

Differential equations for Wilson loops in ABJM theory

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We derive a system of differential equations which are satisfied by the vacuum expectation values (vevs) of BPS Wilson loops and 't Hooft coupling of ABJM theory. They are Picard-Fuchs equations of an algebraic curve defined by the derivative of the planar resolvent of the corresponding matrix model. The weak and strong coupling behaviors can be reproduced by their local solutions around regular singularities. We also obtain a recursion relation which can be used to determine the planar vevs of BPS Wilson loops in arbitrary representations.

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I. INTRODUCTION

ABJM theory was proposed in [1] as an example of the boundary CFT for AdS/CFT correspondence [2]. This is an $\mathcal{N} = 6$ superconformal Chern-Simons theory coupled to matter fields in three dimensions. The gauge group is $U(N) \times U(N)$, and the levels of the first and the second unitary factors are k and $-k$, respectively. The theory can be extended to the one with the gauge group $U(M) \times U(N)$. This is often called ABJ theory, and it is extensively studied in [3]. Note that the Lagrangian of ABJ theory had already appeared previously in [4], based on the construction of [5].

Soon after the appearance of ABJM theory, Wilson loops which preserves some amount of supersymmetry were constructed. The first example of the BPS Wilson loops preserve 1/6 of the supersymmetry [6–8], which is a natural extension of the 1/2 BPS Wilson loop [9,10] in $\mathcal{N} = 4$ super-Yang-Mills theory in four dimensions. The perturbative calculation of its vacuum expectation value (vev) in the 't Hooft coupling $\lambda := N/k$ was also performed, resulting in

$$W = 1 + \frac{5}{6}\pi^2\lambda^2 + \mathcal{O}(\lambda^4). \quad (1.1)$$

This calculation was extended to higher orders in [11]. The second example is the 1/2 BPS Wilson loop constructed in [12]. This is constructed using also fermionic fields. The structure is more complicated than the 1/6 BPS Wilson loops, and it is understood in terms of a supergroup [12,13].

There is a review [14] on Wilson loops in theories including ABJM theory.

The investigation of ABJM theory was accelerated by the localization formulas obtained in [15]. The resulting expression for the partition function resembles the one for matrix models, and therefore the matrix model techniques were applied to ABJM theory [16–18]. Remarkably, it was noticed in [17] that there is a relation between the localized partition function and a topological string theory, through a matrix model of [19], which is then used to analyze the strong coupling limit of the BPS Wilson loops and the free energy [17,18]. The leading behavior of Wilson loops for large λ turns out to be

$$W \sim e^{\pi\sqrt{2\lambda}}, \quad (1.2)$$

which reproduces the dual gravity calculation [6–8] for the 1/2 BPS case.¹ The 't Hooft coupling λ and the free energy are determined as a solution of the Picard-Fuchs equations related to the background geometry of the topological string theory.

There is another powerful machinery for analyzing ABJM theory and some other Chern-Simons-matter theories, known as the Fermi gas formalism, introduced in [21]. This machinery is so powerful that even a nonperturbative structure of the theory can be analyzed precisely. The remarkable results on the free energy of ABJM theory are summarized in the reviews [22,23]. The Fermi gas formalism can also be applied to the BPS Wilson loops [24,25].

The relation between ABJM theory and the topological string theory has achieved remarkable success so far, so it is

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¹The gravity dual of the 1/6 BPS Wilson loops were conjectured [6,8] to be a smeared string which gives the same behavior as (1.2). Later, a 1/6 BPS classical string solution was constructed in [20], but it turned out that it corresponds to a different Wilson loop operator.

naturally expected that similar relations might also be available in other Chern-Simons-matter theories. Recently, it was conjectured in [26] that topological string theories can be related systematically to some quantum mechanical systems including ones appearing in the Fermi gas formalism. This is called TS/ST correspondence. See [27] for a review. However, it seems that the existence of a general relation to Chern-Simons-matter theories is still unclear.

In this paper, we derive a system of differential equations satisfied by the BPS Wilson loops and the 't Hooft coupling of ABJM theory. This is achieved by considering an algebraic curve defined by the derivative of the resolvent of a matrix model for ABJM theory. It was observed in [28] that the derivative of the resolvent is much simpler than the resolvent itself. Note that the curve in this paper is different from the one discussed in [17]. The Wilson loops and the 't Hooft coupling can be written as integrals along a closed contour on the curve. This enables us to obtain the Picard-Fuchs equations for them. We analyze the solution of the equations for a weak and strong coupling limit, and reproduce the known results. As a by-product, we find a recursion relation for quantities defined as in (3.16) below, which can be used to obtain vevs of BPS Wilson loops in higher representations in the planar limit. Note that Picard-Fuchs equations are also employed in the context of Seiberg-Witten theory. See for example [29].

Our analysis is confined in the planar limit, so we cannot add any new findings to the existing results which are far more precise and deep, like the ones in [22,23]. Instead, our aim is to explore Chern-Simons-matter theories as general as possible. Since our derivation of the Picard-Fuchs equations does not rely on topological string theory, we expect that it can be applied to more general theories even though there is no known relation to string theory. The investigation on general Chern-Simons-matter theories would provide more insights into the presence/absence of the gravity dual, for example, which would characterize the family of theories with gravity duals. The shape of a thing is determined by its boundary.

This paper is organized as follows. In Sec. II, to illustrate our analysis, we consider pure Chern-Simons theory. For this theory, the vev of the Wilson loop is known exactly [30]. After a short summary on the matrix model of pure Chern-Simons theory in Sec. II A, we derive the Picard-Fuchs equations in Sec. II B. The equation for the Wilson loop can be solved easily. The solution is a function of a parameter which is related to the 't Hooft coupling given by solving another Picard-Fuchs equation. In Sec. II C, we also obtain the recursion relation mentioned above, whose solution is given as a generating function. Then, we analyze ABJM theory in Sec. III. The derivation of the Picard-Fuchs equations for the Wilson loops and the 't Hooft coupling for ABJM theory in Sec. III A is parallel to the one for pure Chern-Simons theory, but it is more complicated. As a result, we find that the Wilson loops and

the 't Hooft coupling are given in terms of solutions of (inhomogeneous) Heun equations. The local solutions around regular singularities of the equations reproduce the known weak and strong coupling expansions of the Wilson loop. The recursion relation for ABJM theory is obtained in Sec. III B which turns out to be a rather simple 5-term relation. Section IV is devoted to discussion. In Appendix A, we summarize the derivation of the curves, on which our analysis is based, purely within the matrix model context. In Appendix B, we estimate the behavior of the 't Hooft coupling in the parameter region corresponding to the strong coupling.

II. PURE CHERN-SIMONS THEORY

The quantum field theory for the Chern-Simons action with a non-Abelian gauge group was defined in [30]. We call this pure Chern-Simons theory. This is a three-dimensional gauge theory defined on a manifold M . This theory is independent of the metric on M , and therefore it is a topological field theory. Indeed, the Chern-Simons action is written without using the metric. A metric dependence may arise from the gauge fixing, but it was shown in [30] that it can be canceled by a counterterm.

The partition function of pure Chern-Simons theory can be calculated exactly by using the topological properties of the theory. The important observables of this theory are the Wilson loops. The correlation functions of the Wilson loops can also be calculated, and they give topological invariants of knots and links in M . For example, when the gauge group is $SU(N)$ and the level is k , the vev of the Wilson loop in the fundamental representation for an unknotted circle in S^3 is given as

$$\langle W \rangle = \frac{q^{N/2} - q^{-N/2}}{q^{1/2} - q^{-1/2}}, \quad q := e^{2\pi i/(k+N)}. \quad (2.1)$$

It was found in [31] that the exact expression for the partition function can be rewritten as a finite-dimensional integral. For the case when the gauge group is $U(N)$ and the manifold M is S^3 , the resulting integral expression is proportional to (2.2) below, up to a shift of the level k due to quantum corrections. The integral resembles the partition function of a Hermitian matrix model, but the interaction between the eigenvalues is more complicated.

The matrix model of [31] was also derived in the context of string theory in [32]. In this derivation, pure Chern-Simons theory is realized on D-branes in the topological A-model which wraps S^3 in the deformed conifold [33]. By mirror symmetry, this corresponds to D-branes in the topological B-model which wraps $\mathbb{P}^1$ in the mirror geometry. According to [34], the world-volume theory on the D-branes is a matrix model which turns out to coincide with the one of [31]. The planar resolvent of the matrix model was obtained in [32] which reproduces the mirror geometry mentioned above via the large N duality [35]. The free

energy of the matrix model is determined explicitly by solving the Picard-Fuchs equations for the geometry obtained from the planar resolvent.

It is now known that the matrix model of [31] can be obtained within the context of quantum field theory by using the supersymmetric localization [15]. By this method, the vevs of BPS Wilson loops are also given as finite-dimensional integrals, in addition to the partition function. Note that there is no shift of the level k in the formulas in [15]. It is because the theory considered in [15] is actually an $\mathcal{N} = 2$ version of pure Chern-Simons theory in which quantum corrections to the level cancel among them [36].

A. Planar limit

Consider $\mathcal{N} = 2$ pure Chern-Simons theory on S^3 with the gauge group $U(N)$ and the level k . As reviewed above, the partition function is reduced to

$$Z = \int du \exp \left(\sum_{i=1}^N \frac{ik}{4\pi} u_i^2 \right) \prod_{i < j} \sinh^2 \frac{u_i - u_j}{2}. \quad (2.2)$$

The integral in the right-hand side is a function of k defined on the upper-half plane. The partition function is then defined by its boundary value.

We focus our attention on the planar limit of the theory defined as

$$N, k \rightarrow \infty, \quad t := \frac{2\pi i N}{k} \text{ fixed}. \quad (2.3)$$

In the following, we call t the 't Hooft coupling, instead of λ defined above. In this limit, the saddle-point approximation becomes exact. Then, various observables of the theory are determined by an appropriate solution of the saddle-point equations

$$\frac{k}{2\pi i} u_i = \sum_{j \neq i} \coth \frac{u_i - u_j}{2}. \quad (2.4)$$

Let $\{\bar{u}_i\}$ be a solution. The definition (2.2) of the integral shows that $\bar{u}_i$ will naturally distribute around $u_i = 0$ when k is large. This is indeed the relevant saddle point which correctly reproduces known results of pure Chern-Simons theory. Other solutions of the saddle-point equations are discussed in [37,38].

We are interested in the vev of the BPS Wilson loop in the fundamental representation. In the planar limit, it is given in terms of $\bar{u}_i$ as

$$W = \frac{1}{N} \sum_{i=1}^N e^{\bar{u}_i}. \quad (2.5)$$

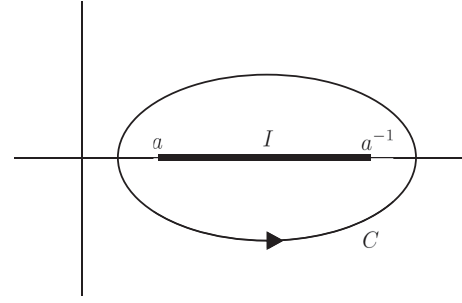


FIG. 1. The resolvent $v(z)$ has a branch cut on the line segment I whose endpoints are at $z = a$ and $z = a^{-1}$. The integration contour C encircles I counterclockwise. The value a can be complex, so that I does not necessarily lie on the real axis in general.

We always assume that the planar limit is implicitly taken for expressions like this one.

The information of the theory in the planar limit is encoded into the resolvent

$$v(z) := \frac{t}{N} \sum_{i=1}^N \frac{z + z_i}{z - z_i}, \quad z_i := e^{\bar{u}_i}. \quad (2.6)$$

This has a branch cut on a line segment I on which z_i condense. By the symmetry of the saddle-point equations (2.4), this satisfies $v(z^{-1}) = -v(z)$. The endpoints of I are therefore at $z = a, a^{-1}$ for some $a \in \mathbb{C}$. We assume $|a| \leq 1$. See Fig. 1.

The resolvent is determined explicitly in [32]. In our notation, it is given as

$$v(z) = 2 \log \frac{z + 1 - \sqrt{(z-a)(z-a^{-1})}}{\sqrt{a} + \sqrt{a^{-1}}}. \quad (2.7)$$

Physical observables can be written in terms of $v(z)$. The Wilson loop (2.5) is given as

$$W = \frac{1}{2t} \int_C \frac{dz}{2\pi i} v(z), \quad (2.8)$$

where C is a closed contour which encircles the branch cut I counterclockwise, as depicted in Fig. 1. The 't Hooft coupling t is also given as

$$t = \frac{1}{2} \int_C \frac{dz}{2\pi i} \frac{v(z)}{z}. \quad (2.9)$$

It turns out that the derivative of $v(z)$ has a simpler form [28],

$$z v'(z) - 1 = - \frac{z - 1}{\sqrt{(z-a)(z-a^{-1})}}. \quad (2.10)$$

This is a solution of an algebraic equation,

$$\omega^2 = \frac{(z-1)^2}{(z-a)(z-a^{-1})}. \quad (2.11)$$

Note that this equation can be derived directly from the saddle-point equations (2.4). See Appendix A. The solution $\omega(z)$ plays the same role as the resolvent $v(z)$. For example, the integration by parts in (2.8) gives

$$W = -\frac{1}{2t} \int_C \frac{dz}{2\pi i} \omega(z). \quad (2.12)$$

In the following, we also consider generalized quantities defined as

$$W_n := \frac{1}{N} \sum_{i=1}^N e^{n\tilde{u}_i}. \quad (2.13)$$

These can be used to obtain the vev of a BPS Wilson loop in a general representation in the planar limit [39]. These also have integral expressions as

$$W_n = -\frac{1}{2nt} \int_C \frac{dz}{2\pi i} z^{n-1} \omega(z). \quad (2.14)$$

The integration by parts in (2.9) gives

$$t = -\frac{1}{4\pi i} \int_C \frac{dz}{z} \log z \omega(z). \quad (2.15)$$

It turns out that the introduction of a variable

$$x := z + z^{-1} \quad (2.16)$$

is convenient for the following analysis. In terms of x , Eq. (2.11) becomes

$$y^2 = \frac{x-2}{x-\alpha}, \quad \alpha := a + a^{-1}. \quad (2.17)$$

The algebraic curve defined by this equation will play an important role in the following. The Wilson loop can be written as

$$W = -\frac{1}{4t} \int_{x(C)} \frac{dx}{2\pi i} y(x), \quad (2.18)$$

where $x(C)$ is the image on the curve of the contour C in the z plane. Rewriting W_n is not easy. Instead, we consider

$$\pi_n := \int_{x(C)} \frac{dx}{2\pi i} x^n y(x). \quad (2.19)$$

These are linear combinations of W_n . The expression of W_n in terms of π_n will be given later.

B. Picard-Fuchs equations

We observed that the Wilson loop W and the 't Hooft coupling t are given as integrals along a closed contour on an algebraic curve. This suggests that they may satisfy some differential equations, which we call Picard-Fuchs equations. We will derive these equations for W and t from the curve (2.17) and analyze their solutions. A different equation for t obtained in a string theory context is discussed in [32]. In the next subsection, we also determine W_n for general n using the properties of the curve (2.17).

1. Wilson loop

First, we derive a differential equation satisfied by

$$\pi_0 = \int_{x(C)} \frac{dx}{2\pi i} y(x). \quad (2.20)$$

This is a function of the parameter α defined in (2.17). This is related to the Wilson loop W as

$$\pi_0 = -4tW. \quad (2.21)$$

Note that t is also a function of α .

The derivative of π_0 with respect to α is given by $\partial_\alpha y$. The defining equation (2.17) of the curve gives a relation

$$2\partial_\alpha y = \frac{y}{x-\alpha}. \quad (2.22)$$

The integration of both sides gives

$$2\partial_\alpha \pi_0 = \int_{x(C)} \frac{dx}{2\pi i} \frac{y}{x-\alpha}. \quad (2.23)$$

In order to obtain a closed equation for π_0 , we need to rewrite the right-hand side in term of π_0 .

For this purpose, note that Eq. (2.17) also gives

$$2\partial_x y = \frac{y}{x-2} - \frac{y}{x-\alpha}, \quad (2.24)$$

$$2\partial_x(xy) = 2y + 2\frac{y}{x-2} - \alpha\frac{y}{x-\alpha}. \quad (2.25)$$

These relations then give

$$(\alpha-2)\frac{y}{x-\alpha} = 4\partial_x y - 2\partial_x(xy) + 2y. \quad (2.26)$$

Since the integration contour $x(C)$ is closed, the terms $\partial_x y$ and $\partial_x(xy)$ are irrelevant after integration. Finally, we obtain

$$(\alpha-2)\partial_\alpha \pi_0 - \pi_0 = 0. \quad (2.27)$$

2. 't Hooft coupling

Next, we derive another differential equation satisfied by the 't Hooft coupling t . Recall the integral expression

$$t = -\frac{1}{4\pi i} \int_C \frac{dz}{z} \log z y(z + z^{-1}). \quad (2.28)$$

From the relation (2.22), we obtain

$$2\partial_\alpha t = -\frac{1}{4\pi i} \int_C \frac{dz}{z} \log z \frac{y}{x - \alpha}. \quad (2.29)$$

We will rewrite the integrand $\log zy/(x - \alpha)$ up to terms of the form $z\partial_z(\dots)$ which do not contribute to $\partial_\alpha t$.

For this purpose, we start with

$$2\partial_x y = \frac{y}{x - 2} - \frac{y}{x - \alpha} \quad (2.30)$$

which we already used above. Recalling $x = z + z^{-1}$, we notice that

$$2(z - z^{-1})z\partial_z y = -(\alpha - 2)y - (\alpha^2 - 4)\frac{y}{x - \alpha} \quad (2.31)$$

holds. We use this to obtain the desired relation

$$\begin{aligned} 2z\partial_z[(z - z^{-1})\log zy] &= 2\log zxy - (\alpha - 2)\log zy \\ &\quad + 2(z - z^{-1})y \\ &\quad - (\alpha^2 - 4)\log z \frac{y}{x - \alpha}. \end{aligned} \quad (2.32)$$

The integration of both sides gives

$$-\frac{\alpha^2 - 4}{4\pi i} \int_C \frac{dz}{z} \log z \frac{y}{x - \alpha} = 2\tau_1 - (\alpha - 2)t - \pi_0, \quad (2.33)$$

where we have defined

$$\tau_1 := -\frac{1}{4\pi i} \int_C \frac{dz}{z} \log zxy(z + z^{-1}). \quad (2.34)$$

Then, we obtain a differential equation

$$2(\alpha^2 - 4)\partial_\alpha t = 2\tau_1 - (\alpha - 2)t - \pi_0, \quad (2.35)$$

which is, however, not closed for t , even when π_0 is known.

Interestingly, the unknown function τ_1 can be obtained from t as follows. The relation (2.22) also gives

$$2\partial_\alpha(xy) = y + \alpha \frac{y}{x - \alpha}. \quad (2.36)$$

The integration then gives

$$2\partial_\alpha \tau_1 = t + 2\alpha\partial_\alpha t. \quad (2.37)$$

This can be used to eliminate τ_1 from (2.35). As a result, we obtain

$$2(\alpha^2 - 4)\partial_\alpha^2 t + (3\alpha - 2)\partial_\alpha t = -\partial_\alpha \pi_0. \quad (2.38)$$

This is an inhomogeneous equation with the source term given by π_0 .

3. Solution of the equations

Recall that π_0 satisfies

$$(\alpha - 2)\partial_\alpha \pi_0 - \pi_0 = 0. \quad (2.39)$$

The solution is simply

$$\pi_0 = A(\alpha - 2). \quad (2.40)$$

The constant A will be determined shortly. Then t satisfies

$$2(\alpha^2 - 4)\partial_\alpha^2 t + (3\alpha - 2)\partial_\alpha t = -\partial_\alpha \pi_0 = -A. \quad (2.41)$$

This is first order in $\partial_\alpha t$, so this can be solved easily. We obtain

$$t = -A \log(\alpha + 2) + B \arctan \frac{\sqrt{\alpha - 2}}{2} + \log C. \quad (2.42)$$

We determine these constants as follows. Suppose that k is large, corresponding to small t . The definition (2.2) of the partition function implies that the saddle-point solution $\bar{u}_i$ become close to zero. Then, the branch cut of the resolvent $v(z)$ becomes small, which is realized by the limit $a \rightarrow 1$ or $\alpha \rightarrow 2$. The relation (2.21) implies that π_0 should behave as

$$\pi_0 \sim -4t \quad (2.43)$$

for small t . Therefore, we should have $t \sim -A(\alpha - 2)/4$ as the leading term. This information tells us that B must vanish and C is determined such that t vanishes at $\alpha = 2$. As a result, we obtain

$$t = -A \log \frac{\alpha + 2}{4}. \quad (2.44)$$

In order to fix the remaining constant A , we need an input from the curve (2.17). Since the branch cut of $y(z + z^{-1})$ shrinks to a point in the limit $a \rightarrow 1$, the contour integral (2.28) for t only depends on the local behavior of the curve around $z = 1$. This is given as

$$y(z) \sim -1 - \frac{1}{2(z - 1)^2}(\alpha - 2) \quad (2.45)$$

for $\alpha \sim 2$. This implies that the leading term of t is

$$t \sim \frac{1}{4}(\alpha - 2), \quad (2.46)$$

which then implies $A = -1$. In summary, we have found

$$\pi_0 = 2 - \alpha, \quad t = \log \frac{\alpha + 2}{4}. \quad (2.47)$$

As a check, let us compare them with the exact result [15]

$$W = \frac{1}{N} e^{\pi i N/k} \frac{\sin \pi N/k}{\sin \pi/k}. \quad (2.48)$$

Note that this is slightly different from (2.1) due to the absent of the shift of the level k , the inclusion of the framing factor, and the different normalization of the Wilson loop in [15,30]. By eliminating α , we obtain

$$W = \frac{e^t - 1}{t} \quad (2.49)$$

for the Wilson loop. This correctly reproduces the planar limit of (2.48).

C. Recursion relation

Let us reconsider the derivation of the Picard-Fuchs equation for π_0 . We started with

$$2\partial_\alpha y = \frac{y}{x - \alpha}. \quad (2.50)$$

We can parametrize x and y near $x = \alpha$ on the curve (2.17) as

$$x \sim \alpha + \xi^2, \quad y \sim \frac{\sqrt{\alpha - 2}}{\xi}. \quad (2.51)$$

In terms of the local coordinate ξ , we find that $\partial_\alpha y$ has a triple pole at $\xi = 0$. It is holomorphic elsewhere. Similarly,

$$2\partial_x y = \frac{y}{x - 2} - \frac{y}{x - \alpha} \quad (2.52)$$

shows that $\partial_x y$ also has a triple pole at $x = \alpha$. The parametrization around $x = 2$,

$$x \sim 2 + \eta^2, \quad y \sim \frac{\eta}{\sqrt{2 - \alpha}}, \quad (2.53)$$

implies that $\partial_x y$ has a simple pole at $\eta = 0$. Otherwise it is holomorphic.

Let V be a vector space of meromorphic functions on the curve (2.17), each member of which has at most a triple pole at $x = \alpha$ and at most a simple pole at $x = 2$, and is holomorphic elsewhere. It is known that the dimension of V is finite. The multiplication by x does not increase the

orders of poles at $x = 2, \alpha$. This operation may produce poles at $x = \infty$. After subtracting the poles at infinity, the resulting function is again a member of V . For example,

$$2x\partial_x y = 2\frac{y}{x-2} - \alpha\frac{y}{x-\alpha} \in V. \quad (2.54)$$

If there are more than $\dim V$ functions in V , then they are linearly dependent.

Our situation is much simpler. We find that three functions $\partial_\alpha y$, $\partial_x y$, $\partial_x(xy) - y$ form a two-dimensional subspace V_2 of V for which

$$\frac{y}{x-2}, \quad \frac{y}{x-\alpha} \quad (2.55)$$

is a basis. Therefore, there must exist a nontrivial relation among them, namely

$$(\alpha - 2)\partial_\alpha y = 2\partial_x y - (\partial_x(xy) - y), \quad (2.56)$$

from which we can derive the Picard-Fuchs equation (2.27).

In a similar manner, we can obtain an infinite number of nontrivial relations. For example, we find that

$$2\partial_x(x^n y) = S_n + 2^n \frac{y}{x-2} - \alpha^n \frac{y}{x-\alpha} \quad (2.57)$$

holds, where

$$\begin{aligned} S_n &:= 2nx^{n-1}y + \frac{x^n - 2^n}{x-2}y - \frac{x^n - \alpha^n}{x-\alpha}y \\ &= 2nx^{n-1}y + \sum_{k=0}^{n-1} (2^{n-1-k} - \alpha^{n-1-k})x^k y. \end{aligned} \quad (2.58)$$

All of the functions $\partial_x(x^n y) - S_n$ therefore belong to V_2 , so any three of them are linearly dependent. We can use these relations to obtain a recursion relation which determines π_n for general n as follows.

The relation (2.57) can be organized in the form

$$\begin{bmatrix} \partial_x(x^n y) - S_n \\ \partial_x(x^{n+1} y) - S_{n+1} \end{bmatrix} = \begin{bmatrix} 2^n & \alpha^n \\ 2^{n+1} & \alpha^{n+1} \end{bmatrix} \begin{bmatrix} \frac{y}{x-2} \\ \frac{-y}{x-\alpha} \end{bmatrix}. \quad (2.59)$$

Using an identity

$$(-2\alpha, \alpha + 2) \begin{bmatrix} 2^n & \alpha^n \\ 2^{n+1} & \alpha^{n+1} \end{bmatrix} = (2^{n+2}, \alpha^{n+2}), \quad (2.60)$$

we find

$$2\alpha S_n - (\alpha + 2)S_{n+1} + S_{n+2} = \partial_x(\cdots). \quad (2.61)$$

The integration of both sides gives a three-term recursion relation

$$4\alpha n\pi_{n-1} - (2(\alpha+2)(n+1) + \alpha - 2)\pi_n + 2(n+2)\pi_{n+1} = 0. \quad (2.62)$$

We need to determine π_1 for this recursion relation to work. Note that the above calculation with $n = 0$ gives

$$-(\alpha+2)S_1 + S_2 = -2(\alpha+2)y + 4xy + (2-\alpha)y = \partial_x(\dots). \quad (2.63)$$

This implies

$$\pi_1 = \frac{1}{4}(3\alpha+2)\pi_0 = \frac{1}{4}(3\alpha+2)(2-\alpha). \quad (2.64)$$

Then, the recursion relation gives

$$\pi_2 = -\frac{1}{6}(4\alpha\pi_0 - (5\alpha+6)\pi_1) = -\frac{1}{8}(\alpha-2)(5\alpha^2+4\alpha+4) \quad (2.65)$$

and so on.

Remarkably, the generating function of π_n defined as

$$g(t) := \sum_{n=0}^{\infty} \pi_n t^{n+1} \quad (2.66)$$

can be determined explicitly. The recursion relation (2.62) implies that $g(t)$ satisfies

$$2(2t-1)(\alpha t-1)\partial_t g - (\alpha-2)g = -(3\alpha+2)t+2)\pi_0 + 4t\pi_1. \quad (2.67)$$

The solution satisfying $g(0) = 0$ is simply

$$g(t) = 2 - 2\sqrt{\frac{1-2t}{1-\alpha t}}. \quad (2.68)$$

This function is actually related to $y(x)$. By the definition of $g(t)$, we have

$$g(t) = \int_{x(C)} \frac{dx}{2\pi i} \frac{t}{1-tx} y(x) = 2 + 2y(t^{-1}). \quad (2.69)$$

This gives the expressions of W_n in terms of π_n . Recall that $y(x)$ is obtained from the resolvent $v(z)$ defined as in (2.6) which has the expansion

$$v(z) = t + 2 \sum_{n=1}^{\infty} t W_n z^{-n} \quad (2.70)$$

given in terms of W_n . Then, $y(x)$ is given as

$$2 + 2y(x) = -4 \sum_{n=1}^{\infty} n t W_n z^{-n}. \quad (2.71)$$

By comparing this expansion with

$$g(x^{-1}) = \sum_{n=0}^{\infty} \pi_n (z + z^{-1})^{n+1}, \quad (2.72)$$

we obtain the desired relations. For example, we obtain

$$W_1 = -\frac{1}{4t}\pi_0, \quad W_2 = -\frac{1}{8t}\pi_1, \quad W_3 = -\frac{1}{12t}(\pi_2 - \pi_0), \quad (2.73)$$

and so on. Note that the integral expressions (2.14) and (2.19) for W_n and π_n , respectively, are formally independent of the theory under consideration. Therefore, the same relation between W_n and π_n can be also used for ABJM theory.

In fact, the information on W_n can be obtained more easily and more explicitly for pure Chern-Simons theory. Recall

$$z v'(z) - 1 = -\frac{z-1}{\sqrt{(z-a)(z-a^{-1})}}. \quad (2.74)$$

Using the expansion (2.70), we find

$$t W_n = \frac{1}{2n} \left[P_n\left(\frac{\alpha}{2}\right) - P_{n-1}\left(\frac{\alpha}{2}\right) \right], \quad (2.75)$$

where $P_n(x)$ are Legendre polynomials.

In the next section, we investigate ABJM theory for which the Picard-Fuchs equations may give us results which are not easily deduced from the resolvent.

III. ABJM THEORY

In this section, we extend the investigation of Picard-Fuchs equations to ABJM theory. As a starting point, we consider a more general theory including ABJ theory and Gaiotto-Tomasiello (GT) theory [40]. The gauge group of the theory is $U(N_1) \times U(N_2)$, and the levels are k_1 and k_2 . ABJM theory corresponds to $N_1 = N_2$ and $k_1 = -k_2$.

The partition function is reduced to [15]

$$Z = \int d\mathbf{u} d\tilde{\mathbf{u}} \exp \left[\sum_{i=1}^{N_1} \frac{ik_1}{4\pi} (u_{1i})^2 + \sum_{j=1}^{N_2} \frac{ik_2}{4\pi} (u_{2j})^2 \right] \times \frac{\prod_{i < i'} \sinh^2 \frac{u_{1i} - u_{1i'}}{2} \prod_{j < j'} \sinh^2 \frac{u_{2j} - u_{2j'}}{2}}{\prod_{i=1}^{N_1} \prod_{j=1}^{N_2} \cosh^2 \frac{u_{1i} - u_{2j}}{2}}. \quad (3.1)$$

As for pure Chern-Simons theory, this integral is a function of k_1 and k_2 which is defined when both k_1 and k_2 are in the upper-half plane.

In order to define the planar limit, we introduce an auxiliary parameter k which is taken to be large. The parameters N_1, N_2, k_1, k_2 are assumed to be proportional to k . In other words, we take the large k limit while keeping the ratios

$$t_1 := \frac{2\pi i N_1}{k}, \quad t_2 := \frac{2\pi i N_2}{k}, \quad \kappa_1 := \frac{k_1}{k}, \quad \kappa_2 := \frac{k_2}{k} \quad (3.2)$$

fixed. For ABJM theory, we have $t_1 = t_2$. In the following, we choose k such that $\kappa_1 = -\kappa_2 = 1$ holds.

We are interested in the vevs of BPS Wilson loops. As reviewed in the Introduction, there are bosonic BPS Wilson loops [6–8] in the fundamental representation of $U(N_1)$ or $U(N_2)$. Their vevs are given in terms of the saddle-point $\{\bar{u}_{1i}, \bar{u}_{2j}\}$ of the integral (3.1) as

$$W_1 = \frac{1}{N_1} \sum_{i=1}^{N_1} e^{\bar{u}_{1i}}, \quad W_2 = \frac{1}{N_2} \sum_{j=1}^{N_2} e^{\bar{u}_{2j}}. \quad (3.3)$$

There is also a fermionic BPS Wilson loop [12]. The vev turns out to be given as $N_1 W_1 + N_2 W_2$. In the planar limit, it is more appropriate to consider

$$\mathcal{W} = t_1 W_1 + t_2 W_2. \quad (3.4)$$

In order to analyze these observables, we define the resolvents

$$v_1(z) := \frac{t_1}{N_1} \sum_{i=1}^{N_1} \frac{z + z_{1i}}{z - z_{1i}}, \quad z_{1i} := e^{\bar{u}_{1i}}, \quad (3.5)$$

$$v_2(z) := \frac{t_2}{N_2} \sum_{j=1}^{N_2} \frac{z + z_{2j}}{z - z_{2j}}, \quad z_{2j} := -e^{\bar{u}_{2j}}. \quad (3.6)$$

Interestingly, a combination $v(z) := v_1(z) - v_2(z)$ of them can be determined explicitly as [17,19]

$$v(z) = 2 \log \left[\frac{\sqrt{(z-b)(z-b^{-1})} - \sqrt{(z-a)(z-a^{-1})}}{\sqrt{a+a^{-1}-b-b^{-1}}} \right]. \quad (3.7)$$

This satisfies $v(z^{-1}) = -v(z)$. The function $v(z)$ has two branch cuts on the line segments I_1 and I_2 on which z_{1i} and z_{2j} , respectively, condense. See Fig. 2. We assume $|a|, |b| \leq 1$.

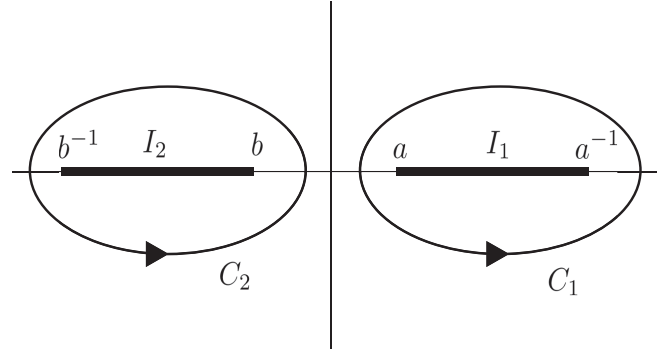


FIG. 2. The resolvent $v(z)$ has two branch cuts as shown in the figure. The integration contour C_1 encircles the cut I_1 counter-clockwise, and C_2 encircles I_2 similarly. The cuts do not necessarily lie on the real axis.

Instead of $v(z)$, it is easier to analyze its derivative

$$zv'(z) - 1 = -\frac{z^2 - 1}{\sqrt{(z-a)(z-a^{-1})(z-b)(z-b^{-1})}}. \quad (3.8)$$

As for pure Chern-Simons theory, we introduce the variable $x = z + z^{-1}$ and consider an algebraic curve defined by

$$y^2 = \frac{(x-2)(x+2)}{(x-\alpha)(x-\beta)}, \quad (3.9)$$

where $\alpha = a + a^{-1}$ and $\beta = b + b^{-1}$. The derivative $zv'(z)$ is then given in terms of $y(x)$ as

$$zv'(z) = 1 + y(z + z^{-1}). \quad (3.10)$$

A derivation of the curve from the saddle point of (3.1) is summarized in Appendix A.

The vevs of the BPS Wilson loops have integral representations

$$W_1 = -\frac{1}{4t_1} \int_{C_1} \frac{dx}{2\pi i} y(x), \quad W_2 = -\frac{1}{4t_2} \int_{C_2} \frac{dx}{2\pi i} y(x), \quad (3.11)$$

and

$$\mathcal{W} = -\frac{1}{4} \int_{C_1+C_2} \frac{dx}{2\pi i} y(x). \quad (3.12)$$

The integration contours C_1 and C_2 are depicted in Fig. 2. Note that these Wilson loops are given by essentially the same integral, and the different choice of the integration contour gives the different Wilson loop.

The integral representations for the 't Hooft couplings

$$\begin{aligned} t_1 &= -\frac{1}{4\pi i} \int_{C_1} \frac{dz}{z} \log z y(z + z^{-1}), \\ t_2 &= \frac{1}{4\pi i} \int_{C_2} \frac{dz}{z} \log z y(z + z^{-1}) \end{aligned} \quad (3.13)$$

can be used to relate the parameters α, β to t_1, t_2 .

In the following, we focus our attention to ABJM theory. Accordingly, we will restrict the parameter space $\mathbb{C}^2$ to a subspace, usually called the ABJM slice. This corresponds to $t_1 = t_2$. It turns out that $t_1 - t_2$ can be given simply as [17]

$$t_1 - t_2 = \frac{1}{2} \int_0^\infty \frac{dz}{z} (y(z + z^{-1}) + 1) = \log \frac{\alpha - \beta}{4}. \quad (3.14)$$

This implies that the ABJM slice is parametrized by $\gamma \in \mathbb{C}$ as

$$\alpha = 2 + \gamma, \quad \beta = -2 + \gamma. \quad (3.15)$$

In the following, we denote $t = t_1 = t_2$ for simplicity.

A. Picard-Fuchs equations

We derive the Picard-Fuchs equations for the Wilson loops and the 't Hooft coupling. As for pure Chern-Simons theory, we consider instead

$$\pi_n = \int_{x(C)} \frac{dx}{2\pi i} x^n y(x). \quad (3.16)$$

It turns out that π_n satisfy a system of differential equations in which π_n couple among them. The equations are independent of the choice of the integration contour C . As a result, the vevs of the bosonic and fermionic Wilson loops are solutions of the same equation. A geometric description of W_n which are linear combinations of π_n is given in [24].

1. A basis of functions

From (3.9), we obtain

$$2\partial_x y = \frac{y}{x-2} + \frac{y}{x+2} - \frac{y}{x-\alpha} - \frac{y}{x-\beta}. \quad (3.17)$$

Based on the experience in pure Chern-Simons theory, we should consider a four-dimensional subspace V_4 , spanned by

$$\frac{y}{x-2}, \quad \frac{y}{x+2}, \quad \frac{-y}{x-\alpha}, \quad \frac{-y}{x-\beta} \quad (3.18)$$

of the vector space of meromorphic functions on the curve (3.9). We would like to express these basis functions as linear combinations of terms of the form $x^n y$, up to total derivatives, like the expression (2.26) for pure Chern-Simons theory.

For this purpose, we use

$$\begin{aligned} 2\partial_x(x^n y) &= 2nx^{n-1}y + \frac{x^n - 2^n}{x-2}y + \frac{x^n - (-2)^n}{x+2}y - \frac{x^n - \alpha^n}{x-\alpha}y - \frac{x^n - \beta^n}{x-\beta}y \\ &\quad + 2^n \frac{y}{x-2} + (-2)^n \frac{y}{x+2} - \alpha^n \frac{y}{x-\alpha} - \beta^n \frac{y}{x-\beta}, \end{aligned} \quad (3.19)$$

which is derived from (3.17). Note that the first line in the right-hand side is

$$\frac{x^n - 2^n}{x-2}y + \frac{x^n - (-2)^n}{x+2}y - \frac{x^n - \alpha^n}{x-\alpha}y - \frac{x^n - \beta^n}{x-\beta}y = \sum_{k=0}^{n-1} (2^{n-1-k} + (-2)^{n-1-k} - \alpha^{n-1-k} - \beta^{n-1-k})x^k y. \quad (3.20)$$

We use four of these relations corresponding to $n = 0, 1, 2, 3$ to rewrite the basis functions (3.18). In order to organize these equations, we introduce vectors

$$Y := \begin{bmatrix} y \\ xy \\ x^2 y \\ x^3 y \end{bmatrix}, \quad B := \begin{bmatrix} \frac{y}{x-2} \\ \frac{y}{x+2} \\ \frac{-y}{x-\alpha} \\ \frac{-y}{x-\beta} \end{bmatrix}. \quad (3.21)$$

We find that they satisfy

$$2\partial_x Y = TY + UB, \quad (3.22)$$

where

$$T := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ -\alpha - \beta & 4 & 0 & 0 \\ 8 - \alpha^2 - \beta^2 & -\alpha - \beta & 6 & 0 \end{bmatrix}, \quad U := \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & -2 & \alpha & \beta \\ 2^2 & (-2)^2 & \alpha^2 & \beta^2 \\ 2^3 & (-2)^3 & \alpha^3 & \beta^3 \end{bmatrix}. \quad (3.23)$$

Then, the basis functions are given as

$$B = -U^{-1}TY + 2U^{-1}\partial_x Y. \quad (3.24)$$

Note that U is invertible if and only if the Vandermonde determinant $\det U$ is nonzero. When $\det U = 0$, the dimension of the vector space V_4 decreases, and the curve (3.9) degenerates. We will see that this degeneration corresponds to singularities of the resulting Picard-Fuchs equations.

Four of them corresponding to $n = 0, 1, 2, 3$ can be organized as

$$2\partial_\gamma Y = \tilde{T}Y + \tilde{U}B, \quad (3.28)$$

where

$$\tilde{T} := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ \alpha + \beta & 2 & 0 & 0 \\ \alpha^2 + \beta^2 & \alpha + \beta & 2 & 0 \end{bmatrix}, \quad \tilde{U} := \begin{bmatrix} 0 & 0 & -1 & -1 \\ 0 & 0 & -\alpha & -\beta \\ 0 & 0 & -\alpha^2 & -\beta^2 \\ 0 & 0 & -\alpha^3 & -\beta^3 \end{bmatrix}. \quad (3.29)$$

2. Wilson loops

We restrict our analysis on the ABJM slice

$$\alpha = 2 + \gamma, \quad \beta = -2 + \gamma. \quad (3.25)$$

The derivative of the Wilson loop W , or π_0 , with respect to γ is given by

$$2\partial_\gamma y = \frac{y}{x - \alpha} + \frac{y}{x - \beta}. \quad (3.26)$$

The relation (3.24) implies that π_0 couples to other π_n . Therefore, we also need to consider $\partial_\gamma \pi_n$, or $\partial_\gamma(x^n y)$ which are given as

$$2\partial_\gamma(x^n y) = \frac{x^n - \alpha^n}{x - \alpha} y + \frac{x^n - \beta^n}{x - \beta} y + \alpha^n \frac{y}{x - \alpha} + \beta^n \frac{y}{x - \beta}. \quad (3.27)$$

Combining this equation with (3.24), we obtain

$$2\partial_\gamma Y = \tilde{T}Y - \tilde{U}U^{-1}TY + 2\tilde{U}U^{-1}\partial_x Y. \quad (3.30)$$

The integration of both sides gives a system of differential equations for $\pi_0, \dots, \pi_3$. The last term in the right-hand side is irrelevant. We find

$$\tilde{T} - \tilde{U}U^{-1}T = \frac{2}{\gamma(\gamma - 4)(\gamma + 4)} \begin{bmatrix} 8 - \gamma^2 & 8\gamma & -6 & 0 \\ -4\gamma & -16 + 5\gamma^2 & -3\gamma & 0 \\ 32 - 4\gamma^2 & -16\gamma + 3\gamma^3 & -24 & 0 \\ 48\gamma - 4\gamma^3 & -64 + 20\gamma^2 & -76\gamma + 4\gamma^3 & 0 \end{bmatrix}. \quad (3.31)$$

This shows that π_0, π_1, π_2 decouple from π_3 .

After a slight modification, we obtain

$$\partial_\gamma \Pi = \frac{1}{\gamma(\gamma - 4)(\gamma + 4)} \mathcal{A}\Pi, \quad (3.32)$$

where

$$\Pi := \begin{bmatrix} \pi_0 \\ \pi_1 \\ \pi_2 - \gamma\pi_1 \end{bmatrix}, \quad \mathcal{A} := \begin{bmatrix} 8 - \gamma^2 & 2\gamma & -6 \\ -4\gamma & -16 + 2\gamma^2 & -3\gamma \\ 32 & -8\gamma & -24 + 3\gamma^2 \end{bmatrix}. \quad (3.33)$$

This is a Fuchsian system of differential equations with the regular singularities at $\gamma = 0, \pm 4, \infty$. The remaining π_3 is then given in terms of π_0, π_1, π_2 as

$$\partial_\gamma \pi_3 = \frac{1}{\gamma(\gamma-4)(\gamma+4)} [(48\gamma - 4\gamma^3)\pi_0 + (-64 + 20\gamma^2)\pi_1 + (-76\gamma + 4\gamma^3)\pi_2]. \quad (3.34)$$

Note that the regular singularities correspond to

$$\begin{aligned} \gamma = 0 & \Leftrightarrow \alpha = 2, & \beta = -2, \\ \gamma = 4 & \Leftrightarrow \alpha = 6, & \beta = 2, \\ \gamma = -4 & \Leftrightarrow \alpha = -2, & \beta = -6, \\ \gamma = \infty & \Leftrightarrow \alpha = \infty, & \beta = \infty. \end{aligned} \quad (3.35)$$

They correspond to various degenerations of the curve (3.9). These points on the ABJM slice are discussed in [17,18]. For example, $\gamma = 0$ corresponds to the weak coupling limit, while $\gamma = \infty$ describes the strong coupling limit or the gravity dual.

The Fuchsian system (3.32) can be reduced to a third-order differential equation for π_0 only, which is easier to deal with. As a result, we obtain

$$\gamma(\gamma-4)(\gamma+4)\partial_\gamma^3 \pi_0 + 2\gamma^2\partial_\gamma^2 \pi_0 - \gamma\partial_\gamma \pi_0 + \pi_0 = 0. \quad (3.36)$$

The solution of this equation then gives π_1 and π_2 as

$$\pi_1 = -\frac{1}{2} [\gamma(\gamma-4)(\gamma+4)\partial_\gamma^2 \pi_0 - 2\gamma\pi_0], \quad (3.37)$$

$$\begin{aligned} \pi_2 = \gamma\pi_1 - \frac{1}{6} [\gamma^2(\gamma-4)(\gamma+4)\partial_\gamma^2 \pi_0 \\ + \gamma(\gamma-4)(\gamma+4)\partial_\gamma \pi_0 - (\gamma^2 + 8)\pi_0]. \end{aligned} \quad (3.38)$$

We notice that $\pi_0 = \gamma$ is a solution of (3.36). This solution corresponds to the vev $\mathcal{W}$ of the fermionic BPS Wilson loop which is proportional to $\alpha + \beta$ [17]. This is actually expected since, as was mentioned in the beginning of this subsection, the vevs of both the bosonic and the fermionic Wilson loops are solutions of the same equation (3.36).

Using this simple solution, we can further reduce the equation (3.36). Let $\pi_0 = \gamma\tilde{\pi}_0$. Then, we find that $\pi := \partial_\gamma \tilde{\pi}_0$ satisfies

$$\gamma(\gamma-4)(\gamma+4)\partial_\gamma^2 \pi + (5\gamma^2 - 48)\partial_\gamma \pi + 3\gamma\pi = 0. \quad (3.39)$$

This is a Heun equation. For a review, see [41] for example. Indeed, by introducing new variables $\gamma = 4z$ and $\pi = 16y$, we obtain the standard form of the Heun equation

$$\partial_z^2 y + \left[\frac{3}{z} + \frac{1}{z-1} + \frac{1}{z+1} \right] \partial_z y + \frac{3z}{z(z-1)(z+1)} y = 0. \quad (3.40)$$

Let f and g be a basis of the space of solutions of (3.39). Then, π_0 is given by their integrals as

$$\pi_0 = A\gamma + B\gamma \int d\gamma f(\gamma) + C\gamma \int d\gamma g(\gamma) \quad (3.41)$$

with appropriate constants A, B, C .

3. 't Hooft coupling

Next, we derive the Picard-Fuchs equation for the 't Hooft coupling t . As for π_n , we need to introduce auxiliary quantities

$$\tau_n := -\frac{1}{4\pi i} \int_C \frac{dz}{z} \log z x^n y(x), \quad \tau_0 = t \quad (3.42)$$

in order to obtain a closed system of equations.

The derivation is similar to the one for pure Chern-Simons theory. The counterpart of (2.32) for ABJM theory turns out to be

$$\begin{aligned} 2z\partial_z [(z-z^{-1}) \log z y] \\ = 2 \log z xy - (\alpha + \beta) \log z y + 2(z-z^{-1})y \\ - (\alpha^2 - 4) \frac{\log z}{x-\alpha} y - (\beta^2 - 4) \frac{\log z}{x-\beta} y, \end{aligned} \quad (3.43)$$

$$\begin{aligned} 2z\partial_z [(z-z^{-1}) \log z xy] \\ = 4 \log z x^2 y - (\alpha + \beta) \log z xy \\ - (\alpha^2 + \beta^2) \log z y + 2(z-z^{-1})xy \\ - \alpha(\alpha^2 - 4) \frac{\log z}{x-\alpha} y - \beta(\beta^2 - 4) \frac{\log z}{x-\beta} y. \end{aligned} \quad (3.44)$$

Note that terms proportional to $y/(x \mp 2)$ do not appear in the right-hand side of the above relations, although they appear in $\partial_x y$. Therefore, these two equations are enough to rewrite the functions

$$\frac{\log z}{x-\alpha}y, \quad \frac{\log z}{x-\beta}y, \quad (3.45)$$

by terms whose integration give τ_n and π_n . Since there appear terms $\log zx^n y$ with $n = 0, 1, 2$, we need to consider the following three γ -derivatives:

$$2\partial_\gamma(\log zy) = \frac{\log z}{x-\alpha}y + \frac{\log z}{x-\beta}y, \quad (3.46)$$

$$2\partial_\gamma(\log zxy) = 2\log zy + \alpha\frac{\log z}{x-\alpha}y + \beta\frac{\log z}{x-\beta}y, \quad (3.47)$$

$$2\partial_\gamma(\log zx^2y) = 2\log zxy + (\alpha + \beta)\log zy + \alpha^2\frac{\log z}{x-\alpha}y + \beta^2\frac{\log z}{x-\beta}y, \quad (3.48)$$

for obtaining a closed set of equations. Eliminating the functions (3.45) from the right-hand side and integrating both sides, we obtain a system of differential equations for $\tau_0 = t, \tau_1, \tau_2$.

As a result, we find

$$\begin{aligned} \partial_\gamma \begin{bmatrix} \tau_0 \\ \tau_1 \\ \tau_2 \end{bmatrix} &= \frac{1}{\gamma(\gamma-4)(\gamma+4)} \begin{bmatrix} 8-\gamma^2 & 5\gamma & -4 \\ -4\gamma & -8+3\gamma^2 & -2\gamma \\ 32-4\gamma^2 & -12\gamma+2\gamma^3 & -16 \end{bmatrix} \begin{bmatrix} \tau_0 \\ \tau_1 \\ \tau_2 \end{bmatrix} \\ &+ \frac{1}{2\gamma(\gamma-4)(\gamma+4)} \begin{bmatrix} -3\gamma & 2 \\ 8-2\gamma^2 & \gamma \\ 4\gamma-\gamma^3 & 8 \end{bmatrix} \begin{bmatrix} \pi_0 \\ \pi_1 \end{bmatrix}. \end{aligned} \quad (3.49)$$

These equations can be reduced to

$$4\gamma^2(\gamma-4)(\gamma+4)\partial_\gamma^3\tau_0 + 16\gamma(\gamma-2)(\gamma+2)\partial_\gamma^2\tau_0 + 8(\gamma^2+2)\partial_\gamma\tau_0 = -(3\gamma^2+16)\partial_\gamma^2\pi_0 - \gamma\partial_\gamma\pi_0 + \pi_0. \quad (3.50)$$

This is an inhomogeneous Heun equation for $\partial_\gamma\tau_0$, and the source term is given by π_0 . Note that the right-hand side can be simplified as

$$-\gamma(3\gamma^2+16)\partial_\gamma\pi - (7\gamma^2+32)\pi. \quad (3.51)$$

For completeness, we give the formulas for τ_1 and τ_2 as

$$\begin{aligned} \tau_1 &= -2\gamma(\gamma-4)(\gamma+4)\partial_\gamma^2\tau_0 - (3\gamma^2-16)\partial_\gamma\tau_0 + \gamma\tau_0 \\ &- (\gamma-4)(\gamma+4)\partial_\gamma^2\pi_0 - 2\gamma\partial_\gamma\pi_0 + \frac{3}{2}\pi_0, \end{aligned} \quad (3.52)$$

$$\begin{aligned} \tau_2 &= -\frac{5}{2}\gamma^2(\gamma-4)(\gamma+4)\partial_\gamma^2\tau_0 - 4\gamma(\gamma^2-6)\partial_\gamma\tau_0 + (\gamma^2+2)\tau_0 \\ &- \frac{11}{8}\gamma(\gamma-4)(\gamma+4)\partial_\gamma^2\pi_0 - \frac{5}{2}\gamma^2\partial_\gamma\pi_0 + \frac{7}{4}\gamma\pi_0. \end{aligned} \quad (3.53)$$

The physical meaning of these quantities are not clear.

4. Local solutions of the equations

We investigate the local solutions of the equations

$$\gamma(\gamma-4)(\gamma+4)\partial_\gamma^2\pi + (5\gamma^2-48)\partial_\gamma\pi + 3\gamma\pi = 0 \quad (3.54)$$

and

$$4\gamma^2(\gamma-4)(\gamma+4)\partial_\gamma^3t + 16\gamma(\gamma-2)(\gamma+2)\partial_\gamma^2t + 8(\gamma^2+2)\partial_\gamma t = -\gamma(3\gamma^2+16)\partial_\gamma\pi - (7\gamma^2+32)\pi \quad (3.55)$$

around the regular singularities $\gamma = 0$ and $\gamma = \infty$, corresponding to the perturbative limit and the dual gravity limit, respectively. The other regular singularities $\gamma = \pm 4$ correspond to the ones investigated in [18]. They are related to pure Chern-Simons theory, which can be understood by comparing the curves (2.17) and (3.9).

Recall that π_0 is given in terms of a basis of solutions f and g of (3.54) as

$$\pi_0 = A\gamma + \gamma \int d\gamma\pi(\gamma), \quad \pi = Bf + Cg. \quad (3.56)$$

(i) $\gamma = 0$. First, we consider the local solution of (3.54) around $\gamma = 0$. The characteristic exponents at $\gamma = 0$ are 0 and -2 . The local solution for the exponent 0 is a power series given as

$$f_0 = 1 + \frac{3}{128}\gamma^2 + \frac{15}{16384}\gamma^4 + \mathcal{O}(\gamma^6). \quad (3.57)$$

The other solution for the exponent -2 has a logarithmic part. It is given as

$$g_0 = \gamma^{-2} \left(1 - \frac{7}{16384}\gamma^4 - \frac{21}{1048576}\gamma^6 \right) - \frac{1}{32}f_0 \log \gamma + \mathcal{O}(\gamma^6). \quad (3.58)$$

Then, π_0 is given as

$$\begin{aligned} \pi_0 &= A_0\gamma + B_0 \left(\gamma^2 + \frac{1}{128}\gamma^4 + \frac{3}{16384}\gamma^6 \right) + C_0 \left[-1 + \frac{1}{32}\gamma^2 - \frac{1}{16384}\gamma^4 - \frac{3}{1048576}\gamma^6 \right. \\ &\quad \left. - \frac{1}{32} \left(\gamma^2 + \frac{1}{128}\gamma^4 + \frac{3}{16384}\gamma^6 \right) \log \gamma \right] + \mathcal{O}(\gamma^8). \end{aligned} \quad (3.59)$$

Since π_0 is related to the Wilson loop as $\pi_0 = -4tW$, this should vanish at $\gamma = 0$. Therefore, we set $C_0 = 0$, which eliminates the logarithmic part. Then, we obtain

$$\pi_0 = A_0\gamma + B_0\gamma^2 + \frac{B_0}{128}\gamma^4 + \mathcal{O}(\gamma^6), \quad (3.60)$$

$$\pi_1 = 16B_0\gamma + A_0\gamma^2 + \frac{3B_0}{4}\gamma^3 + \mathcal{O}(\gamma^5), \quad (3.61)$$

$$\pi_2 = 4A_0\gamma + 28B_0\gamma^2 + A_0\gamma^3 + \frac{9B_0}{32}\gamma^4 + \mathcal{O}(\gamma^6). \quad (3.62)$$

From the definition of π_n , we should have

$$\pi_0 \sim -4t, \quad \pi_1 \sim -8t, \quad \pi_2 \sim -16t, \quad (3.63)$$

for small γ . This implies $B_0 = A_0/8$. The remaining constant A_0 will be determined shortly.

Next, we consider the second equation (3.55). We find that

$$t = A_0 \left(-\frac{1}{4}\gamma - \frac{1}{768}\gamma^3 - \frac{9}{327680}\gamma^5 \right) + \mathcal{O}(\gamma^7) \quad (3.64)$$

is a solution. To obtain the general solution of (3.55), we have to solve the following homogeneous equation:

$$4\gamma^2(\gamma - 4)(\gamma + 4)\partial_\gamma^2\tau + 16\gamma(\gamma - 2)(\gamma + 2)\partial_\gamma\tau + 8(\gamma^2 + 2)\tau = 0. \quad (3.65)$$

The integral of τ can contribute to t . The characteristic exponents of this equation are $\pm\frac{1}{2}$. The local solutions are

$$\tau_+ = \gamma^{\frac{1}{2}} \left(1 + \frac{5}{128}\gamma^2 + \frac{63}{32768}\gamma^4 \right) + \mathcal{O}(\gamma^{\frac{13}{2}}), \quad (3.66)$$

$$\tau_- = \gamma^{-\frac{1}{2}} \left(1 + \frac{3}{128}\gamma^2 + \frac{35}{32768}\gamma^4 \right) + \mathcal{O}(\gamma^{\frac{11}{2}}). \quad (3.67)$$

These are not holomorphic at $\gamma = 0$. If they would contribute to t , then the required behaviors (3.63) could not be satisfied. Therefore, (3.64) is the solution describing the perturbative limit. Note that there was also an additive integration constant which was fixed by requiring $t(0) = 0$.

The constant A_0 was not fixed by the above arguments. It is determined by the local behavior of the curve (3.9), as for pure Chern-Simons theory. We find that the behavior of $y(x)$ near $z = 1$ is

$$y(z + z^{-1}) \sim -1 - \frac{1}{2(z-1)^2}\gamma \quad (3.68)$$

for small γ . This gives $t \sim \gamma/4$, implying $A_0 = -1$.

In summary, we have obtained

$$\pi_0 = -\gamma - \frac{1}{8} \int d\gamma f_0(\gamma) = -\gamma - \frac{1}{8}\gamma^2 - \frac{1}{1024}\gamma^4 + \mathcal{O}(\gamma^6), \quad (3.69)$$

$$t = \frac{1}{4}\gamma + \frac{1}{768}\gamma^3 + \frac{9}{327680}\gamma^5 + \mathcal{O}(\gamma^7). \quad (3.70)$$

Indeed, this reproduces the known result [6–8]

$$\begin{aligned} W &= e^{t/2} \left(1 - \frac{5}{24}t^2 \right) + \mathcal{O}(t^3) \\ &= e^{\pi i \lambda} \left(1 + \frac{5}{6}\pi^2 \lambda^2 \right) + \mathcal{O}(\lambda^3), \quad \lambda = \frac{N}{k}, \end{aligned} \quad (3.71)$$

including the framing factor. A higher-order extension of [6–8] was obtained in [11].

(ii) $\gamma = \infty$. Next, we analyze the local solution around $\gamma = \infty$ which describes the gravity dual. For this purpose, we introduce a new variable $\xi := \gamma^{-1}$ and rewrite Eqs. (3.54) and (3.55) in terms of ξ . Then, the equations to be solved become

$$\xi^2(1 - 16\xi^2)\partial_\xi^2\pi + \xi(16\xi^2 - 3)\partial_\xi\pi + 3\pi = 0 \quad (3.72)$$

and

$$\begin{aligned} 4\xi^2(1 - 16\xi^2)\partial_\xi^2\tau - \xi(8 + 64\xi^2)\partial_\xi\tau + 8(1 + 2\xi^2)\tau \\ = \xi(3 + 16\xi^2)\partial_\xi\pi - (7 + 32\xi^2)\pi, \end{aligned} \quad (3.73)$$

where $\tau = \partial_\gamma t$.

We start with Eq. (3.72). The characteristic exponents at $\xi = 0$ turn out to be 1 and 3. The local solution for the exponent 3 is

$$f_\infty = \xi^3 + 6\xi^5 + 60\xi^7 + \mathcal{O}(\xi^9). \quad (3.74)$$

The other solution for the exponent 1 is

$$g_\infty = \xi - 28\xi^5 - 336\xi^7 - 8f_\infty \log \xi + \mathcal{O}(\xi^9). \quad (3.75)$$

Then, π_0 is given as

$$\begin{aligned} \pi_0 &= \frac{A_\infty}{\xi} + B_\infty \left(-\frac{1}{2}\xi - \frac{3}{2}\xi^3 - 10\xi^5 \right) + C_\infty \left[-\frac{\log \xi}{\xi} - 2\xi + 4\xi^3 + \frac{128}{3}\xi^5 \right. \\ &\quad \left. + 8 \left(\frac{1}{2}\xi + \frac{3}{2}\xi^3 + 10\xi^5 \right) \log \xi \right] + \mathcal{O}(\xi^7 \log \xi). \end{aligned} \quad (3.76)$$

Recall that we have determined the local behavior of π_0 around $\gamma = 0$ completely as (3.69). Since the connection formula for the Heun equation is known [42], the local solution (3.69) can be continued analytically to $\gamma = \infty$. This may fix the constants $A_\infty, B_\infty, C_\infty$ in (3.76). In the following, we instead determine the coefficients again by using the curve (3.9) as a substitute of the connection formula.

Next, we consider Eq. (3.73). The corresponding homogeneous equation for τ has the characteristic exponents 1 and 2. The homogeneous solution for the exponent 2 is

$$h(\xi) = \xi^2 + 10\xi^4 + 126\xi^6 + \mathcal{O}(\xi^8). \quad (3.77)$$

The other homogeneous solution for the exponent 1 is

$$\tilde{h}(\xi) = \xi + 6\xi^3 + 70\xi^5 + 924\xi^7 + \mathcal{O}(\xi^9). \quad (3.78)$$

We find that

$$i(\xi) = C_\infty \tilde{h}(\xi) \log \xi + \frac{B}{4} \xi^3 + \frac{204B_\infty - 664C_\infty}{48} \xi^5 + \mathcal{O}(\xi^7) \quad (3.79)$$

is a solution of the inhomogeneous equation (3.73). Therefore, as long as C_∞ is nonvanishing, τ behaves as

$$\tau \sim C_\infty \xi \log \xi = -C_\infty \frac{\log \gamma}{\gamma} \quad (3.80)$$

for small ξ or large γ . Note that the homogeneous solutions $h(\xi)$ and $\tilde{h}(\xi)$ give only subleading terms. Since $\tau = \partial_\gamma t$, we find

$$t \sim -\frac{C_\infty}{2} (\log \gamma)^2. \quad (3.81)$$

Inverting this relation, we obtain

$$\gamma \sim \exp \left(\sqrt{-\frac{2t}{C_\infty}} \right). \quad (3.82)$$

Then, the expression (3.76) for π_0 implies

$$(4\alpha\beta, -4(\alpha + \beta), 4 - \alpha\beta, \alpha + \beta) \begin{bmatrix} 2^n & (-2)^n & \alpha^n & \beta^n \\ 2^{n+1} & (-2)^{n+1} & \alpha^{n+1} & \beta^{n+1} \\ 2^{n+2} & (-2)^{n+2} & \alpha^{n+2} & \beta^{n+2} \\ 2^{n+3} & (-2)^{n+3} & \alpha^{n+3} & \beta^{n+3} \end{bmatrix} = (2^{n+4}, (-2)^{n+4}, \alpha^{n+4}, \beta^{n+4}). \quad (3.89)$$

This can be used to obtain

$$-4\alpha\beta\mathcal{S}_n + 4(\alpha + \beta)\mathcal{S}_{n+1} + (\alpha\beta - 4)\mathcal{S}_{n+2} - (\alpha + \beta)\mathcal{S}_{n+3} = -\mathcal{S}_{n+4} + \partial_x(\cdots). \quad (3.90)$$

$$\log \pi_0 \sim \sqrt{-\frac{2t}{C_\infty}}. \quad (3.83)$$

The constant C_∞ is indeed nonvanishing. This can be obtained from the curve (3.9) as

$$C_\infty = \frac{2}{\pi i}. \quad (3.84)$$

See Appendix B for details. This reproduces the dual gravity calculation [6–8]

$$\log W \sim \log \pi_0 \sim \pi \sqrt{2\lambda}. \quad (3.85)$$

B. Recursion relation

We have obtained a system of differential equations for π_n with $n \leq 3$. The other π_n may also satisfy suitable differential equations. It is much easier to determine them by using a recursion relation, as for pure Chern-Simons theory.

Recall that we have

$$2\partial_x(x^n y) = \mathcal{S}_n + 2^n \frac{y}{x-2} + (-2)^n \frac{y}{x+2} - \alpha^n \frac{y}{x-\alpha} - \beta^n \frac{y}{x-\beta}, \quad (3.86)$$

where

$$\mathcal{S}_n := 2nx^{n-1}y + \frac{x^n - 2^n}{x-2} + \frac{x^n - (-2)^n}{x+2} - \frac{x^n - \alpha^n}{x-\alpha} - \frac{x^n - \beta^n}{x-\beta}. \quad (3.87)$$

We arrange these relations as

$$\begin{bmatrix} 2\partial_x(x^n y) - \mathcal{S}_n \\ 2\partial_x(x^{n+1} y) - \mathcal{S}_{n+1} \\ 2\partial_x(x^{n+2} y) - \mathcal{S}_{n+2} \\ 2\partial_x(x^{n+3} y) - \mathcal{S}_{n+3} \end{bmatrix} = \begin{bmatrix} 2^n & (-2)^n & \alpha^n & \beta^n \\ 2^{n+1} & (-2)^{n+1} & \alpha^{n+1} & \beta^{n+1} \\ 2^{n+2} & (-2)^{n+2} & \alpha^{n+2} & \beta^{n+2} \\ 2^{n+3} & (-2)^{n+3} & \alpha^{n+3} & \beta^{n+3} \end{bmatrix} \begin{bmatrix} \frac{y}{x-2} \\ \frac{y}{x+2} \\ \frac{-y}{x-\alpha} \\ \frac{-y}{x-\beta} \end{bmatrix}. \quad (3.88)$$

We find an identity

The integration gives a rather simple five-term recursion relation

$$(2n+10)\pi_{n+4} - (2n+9)(\alpha+\beta)\pi_{n+3} + [(2n+8)\alpha\beta - 2(2n+4)]\pi_{n+2} + 4(2n+3)(\alpha+\beta)\pi_{n+1} - 4(2n+2)\pi_n = 0. \quad (3.91)$$

This relation determines all π_n with $n \geq 4$ in terms of $\pi_0, \dots, \pi_3$.

We can show that the generating function $g(t)$ of π_n defined as in (2.66) satisfies a first-order differential equation which is, however, rather complicated. For completeness, we show the equation below:

$$p\partial_t g + qg = r, \quad (3.92)$$

where

$$p = 2 - 2(\alpha+\beta)t + (2\alpha\beta - 4)t^2 + 8(\alpha+\beta)t^3 - 8t^4, \quad (3.93)$$

$$q = -(\alpha+\beta) + (2\alpha\beta + 4)t - 4(\alpha+\beta)t^2, \quad (3.94)$$

$$r = (8\alpha\beta + 10)\pi_0 + [(8\alpha\beta + 8)\pi_1 - 3(\alpha+\beta)\pi_0]t + \{6\pi_2 - 5(\alpha+\beta)\pi_1 + [12(\alpha+\beta) - 4\alpha\beta]\}t^2 + [8\pi_3 - 7(\alpha+\beta)\pi_2 - 2\alpha\beta\pi_1 - 8(\alpha+\beta)\pi_0]t^3. \quad (3.95)$$

IV. DISCUSSION

We have derived the Picard-Fuchs equations from the curves (2.17) and (3.9) for pure Chern-Simons theory and ABJM theory, respectively, which are satisfied by the BPS Wilson loops and the 't Hooft coupling. The curves are obtained from the derivative of the resolvents of the theories which have simpler forms compared with the resolvents themselves, and therefore, the analysis becomes simpler. We have checked that the Picard-Fuchs equations give the known results correctly. We have also obtained recursion relations for the quantities π_n from which we can obtain the planar vevs of Wilson loops in an arbitrary representation.

Since our analysis does not rely on any relation to (topological) string theory, we expect that it can be extended to more general Chern-Simons-matter theories, irrespective of whether there exist gravity duals for them. The exploration of various theories may help us to characterize the family of Chern-Simons-matter theories with gravity duals.

An immediate extension of our analysis will be to consider ABJ theory. This requires us to vary two parameters α and β independently. This results in Picard-Fuchs equations which are a system of partial differential equations. In this context, we are interested in the following two

issues. The one is the conjecture in [3] which claims that the theory does not exist as a unitary theory when the 't Hooft couplings do not satisfy a certain condition. It is interesting to see whether and how it appears in the Picard-Fuchs equations. The other one is the strong coupling limit. In [17], it is observed that the strong coupling behavior of the Wilson loops may depend on how we take the strong coupling limit. This should be reflected in the local behavior of the Picard-Fuchs equations.

It is known [43] that a similar phenomenon occurs in Chern-Simons theory with flavors in which the supersymmetry is broken in some parameter region. This case is much simpler than ABJ theory, and the analysis of this theory will be helpful to understand ABJ theory. It will be also interesting to reconsider ABJM theory with flavors, which is discussed in [44,45], in light of our analysis.

Another interesting extension will be to consider GT theory. In this theory, it is known [46] that the strong coupling behavior of the Wilson loop is qualitatively different from the one in ABJM theory. It is interesting to clarify how the difference arises. One more direction of the extension is to change the number of bifundamental matters. Such theories are discussed in [28]. Comparing these extensions, we hope to understand how information on various physically interesting limits is encoded in the analytic properties of the Picard-Fuchs equations.

DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: DERIVATION OF THE CURVES

In this appendix, we summarize how to derive the curves (2.17) and (3.9) defined by the derivative $zv'(z)$ of the resolvent $v(z)$, without determining $v(z)$ itself. In fact, $zv'(z)$ has a simpler structure than $v(z)$.

First, we consider the matrix model for pure Chern-Simons theory to illustrate the derivation. The saddle-point equations (2.4) can be rewritten in terms of the resolvent (2.6) as

$$2 \log x = v(x_+) + v(x_-), \quad x \in I, \quad (A1)$$

where I is the branch cut of $v(z)$. This equation can be simplified by differentiating both sides by x . As a result, we obtain

$$\omega(x_+) + \omega(x_-) = 0, \quad \omega(z) := zv'(z) - 1. \quad (A2)$$

This implies that $\omega(z)$ changes its sign when it passes the branch cut I . Due to the Wigner law for $v(z)$, $\omega(z)$ behaves as

$$\omega(z) \sim \frac{c}{\sqrt{z-a}} \quad (\text{A3})$$

around $z = a$, and similarly for $z = a^{-1}$. The definition (2.6) of $v(z)$ implies $\omega(\infty) = -1$.

From these analytic properties, we conclude that $\omega(z)^2$ is a rational function on $\mathbb{C}$ with simple poles at $z = a, a^{-1}$. The asymptotic value of $\omega(z)$ implies

$$\omega(z)^2 = \frac{z^2 + Az + B}{(z-a)(z-a^{-1})}. \quad (\text{A4})$$

We have the inversion symmetry $\omega(z^{-1}) = \omega(z)$ followed from the symmetry of the saddle-point equations (2.4). This implies $B = 1$. If the numerator of the right-hand side is not a complete square, then $\omega(z)$ would have extra branch points other than $z = a, a^{-1}$. We are interested in the single-cut solution, so we should choose $A = \pm 2$.

To fix the sign of A , recall that the nonanalytic part of $v(z)$ should be proportional to $\sqrt{(z-a)(z-a^{-1})}$. In the limit $a \rightarrow 1$, this becomes $z - 1$, and therefore $\omega(z)$ should not have any pole at $z = 1$ in the limit. This can be achieved only when we choose $A = -2$. Now we conclude that $\omega(z)$ satisfies

$$\omega(z)^2 = \frac{(z-1)^2}{(z-a)(z-a^{-1})}, \quad (\text{A5})$$

which reproduces the curve (2.17).

The curve (3.9) for ABJM theory can be derived similarly. The resolvents $v_1(z)$ and $v_2(z)$ defined in (3.5) and (3.6) satisfy

$$2 \log x = v_1(x_+) + v_2(x_-) - 2v_2(x), \quad x \in I_1 \quad (\text{A6})$$

$$-2 \log(-\tilde{x}) = v_2(\tilde{x}_+) + v_2(\tilde{x}_-) - 2v_1(\tilde{x}), \quad \tilde{x} \in I_2 \quad (\text{A7})$$

By the differentiation, we obtain

$$\omega(x_+) + \omega(x_-) = 0, \quad \omega(\tilde{x}_+) + \omega(\tilde{x}_-) = 0, \quad (\text{A8})$$

where

$$\omega(z) := zv'_1(z) - zv'_2(z) - 1. \quad (\text{A9})$$

These equations, the Wigner law, the asymptotic value, and the absence of extra branch points imply

$$\omega(z)^2 = \frac{(z^2 + Az + B)^2}{(z-a)(z-a^{-1})(z-b)(z-b^{-1})}. \quad (\text{A10})$$

The inversion symmetry of $\omega(z)$ requires

$$B = \pm 1, \quad A = AB. \quad (\text{A11})$$

By the same argument as for pure Chern-Simons theory given above, $\omega(z)^2$ must be holomorphic in the limit $a \rightarrow 1$ and $b \rightarrow -1$ simultaneously. This can be realized only when we choose $A = 0$ and $B = -1$. We conclude that $\omega(z)$ satisfies

$$\omega(z)^2 = \frac{(z^2 - 1)^2}{(z-a)(z-a^{-1})(z-b)(z-b^{-1})}. \quad (\text{A12})$$

This is the curve (3.9) for ABJM theory.

APPENDIX B: STRONG COUPLING BEHAVIOR

In this appendix, we calculate the leading term of the 't Hooft coupling of ABJM theory in the strong coupling limit. This corresponds to extracting a divergent contribution in the limit from the integral expression

$$t = -\frac{1}{4\pi i} \int_C \frac{dz}{z} \log z y(z + z^{-1}), \quad (\text{B1})$$

where $y(x)$ defines the curve (3.9).

It is instructive to perform a similar calculation for pure Chern-Simons theory. The 't Hooft coupling is given as

$$t = \frac{1}{4\pi i} \int_C \frac{dz}{z} \log z \frac{z-1}{\sqrt{(z-a)(z-a^{-1})}}. \quad (\text{B2})$$

The integration contour C is depicted in Fig. 1. We are interested in a limit $\alpha \rightarrow \infty$ which was found as one of the singularities of the Picard-Fuchs equation (2.27). This limit corresponds to $a \rightarrow 0$.

In the limit, the contour is pinched by the branch point at $z = a$ and the singularity at $z = 0$, resulting in a divergent behavior. Since the integral expression is invariant under $z \rightarrow z^{-1}$, we see that another pinching occurs at $z = \infty$. To simplify the calculation, we use this symmetry to rewrite the above integral as

$$t = -\frac{1}{2\pi i} \int_{C'} \frac{dz}{z} \log z \frac{z-1}{\sqrt{(z-a)(z-a^{-1})}}, \quad (\text{B3})$$

where the contour C' is depicted in the left of Fig. 3.

We are interested in the leading term for small a . In order to calculate this term, we deform the contour C' as in the right of Fig. 3. Then we find that only the part $C_{a/2}$ gives a divergent behavior for t . By rescaling the variable z , the corresponding integral can be estimated as

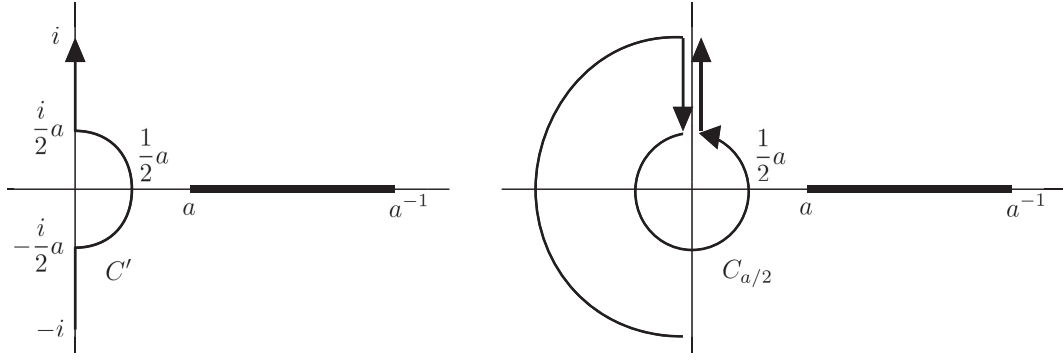


FIG. 3. Deformations of the integration contour. The integration along the contour C' is $-\frac{1}{2}$ times the original integral. The contour in the right figure is a deformation of C' . The leading contribution comes from the circular part $C_{a/2}$.

$$t \sim -\frac{1}{2\pi i} \int_{C_1} \frac{dz}{z} \log a \frac{az - 1}{\sqrt{(az - a)(az - a^{-1})}} \\ = -\log a \sim \log \alpha, \quad (\text{B4})$$

where C_1 is the unit circle with the counterclockwise direction. This indeed reproduces the large α behavior of t derived from the exact result (2.47). The point of this calculation is that the pinching of the integration contour gives a divergent factor $\log \alpha$.

Now, we consider the 't Hooft coupling

$$t = \frac{1}{4\pi i} \int_C \frac{dz}{z} \log z \frac{z^2 - 1}{\sqrt{(z - a)(z - a^{-1})(z - b)(z - b^{-1})}} \quad (\text{B5})$$

of ABJM theory in the limit $\gamma \rightarrow \infty$. We choose C_1 in Fig. 2 for the contour C . Recall that γ determines a and b by

$$a + a^{-1} = \alpha = 2 + \gamma, \quad b + b^{-1} = \beta = -2 + \gamma \quad (\text{B6})$$

on the ABJM slice. For large γ , this implies

$$a \sim \frac{1}{\gamma} z_-, \quad b \sim \frac{1}{\gamma} z_+, \quad z_{\pm} := 1 \pm \frac{2}{\gamma}. \quad (\text{B7})$$

Note that both a and b approach the origin where the integrand is singular.

As for pure Chern-Simons theory, t can be rewritten as

$$t = -\frac{1}{2\pi i} \int_{C'} \frac{dz}{z} \log z \frac{z^2 - 1}{\sqrt{(z - a)(z - a^{-1})(z - b)(z - b^{-1})}} \quad (\text{B8})$$

for the contour C' depicted in Fig. 3. The divergent behavior then appears from the pinching at $z = 0$. By rescaling the variable z , we find

$$t \sim -\frac{1}{2\pi i} \int_{C_\gamma} \frac{dz}{z} \log \frac{1}{\gamma} \\ \times \frac{z^2/\gamma^2 - 1}{\sqrt{(z - z_-)(z/\gamma^2 - z_-^{-1})(z - z_+)(z/\gamma^2 - z_+^{-1})}} \quad (\text{B9})$$

for large γ . The contour C_γ is depicted in Fig. 4. Note that the pinching at $z = 0$ in the original expression is replaced with a divergent factor $\log \gamma$.

Interestingly, there is another source of divergence. The contour C_γ is pinched by the branch points at $z = z_{\pm}$ both of which approach $z = 1$. The divergent part can be estimated as

$$t \sim -\frac{1}{2\pi i} \log \gamma \int_{1-i}^{1+i} \frac{dz}{\sqrt{(z - z_-)(z - z_+)}} \sim -\frac{1}{\pi i} (\log \gamma^2). \quad (\text{B10})$$

This implies that one of the integration constants appearing in (3.81) turns out to be

$$C_\infty = \frac{2}{\pi i}. \quad (\text{B11})$$

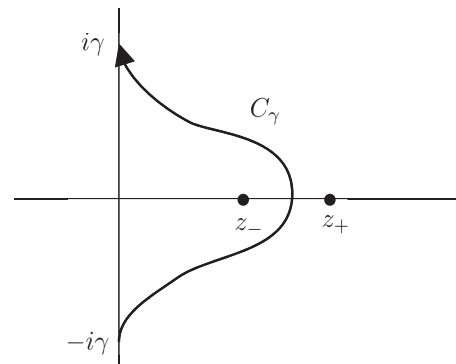


FIG. 4. The contour C_γ is schematically depicted. The leading contribution comes from the part of C_γ near $z = z_{\pm}$, so the details of the contour and the branch cuts are not relevant.

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