chi_1^{(a)}, \chi_1^{(b)}, B \rangle + (1 \leftrightarrow 2),$$

$$\chi_i^{(a)} = z_{i,0} (\mu_{i\bar{i}} - \mu_{12}), \quad \chi_i^{(b)} = z_{i,0} (\mu_{ii} - \mu_{i\bar{i}}). \quad (33)$$

The relations required to express the $\mathcal{C}$ matrix are again easy to determine by inspection. They are given by

$$z_{i,0} \partial_{i,0} B = z_{i,0} w_0 \cdot \Lambda_i + \chi_i^{(a)}, \quad (34)$$

$$\tilde{z} \partial_{\tilde{z}} B = 2B + \ell_1 \cdot \Lambda_1 + \ell_2 \cdot \Lambda_2 - \chi_1^{(a)} - \chi_2^{(a)}, \quad (35)$$

alongside Eq. (19) for $j > 0$. Given the generating sets in Eqs. (28) and (33), the syzygy matrices $\mathcal{R}$ can easily be computed with Groebner basis techniques.

pp → t̄tH pentabox study. We have applied the Landau decomposition strategy to the set of five-point diagrams within the pentabox topology of Fig. 3(a), which we depict in Fig. 3. In all cases, similar to the double box example, we confirm that the nonreduced locus of $J_{\text{syz}}^{\Gamma_1}$ does not intersect

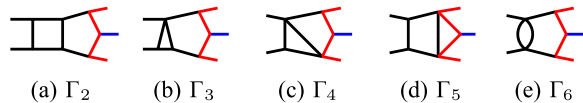


FIG. 3. Remaining five-point topologies contributing to the leading-color, light-quark-loop contributions to $pp \rightarrow t\bar{t}H$ pentabox topology (a, b, c, d, e).

the maximal cut, therefore implying Eq. (10). As such, we could construct all Landau decompositions with the strategy described for the double box. Moreover, we explicitly confirm the equality of Eq. (8) computationally.

We find that the universality of infrared singularities is reflected in our construction: all (SINGULAR) minimal primes that we encounter are either J_{4d} or are captured by the graphs in Fig. 2. Moreover, we are able to recycle ingredients between topologies. Firstly, the ideal in Eq. (28) arises as a prelocalization on the soft singularity for an exchange between the two massless legs also in the pentabox, Γ_2 , and the box triangle, Γ_5 . Secondly, for each of Γ_4 , Γ_5 , and Γ_6 , the singularity where the central line is soft results in the same prelocalization. In both cases, one can easily reuse the building blocks of the Fitting construction. In total, only seven nontrivially distinct constructions arise. For the interested reader, we present results of the calculations of the $\mathcal{C}$ and $\mathcal{R}$ matrices for all graphs in Fig. 3. in the Supplemental Material [49]. We note that, in contrast to commonly used computational methods, no degree bound was used in our calculation. Therefore, our (critical) syzygy solutions are proven to be complete.

Discussion and outlook. In this work, we have studied Feynman integral reduction with syzygy methods. We applied advanced methods from commutative algebra to construct analytic generating sets of syzygies that are essential for efficient computation of Feynman integrals for collider physics. Our formulas are highly compact, and physically transparent: they are organized by the Landau

locus. Strikingly, this highlights how Feynman integral reduction is controlled by infrared singularities. Interestingly, we have observed that, once syzygies associated to a Landau locus component have been constructed, they can be applied to other diagrams. This suggests a universality of our syzygy solutions, which we expect can be applied to amplitude calculations for collider processes beyond those studied in this work.

Given our results, an important next step is to understand how to construct a minimal set of total derivatives suitable for integrand-level reduction [19,38]. Indeed, the compact nature of our results hints at the tantalizing possibility of constructing stable, floating-point reduction methods in which there has been renewed interest [38,53–55]. Altogether, we expect our approach to be of crucial importance for future calculations of LHC scattering amplitudes, enabling progress both beyond $2 \rightarrow 3$ kinematics and at higher loop orders.

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Data availability. There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

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- [1] K. G. Chetyrkin and F. V. Tkachov, Integration by parts: The algorithm to calculate β -functions in 4 loops, *Nucl. Phys. B* **192**, 159 (1981).
 - [2] A. V. Kotikov, Differential equations method: New technique for massive Feynman diagrams calculation, *Phys. Lett. B* **254**, 158 (1991).
 - [3] E. Remiddi, Differential equations for Feynman graph amplitudes, *Nuovo Cimento Soc. Ital. Fis.* **110A**, 1435 (1997).
 - [4] T. Gehrmann and E. Remiddi, Differential equations for two-loop four-point functions, *Nucl. Phys. B* **580**, 485 (2000).
 - [5] J. M. Henn, Multiloop integrals in dimensional regularization made simple, *Phys. Rev. Lett.* **110**, 251601 (2013).
 - [6] S. Laporta, High-precision calculation of multiloop Feynman integrals by difference equations, *Int. J. Mod. Phys. A* **15**, 5087 (2000).
 - [7] P. Maierhöfer, J. Usovitsch, and P. Uwer, KIRA—A Feynman integral reduction program, *Comput. Phys. Commun.* **230**, 99 (2018).
 - [8] J. Klappert, F. Lange, P. Maierhöfer, and J. Usovitsch, Integral reduction with KIRA 2.0 and finite field methods, *Comput. Phys. Commun.* **266**, 108024 (2021).

- [9] A. V. Smirnov, Algorithm FIRE—Feynman integral REDuction, *J. High Energy Phys.* **10** (2008) 107.
- [10] A. V. Smirnov and M. Zeng, FIRE 7: Automatic reduction with modular approach, [arXiv:2510.07150](https://arxiv.org/abs/2510.07150).
- [11] T. Peraro, FINITEFLOW: multivariate functional reconstruction using finite fields and dataflow graphs, *J. High Energy Phys.* **07** (2019) 031.
- [12] X. Guan, X. Liu, Y.-Q. Ma, and W.-H. Wu, BLADE: A package for block-triangular form improved Feynman integrals decomposition, *Comput. Phys. Commun.* **310**, 109538 (2025).
- [13] R. N. Lee, Presenting LITERED: A tool for the loop InTEgrals REDuction, [arXiv:1212.2685](https://arxiv.org/abs/1212.2685).
- [14] R. N. Lee, LITERED 1.4: A powerful tool for reduction of multiloop integrals, *J. Phys. Conf. Ser.* **523**, 012059 (2014).
- [15] Z. Wu, J. Boehm, R. Ma, H. Xu, and Y. Zhang, NEATIBP 1.0, a package generating small-size integration-by-parts relations for Feynman integrals, *Comput. Phys. Commun.* **295**, 108999 (2024).
- [16] Z. Wu, J. Böhm, R. Ma, J. Usovitsch, Y. Xu, and Y. Zhang, Performing integration-by-parts reductions using NEATIBP 1.1 + KIRA, *Comput. Phys. Commun.* **316**, 109798 (2025).
- [17] J. Gluza, K. Kajda, and D. A. Kosower, Towards a basis for planar two-loop integrals, *Phys. Rev. D* **83**, 045012 (2011).
- [18] K. J. Larsen and Y. Zhang, Integration-by-parts reductions from unitarity cuts and algebraic geometry, *Phys. Rev. D* **93**, 041701 (2016).
- [19] H. Ita, Two-loop integrand decomposition into master integrals and surface terms, *Phys. Rev. D* **94**, 116015 (2016).
- [20] A. von Manteuffel and R. M. Schabinger, A novel approach to integration by parts reduction, *Phys. Lett. B* **744**, 101 (2015).
- [21] T. Peraro, Scattering amplitudes over finite fields and multivariate functional reconstruction, *J. High Energy Phys.* **12** (2016) 030.
- [22] S. Mizera, Scattering amplitudes from intersection theory, *Phys. Rev. Lett.* **120**, 141602 (2018).
- [23] P. Mastrolia and S. Mizera, Feynman integrals and intersection theory, *J. High Energy Phys.* **02** (2019) 139.
- [24] S. Mizera and A. Pokraka, From infinity to four dimensions: Higher residue pairings and Feynman integrals, *J. High Energy Phys.* **02** (2020) 159.
- [25] H. Frellesvig, F. Gasparotto, S. Laporta, M. K. Mandal, P. Mastrolia, L. Mattiazzi, and S. Mizera, Decomposition of Feynman integrals by multivariate intersection numbers, *J. High Energy Phys.* **03** (2021) 027.
- [26] S. Caron-Huot and A. Pokraka, Duals of Feynman integrals. Part I. Differential equations, *J. High Energy Phys.* **12** (2021) 045.
- [27] S. Caron-Huot and A. Pokraka, Duals of Feynman integrals. Part II. Generalized unitarity, *J. High Energy Phys.* **04** (2022) 078.
- [28] L. D. Landau, On the analytic properties of vertex parts in quantum field theory, *Zh. Eksp. Teor. Fiz.* **37**, 62 (1960) [*J. Exp. Theor. Phys.* **13**, 181 (1959)].
- [29] M. Helmer and F. Tellander, Spectral decomposition of Euler-Mellin integrals, [arXiv:2505.12458](https://arxiv.org/abs/2505.12458).
- [30] M. Helmer and F. Tellander, Geometric singularities of Feynman integrals, *Phys. Rev. D* **112**, 065019 (2025).
- [31] S. Mizera and S. Telen, Landau discriminants, *J. High Energy Phys.* **08** (2022) 200.
- [32] C. Fevola, S. Mizera, and S. Telen, Landau singularities revisited: Computational algebraic geometry for Feynman integrals, *Phys. Rev. Lett.* **132**, 101601 (2024).
- [33] C. Fevola, S. Mizera, and S. Telen, Principal Landau determinants, *Comput. Phys. Commun.* **303**, 109278 (2024).
- [34] M. Correia, M. Giroux, and S. Mizera, SOFIA: Singularities of Feynman integrals automatized, *Comput. Phys. Commun.* **320**, 109970 (2026).
- [35] M. Helmer, G. Papathanasiou, and F. Tellander, Landau singularities from Whitney stratifications, *Phys. Rev. D* **114**, L011901 (2026).
- [36] C. Dlapa, M. Helmer, G. Papathanasiou, and F. Tellander, Symbol alphabets from the Landau SINGULAR locus, *J. High Energy Phys.* **10** (2023) 161.
- [37] Z. Bern, M. Enciso, H. Ita, and M. Zeng, Dual conformal symmetry, integration-by-parts reduction, differential equations and the nonplanar sector, *Phys. Rev. D* **96**, 096017 (2017).
- [38] S. Abreu, F. Febres Cordero, H. Ita, M. Jaquier, B. Page, and M. Zeng, Two-loop four-gluon amplitudes from numerical unitarity, *Phys. Rev. Lett.* **119**, 142001 (2017).
- [39] Y. Zhang, Lecture notes on multi-loop integral reduction and applied algebraic geometry, [arXiv:1612.02249](https://arxiv.org/abs/1612.02249).
- [40] J. Böhm, A. Georgoudis, K. J. Larsen, M. Schulze, and Y. Zhang, Complete sets of logarithmic vector fields for integration-by-parts identities of Feynman integrals, *Phys. Rev. D* **98**, 025023 (2018).
- [41] J. Bosma, K. J. Larsen, and Y. Zhang, Differential equations for loop integrals in Baikov representation, *Phys. Rev. D* **97**, 105014 (2018).
- [42] R. M. Schabinger, A new algorithm for the generation of unitarity-compatible integration by parts relations, *J. High Energy Phys.* **01** (2012) 077.
- [43] S. Abreu, G. De Laurentis, H. Ita, M. Klinkert, B. Page, and V. Sotnikov, Two-loop QCD corrections for three-photon production at hadron colliders, *SciPost Phys.* **15**, 157 (2023).
- [44] B. Agarwal, S. P. Jones, and A. von Manteuffel, Two-loop helicity amplitudes for $gg \rightarrow ZZ$ with full top-quark mass effects, *J. High Energy Phys.* **05** (2021) 256.
- [45] D. Bendle, J. Böhm, W. Decker, A. Georgoudis, F.-J. Pfreundt, M. Rahn, P. Wasser, and Y. Zhang, Integration-by-parts reductions of Feynman integrals using SINGULAR and GPI-SPACE, *J. High Energy Phys.* **02** (2020) 079.
- [46] B. Page and Q. Song, Critical points and syzygies for Feynman integrals, *J. High Energy Phys.* **02** (2026) 004.
- [47] M. F. Atiyah and I. G. Macdonald, *Introduction to Commutative Algebra* (Addison-Wesley Publishing Company, Reading, MA, 1969).
- [48] D. Eisenbud, *Commutative Algebra: With a View Toward Algebraic Geometry* (Springer Science & Business Media, New York, 1995), [10.1007/978-1-4612-5350-1](https://doi.org/10.1007/978-1-4612-5350-1).
- [49] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/fbmy-g7tk> for the proof of our main result, remaining examples, and explicit expressions for the generators of the prelocalization ideals and the $\mathcal{R}$ matrices.

- [50] P. A. Baikov, Explicit solutions of the three loop vacuum integral recurrence relations, *Phys. Lett. B* **385**, 404 (1996).
- [51] S. Caron-Huot, M. Correia, and M. Giroux, Recursive landau analysis, *Phys. Rev. Lett.* **135**, 131603 (2025).
- [52] W. Decker, G.-M. Greuel, G. Pfister, and H. Schönemann, SINGULAR 4-2-1—A computer algebra system for polynomial computations, <http://www.singular.uni-kl.de> (2021).
- [53] S. Abreu, J. Dormans, F. Febres Cordero, H. Ita, M. Kraus, B. Page, E. Pascual, M. S. Ruf, and V. Sotnikov, CARAVEL: A c++ framework for the computation of multi-loop amplitudes with numerical unitarity, *Comput. Phys. Commun.* **267**, 108069 (2021).
- [54] S. Pozzorini, N. Schär, and M. F. Zoller, Two-loop tensor integral coefficients in OPENLOOPS, *J. High Energy Phys.* **05** (2022) 161.
- [55] G. Bevilacqua, D. Canko, C.G. Papadopoulos, and A. Spourdalakis, Towards numerical two-loop integrand reduction, *J. High Energy Phys.* **11** (2025) 069.