

Generalizing deconfined criticality to 3D N -flavor $SU(2)$ quantum chromodynamics on the fuzzy sphere

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The infrared behavior of gauge theories coupled to matter remains an open problem in quantum field theory. For a given gauge group, such theories are expected to flow to an interacting conformal fixed point over a range of fermion or scalar flavors, known as the “conformal window.” Their nature is important for understanding critical phases and phase transitions beyond the Landau paradigm like the deconfined quantum critical point yet remains challenging for conventional nonperturbative approaches. In this work, we study a family of fuzzy-sphere models corresponding to nonlinear sigma models with $Sp(N)$ global symmetry extended to the strongly coupled region. These theories are expected have an infrared fixed point described by $SU(2)$ quantum chromodynamics in three space-time dimensions with N flavors of fermions. They can be viewed as a generalization of the $SO(5)$ deconfined quantum critical point, corresponding to $N = 2$. We investigate them using quantum Monte Carlo for N up to 16. We find evidence that for $N \geq 4$ the phase diagram contains a critical phase that appears to be absent for $N = 2$. Within this phase, we measure the two-point correlation function and the excitation spectrum, which exhibit emergent conformal symmetry. We also extract the scaling dimension Δ_ϕ of a leading operator and find consistency with large- N expectations.

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I. INTRODUCTION

Critical phenomena account for one of the central topics in modern condensed matter physics [1,2]. They appear widely in quantum phase transitions, i.e., singularities of ground state upon tuning a parameter in the Hamiltonian, exhibiting power-law correlations and scale invariance, and extend beyond physics to complex systems. The seemingly complicated phenomena can be governed by simple underlying theories, a powerful idea known as universality: Distinct critical systems share the same universal numbers, i.e., critical exponents, regardless of microscopic details. At low energies, critical points often exhibit an emergent conformal symmetry—they become invariant under transformations that locally preserve angles while changing

shapes. Conformal field theory (CFT) captures the essence of the resulting universal behavior [1–3] and produces many predictions. CFT also connects with high-energy physics, providing insights into string theory [4], quantum gravity [5], renormalization group (RG) flow [6], etc. Many CFTs are well understood in 2D [7–9], but their solutions in 3D remain difficult, although some conformal data have been determined at high precision by conformal bootstrap [10,11].

One fertile ground for discovering exotic 3D CFTs is phase transitions beyond Landau’s paradigm of spontaneous symmetry breaking. A pioneering example of transitions beyond Landau is the deconfined quantum critical point (DQCP) originally proposed as direct continuous transition between the Néel antiferromagnet (AFM) and a valence-bond solid (VBS) phase [12–17] (also see a recent review [18]). It also features several interesting properties, including an enhanced $SO(5)$ symmetry at the transition point [19]. Several field theory descriptions in terms of dual critical gauge theories have been proposed, such as 3D quantum electrodynamics (QED_3) with a $U(1)$ gauge field coupled to two complex scalars [12] or the 3D quantum chromodynamics (QCD_3) with a $SU(2)$ gauge field coupled

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to two fundamental fermions [20]. However, numerical studies have shown that the DQCP is weakly first order [19,21,22], plausibly being pseudocritical, with approximate conformal symmetry plausibly emerging from nearby complex fixed points [20,23,24]. A natural question is therefore whether this setup can be generalized to allow a continuous phase transition, described by a genuine CFT. One such attempt has been the transition between the $SU(N)$ AFM and a VBS [25–28] phase, which is shown to be continuous at large enough N and described by scalar QED₃ with N scalars (i.e., $\mathbb{CP}^{N-1}$ theory). With the help of the field theory descriptions, we can consider another candidate: the $SU(2)$ QCD₃ with N fermions with a $Sp(N)/\mathbb{Z}_2$ global symmetry; the original AFM-VBS DQCP corresponds to the $N = 2$ case.

On the other hand, the nature of gauge theories has been a long-standing problem in the study of quantum fields. Gauge theories provide the successful theoretical framework for the dynamics of elementary particles. Furthermore, coupling matter to gauge fields is one of a few known ways to construct interacting critical theories. For a given gauge group, the RG flow lands on an interacting infrared CFT only at a certain range of flavor number N of the matter coupled to the gauge field instead of confinement or spontaneous symmetry breaking [29]. The determination of this range, known as the conformal window, is an outstanding problem in quantum field theory. In four dimensions, the conformal window is bounded from above and below $N_{c1} < N < N_{c2}$. In 3D, the gauge theory is conformal when the number of flavors exceeds a critical value $N > N_c$. The conformal window of the QED₃ has been extensively studied by lattice-based method [30–35] and dimension and large-flavor-number expansion [36–47]; the lattice calculation of $SU(2)$ QCD₃ [48,49] shows that N_c lies between 4 and 6; $SU(N)$ QCD₃ at large N_c has also been studied [33].

The fuzzy-sphere regularization [50] has emerged as a new powerful method to study 3D CFTs. By studying interacting quantum systems on the fuzzy (noncommutative) sphere, the method realizes $(2+1)$ D quantum phase transitions on the geometry $S^2 \times \mathbb{R}$. Compared with conventional lattice simulations, this is a regularization scheme in the continuum [51] that preserves the rotation symmetry exactly, which allows us to observe the emergent conformal symmetry directly and extract the conformal data efficiently. The power of this approach has been demonstrated in the context of the 3D Ising transition [52–62], establishing the emergent conformal symmetry and calculating a wealth of conformal data at high precision. This approach has since been applied to various defects, boundaries [63–67] and various other CFTs [21,68–83].

Notably, the $SO(5)$ DQCP has been studied on a fuzzy-sphere setup with $N_f = 4$ flavors of fermions with a $Sp(2)/\mathbb{Z}_2$ flavor symmetry at half filling [21,51,84]. The correspondence with DQCP can be seen through matching

the anomaly of the nonlinear sigma model (NLSM) on the Grassmannian $Sp(2)/(Sp(1) \times Sp(1)) \equiv S^4$ with a level-1 Wess-Zumino-Witten (WZW) term [19,85]. A natural generalization would be $N_f = 2N$ flavors of fermions with $Sp(N)^1$ symmetry at half filling and even N . This setup realizes the NLSM-WZW on

$$\frac{Sp(N)}{Sp(N/2) \times Sp(N/2)}$$

extended to the strongly coupled region, which matches with the N -flavor $SU(2)$ QCD₃ [69]. A small value $N = 4$ has been studied through exact diagonalization (ED) and hint of conformal symmetry has been observed. However, due to the exponentially divergent computational cost, larger N are inaccessible through ED and density matrix renormalization group.

These models are well suited for auxiliary-field quantum Monte Carlo (QMC) sampling, however. As we will demonstrate below in more detail, the effective Hamiltonian for any given auxiliary-field configuration is manifestly $SU(N) \subset Sp(N)$ symmetric for even N and therefore block diagonal.² This implies that, for example, the weight of a configuration takes the form $\det(M)^N$, where M represents the fermion determinant of a flavor pair. Consequently, the computational effort is essentially independent of N . Furthermore, a particle-hole symmetry at half filling guarantees the absence of the notorious fermion sign problem, allowing large-scale QMC simulations [51,86].

In this work, we use such QMC simulations in the fuzzy-sphere setup to study the $Sp(N)$ -symmetric models at arbitrary N with a particular focus on the conformal window and the associated possibility of generalized deconfined criticality. We first give theoretical details about the fuzzy-sphere model and match it with the $SU(2)$ QCD₃ through NLSM-WZW, as well as observables of interest. Then we present the numerical results for the phase diagram, the conformal correlators, the scaling dimension Δ_ϕ of a leading operator, and the state-operator correspondence. We conclude with a discussion on the implications and prospective extensions of our work.

II. MODEL AND METHOD

A. Fuzzy-sphere model

To construct the NLSM on the Grassmannian $Sp(N)/(Sp(N/2) \times Sp(N/2))$ with the fuzzy-sphere regularization, we start with $N_f = 2N$ flavors of fermions moving on a sphere with a $4\pi s$ monopole at its center.

¹Here we adopt a notation that $Sp(N)$ denotes the group of $2N \times 2N$ unitary symplectic matrices.

²Note that some symmetries of the Hamiltonian may be broken for a given auxiliary-field configuration and will only be restored upon averaging over the configuration space.

Due to the presence of the monopole, the single-particle eigenstates form highly degenerate quantised Landau levels. The single-particle (noninteracting) ground state, i.e., the lowest Landau level (LLL), has a degeneracy $N_{\text{orb}} = 2s + 1$ for each flavor. We partially fill the LLL and set the gap between the LLL and higher Landau levels to be much larger than any other energy scale in the system. In this case, we can effectively project the system into the LLL. After the projection, the fermion operator $\psi^i(\mathbf{r})$ ($i = 1, \dots, N_f$) can be expressed in terms of the creation operators on the LLL:

$$\psi_i^\dagger(\mathbf{r}) = \frac{1}{R} \sum_{m=-s}^s Y_{sm}^{(s)}(\mathbf{r}) c_{mi}^\dagger, \quad (1)$$

where $Y_{sm}^{(s)}(\mathbf{r})$ are the monopole spherical harmonics, $m = -s, \dots, s$ labels the Landau orbitals, and the radius of the sphere is taken as $R = N_{\text{orb}}^{1/2}$.

The model possesses a maximal global symmetry of $\text{SU}(2N)$ alongside the $\text{U}(1)$ charge conservation that decouples at the critical point. To construct an interaction Hamiltonian that breaks the global symmetry from $\text{SU}(2N)$ to $\text{Sp}(N)$, we consider the $\text{Sp}(N)$ -invariant fermion bilinears:

- (1) the fermion density $n(\mathbf{r}) = \psi_i^\dagger(\mathbf{r})\psi^i(\mathbf{r})$ which is $\text{SU}(2N)$ invariant, and
- (2) the pairing operator $\Delta(\mathbf{r}) = \psi^i(\mathbf{r})\Omega_{ij}\psi^j(\mathbf{r})/2$, where $\Omega = \mathbb{I}_N \otimes i\sigma^y$, which is $\text{Sp}(N)$ invariant but not $\text{SU}(2N)$ invariant.

The Hamiltonian consists of the local density-density interaction and pair-pair interaction [21,69]

$$H = \int d^2\mathbf{r} \left(U n(\mathbf{r})^2 - \frac{V}{N} \Delta^\dagger(\mathbf{r}) \Delta(\mathbf{r}) \right). \quad (2)$$

After the projection onto the LLL, we express the model in terms of the fermion operators c_m^i and $c_{m,i}^\dagger$, where m labels the Landau orbital, through the angular components of n and Δ :

$$\begin{aligned} n(\mathbf{r}) &= \frac{1}{R^2} \sum_{l=0}^{2s} \sum_{m=-l}^l n_{lm} Y_{lm}(\mathbf{r}), \\ \Delta(\mathbf{r}) &= \frac{1}{R^2} \sum_{m=-2s}^{2s} \Delta_{2s,m} Y_{2s,m}^{(2s)}(\mathbf{r}), \\ H &= \frac{4\pi U}{R^2} \sum_{lm} n_{lm}^\dagger n_{lm} - \frac{4\pi V}{NR^2} \sum_m \Delta_{2s,m}^\dagger \Delta_{2s,m}, \end{aligned} \quad (3)$$

where the components n_{lm} and $\Delta_{2s,m}$ take the form

$$\begin{aligned} n_{lm} &= \sum_{m_1 m_2} \Lambda_{m_1 m_2}^{(n)lm} c_{im_1}^\dagger c_{im_2}^i, \\ \Delta_{2s,m} &= \sum_{m_1 m_2} \Lambda_{m_1 m_2}^{(\Delta)2s,m} c_{im_1}^\dagger \Omega_{ij} c_{im_2}^j \end{aligned} \quad (4)$$

in the orbital space, and we give the expression for coefficients $\Lambda_{m_1 m_2}^{(n)lm}$ and $\Lambda_{m_1 m_2}^{(\Delta)2s,m}$ in Appendix A. Apart from the $\text{SO}(3)$ rotation symmetry and the $\text{Sp}(N)$ global symmetry, this model also has an antiunitary particle-hole symmetry

$$\mathcal{P}: c_m^j \mapsto c_{mk}^\dagger \Omega^{jk}, \quad i \mapsto -i. \quad (5)$$

In the following sections, we show that this model is most likely to realize the $\text{SU}(2)$ QCD₃ with N flavors of fundamental fermions. We do this in two steps: In Sec. II B, we show that the fuzzy-sphere model can be described by a NLSM with WZW level 1; in Sec. II C, we discuss the phase diagram of the NLSM, and we especially show that at large enough N , it contains a stable fixed point corresponding to the QCD.

B. Matching the WZW level of NLSM

In this section, we show that the model on the lowest Landau level matches the NLSM on the Grassmannian

$$\frac{\text{Sp}(N)}{\text{Sp}(N/2) \times \text{Sp}(N/2)} \quad (6)$$

with WZW level 1; i.e., they have the same symmetry and anomaly. Its action is

$$\begin{aligned} S_{\text{NLSM}}[\mathbf{Q}] &= \frac{1}{g} \int d^3x \text{tr}(\partial^\mu \mathbf{Q})^2 + k \Gamma_{\text{WZW}}[\mathbf{Q}], \\ \Gamma_{\text{WZW}}[\mathbf{Q}] &= \frac{2\pi i}{(16\pi)^2} \int_0^1 du \int d^3x \\ &\quad \times \epsilon^{\mu\nu\rho\sigma} \text{tr}(\tilde{\mathbf{Q}} \partial_\mu \tilde{\mathbf{Q}} \partial_\nu \tilde{\mathbf{Q}} \partial_\rho \tilde{\mathbf{Q}} \partial_\sigma \tilde{\mathbf{Q}}), \end{aligned} \quad (7)$$

where g is the stiffness of the NLSM, k is the WZW level, $x = (\tau, \mathbf{r})$, and $\mathbf{Q}(x)$ is a $2N \times 2N$ matrix field that parametrizes the Grassmannian:

$$\mathbf{Q} = \mathbf{g} \Sigma \mathbf{g}^{-1}, \quad \mathbf{g} \in \text{Sp}(N), \quad \Sigma = \begin{pmatrix} \mathbb{I}_N & 0 \\ 0 & -\mathbb{I}_N \end{pmatrix}. \quad (8)$$

The WZW term is expressed by extending it into an auxiliary fourth dimension $\tilde{\mathbf{Q}}(x, u)$ parametrized by $0 \leq u \leq 1$ with boundary condition

$$\tilde{\mathbf{Q}}(x, u = 1) = \mathbf{Q}(x), \quad \tilde{\mathbf{Q}}(x, u = 0) = \Sigma.$$

We shall derive the same action from the fuzzy-sphere model. The calculation for the specific case $N = 2$ has

already been performed by Lee and Sachdev [85]. Here we sketch the process and leave the details to Appendix B. We start from free fermions on the lowest Landau level coupled to the matrix field $\mathbf{Q}(x)$:

$$S[\psi, \bar{\psi}, \mathbf{Q}] = \int d^3x \bar{\psi}_i \frac{\partial \psi^i}{\partial \tau} + \frac{\text{tr}(\partial^\mu \mathbf{Q})^2}{2g_0} - \lambda \bar{\psi}_i Q^i_j \psi^j. \quad (9)$$

The fermion fields are projected to the lowest Landau level:

$$\psi^i(x) = \sum_m \phi_m(\mathbf{r}) c_m^i(\tau),$$

where m is the orbital index and $\phi_m(\mathbf{r})$ is the single-particle wave function of the lowest Landau level. Physically, the coupling between fermions to $\mathbf{Q}$ captures the four-fermion interactions—integrating out the Q field results in an effective four-fermion term similar to Eq. (2) to the leading order, together with interactions with more fermions at higher order.

Integrating out the fermions gives an effective action $S_{\text{eff}}[\mathbf{Q}]$ of the matrix field $\mathbf{Q}$. In the long-wavelength limit, this effective action is equivalent to S_{NLSM} in Eq. (7) with WZW level $k = 1$. This can be derived perturbatively by expanding $\mathbf{Q}$ in vicinity of a reference configuration which we take as Σ :

$$\mathbf{Q} = \Sigma + \delta\mathbf{Q}.$$

To the second order of $\delta\mathbf{Q}$, S_{eff} gives the self-energy correction for $\mathbf{Q}$; to the fourth order, the box diagram gives the WZW term that we desire:

$$\begin{aligned} \lim_{q \rightarrow 0} \text{Diagram} &= \frac{3i}{64\pi} \int d^3x \text{tr}(\Sigma \delta\mathbf{Q} \partial_0 \delta\mathbf{Q} \partial_1 \delta\mathbf{Q} \partial_2 \delta\mathbf{Q}) \\ &= \Gamma_{\text{WZW}}[\mathbf{Q}], \end{aligned} \quad (10)$$

where the solid arrows and the dashed lines denote, respectively, the propagator of fermions and matrix fields. In Appendix B, we perform the detailed calculation on a large flat space with periodic boundary condition. We expect the result to extend to the LLL on the sphere, as the matching of the WZW level is not sensitive to the geometric details.

C. Connection between NLSM-WZW and QCD₃

Now that we have matched the fuzzy-sphere model and the NLSM-WZW, we discuss the phase diagram of the NLSM. We note that although the NLSM can only describe the spontaneous symmetry breaking (SSB) in the

renormalizable region, by extending it to the strongly coupled region, its phase diagram may contain critical gauge theories [87,88]. These gauge theories have the same symmetry and anomaly as the fuzzy-sphere model and are thus natural end points of RG flow for the fuzzy-sphere model.

A natural candidate in the phase diagram is SU(2) QCD₃, i.e., SU(2) gauge field coupled to N flavors of fermions in the fundamental representation. To see this, we consider N fermions ψ^{ias} in the bifundamental representation of $\text{Sp}(N)$ flavor symmetry and the SU(2) gauge group, where the three indices are, respectively, flavor, gauge and spinor.³ We then couple it to a bosonic mass field $\mathbf{Q}$ in the $\text{Sp}(N)$ Grassmannian through $\bar{\Psi}\mathbf{Q}\Psi = \bar{\psi}_{ias} Q^i_j \psi^{jas}$. Condensing $\mathbf{Q}$ gives the spontaneous-symmetry-broken phase described by the NLSM; on the other hand, when $\mathbf{Q}$ is gapped, we recover the Lagrangian of the QCD.

The phase diagram then depends on the fate of the QCD in the infrared. It is generally believed that SU(2) QCD₃ flows to a conformal fixed point at large enough N . In this range, the phase diagram contains three fixed points [Fig. 1(a)]: a stable fixed point of the spontaneous-symmetry-breaking phase, a stable conformal fixed point of the critical SU(2) QCD, and an unstable conformal fixed point corresponding to their phase transition, described by SU(2) QCD coupled to a Gross-Neveu-Yukawa (GNY) field in the adjoint representation of the $\text{Sp}(N)$. In the large- N limit, the operator spectrum of the stable QCD fixed point has been studied using the standard perturbative computation technique [89].⁴

$$\begin{aligned} \Delta_{S^-} &= 2 + \frac{64}{3\pi^2} \frac{1}{N} + \mathcal{O}(N^{-2}), \\ \Delta_\phi &= 2 - \frac{32}{3\pi^2} \frac{1}{N} + \mathcal{O}(N^{-2}). \end{aligned} \quad (11)$$

Specifically, the scaling dimension for the bilinear $S^- = \psi^i \psi^j \Omega_{ij}$ with odd parity in the singlet representation of $\text{Sp}(N)$, and

$$\phi^{[ij]} = \psi^{[i} \psi^{j]} - (\text{trace}) \quad (12)$$

in the traceless antisymmetric rank-2 tensor representation

³The fermions here are different from the fermions in the microscopic model in Sec. II B. The Lagrangian reads

$$\mathcal{L}[\psi^{ia}, A_\mu] = \psi^{ias_1} \Omega_{ij} \Omega_{ab} \Omega_{s_1 s'} i(\gamma^\mu)^{s'}_{s_2} (D_{A,\mu} \psi)^{jbs_2},$$

where the covariant derivative $D_{A,\mu} = \partial_\mu - iA_\mu$, the $\text{Sp}(N)$ flavor index $i, j = 1, \dots, 2N$, the $\text{Sp}(1) \cong \text{SU}(2)$ gauge index $a, b = 1, 2$, and the $\text{Sp}(1) \cong \text{SU}(2)$ spinor index $s = 1, 2$. From this Lagrangian, the $\text{Sp}(N)$ symmetry is explicit.

⁴Note that N in our convention is twice the N in their convention.

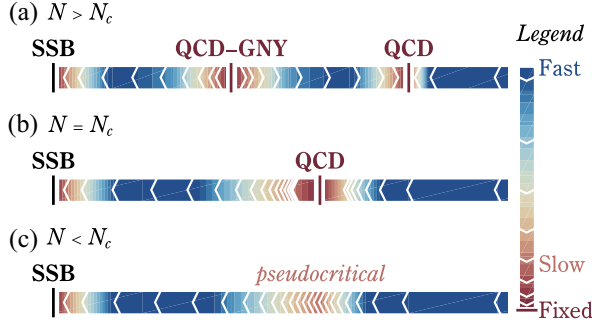


FIG. 1. The putative RG-flow diagram of the NLSM extended to the strongly coupled region. The color and the intervals between arrows mark the rate of the RG flow. The red and black bars denote the conformal and nonconformal fixed points, respectively.

of $\text{Sp}(N)$, where the contraction of the other indices is omitted. Also note that this singlet, with odd parity, corresponds to the singlet fermion mass, while the parity-even singlet S^+ that controls the RG flow in Fig. 1(a) can be written as a four-fermion operator in the Lagrangian.

D. Conformal window of gauge theories

As we decrease N , the two fixed points QCD and QCD-GNY approach each other, and at a critical N_c , they collide into one fixed point [Fig. 1(b)]. This critical N_c is signaled by an exactly marginal singlet $\Delta_{S^+} = 3$ that controls the phase diagram. When we further decrease N to $N < N_c$, the fixed point becomes two conjugate complex fixed points [Fig. 1(c)], and the QCD flows directly to the SSB fixed point. The complex fixed points are described by complex CFTs. If they are sufficiently close to the real axis, the RG flow on the real axis in their vicinity slows down significantly. This is known as pseudocriticality and is studied in deconfined criticality on the fuzzy sphere [21] which corresponds to $N = 2$ here, as well as the 2D five-state Potts model [23,24].

A similar scenario holds for 3D critical gauge theories involving N fermions (or critical scalars) coupled to a more general dynamical gauge field [e.g., $\text{SU}(k)$, $\text{U}(k)$, $\text{Sp}(k)$]. For each gauge group, there exists a critical value of flavor number N_c above which the critical gauge theory flows to a conformal fixed point. The $N > N_c$ region where the 3D gauge theory flows to an interacting conformal fixed point is known as the conformal window. For QCD in 4D, there exist two critical N_c 's: $N < N_{c1}$ flows to SSB, $N_{c1} < N < N_{c2}$ flows to an interacting conformal fixed point, and $N > N_{c2}$ flows to a free fixed point. The conformal window problem, i.e., determining the exact value of the critical flavor number, has been an outstanding problem for condensed matter and high-energy physicists. In 3D, the QCDs are interacting throughout $N > N_c$ even at large N . In this paper, we shall focus on the $N > N_c$ case and

examine the evidence for conformal symmetry at the stable QCD fixed point, as well as obtain its conformal data.

E. Method and observables

Here we give the general approach and some model-specific aspects to studying the $\text{Sp}(N)$ symmetric NLSM through auxiliary-field quantum Monte Carlo for arbitrary N . To facilitate the Hubbard-Stratonovich decoupling, we write the Hamiltonian, Eq. (3), as a sum of perfect squares of Hermitian operators:

$$H = \frac{4\pi U}{R^2} \frac{1}{2} \sum_{l,m} \left(e^{i\frac{\pi}{4}} n_{lm}^\dagger + e^{-i\frac{\pi}{4}} n_{lm} \right)^2 - \frac{4\pi V}{R^2} \frac{1}{4N} \sum_m \left[\left(\Delta_{2s,m}^\dagger + \Delta_{2s,m} \right)^2 + (i\Delta_{2s,m}^\dagger - i\Delta_{2s,m})^2 \right]. \quad (13)$$

From this point on, we fix $U = 1$ as the unit of energy. We can next readily introduce auxiliary fields, $\eta_{lm,\tau}^z$, $\eta_{m,\tau}^+$, and $\eta_{m,\tau}^-$, for three kinds of perfect squares, respectively, on every imaginary-time slice τ in the Trotterized partition sum $Z = \text{Tr} \prod_\tau e^{-\Delta\tau H}$. The attentive reader may have noticed that the fermion bilinears coupling to $\eta_{m,\tau}^+$ and $\eta_{m,\tau}^-$ do not preserve the fermion particle number. Hence, a partial particle-hole transformation, $c_m^j \mapsto c_{mj}^\dagger$ for $j = N+1, \dots, 2N$, is performed to restore the fermion particle number in this new “computational” basis, $\tilde{c}_m^{j,\uparrow} = c_m^j$ and $\tilde{c}_m^{j,\downarrow} = c_{m,j+N}^\dagger$ with $j = 1, \dots, N$. Equation (4) is expressed in the new basis as

$$n_{lm} = \sum_{m_1 m_2} \Lambda_{m_1 m_2}^{(n)lm} \left(\tilde{c}_{m_1,j,\uparrow}^\dagger \tilde{c}_{m_2}^{j,\uparrow} - \tilde{c}_{m_2,j,\downarrow}^\dagger \tilde{c}_{m_1}^{j,\downarrow} \right), \quad \Delta_{2s,m} = \sum_{m_1 m_2} \Lambda_{m_1 m_2}^{(\Delta)2s,m} \tilde{c}_{m_2,j,\downarrow}^\dagger \tilde{c}_{m_1}^{j,\uparrow}, \quad \Delta_{2s,m}^\dagger = \sum_{m_1 m_2} \Lambda_{m_1 m_2}^{(\Delta)2s,m} \tilde{c}_{m_1,j,\uparrow}^\dagger \tilde{c}_{m_2}^{j,\downarrow}. \quad (14)$$

For any given auxiliary-field configuration, the partition sum is $\text{SU}(N)$ symmetric⁵ and therefore exhibits N identical $2N_{\text{orb}} \times 2N_{\text{orb}}$ blocks. In a typical model with short-range interaction, the numerical effort of the QMC algorithm scales cubically with the number of fermions per block. However, this is assuming that the auxiliary fields couple to local operators or, more precisely, that the rank of those operators is independent of system size and typically rather small. In the fuzzy-sphere model, the form factors Λ generically exhibit N_{orb} nonzero eigenvalues such that their rank scales with the system size as well. Consequently, the numerical effort here scales as $(2N_{\text{orb}})^4 L_{\text{trot}}$. While N does

⁵Note that the form factors are independent of the flavor index.

not directly affect the computational cost, long autocorrelation times due to a low acceptance rate could indirectly extend the run-time. Nevertheless, we find that this model shows only a weak dependence of the acceptance rate ($\sim 40\%$ – 50%) on N (Appendix C, Fig. 8), making it well suited for QMC simulations at intermediate to large values of N . Lastly, we are employing a higher-order Trotter decomposition to improve the systematic Trotter error [90,91].

In contrast to ED or density matrix renormalization group calculations where one can directly access the energies of the lowest eigenstates, for QMC the critical data must be measured through correlation functions of (local) observables. On the fuzzy sphere, the simplest observables are the fermion-bilinear density operators. Here, the bilinears can be organized in different representations of the $\text{Sp}(N)$ global symmetry, namely, the singlet S , the antisymmetric traceless rank-2 tensor A and symmetric rank-2 tensor T :

$$n_S(\mathbf{r}) = \psi_i^\dagger \psi^i, \quad (15a)$$

$$n_A^{[ij]}(\mathbf{r}) = \psi_k^\dagger \Omega^{ki} \psi^j - \psi_k^\dagger \Omega^{kj} \psi^i - \frac{1}{N} \Omega^{ij} \psi_k^\dagger \psi^k, \quad (15b)$$

$$n_T^{(ij)}(\mathbf{r}) = \psi_k^\dagger \Omega^{ki} \psi^j + \psi_k^\dagger \Omega^{kj} \psi^i. \quad (15c)$$

Under the particle-hole symmetry $\mathcal{P}$, n_S and n_A are odd while n_T is even. The density operators can be expressed in terms of the angular components $n_{R,lm}$:

$$n_{O,lm} = \int d^2\mathbf{r} \bar{Y}_{lm}(\mathbf{r}) n_O(\mathbf{r}),$$

$$n_O(\mathbf{r}) = \frac{1}{R^2} \sum_{lm} n_{O,lm} Y_{lm}(\mathbf{r}). \quad (16)$$

where $O = S, A, T$ is the $\text{Sp}(N)$ representation.

With quantum Monte Carlo, we can measure their real-space equal-time two-point functions:

$$C_O(\gamma_{12}) = \langle (n_O(\mathbf{r}_1))^\dagger n_O(\mathbf{r}_2) \rangle, \quad (17)$$

where γ_{12} is the angular distance between the two points. It can be decomposed into the linear combination of the correlators of the angular components $\langle n_{O,lm}^\dagger n_{O,lm} \rangle$. The details are given in Appendix A. We can also measure the imaginary-time-displaced correlator of a given angular component:

$$C_{O,l}(\tau) = \langle (n_{O,l0}(\tau))^\dagger n_{O,l0}(0) \rangle. \quad (18)$$

Here we have taken $m = 0$ because all the choices of m are equivalent due to the rotation symmetry.

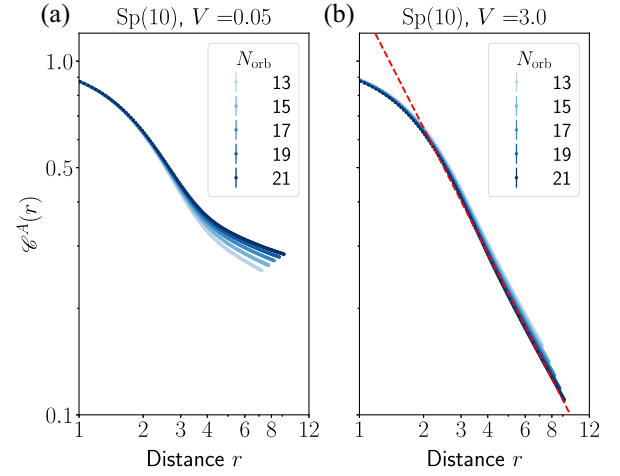


FIG. 2. The equal-time correlation functions in the $\text{Sp}(10)$ model at (a) $V = 0.05$ in the spontaneous-symmetry-broken phase and (b) $V = 3.0$ in the critical QCD phase. The linear distance is taken as $r = 2R \sin(\gamma_{12}/2)$. The values of $C^a(r)$ are scaled so that $C^a(0) = 1$ for each of the system sizes. The red dashed line in (b) marks the a power-law fit to data for $r > 2.5$.

III. RESULTS

A. Phase diagram

We first determine the phase diagram through the equal-time correlation functions and related RG-invariant dimensionless quantities, in parallel with the standard analysis used in lattice models.

The order parameter that captures the symmetry-breaking pattern on the Grassmannian (6) is the antisymmetric density operator (15b) $n_{[ij]}^A$. Taking the $\text{Sp}(10)$ theory as an example, we calculate its two-point function (17) $C_A(\gamma_{12})$ and normalize it by its value at zero distance $C_A(\gamma_{12} = 0)^6$:

$$\frac{C_A(r)}{C_A(0)} = \frac{\langle n_{A,[ij]}(\mathbf{r}_1) n_{A,[ij]}(\mathbf{r}_2) \rangle}{\langle n_A(\mathbf{r})^2 \rangle}.$$

We plot it in Fig. 2 as a function of the spatial distance $r = 2R \sin(\gamma_{12}/2)$ for different system sizes, where γ_{12} is the angular distance between the two points and $R = \sqrt{N_{\text{orb}}}$. We identify two likely distinct phases: At $V = 0.05$, the correlation function tends to remain finite at long distance, indicating a SSB phase; at $V = 3$, the data at large distance converge to a power law, indicating a critical phase with scale invariance. This phase diagram is consistent with the conformal window depicted in Fig. 1(a). The critical phase flows to the stable QCD fixed point.

⁶The zero-distance correlator $C_A(0)$, which diverges in the CFT limit, is instead controlled by model-dependent short-range behavior in the present model and thus saturates to a constant in the thermodynamic limit.

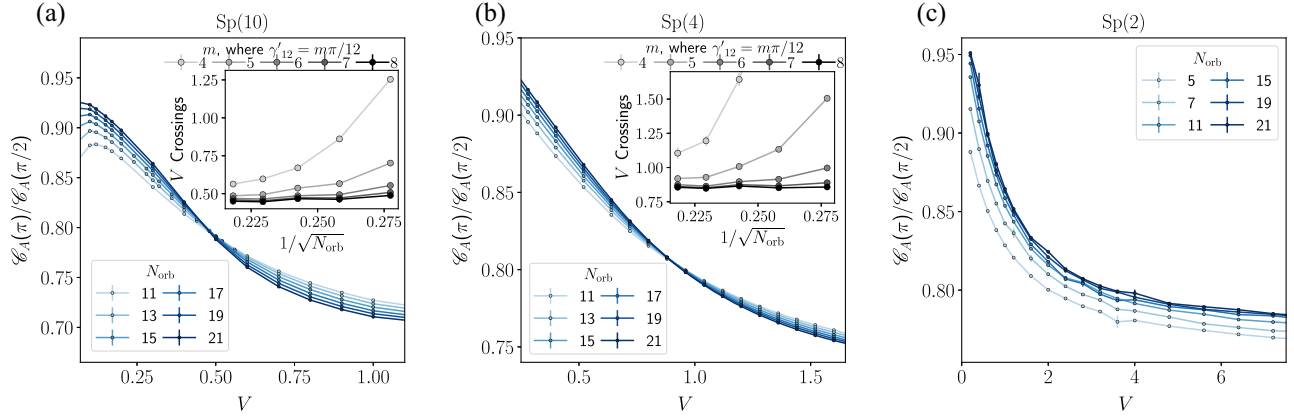


FIG. 3. Main: RG-invariant observables $C_A(\pi)/C_A(\pi/2)$ as a function of the coupling V for Sp(10), Sp(4), and Sp(2) models. Insets of (b) and (c): the V value at which the RG-invariant observable $C_A(\pi)/C_A(\gamma'_{12} = m\pi/12)$ has a crossing as a function of $1/\sqrt{N_{\text{orb}}}$ for pairs $(N_{\text{orb}} - 2, N_{\text{orb}})$.

To determine the critical point between the two phases, we examine the ratio of the correlator at two distinct distances:

$$\lambda = C_A(\gamma_{12})/C_A(\gamma'_{12}).$$

This quantity is dimensionless and is RG invariant at fixed points. Similar to the Binder cumulant [92], this ratio approaches distinct values in different phases and exhibits crossings for different system sizes at a critical point [93]. We take $\gamma_{12} = \pi$ in the numerator as the antipodal correlation. For both Sp(10) and Sp(4), we observe such an intersection marking a critical point V_c [Figs. 3(a) and 3(b)]. At $V < V_c$, λ increases with system size and approaches unity as expected for the symmetry-broken phase; for $V > V_c$, λ decreases with system size and converges to a different finite value. We extract the crossing point V_c for different size pairs $(N_{\text{orb}} - 2, N_{\text{orb}})$ at multiple choices of γ'_{12} [Figs. 3(a) and 3(b), insets].⁷ For all $\gamma'_{12} = m\pi/12$ ($m = 4, 5, 6, 7, 8$), the crossing point converges to the same value— $V_c \approx 0.5$ for the Sp(10) model and $V_c \approx 0.8$ for the Sp(4) model—with increasing system size. This result suggests a continuous phase transition with scale invariance at the crossing point separating a SSB phase and a critical QCD phase [Fig. 1(a)]. For comparison, λ increases monotonically with system size at all parameter ranges with no crossing point in the Sp(2) model [Fig. 3(c)], suggesting a symmetry-broken phase throughout the phase diagram. This is consistent with the pseudocriticality for the SO(5) DQCP [21] [Fig. 1(c)].

B. Conformal correlator and scaling dimensions

Given the evidence for a scale-invariant critical point, we present evidence for the conformal symmetry and extract

⁷The crossing points and their errors are estimated using a polynomial fit of the curves in their vicinity.

the conformal data in the critical QCD phase $V > V_c$ from the same equal-time two-point function for the Sp(10) and Sp(4) models.

At a conformal fixed point, any gapless local observables $n_O(\mathbf{r})$ on the fuzzy sphere can be written as the linear combination of CFT operators in the same representation of global symmetry and parity. To the lowest order, n_A and n_T are used to express the fermion bilinear operator (12) in the CFT and the temporal component of the conserved symmetry current:

$$\begin{aligned} n_A(\mathbf{r}) &= \text{const} \times \phi(\mathbf{r}) + \dots, \\ n_T(\mathbf{r}) &= \text{const} \times J^T(\mathbf{r}) + \dots \end{aligned} \quad (19)$$

To the leading order, the two-point correlator $C_A(\gamma_{12})$ should behave like a power law [1,53]:

$$\begin{aligned} C_A(\gamma_{12}) &= \langle n_A(\mathbf{r}_1) n_A(\mathbf{r}_2) \rangle \\ &\stackrel{\text{CFT}}{=} \text{const} \times \langle \phi(\hat{\mathbf{n}}_1) \phi(\hat{\mathbf{n}}_2) \rangle_{S^2 \times \mathbb{R}} \\ &= \text{const} \times R^{-2\Delta_\phi} \left(2 \sin \frac{\gamma_{12}}{2} \right)^{-2\Delta_\phi}, \end{aligned} \quad (20)$$

where $\hat{\mathbf{n}}_i = \mathbf{r}_i/r_i$ is the unit vector. We note that the R dependence is a consequence of scale invariance, while the γ_{12} dependence is a consequence of conformal invariance. We measure this two-point function through the equal-time correlator in the QMC. To compare with the conformal form, we cancel the constant factor by normalizing $C_A(\gamma_{12})$ by its value at $\gamma_{12} = \pi$ and compare $C_A(\gamma_{12})/C_A(\pi)$ with the conformal correlator $(\sin \frac{\gamma_{12}}{2})^{-2\Delta_\phi}$. With increasing system size, the measured correlator on the fuzzy sphere converges toward the conformal correlator [Figs. 4(a) and 4(d)]. While we focus on the stable conformal phase at large V here, equivalent data for the critical point at V_c are shown in the Appendix E.

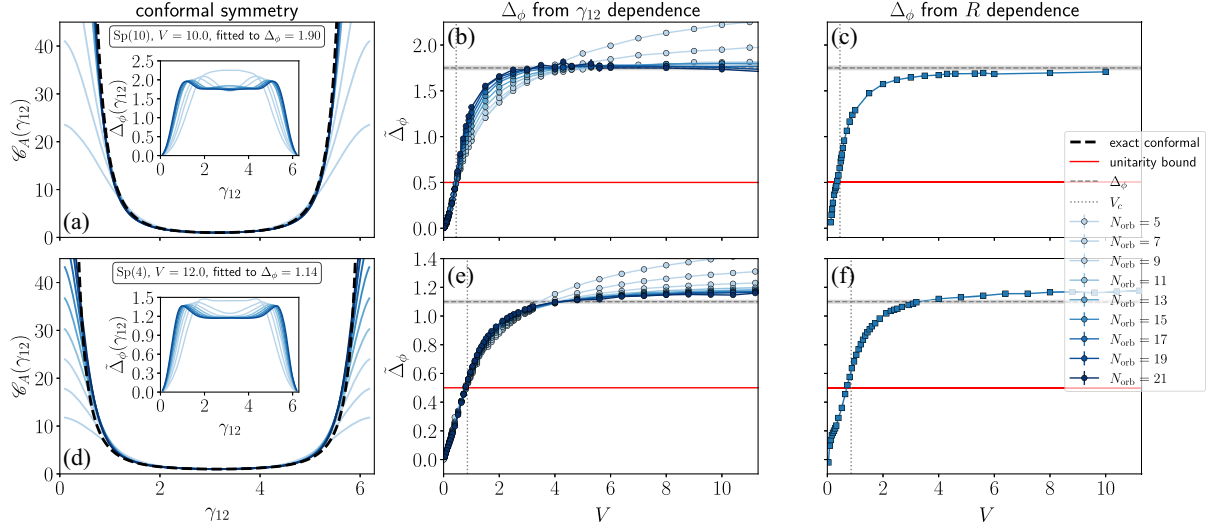


FIG. 4. The conformal correlator $\mathcal{C}_A(\gamma_{12})$ and the extracted scaling dimension Δ_ϕ for (a)–(c) the $\text{Sp}(10)$ model and (d)–(f) the $\text{Sp}(4)$ model. (a),(d) The real-space correlator (20) $\mathcal{C}_A(\gamma_{12})$ of the density operator n_A as a function of the angular distance γ_{12} , measured at (a) $V = 10$ and (d) $V = 12$. The dashed lines mark the best-fit conformal 2-pt function. Insets of (a) and (d): the scaling dimension extracted through Eq. (21) as a function of γ_{12} . The dashed line marks the best-fit conformal 2-pt function. (b),(e) The scaling dimension Δ_ϕ as a function of V extracted through γ_{12} dependence (22). (c),(f) The scaling dimension Δ_ϕ as a function of V extracted through R dependence (21). The dashed grid line and the gray shade mark the finite-size scaling result of Δ_ϕ and its error bar (see Appendix D). The red grid line marks the unitarity bound $\Delta = 1/2$.

As an important piece of conformal data, the scaling dimension Δ_ϕ can be extracted from the conformal correlator. Starting from Eq. (20), we employ two methods.

- (1) Fix the size R and examine the dependence on the angular distance γ_{12} . The Δ_ϕ can be measured through the logarithmic derivative of the correlator

$$\begin{aligned}\tilde{\Delta}_\phi(\gamma_{12}, V, R) &= -\frac{1}{2} \frac{\partial \log \mathcal{C}_A(\gamma_{12})}{\partial \log r_{12}} \\ &= -\tan \frac{\gamma_{12}}{2} \frac{\partial \log \mathcal{C}_A(\gamma_{12})}{\partial \gamma_{12}},\end{aligned}\quad (21)$$

where $r_{12} = 2R \sin(\gamma_{12}/2)$ is the conformal distance. We emphasize that this method of extracting Δ_ϕ utilizes the power of conformal symmetry, such that the traditional finite-size scaling with respect to system size is not needed [1].

- (2) Fix the angular distance γ_{12} and examine the dependence on the size R . At a certain γ_{12} ,

$$\mathcal{C}_A(\gamma_{12}) \sim R^{-2\Delta_\phi}, \quad (22)$$

so a power-law fit of $\mathcal{C}_A$ as a function of $R = N_{\text{orb}}^{1/2}$ directly yields Δ_ϕ .

The $\tilde{\Delta}_\phi$ measured through γ_{12} dependence is in general a function of angle γ_{12} , parameter V and system size R . As a quantitative check that the measured correlator exhibits conformal symmetry, we first show that its value is approximately independent of γ_{12} over a wide range of

distances, with deviations appearing only in the short-distance region $\gamma_{12} \ll 1$ dominated by the regulator [Figs. 4(a) and 4(d), insets]. We find that, in extracting Δ_ϕ from Eq. (21), γ_{12} slightly smaller than π empirically suffers from the small finite-size effect and QMC statistical error. Hereafter, we fix $\gamma_{12} = 7\pi/9$. We examine the dependence of $\tilde{\Delta}_\phi$ on V [Figs. 4(b) and 4(e)]. For different values of V within the critical QCD phase, the extracted scaling dimension scales toward a common value in the thermodynamic limit. Performing a finite-size scaling analysis across multiple V values, we obtain $\Delta_\phi = 1.10(1)$ for $\text{Sp}(4)$ and $1.75(2)$ for $\text{Sp}(10)$. Further detail of the finite-size scaling is provided in the Appendix C. As an additional consistency check, in the symmetry-broken phase, this analysis yields $\tilde{\Delta}_\phi < 1/2$, violating the unitarity bound as expected. From the alternative method based on R dependence, the fitted scaling dimension Δ_ϕ is consistent with those extracted from γ_{12} dependence [Figs. 4(c) and 4(f)].

As discussed above, the computational cost in the QMC simulations depends only weakly on the number N of flavors. Thanks to this advantage, we are able to access the model at large N up to $N = 16$. For each N , we extract the scaling dimension Δ_ϕ through Eq. (21); for details see Appendix D. With increasing N , the results approach the large- N expansion (Fig. 5). A previous study on lattice gauge theory using Wilson fermions, where the $\text{Sp}(N)$ symmetry arises in the continuum, finds the critical exponents $\Delta_\phi = 1.62(8)$ for $N = 8$ and $\Delta_\phi = 1.52(6)$ for $N = 12$ [49]. Within statistical uncertainty, these previous

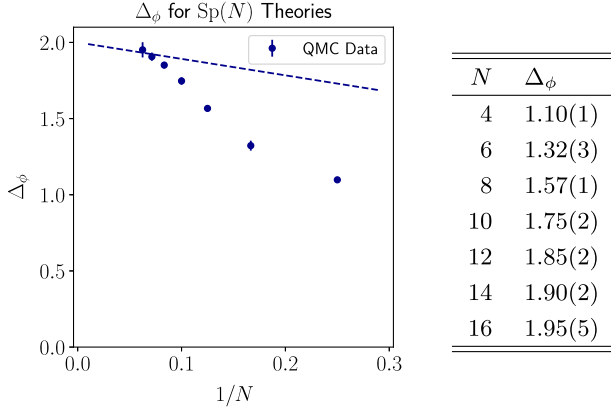


FIG. 5. The scaling dimension Δ_ϕ as a function of N calculated from the conformal correlator. The dashed line is the large- N expansion result [89].

simulation show no strong dependence of Δ_ϕ on N , in contrast to our data.

We have also checked the correlator in the symmetric representation T , which corresponds to the two-point function of the conserved symmetry current J^μ with scaling dimension $\Delta_J = 2$:

$$\mathcal{C}_T(\gamma_{12}) \stackrel{\text{CFT}}{\sim} \langle J^\tau(\hat{n}_1) J^\tau(\hat{n}_2) \rangle_{S^2 \times \mathbb{R}} \sim \left(2R \sin \frac{\gamma_{12}}{2} \right)^{-4}, \quad (23)$$

where $\sim$ denotes the omission of a constant factor. We can extract the scaling dimension Δ_J in a similar fashion as Eqs. (21) and (22). In the critical QCD phase, the scaling dimension approaches $\Delta_J = 2$ in the thermodynamic limit as expected (Fig. 6). When crossing the critical point from the SSB phase into the critical QCD phase, Δ_J first increases from $\Delta_J < 2$ to $\Delta_J > 2$ and then gradually relaxes back to 2 deeper in the conformal phase. Notably, the coupling at which Δ_J crosses 2 lies close to the critical point extracted from the RG-invariant analysis in Fig. 3, which is suggestive of a conformal transition.

C. State-operator correspondence

On the fuzzy sphere, we can make use of the state-operator correspondence to further verify conformal symmetry and extract scaling dimensions. Each eigenstate of the Hamiltonian at the conformal fixed point has a one-to-one correspondence to a CFT local operator. The state and the operator have corresponding Lorentz spin and representation under global symmetry, and the excited energy is proportional to the scaling dimension

$$E_\Phi - E_0 = \frac{v}{R} \Delta_\Phi, \quad (24)$$

where v is a model-dependent light speed. For quantum Monte Carlo, the excited states can be accessed through the time-displaced correlation functions of the angular

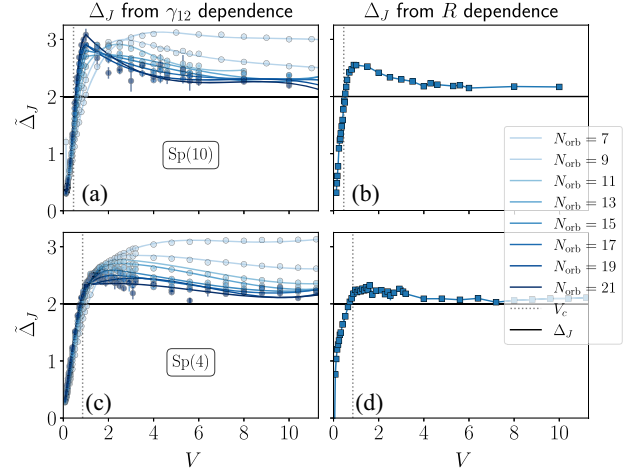


FIG. 6. The scaling dimension Δ_ϕ as a function of V calculated through conformal 2-pt correlator $\mathcal{C}_T$ in the (a),(b) Sp(10) and (c),(d) Sp(4) models. Panels (a) and (c) are extracted through γ_{12} dependence; (b) and (d) are extracted through R dependence. The black line marks the expected value $\Delta_J = 2$. The dashed gray line marks the critical point extracted through crossing in Fig. 3.

components of the density operator $n_{O,l}$ in certain representation $O = S, A, T$ of the $\text{Sp}(N)$ symmetry. At zero temperature, it can be expressed in terms of the spectral decomposition

$$\begin{aligned} \mathcal{C}_{O,l}(\tau) &= \langle (n_{O,l}(\tau))^\dagger n_{O,l}(0) \rangle \\ &= \sum |\langle \Phi_{O,l} | n_{O,l} | 0 \rangle|^2 e^{-(E_\Phi - E_0)\tau}, \end{aligned} \quad (25)$$

where the sum is taken over the excited states $|\Phi_{O,l}\rangle$ in the O representation with Lorentz spin l . To the leading order at large τ , the rate of the exponential decay yields the scaling dimension $\Delta_{O,l}$ of the lowest scaling operator in the corresponding sector:

$$\mathcal{C}_{O,l}(\tau) = \text{const} \times e^{-(v/R)\tau \Delta_{O,l}} + \text{subleading}. \quad (26)$$

The subleading terms encode other operators in the same representation with the same Lorentz spin, e.g., certain descendants like $\partial_\mu \partial^\mu O_l$ which have a higher scaling dimension and therefore decay faster. In the antisymmetric representation A , the lowest states are the primary ϕ and its descendants $\partial^{\mu_1} \dots \partial^{\mu_l} \phi$; in the symmetric representation T , the lowest states are the conserved symmetry current J^μ and its descendants $\partial^{\mu_1} \dots \partial^{\mu_{l-1}} J^{\mu_l}$. Conformal symmetry thus predicts

$$\Delta_{A,l} = \Delta_\phi + l, \quad \Delta_{T,l} = 1 + l.$$

We perform the analysis for the Sp(10) and Sp(4) models and extract $\Delta_{A,l}$ for $l = 0, 1, 2, 3$ and $\Delta_{T,l}$ for $l = 1, 2, 3$ (Fig. 7) in the conformal phase. The model-dependent velocity v is fixed by calibrating the lowest antisymmetric

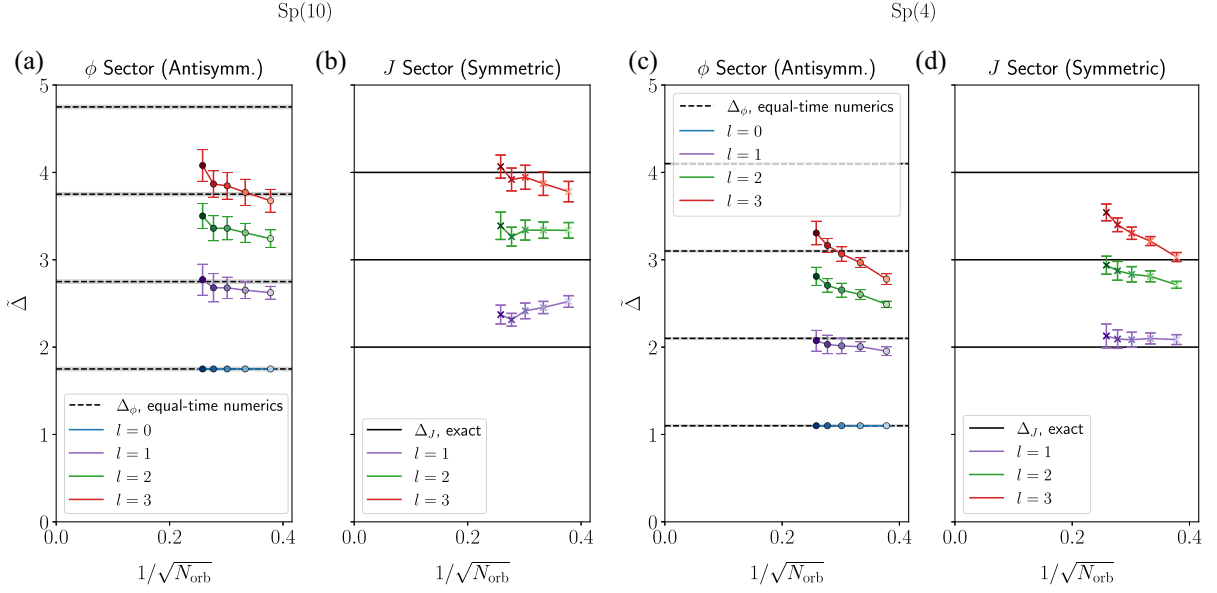


FIG. 7. The operator spectrum by state-operator correspondence for the lightest operators in each symmetry sector as a function at different system size N_{orb} . The data are extracted through the time-displaced correlator for (a),(b) the $\text{Sp}(10)$ model at $V = 5.0$ and (c), (d) the $\text{Sp}(4)$ model at $V = 6.0$. Panels (a) and (c) plot the multiplet $\partial^l \phi$ of ϕ in the A sector; (b) and (d) plot the multiplet $\partial^{l-1} J$ of J in the T sector.

gap $\Delta_{A,0} = \Delta_\phi$, using the value obtained from equal-time conformal correlators. With increasing system size, the extracted scaling dimensions in both sectors for both models approach the conformal predictions, providing further evidence for emergent conformal symmetry.

IV. DISCUSSION

In this paper, we have identified a family of CFTs on the fuzzy sphere which putatively corresponds to the $\text{SU}(2)$ QCD with N flavors of fermions in three space-time dimensions. It is a generalization of the $\text{SO}(5)$ deconfined criticality at $N = 2$. On the fuzzy sphere, we construct a model with $N_f = 2N$ flavors of fermions and $\text{Sp}(N)$ global symmetry. We show that the model matches the symmetry and anomaly with a $\text{Sp}(N)$ -symmetric NLSM with a level-1 WZW topological term. Extending the NLSM to the strongly coupled region, its phase diagram depends on N . For $N > N_c$, defining the conformal window, the theory exhibits a symmetry-broken phase and a critical QCD phase separated by a continuous phase transition. For $N < N_c$, the phase diagram contains only a symmetry-broken phase and may display pseudocritical behavior in certain parameter regions, as putatively expected for the DQCP.

This fuzzy-sphere model is free of the sign problem and therefore accessible with large-scale quantum Monte Carlo simulations. For $N \geq 4$, we find evidence for the phase diagram containing a symmetry-broken phase and a critical QCD phase separated by a continuous phase transition. By computing real-space equal-time correlators as well as

time-displaced correlators that probe excited states, we identify signatures of conformal symmetry in the critical phase, including conformal two-point functions and integer-spaced conformal multiplets. From the conformal correlator, we extract the scaling dimension Δ_ϕ of the leading operator in the rank-2 traceless antisymmetric tensor representation, which serves as the order parameter for the phase transition. Notably, N enters the QMC formulation as a parameter instead of the number of fermions simulated, allowing us to reach up to $N = 16$. Our results on the fuzzy sphere in the large N are consistent with the perturbative large- N calculation.

This agreement between fuzzy sphere and large- N expansion provides additional evidence that the conformal field theory observed on the fuzzy sphere indeed corresponds to $\text{SU}(2)$ QCD₃. In earlier fuzzy-sphere studies of the $\text{SO}(5)$ DQCP [21] and $\text{Sp}(N)$ CFTs [69], the matching with candidate Lagrangian descriptions was limited to the global symmetries and anomalies. Due to the strongly coupled nature of these theories, a more quantitative comparison was not previously available. Our work represents the first extension of the fuzzy-sphere approach into the large- N region, where the CFT becomes weakly coupled and the perturbative large- N expansion is quantitatively reliable. In this region, we are able to demonstrate agreement between fuzzy-sphere results and large- N predictions, providing evidence that the family of $\text{Sp}(N)$ -symmetric fuzzy-sphere models realizes the candidate Lagrangian description of $\text{SU}(2)$ QCD₃. Meanwhile, in the large- N limit, the fuzzy-sphere model may exhibit semiclassical behavior, and it may be interesting to

investigate it perturbatively using approaches like saddle-point expansion.

Returning to the strongly coupled region at finite N , we have presented evidence that $SU(2)$ QCD₃ is conformal for flavor number $N \geq 4$. Combined with previous results indicating that the DQCP at $N = 2$ is not genuinely critical, our findings tentatively suggest that the boundary of the conformal window lies in the range $2 < N_c < 4$. A sharper diagnostic of the conformal window is provided by the scaling dimension of the singlet operator S_+ , which governs the RG flow in the phase diagram of the extended NLSM (Fig. 1). For $N > N_c$, S_+ is irrelevant at the stable QCD fixed point and with $\Delta_{S_+} > 3$; for $N < N_c$, S_+ becomes relevant with $\Delta_{S_+} < 3$ [21]; and at the critical value $N = N_c$, S_+ is exactly marginal with $\Delta_{S_+} = 3$. Determining the scaling dimension Δ_{S_+} would therefore provide a decisive criterion for the conformal-window boundary N_c . However, the operator S_+ can be realized only by a four-fermion term rather than a two-fermion density operator, and its measurement requires more sophisticated QMC techniques. Since N enters the QMC formulation as a continuous parameter, it is in principle possible to analytically continue N to non-even-integer values and determine $\Delta_{S_+}(N)$ as a smooth function of N , opening the possibility of directly observing $\Delta_{S_+}(N_c) = 3$.

While this work focuses on the conformal window of $SU(2)$ QCD₃, analogous conformal-window problems arise more broadly in three-dimensional critical gauge theories involving multiple flavors of fermions or scalars coupled to gauge fields, such as $SU(k)$, $U(k)$, or $Sp(k)$. Some of these theories may be realisable on the fuzzy sphere through appropriate NLSM-WZW constructions, while others, such as $U(k)$ gauge theories, have additional challenges realizing their $U(1)$ flux symmetry. Further discussion is provided in the Appendix B. An especially important open problem is the conformal window of QED₃ with N flavors of Dirac fermions, which is directly tied to the nature of the Dirac spin liquid at $N = 4$.

The auxiliary-field QMC simulations were carried out with the ALF package [94–96]. The WIGXJPF library was used for the $3j$ symbols [97].

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DATA AVAILABILITY

The data that support the findings of this manuscript are openly available at Zenodo in Ref. [98].

APPENDIX A: THE ANGULAR COMPONENTS OF THE DENSITY OPERATORS

The density operator is local fermion bilinear

$$n_M(\hat{\mathbf{n}}) = \psi_i^\dagger(\hat{\mathbf{n}}) M^i_j \psi^j(\hat{\mathbf{n}}). \quad (\text{A1})$$

Here, the matrix insertion M puts the density operators in a certain representation of the flavor symmetry.

Like the fermion operator, the density operator can also be expressed in the orbital space

$$n_M(\hat{\mathbf{n}}) = \frac{1}{R^2} \sum_{lm} Y_{lm}(\hat{\mathbf{n}}) n_{M,lm}. \quad (\text{A2})$$

Conversely,

$$\begin{aligned} n_{M,lm} &= R^2 \int d^2\hat{\mathbf{n}} \bar{Y}_{lm}(\hat{\mathbf{n}}) n_M(\hat{\mathbf{n}}) \\ &= \int d^2\hat{\mathbf{n}} \bar{Y}_{lm}(\hat{\mathbf{n}}) \left(\sum_{m_1} \bar{Y}_{sm_1}^{(s)}(\hat{\mathbf{n}}) c_{m_1 i}^\dagger \right) M^i_j \left(\sum_{m_2} Y_{sm_2}^{(s)}(\hat{\mathbf{n}}) c_{m_2}^j \right) \\ &= \sum_{m_1 m_2} c_{m_1 i}^\dagger M^i_j c_{m_2}^j \Lambda_{m_1 m_2}^{(n)lm}, \end{aligned} \quad (\text{A3})$$

$$\Lambda_{m_1 m_2}^{(n) lm} = \int d^2 \hat{\mathbf{n}} \bar{Y}_{lm}(\hat{\mathbf{n}}) \bar{Y}_{sm_1}^{(s)}(\hat{\mathbf{n}}) Y_{sm_2}^{(s)}(\hat{\mathbf{n}}) \\ = \delta_{m+m_1, m_2} (-1)^{s+m_2} (2s+1) \sqrt{\frac{2l+1}{4\pi}} \begin{pmatrix} s & l & s \\ m_1 & m & -m_2 \end{pmatrix} \begin{pmatrix} s & l & s \\ s & 0 & -s \end{pmatrix}, \quad (\text{A4})$$

where $(\begin{smallmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{smallmatrix})$ is the $3j$ symbol. In this way, we have fully expressed the density operator in terms of the operators in the orbital space $c_{mf}^{(\dagger)}$.

Similarly, we can decompose the pairing operators in terms of the monopole spherical harmonics

$$\Delta(\mathbf{r}) = \frac{1}{R^2} \sum_{lm} Y_{lm}^{(2s)} \Delta_{lm}, \\ \Delta_{2s, m} = \sum_{m_1 m_2} c_{m_1}^i \Omega_{ij}^j c_{m_2}^j \Lambda^{(\Delta) 2s, m}_{m_1 m_2}, \\ \Lambda^{(\Delta) 2s, m}_{m_1 m_2} = \delta_{m_1+m_2, m} (-1)^m \frac{2s+1}{\sqrt{4\pi}} \begin{pmatrix} s & s & 2s \\ m_1 & m_2 & -m \end{pmatrix}. \quad (\text{A5})$$

Here the only nonvanishing component is $l = 2s$. From these components we can build the Hamiltonian Eq. (3).

We also write the correlation function of two points on the unit sphere in terms of the angular components

$$\langle 0 | n_M(\mathbf{r}_1) n_M(\mathbf{r}_2) | 0 \rangle \\ = \sum_{l m_1 m_2} \frac{1}{R^4} \langle n_{M, l m_1} n_{M, l m_2} \rangle Y_{l m_1}(\mathbf{r}_1) Y_{l m_2}(\mathbf{r}_2) \\ = \sum_l \frac{2l+1}{R^2} P_l(\cos \theta_{12}) \langle n_{M, l 0} n_{M, l 0} \rangle. \quad (\text{A6})$$

APPENDIX B: THE DERIVATION OF THE WZW LEVEL

1. General principle

We start from N_f flavors of free fermions on the lowest Landau level on a large torus; the action reads

$$Z = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{-S_\psi}, \\ S_\psi[\psi, \bar{\psi}] = \int d^2 \mathbf{r} d\tau \bar{\psi}_a(\mathbf{r}, \tau) \frac{\partial}{\partial \tau} \psi^a(\mathbf{r}, \tau). \quad (\text{B1})$$

The fermion fields are projected to the lowest Landau level:

$$\psi^a(\mathbf{r}, \tau) = \frac{1}{\sqrt{L_y}} \sum_{p_y} e^{ip_y y} \phi_{p_y}(x) c_{p_y}^a(\tau). \quad (\text{B2})$$

The orbitals are labeled by the momentum along y direction $p_y = 2\pi m/L_y$, where $m = 1, \dots, N_\phi = V/2\pi = L_x L_y/2\pi$,

and the wave function in the limit of large V is

$$\phi_{p_y}(x) = \frac{1}{\pi^{1/4}} \exp \left[-\frac{1}{2} (x - p_y)^2 \right]. \quad (\text{B3})$$

We then couple the free fermions to a matrix field $\hat{\mathbf{Q}}(\mathbf{r}, \tau)$ that lives on a Grassmannian:

$$Z = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}\hat{\mathbf{Q}} e^{-S_\psi - S_Q}, \\ S_Q[\psi, \bar{\psi}, \hat{\mathbf{Q}}] = -\lambda \int d^2 \mathbf{r} d\tau \bar{\psi}_a \hat{Q}_a^b \psi^b. \quad (\text{B4})$$

The standard parametrization of the Grassmannian is

$$\mathbf{Q} = \mathbf{g} \Sigma \mathbf{g}^{-1} \in \frac{G(N)}{G(N) \times G(N-M)}, \\ \mathbf{g} \in G(N), \quad \Sigma = \begin{pmatrix} \mathbb{I}_M & 0 \\ 0 & -\mathbb{I}_{N-M} \end{pmatrix}, \\ G(N) = \text{Sp}(N/2), \text{U}(N), \text{O}(N). \quad (\text{B5})$$

In the vicinity of Σ , $\mathbf{Q}$ can be expanded in terms of the generators $\mathbf{t}_i$ of $\mathfrak{g}(N)$:

$$\mathbf{g} = 1 + \epsilon^i \mathbf{t}_i, \\ \mathbf{Q} = (1 + \epsilon^i \mathbf{t}_i) \Sigma (1 - \epsilon^i \mathbf{t}_i) \\ = \Sigma + \epsilon^i \mathbf{A}_i, \quad \mathbf{A}_i = [\mathbf{t}_i, \Sigma]. \quad (\text{B6})$$

On the other hand, we consider a general setup where the fermions on the LLL do not have to be N flavor in the fundamental representation. For example, N_f can be a multiplication of N , and the $N_f \times N_f$ matrix field $\hat{\mathbf{Q}}$ depends on the physical setup of the fermions. The hat denotes that $\hat{\mathbf{Q}}$ may be different from the standard parametrization. We list several examples.

- (i) For $N_f = N$ flavors of fundamental fermions, $\hat{\mathbf{Q}} = \mathbf{Q}$;
 - (ii) for $N_f = 2N$ flavors of fermions containing two copies of fundamental, $\hat{\mathbf{Q}} = \mathbf{Q} \oplus \mathbf{Q}$;
 - (iii) for $N_f = 2N$ flavors of fermions containing N fundamental and N antifundamental, $\hat{\mathbf{Q}} = \mathbf{Q} \oplus \mathbf{Q}^\dagger$.
- Physically, this coupling captures certain four-fermion interaction. Expanding this interaction in the momentum space,

$$S_Q[c, \bar{c}, \hat{\mathbf{Q}}] = -\lambda \int d\tau \sum_{p_y, \mathbf{q}} F(p_y, \mathbf{q}) \bar{c}_{a, p_y + q_y/2}(\tau) \hat{Q}^a_b(\mathbf{q}, \tau) c^b_{p_y - q_y/2}(\tau),$$

$$F(p_y, \mathbf{q}) = \int dx e^{iq_x x} \bar{\phi}_{p_y + q_y/2}(x) \phi_{p_y - q_y/2}(x) = e^{-q^2/4} e^{-iq_x p_y}, \quad (\text{B7})$$

where we have used the Fourier transformation

$$c_{p_y}(\tau) = \int \frac{d\omega}{2\pi} e^{i\omega\tau} c_{p_y}(\omega),$$

$$\hat{Q}^a_b(\mathbf{r}, \tau) = \frac{1}{V} \sum_{\mathbf{q}} \hat{Q}^a_b(\mathbf{q}, \tau) e^{i\mathbf{q}\cdot\mathbf{r}}$$

$$= \int \frac{d\nu}{2\pi} \frac{1}{V} \sum_{\mathbf{q}} \hat{Q}^a_b(\mathbf{q}, \nu) e^{i\mathbf{q}\cdot\mathbf{r} + i\nu\tau}.$$

The field $\hat{\mathbf{Q}}(\mathbf{r}, \tau)$ can be expanded into the infinitesimal fields

$$\hat{\mathbf{Q}}(\mathbf{r}, \tau) = \hat{\Sigma}(\mathbf{r}, \tau) + \phi^i(\mathbf{r}, \tau) \hat{\mathbf{A}}_i. \quad (\text{B8})$$

In terms of ϕ^i ,

$$S[c, \bar{c}, \phi] = \int d\tau \left[\sum_{p_y} \bar{c}_{a, p_y} \left(\delta^a_b \frac{\partial}{\partial \tau} - \lambda \hat{\Sigma}^a_b \right) c^b_{p_y} \right. \\ \left. - \lambda \sum_{p_y, \mathbf{q}} F(p_y, \mathbf{q}) \bar{c}_{a, p_y + q_y/2} \phi^i (\hat{\mathbf{A}}_i)^a_b c^b_{p_y - q_y/2} \right]. \quad (\text{B9})$$

Regarding the first term as the free action and the second term as the perturbation, the Feynman rules read⁸

$$a \xrightarrow{p} b = G^a_b(\omega)$$

$$\mathbf{G}(\omega) = (i\omega - \lambda \hat{\Sigma})^{-1}$$

$$I \xrightarrow{p} \begin{array}{c} p - \frac{1}{2}q \\ \nearrow a \\ \searrow b \\ p + \frac{1}{2}q \end{array} = (\Gamma_i)^a_b(p_y, \mathbf{q}) \quad (\text{B10})$$

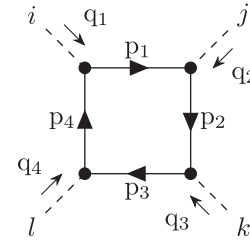
$$\Gamma_i(p_y, \mathbf{q}) = F(p_y, \mathbf{q}) \hat{\mathbf{A}}_i$$

To obtain an effective action with respect to ϕ^i , we integrate out $c, \bar{c}$:

⁸Here we adopt a shorthand notation that $p = (\omega, p_y)$ and $q = (\nu, \mathbf{q})$.

$$Z = \int \mathcal{D}\bar{c} \mathcal{D}c \mathcal{D}\phi e^{-S[c, \bar{c}, \phi]} = \int \mathcal{D}\phi e^{-S_{\text{eff}}[\phi]}. \quad (\text{B11})$$

To the second order, S_{eff} contains the kinetic term in the NLSM - ; to the fourth order, it contains the box diagram that corresponds to the WZW term that we desire:



$$(\text{B12})$$

To compare this box diagram term with the WZW term in the NLSM, we first express the WZW action in terms of the ϕ^i fields. In the NLSM, the matrix field $\mathbf{Q}(\mathbf{r}, \tau)$ is extended into auxiliary fourth dimension $\tilde{\mathbf{Q}}(\mathbf{r}, \tau, u)$ parametrized by $0 \leq u \leq 1$ with boundary condition

$$\tilde{\mathbf{Q}}(\mathbf{r}, \tau, u = 1) = \mathbf{Q}(\mathbf{r}, \tau), \quad \tilde{\mathbf{Q}}(\mathbf{r}, \tau, u = 0) = \Sigma.$$

Specifically, we take

$$\tilde{\mathbf{Q}}(\mathbf{r}, \tau, u) = \Sigma + u\delta\mathbf{Q} = \Sigma + u\phi^i \mathbf{A}_i, \quad (\text{B13})$$

$$S_{\text{WZW}} = i\kappa \int_0^1 du \int d^2\mathbf{r} d\tau \epsilon^{\mu\nu\rho\sigma} \text{tr}(\tilde{\mathbf{Q}} \partial_\mu \tilde{\mathbf{Q}} \partial_\nu \tilde{\mathbf{Q}} \partial_\rho \tilde{\mathbf{Q}} \partial_\sigma \tilde{\mathbf{Q}})$$

$$= 24i\kappa \int_0^1 du \int d^2\mathbf{r} d\tau \text{tr}(\tilde{\mathbf{Q}} \partial_x \tilde{\mathbf{Q}} \partial_y \tilde{\mathbf{Q}} \partial_\tau \tilde{\mathbf{Q}} \partial_u \tilde{\mathbf{Q}})$$

$$= 24i\kappa \int_0^1 u^3 du \int d^2\mathbf{r} d\tau \text{tr}(\Sigma \partial_x \delta\mathbf{Q} \partial_y \delta\mathbf{Q} \partial_\tau \delta\mathbf{Q} \delta\mathbf{Q})$$

$$= 6i\kappa \text{tr}(\Sigma \mathbf{A}_i \mathbf{A}_j \mathbf{A}_k \mathbf{A}_l) \int d^2\mathbf{r} d\tau \partial_x \phi^i \partial_y \phi^j \partial_\tau \phi^k \phi^l, \quad (\text{B14})$$

where κ is quantized to be

$$\kappa = \frac{2\pi k}{(16\pi)^2}.$$

On the other hand, the box diagram (B12) gives

$$\begin{aligned}
 S_{\square} &= - \int \frac{d\nu_1}{2\pi} \frac{d\nu_2}{2\pi} \frac{d\nu_3}{2\pi} \sum_{\mathbf{q}_1 \mathbf{q}_2 \mathbf{q}_3} \phi^i(\mathbf{q}_1, \nu_1) \phi^j(\mathbf{q}_2, \nu_2) \phi^k(\mathbf{q}_3, \nu_3) \phi^l(\mathbf{q}_4, \nu_4) \\
 &\quad \times \frac{1}{V} \sum_{p_y} \int \frac{d\omega}{2\pi} \text{tr}(\mathbf{G}(\omega_1) \Gamma_i(p_{1y}, \mathbf{q}_1) \mathbf{G}(\omega_2) \Gamma_j(p_{y,2}, \mathbf{q}_2) \mathbf{G}(\omega_3) \Gamma_k(p_{y,3}, \mathbf{q}_3) \mathbf{G}(\omega_4) \Gamma_l(p_{y,4}, \mathbf{q}_4)) \\
 &= \int \frac{d\nu_1}{2\pi} \frac{d\nu_2}{2\pi} \frac{d\nu_3}{2\pi} \sum_{\mathbf{q}_1 \mathbf{q}_2 \mathbf{q}_3} \mathcal{I}_{ijkl}^{\square}(\{\nu, \mathbf{q}\}) \phi^i(\mathbf{q}_1, \nu_1) \phi^j(\mathbf{q}_2, \nu_2) \phi^k(\mathbf{q}_3, \nu_3) \phi^l(\mathbf{q}_4, \nu_4),
 \end{aligned} \tag{B15}$$

where

$$\begin{aligned}
 \mathbf{q}_4 &= -\mathbf{q}_1 - \mathbf{q}_2 - \mathbf{q}_3, \\
 p_1 &= p + \frac{1}{2} \mathbf{q}_1, \\
 p_2 &= p + \mathbf{q}_1 + \mathbf{q}_2, \\
 p_3 &= p + \mathbf{q}_1 + \mathbf{q}_2 + \frac{1}{2} \mathbf{q}_3, \\
 p_4 &= p - \frac{1}{2} \mathbf{q}_1,
 \end{aligned}$$

and the form factor⁹

$$\begin{aligned}
 \mathcal{I}_{ijkl}^{\square}(\{\nu, \mathbf{q}\}) &= -\frac{1}{V} \sum_{p_y} \int \frac{d\omega}{2\pi} \\
 &\quad \times \text{tr}(\mathbf{G}(\omega_1) \Gamma_i(p_{1y}, \mathbf{q}_1) \mathbf{G}(\omega_2) \Gamma_j(p_{y,2}, \mathbf{q}_2) \\
 &\quad \times \mathbf{G}(\omega_3) \Gamma_k(p_{y,3}, \mathbf{q}_3) \mathbf{G}(\omega_4) \Gamma_l(p_{y,4}, \mathbf{q}_4)).
 \end{aligned} \tag{B16}$$

Writing the WZW term in the same form,

$$\begin{aligned}
 S_{\text{WZW}} &= \int \frac{d\nu_1}{2\pi} \frac{d\nu_2}{2\pi} \frac{d\nu_3}{2\pi} \sum_{\mathbf{q}_1 \mathbf{q}_2 \mathbf{q}_3} \mathcal{I}_{ijkl}^{\text{WZW}}(\{\nu, \mathbf{q}\}) \\
 &\quad \times \phi^i(\mathbf{q}_1, \nu_1) \phi^j(\mathbf{q}_2, \nu_2) \phi^k(\mathbf{q}_3, \nu_3) \phi^l(\mathbf{q}_4, \nu_4), \\
 \mathcal{I}_{ijkl}^{\text{WZW}}(\{\nu, \mathbf{q}\}) &= 6\kappa \text{tr}(\Sigma \mathbf{A}_i \mathbf{A}_j \mathbf{A}_k \mathbf{A}_l) q_{1x} q_{2y} \nu_3.
 \end{aligned} \tag{B17}$$

We need to match $\mathcal{I}^{\square}$ and $\mathcal{I}^{\text{WZW}}$ in the long-wavelength limit $\{\mathbf{q}, \nu\} \rightarrow 0$:

$$6\kappa \text{tr}(\Sigma \mathbf{A}_i \mathbf{A}_j \mathbf{A}_k \mathbf{A}_l) = \frac{\partial}{\partial q_{1x}} \frac{\partial}{\partial q_{2y}} \frac{\partial}{\partial \nu_3} \mathcal{I}_{ijkl}^{\square} \Big|_{\{\mathbf{q}, \nu\}=0}. \tag{B18}$$

⁹Since the integral does not depend on p_y , the sum $\frac{1}{V} \sum_{p_y}$ will give only a factor $1/2\pi$.

2. Models

a. Warm-up: SO(5) deconfined criticality

In the Sp(2) DQCP, the Grassmannian is parametrized by the Γ matrices:

$$\mathbf{Q} = \hat{\mathbf{Q}} = n^i \Gamma_i, \quad n^i n_i = 1. \tag{B19}$$

We choose one component as the reference and the rest as the perturbations:

$$\Sigma = \Gamma_0, \quad \mathbf{A} = (\Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5). \tag{B20}$$

Under this choice, $\text{tr}(\Sigma \mathbf{A}_i \mathbf{A}_j \mathbf{A}_k \mathbf{A}_l) = 4\epsilon_{ijkl}$. By evaluating Eqs. (B16) and (B18), we recover the result

$$\kappa = \frac{2\pi k}{(16\pi)^2}, \quad k = 1.$$

b. Symplectic Grassmannian $\mathbf{G} = \text{Sp}$

We now move on to a more general case of the Grassmannian

$$\frac{\text{Sp}(N)}{\text{Sp}(M) \times \text{Sp}(N-M)}.$$

We consider $N_f = 2N$ fermions in the fundamental representation. The generators in Eq. (B6) are parametrized by

$$\begin{aligned}
 (t_{(a_0 b_0)})^a_b &= \frac{1}{2} (\delta^a_{a_0} \Omega_{b_0 b} + \delta^a_{b_0} \Omega_{a_0 b}), \\
 (A_{(a_0 b_0)})^a_b &= \delta^a_{a_0} \Omega_{b_0 b} - \delta^a_{b_0} \Omega_{a_0 b}, \\
 a_0 &= 1, \dots, 2M, \quad b_0 = 2M+1, \dots, 2N,
 \end{aligned} \tag{B21}$$

where the symplectic two form is taken as

$$\Omega = \begin{pmatrix} i\sigma^2 & & \\ & \ddots & \\ & & i\sigma^2 \end{pmatrix},$$

the index $i = (a_0 b_0)$ denotes that $\mathbf{t}_i$ lives in the symmetric rank-2 tensor representation, and the $A_{(a_0 b_0)}$ outside the range of $a_0 \leq 2M$ and $b_0 > 2M$ vanish. Correspondingly, the ϕ^i fields live in the bifundamental representation of $\text{Sp}(M) \times \text{Sp}(N - M)$. By evaluating Eqs. (B16) and (B18) component by component, we obtain

$$\kappa = \frac{2\pi}{(16\pi)^2}, \quad k = 1.$$

Several comments:

- (1) The $\text{Sp}(2)$ case is also recovered, as ϕ^i lives in the fundamental representation of $\text{O}(4) = \text{Sp}(1) \times \text{Sp}(1)$;
- (2) the theories with different (N, M) have the same WZW level, as the components $\mathcal{I}_{ijkl}^\square$ for smaller N and $N - M$ is a subset of larger ones.

c. Orthogonal Grassmannian $\mathbf{G}=\mathbf{O}$

We consider $N_f = N$ fermions in the vector representation. The generators in Eq. (B6) are parametrized by

$$\begin{aligned} (t_{[a_0 b_0]})_{ab} &= \frac{1}{2}(\delta_{aa_0} \delta_{bb_0} - \delta_{ab_0} \delta_{ba_0}), \\ (A_{[a_0 b_0]})_{ab} &= \delta_{aa_0} \delta_{bb_0} + \delta_{ab_0} \delta_{ba_0}, \\ a_0 &= 1, \dots, M, \quad b_0 = M + 1, \dots, N. \end{aligned} \quad (\text{B22})$$

By evaluating Eqs. (B16) and (B18) component by component, we obtain

$$\kappa = \frac{2\pi}{(16\pi)^2}.$$

Note that this now corresponds to

$$k = 2,$$

because for O and SO Grassmannian the quantization of the WZW term is halved.

d. Unitary Grassmannian $\mathbf{G}=\mathbf{U}$

We consider $N_f = N$ fermions in the fundamental representation. The generators in Eq. (B6) are parametrized by

$$\begin{aligned} (t_{a_0 b_0})^a_b &= \delta_{a_0}^a \delta_b^{b_0}, \\ (A_{a_0 b_0})^a_b &= \begin{cases} 2\delta_{a_0}^a \delta_b^{b_0}, & a_0 = 1, \dots, M, b_0 = M + 1, \dots, N, \\ -2\delta_{a_0}^a \delta_b^{b_0}, & a_0 = M + 1, \dots, N, b_0 = 1, \dots, M. \end{cases} \end{aligned} \quad (\text{B23})$$

By evaluating Eqs. (B16) and (B18) component by component, we obtain

$$\kappa = \frac{2\pi}{(16\pi)^2}, \quad k = 1.$$

We note that the U Grassmannian possesses an additional $\text{U}(1)_m$ symmetry associated with the nontrivial homotopy group $\pi_2 = \mathbb{Z}$. In a CFT arising from the corresponding NLSM, this $\text{U}(1)$ symmetry cannot be identified with the electric $\text{U}(1)_e$, since $\text{U}(1)_e$ is necessarily gapped on the fuzzy sphere.

e. Manipulations of the WZW level

We now consider more general cases beyond N flavors of fermions in the fundamental representation of $\text{G}(N)$ and $\hat{\mathbf{Q}} = \mathbf{Q}$. We discuss how different manipulations can modify the WZW level.

- (1) Complex conjugate does not change the WZW level:

$$\begin{aligned} \hat{\mathbf{Q}}' &= \hat{\mathbf{Q}}^\dagger, & \hat{\mathbf{t}}' &= -\hat{\mathbf{t}}^\dagger, & \hat{\mathbf{A}}' &= \hat{\mathbf{A}}^\dagger, \\ \mathcal{I}'^\square &= \mathcal{I}^\square, & \kappa' &= \kappa. \end{aligned}$$

- (2) Particle-hole transformation inverses the WZW level:

$$\hat{\Sigma}' = -\hat{\Sigma}, \quad \hat{\mathbf{A}}' = -\hat{\mathbf{A}}, \quad \mathcal{I}'^\square = -\mathcal{I}^\square, \quad \kappa' = -\kappa.$$

- (3) Stacking sums over the WZW level:

$$\hat{\mathbf{Q}}' = \hat{\mathbf{Q}}_1 \oplus \hat{\mathbf{Q}}_2, \quad \mathcal{I}'^\square = \mathcal{I}_1^\square + \mathcal{I}_2^\square, \quad \kappa' = \kappa_1 + \kappa_2.$$

3. Unitary Grassmannian: Matching the magnetic $\text{U}(1)_m$

In the $\text{G} = \text{U}$ Grassmannian, there is an additional $\text{U}(1)_m$ symmetry. To realize a CFT on the LLL, we should identify both the $\text{SU}(N)$ and the $\text{U}(1)_m$ as subgroups of the maximal $\text{SU}(N_f)$ global symmetry. To see if the $\text{U}(1)_m$ has the correct winding with $\text{SU}(N)$, we couple the $\text{U}(1)_m$ current to a gauge field a_μ through minimal coupling:

$$\begin{aligned} Z &= \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}\hat{\mathbf{Q}} \mathcal{D}a e^{-S_\psi - S_Q - S_a} \\ S_a[\psi, \bar{\psi}, a] &= - \int d^2\mathbf{r} d\tau a_\mu j^\mu \\ j^0 &= \bar{\psi} \hat{\mathbf{t}}_m \psi, \quad j^{1,2} = 0 \end{aligned} \quad (\text{B24})$$

$$= \delta^\mu_0 T^a_b F(p_y, \mathbf{q})$$

where $\hat{\mathbf{t}}_m$ is the $U(1)_m$ generator.¹⁰ Integrating out the fermions, the gauge field is coupled to the matrix field in the third order through the triangle diagram

(B25)

In the NLSM, this corresponds to coupling the gauge field a_μ to the skyrmion current

$$\begin{aligned}
 S_{\text{skyr}} &= \frac{q_m}{2\pi} \int d^2\mathbf{r} d\tau \epsilon^{\mu\nu\rho} a_\mu \text{tr}(\mathbf{Q} \partial_\nu \mathbf{Q} \partial_\rho \mathbf{Q}) \\
 &= \frac{q_m}{2\pi} \text{tr}(\Sigma \mathbf{A}_i \mathbf{A}_j) \int d^2\mathbf{r} d\tau \epsilon^{\mu\nu\rho} a_\mu \partial_\nu \phi^i \partial_\rho \phi^j \\
 &= \int \frac{d\nu_1}{2\pi} \frac{d\nu_2}{2\pi} \sum_{\mathbf{q}_1 \mathbf{q}_2} \mathcal{I}_{ij}^{\mu, \text{skyr}} \\
 &\quad \times a_\mu(\mathbf{q}_a, \nu_a) \phi^i(\mathbf{q}_1, \nu_1) \phi^j(\mathbf{q}_2, \nu_2), \\
 \mathcal{I}_{ij}^{\mu, \text{skyr}} &= \frac{q_m}{2\pi} \text{tr}(\Sigma \mathbf{A}_i \mathbf{A}_j) \epsilon^{\mu\nu\rho} q_{1\nu} q_{2\rho},
 \end{aligned} \tag{B26}$$

where q_m is the skyrmion charge. Evaluating the triangle diagram Eq. (B25) explicitly,

$$\begin{aligned}
 S_\Delta &= - \int \frac{d\nu_1}{2\pi} \frac{d\nu_2}{2\pi} \sum_{\mathbf{q}_1 \mathbf{q}_2} a_0(\mathbf{q}_a, \nu_a) \phi^i(\mathbf{q}_1, \nu_1) \phi^j(\mathbf{q}_2, \nu_2) \\
 &\quad \times \frac{1}{V} \sum_{p_y} \int \frac{d\omega}{2\pi} \text{tr}(\mathbf{G}(\omega_1) \Gamma_i(p_{1y}, \mathbf{q}_1) \mathbf{G}(\omega_2) \Gamma_i(p_{2y}, \mathbf{q}_2) \mathbf{G}(\omega_3) \hat{\mathbf{t}}_m F(p_{3y}, \mathbf{q}_3)) \\
 &= \int \frac{d\nu_1}{2\pi} \frac{d\nu_2}{2\pi} \sum_{\mathbf{q}_1 \mathbf{q}_2} \mathcal{I}_{ij}^{0, \Delta}(\{\nu, \mathbf{q}\}) a_0(\mathbf{q}_a, \nu_a) \phi^i(\mathbf{q}_1, \nu_1) \phi^j(\mathbf{q}_2, \nu_2), \\
 \mathcal{I}_{ij}^{\mu, \Delta} &= \frac{1}{V} \sum_{p_y} \int \frac{d\omega}{2\pi} \text{tr}(\mathbf{G}(\omega_1) \Gamma_i(p_{1y}, \mathbf{q}_1) \mathbf{G}(\omega_2) \Gamma_i(p_{2y}, \mathbf{q}_2) \mathbf{G}(\omega_3) \hat{\mathbf{t}}_m F(p_{3y}, \mathbf{q}_3)).
 \end{aligned} \tag{B27}$$

Matching $\mathcal{I}^\Delta$ and $\mathcal{I}^{\text{skyr}}$ in the long-wavelength limit, we obtain the skyrmion charge of $U(1)_m$:

$$\frac{q_m}{2\pi} \text{tr}(\Sigma \mathbf{A}_i \mathbf{A}_j) = \frac{\partial}{\partial q_{1x}} \frac{\partial}{\partial q_{2y}} \mathcal{I}_{ij}^{0, \Delta} \Big|_{\{\mathbf{q}, \nu\}=0}. \tag{B28}$$

APPENDIX C: QMC DETAILS

We begin this section by commenting on the form of the density interaction used in the Hubbard-Stratonovich decoupling. Note the relation $n_{lm}^\dagger = (-1)^m n_{l-m}$, thus, we have

$$\begin{aligned}
 &\frac{1}{4} \sum_m [(1+i)n_{lm}^\dagger + (1-i)n_{lm}]^2 \\
 &= \frac{1}{4} \sum_m [(1+i)^2 n_{lm}^\dagger n_{lm}^\dagger + (1-i)^2 n_{lm} n_{lm} \\
 &\quad + (1+i)(1-i)(n_{lm}^\dagger n_{lm} + n_{lm} n_{lm}^\dagger)]
 \end{aligned} \tag{C1}$$

¹⁰The fermions only couple to the temporal component of the gauge field; we expect the Lorentz symmetry to emerge.

$$\begin{aligned}
 &= \frac{1}{2} \sum_m [in_{lm}^\dagger n_{lm}^\dagger - in_{lm} n_{lm} \\
 &\quad + (n_{lm}^\dagger n_{lm} + n_{lm} n_{lm}^\dagger)]
 \end{aligned} \tag{C2}$$

$$\begin{aligned}
 &= \frac{1}{2} \sum_m [in_{l-m} n_{l-m} - in_{lm} n_{lm} \\
 &\quad + (n_{lm}^\dagger n_{lm} + (-1)^m n_{lm} n_{l-m})]
 \end{aligned} \tag{C3}$$

$$= \sum_m n_{lm}^\dagger n_{lm}. \tag{C4}$$

Thus, the density interaction can be written as a perfect square of a Hermitian operator and is thus readily decoupled using a Hubbard-Stratonovich transformation.

As briefly mentioned in the main text, we are using a higher-order Trotter decomposition [91]. Let us assume $H = \sum_{i=1}^N O_i$ to simplify the notation; thus,

$$\begin{aligned}
 Z &= \text{Tr}[e^{-\beta H}] \\
 &= \text{Tr} \left[\prod_{\tau=1}^{L_{\text{tot}}} \prod_{s=1}^2 U_{\tau, s} \right] + \mathcal{O}(\Delta\tau^2), \quad \text{with} \\
 U_{\tau, s} &= e^{d_s \Delta\tau O_N} \dots e^{d_s \Delta\tau O_1} e^{c_s \Delta\tau O_1} \dots e^{c_s \Delta\tau O_N}.
 \end{aligned} \tag{C5}$$

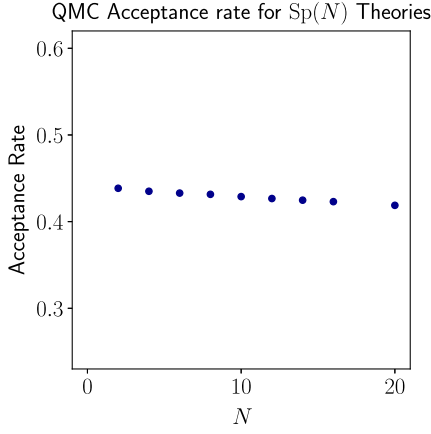


FIG. 8. QMC acceptance rate as a function of N for $\text{Sp}(N)$ theories with $N_{\text{orb}} = 15$. There is very little reduction in acceptance even for $N = 20$. Error bars are smaller than the point size.

The coefficients are $c_1 = d_2 = 0.21178$ and $c_2 = d_1 = 0.28822$. Similar to $e^{A+B} \approx e^{A/2} e^B e^{A/2}$, this decomposition is symmetric, but the c_s and d_s coefficients have been tuned to minimize the prefactor of the leading-order Trotter error [91].

Furthermore, we scale the inverse temperature $\beta = 10s$ with the monopole flux s , i.e., the system size, such that all presented results are ground state properties. We use $\Delta\tau = 0.1$ throughout the manuscript.

In Fig. 8, we plot the average acceptance rate of proposed auxiliary-field updates with respect to the number of flavors N in the $\text{Sp}(N)$ theories at a constant $N_{\text{orb}} = 15$. The acceptance rate roughly stays around 40%–50% and only slightly decreases with increasing N . Hence, this algorithm is well suited to studying large- N theories, retaining the polynomial scaling of sign-problem-free QMC.

APPENDIX D: QMC DATA FOR $\tilde{\Delta}_\phi$ DEEP IN THE CRITICAL PHASE AND FITTING DETAILS

First, we present the $\tilde{\Delta}_\phi$ results for the model with $\text{Sp}(2)$, relevant for $\text{SO}(5)$ deconfined quantum criticality, in

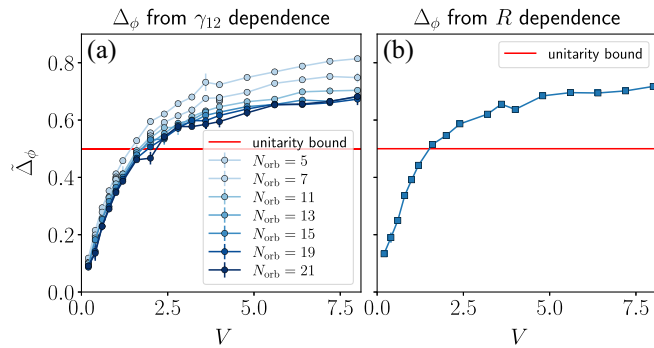


FIG. 9. Extrapolated $\tilde{\Delta}_\phi$ by γ_{12} dependence (a) and R dependence (b). In contrast to the $\text{Sp}(4)$ and $\text{Sp}(10)$ theories, for $\text{Sp}(2)$ we do not see a crossing point.

analogy to Fig. 4 of the main text. In doing so we benchmark the QMC simulations against the exact diagonalization results in [21]. In Fig. 9(a) we show $\tilde{\Delta}_\phi$ from

$$\tilde{\Delta}_\phi(\theta) = -\tan \gamma_{12} \left. \frac{\partial \log \mathcal{C}_A}{\partial \gamma_{12}} \right|_{\gamma_{12}=7\pi/9}, \quad (\text{D1})$$

as a function of V and N_{orb} . The value of $\gamma_{12} = 7\pi/9$ is the same value used in the main text due to its statistical precision and reduced finite-size effects. While these plots in general use the same analysis as those in the main text for $\text{Sp}(4)$ and $\text{Sp}(10)$, they do not show a crossing in Fig. 9(a). Instead, the data systematically drifts toward smaller values with system size, approaching the unitarity bound. This is

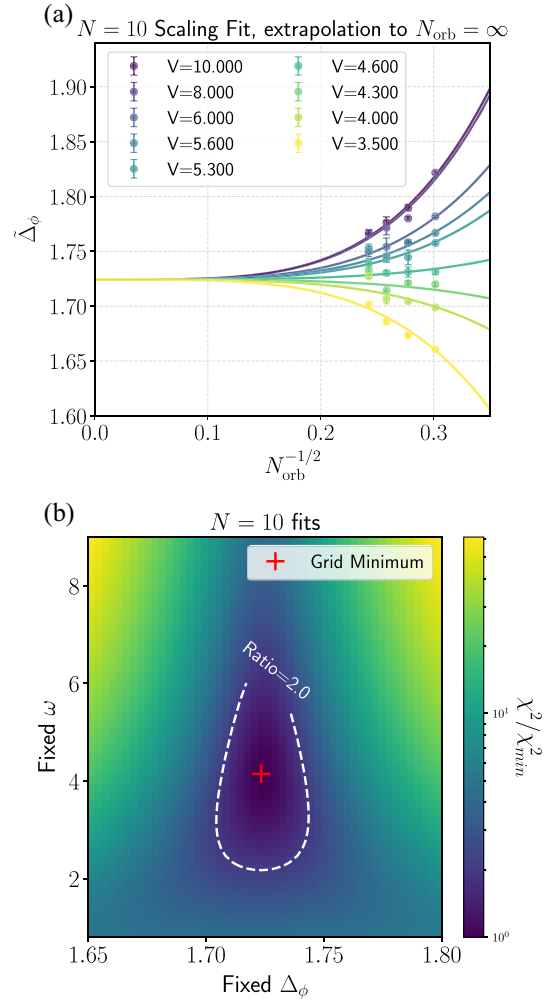


FIG. 10. Fits to determine Δ_ϕ for $N = 10$. The plot to the left shows data that is fit using the ansatz of $\Delta_\phi(V, N_{\text{orb}}) = \Delta_\phi + g(V)N_{\text{orb}}^{-\omega/2}$. The observable used for $\Delta_\phi(V, N_{\text{orb}})$ is the RG-invariant $\partial \mathcal{C}_A \partial \gamma_{12}$ at $\gamma_{12} = 7\pi/9$. The plot to the right gives the chi-square values for fits relative to the best chi-square fit, for a grid of fixed Δ and ω values.

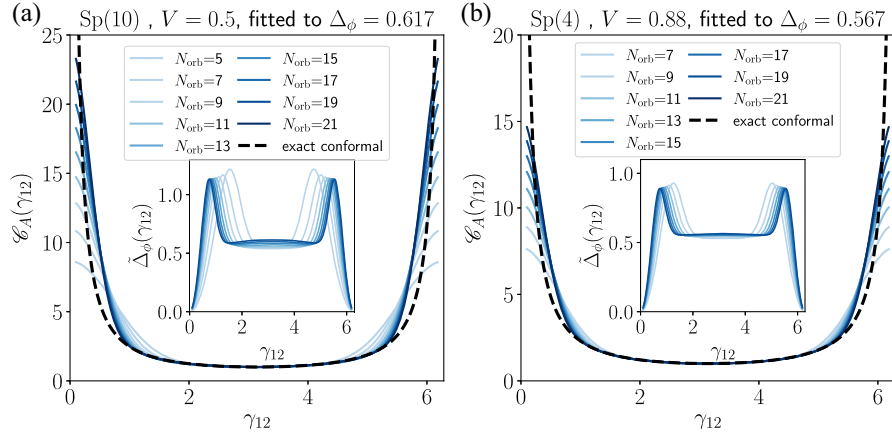


FIG. 11. Data close to criticality for Sp(10) in plot (a) and Sp(4) in plot (b).

consistent with the scenario depicted in Fig. 1, i.e., pseudocriticality ($2 < N_c$), for Sp(2), as found in [21]. Figure 9(b) depicts the same quantity, $\tilde{\Delta}_\phi$, extracted via its size dependence, which is consistent with the data in Fig. 9(a).

Finally, let us discuss the procedure used to determine “the” operator scaling dimension Δ_ϕ for each N , as reported in the table of Fig. 5 of the main text. Here, we consider the $\tilde{\Delta}_\phi$ data for a whole range of values of N_{orb} and V , which are roughly in the critical phase. For example, we consider the range $3.5 \leq V \leq 10$ for Sp(10). Since the lowest irrelevant operator dominates the RG flow toward the stable CFT fixed point, the scaling corrections in its vicinity are given by [77]

$$\tilde{\Delta}_\phi(V, N_{\text{orb}}) = \Delta_\phi + g(V)N_{\text{orb}}^{-\omega/2}. \quad (\text{D2})$$

Hence, we can fit the whole dataset and extract the scaling dimension Δ_ϕ for each Sp(N) symmetry group. An exemplary fit is depicted in Fig. 10(a). We evaluate the standard deviation χ^2 as a function of the fitting parameters Δ_ϕ and ω shown in Fig. 10(b) and use the ratio $\chi^2/\chi^2_{\text{min}} = 2$ to estimate the confidence interval. For Sp(10), we get $\Delta_\phi = 1.75(2)$. A similar analysis is repeated for all other values of N and the extracted scaling dimensions are summarized in Fig. 5 of the main text.

APPENDIX E: QMC DATA FOR $\tilde{\Delta}_\phi$ AT CRITICALITY

In the main text, we analyze RG-invariant correlation ratios to locate the phase transition V_c between the symmetry-broken phase at weak coupling and a symmetric phase at strong coupling. We proceed by identifying the symmetric phase as a stable Sp(N) symmetric CFT. Here, we supplement this discussion by analyzing the conformal symmetry at the critical point. Before we begin this discussion, we note that finite-size effects are expected to be more significant here since with increasing system size, we will ultimately flow to the symmetric or symmetry-broken phase, owing to the repulsive nature of the fixed point, unless we tune V to remain at V_c , which is beyond the scope of this study.

In Fig. 11 shows the correlation function $\mathcal{C}_A(\gamma_{12})$ and the logarithmic derivative as a proxy for the scaling dimension, $\tilde{\Delta}_\phi$, in the inset, analogous to Figs. 4(a) and 4(d) of the main text. We again find the data approach the conformal expectation with increasing system size rather well for both Sp(10) and Sp(4) symmetry. The inset exhibits a plateau for around $\gamma_{12} = \pi$, indicating a power-law correlation function and a scaling dimension $\Delta_\phi \approx 0.6$, consistent with the unitarity bound, $\Delta \geq 0.5$.

In both cases, the $\tilde{\Delta}_\phi(\gamma_{12} = \pi)$ ultimately increases for the largest systems [slightly more prominent for Sp(10)] indicating that V is very close to but slightly above V_c and should approach the scaling dimension of the stable conformal phase in the thermodynamic limit.

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