High-post-Newtonian-order dynamics induced by tail-of-tail interactions: The nongeodesic terms

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We compute the tail-of-tail contribution to the conservative dynamics of eccentric, nonspinning compact binaries to relative 1PN order and to $\mathcal{O}(e_t^{12})$. Using the 1PN quasi-Keplerian dynamics in harmonic coordinates, we derive the Delaunay-averaged Hamiltonian at 5.5PN and 6.5PN order and match it to the effective-one-body description, allowing us to determine the corresponding contributions to the nongeodesic EOB Q potential through the $\mathcal{O}(p_r^{12})$, including the dependence on the symmetric mass ratio up to $\mathcal{O}(\nu^2)$. The terms linear in the mass ratio reproduce the available first-order self-force results, while the quadratic terms provide qualitatively new eccentric second-order self-force predictions arising from the tail-of-tail terms. We independently rederive the averaged Hamiltonian using a Fourier-Bessel decomposition of the hereditary interaction. Applying the first law at fixed orbital frequencies, we recover known contributions to the first-order self-force redshift through $\mathcal{O}(e^{12})$ and obtain the complete tail-of-tail contribution to the second-order inverse redshift at 5.5PN and 6.5PN through $\mathcal{O}(e^{10})$. These results provide weak-field building blocks for future high-order EOB and second-order self-force developments.

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I. INTRODUCTION

In recent years, significant progress on the general-relativistic two-body problem has been driven by the synergy between a diverse set of complementary approaches: post-Newtonian (PN) theory [1], post-Minkowskian (PM) calculations and scattering amplitudes [2–7], effective field theory (EFT) [8–12], gravitational self-force (GSF) [13], and numerical relativity [14]. Frameworks such as the effective-one-body (EOB) formalism [15,16] and the Tutti Frutti approach pioneered in [17–20] allow us to combine information from the individual approaches into a complete description of the binary dynamics. The EOB formalism plays a particularly central role, as its resummed potentials provide a common framework into which PN, PM, and GSF results can be transcribed.

These theoretical programs are increasingly driven by observational demands. The LIGO-Virgo-KAGRA detectors [21–23] have established compact-binary coalescences as routine observations [24–28], and next-generation facilities, such as LISA [29], the Einstein Telescope [30], and Cosmic Explorer [31], will deliver signal-to-noise ratios that place stringent accuracy requirements on waveform models [32–34]. Orbital eccentricity poses a particular challenge in this context, while also serving as one of the most informative tracers of astrophysical binary formation channels [35,36]. Several formation pathways, including dynamical interactions in dense star clusters and galactic nuclei, secular evolution in hierarchical multiples, and interactions in AGN disks, can produce binaries that retain measurable eccentricity in the sensitive band of ground-based detectors [37–45]. There is already mounting evidence for orbital eccentricity in population-level studies [46–48] as well as in individual events, such as the neutron star-black hole binary GW200105 [49–53].

These developments have spurred rapid progress in waveform models for eccentric binaries across all major frameworks. Within the PN framework, ready-to-use inspiral models for precessing-eccentric binaries are now available [54–57]. In the EOB approach, eccentric extensions of both the TEOBResumS and SEOBNR families have reached maturity, with and without spin precession [58–68]. Eccentricity has also been recently incorporated into the phenomenological frameworks [69,70], building on the quasicircular foundations introduced in [71–74]. All of these models ultimately rest on analytic solutions to the two-body dynamics, with their accuracy determined by the available

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PN, mass ratio, and eccentricity information. Extending the conservative dynamics of eccentric binaries to higher PN orders, including the nonlocal hereditary sectors, and obtaining self-force information through second order in the mass ratio are important stepping stones toward improving these models.

Hereditary contributions are of particular interest as they differ qualitatively from instantaneous terms [1]. Beginning at 4PN order, the conservative dynamics acquires nonlocal-in-time contributions from gravitational-wave tails, generated by back-scattering off the curvature sourced by the total mass [75–82]. Starting at 5.5PN order, nonlocal-in-time interactions mediated by tail-of-tail effects also contribute to the conservative dynamics [83–87]. Recent work extended the tail-of-tail analysis to eccentric motion by evaluating the time-split action through relative 1PN order and $\mathcal{O}(e_t^2)$ [87]. The resulting averaged dynamics determines the corresponding 5.5PN and 6.5PN corrections to the EOB potentials A and $\bar{D}$, including terms at $\mathcal{O}(\nu^2)$ [86–88]. This truncation, nevertheless, leaves the nongeodesic contributions to the EOB Hamiltonian undetermined, as the $Q(u, p_r; \nu)$ potential in the Damour-Jaranowski-Schäfer (DJS) gauge [89] only contributes at quartic order in the radial momentum and therefore first contributes at quartic order in eccentricity.

The aim of the present work is to extend the calculations presented in [86,87] to higher orders in eccentricities, allowing us to determine the corrections to the nongeodesic EOB Q potential. Working in harmonic coordinates with 1PN-accurate quasi-Keplerian (QK) orbital dynamics, we extend the Delaunay-averaged tail-of-tail Hamiltonian of Ref. [87] through $\mathcal{O}(e_t^{12})$ at relative 1PN order. Matching to the Delaunay averaged EOB Hamiltonian order-by-order in eccentricity allows us to determine the tail-of-tail contributions to the Q potential through the $\mathcal{O}(p_r^{12})$, at 5.5PN and 6.5PN, including the complete $O(\nu^2)$ dependence. The $O(\nu)$ terms reproduce the known GSF-derived coefficients through p_r^8 [86,88], while the higher-order p_r terms and the $O(\nu^2)$ parts constitute new predictions.

We validate our results through three complementary checks. First, we rederive the averaged Hamiltonian by decomposing the source multipoles into discrete Fourier modes, and then evaluate the hereditary kernels mode-by-mode following the spectral methods of Refs. [90–94]. Second, we use the first law of binary black hole mechanics [95,96], generalized to hereditary interactions in Ref. [97], to compute the tail-of-tail contribution to the averaged redshift at fixed orbital frequencies. At first order in the mass ratio, the resulting expression agrees through $\mathcal{O}(e^{12})$ with the independent GSF calculation of Ref. [98]. Third, a complementary EOB calculation provides a self-consistency check of the second-order self-force (2SF) terms through $\mathcal{O}(e^2)$. As a further result, we obtain the complete half-integer-PN contributions to the 2SF inverse redshift at 5.5PN and 6.5PN through $\mathcal{O}(e^{10})$. These results provide explicit weak-field benchmarks for ongoing extensions of second-order self-force calculations [99–106].

We note that the results presented here should be viewed as building blocks for future developments, rather than as terms that can be immediately incorporated into a complete description of the conservative dynamics. In particular, the tail-of-tail sector is only one contribution to the full two-body Hamiltonian, and the high-order eccentricity and $\mathcal{O}(\nu^2)$ information obtained here at 5.5PN and 6.5PN can only be exploited once the tail-free and first-order-tail contributions are known at comparable PN accuracy. At present, these are only partially available [17–19,107]. A similar comment applies to the 2SF redshift contributions of Sec. VI, which give the tail-of-tail part of the half-integer 2SF sector and should be regarded as targets for future eccentric 2SF calculations.

The paper is organized as follows. Section II summarizes the quasi-Keplerian parametrization and our conventions. Section III introduces the time-split tail-of-tail action and derives the Delaunay-averaged Hamiltonian through $\mathcal{O}(e_t^{12})$. Section IV performs the EOB matching and presents the Q-potential coefficients. Section V develops the spectral cross-validation, and Sec. VI derives the first- and second-order self-force redshift contributions. Throughout, we use units $G = c = 1$, except where $\eta = 1/c$ is retained as a PN bookkeeping parameter. The total mass is denoted $M = m_1 + m_2$, with $\mu = m_1 m_2/M$ the reduced mass, and $\nu = \mu/M$ the symmetric mass ratio.

II. QUASI-KEPLERIAN ORBITS

We will work with ellipticlike orbits using the QK parametrization in harmonic coordinates at 1PN order accuracy [108]. Following [87], we adopt mass-rescaled dimensionless orbital variables, such that $r = r^{\rm phys}/(GM)$, $a_r = a_r^{\rm phys}/(GM)$, $n = n^{\rm phys}(GM/c^3)$. We use $M = m_1 + m_2$, $\mu = m_1 m_2/M$, and $\nu = \mu/M$ for the total mass, reduced mass, and symmetric mass ratio respectively. At this order, the QK parametrization reduces to [108,109]

$$nt = \ell = u - e_t \sin u, \tag{1}$$

$$r = a_r(1 - e_r \cos u), \tag{2}$$

$$\phi = 2K \arctan\left[\left(\frac{1+e_\phi}{1-e_\phi}\right)^{1/2} \tan\frac{u}{2}\right], \tag{3}$$

where $n = 2\pi/P$ with P the period of the radial motion, ℓ is the mean anomaly, u is the eccentric anomaly, $K = 1 + k$ encodes the periastron advance with k the fractional periastron advance per radial period, and a_r, e_t, e_r, and e_ϕ are the semimajor axis and the time, radial, and angular eccentricities, respectively. Unless otherwise stated, all QK elements in this section are defined in harmonic coordinates. The true anomaly $v(u)$ can be related to the eccentric anomaly u through [108]

$$v = 2\arctan\left[\left(\frac{1+e_\phi}{1-e_\phi}\right)^{1/2}\tan\frac{u}{2}\right], \tag{4}$$

such that

$$\cos v = \frac{\cos u - e_\phi}{1 - e_\phi \cos u}, \tag{5}$$

$$\sin v = \frac{\sqrt{1-e_\phi^2}\sin u}{1 - e_\phi \cos u}. \tag{6}$$

At 1PN, the relation between the mean motion and the radial semimajor axis is given by

$$n = a_r^{-3/2}\left[1 + \frac{1}{2}\frac{\eta^2}{a_r}(-9+\nu)\right], \tag{7}$$

where we have introduced $\eta = 1/c$ to track the PN order [87], with η^2 denoting terms at 1PN order. It will also be useful to describe the orbit in terms of the reduced binding energy and angular momentum [86,91,96,110],

$$\varepsilon = -\frac{2E_b}{\mu c^2}, \qquad j = \varepsilon\left(\frac{cL}{GM\mu}\right)^2. \tag{8}$$

Here, E_b denotes the conservative binding energy and L the conservative orbital angular momentum. The QK orbital elements can be expressed in terms of ε and j as [86,108,110]

$$a_r = \frac{1}{\varepsilon}\left[1 - \frac{7-\nu}{4}\eta^2\varepsilon\right], \tag{9}$$

$$n = \varepsilon^{3/2}\left[1 - \frac{15-\nu}{8}\eta^2\varepsilon\right], \tag{10}$$

$$e_t^2 = (1-j) + \eta^2\varepsilon\left[\frac{17-7\nu}{4}j - 2 + 2\nu\right], \tag{11}$$

$$e_r^2 = (1-j) + \eta^2\varepsilon\left[6 - \nu - \frac{15-5\nu}{4}j\right], \tag{12}$$

$$e_\phi^2 = (1-j) + \eta^2\varepsilon\left[6 - \frac{15-\nu}{4}j\right], \tag{13}$$

$$K = 1 + k, \tag{14}$$

where $k = K - 1 = \Omega_\phi/\Omega_r - 1$ is the fractional periastron advance per radial period [110,111], and up to 2PN is given by [110]

$$k = \frac{3\eta^2\varepsilon}{j}\left\{1 + \eta^2\varepsilon\left[-\frac{5-2\nu}{4} + \frac{35-10\nu}{4j}\right]\right\}. \tag{15}$$

Although the orbital dynamics used here are only required through 1PN, we retain the 2PN correction to k as this will be necessary in Sec. VI B, when calculating the second-order self-force correction to the redshift invariants. The energy and angular momentum can be equivalently written in terms of the QK orbital elements,

$$\varepsilon = \frac{1}{a_r} - \frac{7-\nu}{4a_r^2}\eta^2, \tag{16}$$

$$j = (1-e_t^2) + \frac{\eta^2}{a_r}\left[4 + 2(\nu-3)e_t^2 - \frac{7-\nu}{4}(1-e_t^2)\right]. \tag{17}$$

We will use these relations repeatedly when converting between harmonic QK variables, EOB variables, and fixed-frequency orbital parameters.

III. THE TAIL-OF-TAIL ACTION

We now collect the ingredients required to construct the conservative tail-of-tail Hamiltonian through relative 1PN order. Where possible, our notation follows Ref. [87], and we explicitly state the sign and principal-value conventions adopted in this work. We work with the time-split tail-of-tail action [18,86,87]

$$S^{\rm tt}_{\text{time-split}} = -\frac{1}{4}\left(\frac{G\mathcal{M}}{c^3}\right)^2 G\int dt\int_{-\infty}^{\infty}\frac{d\tau}{\tau}\mathcal{G}^{\rm split}(t,\tau), \tag{18}$$

where

$$\mathcal{G}^{\rm split}(t,\tau) = \sum_{l\geq 2}\frac{1}{c^{2l+1}}\left(a_l\beta_l^{(\rm even)}\mathcal{G}_I^l(t,\tau) + \frac{b_l}{c^2}\beta_l^{(\rm odd)}\mathcal{G}_J^l(t,\tau)\right). \tag{19}$$

Here, we have introduced the 1PN accurate ADM mass,

$$\mathcal{M} = M\left(1 - \eta^2\frac{\nu}{2a_r}\right), \tag{20}$$

which will contribute to the ν-dependent part of the relative-1PN tail-of-tail Hamiltonian.

The kernels $\mathcal{G}_X^l$ are defined by

$$\mathcal{G}_I^l(t,\tau) \equiv I_L^{(l+1)}(t)[I_L^{(l+2)}(t+\tau) - I_L^{(l+2)}(t-\tau)], \tag{21}$$

$$\mathcal{G}_J^l(t,\tau) \equiv J_L^{(l+1)}(t)[J_L^{(l+2)}(t+\tau) - J_L^{(l+2)}(t-\tau)], \tag{22}$$

where I_L and J_L are the Multipolar-post-Minkowskian (MPM) Blanchet-Damour-Iyer multipole moments [75,77,112,113], and the superscripts denote the number of time derivatives applied to the multipole moment. This

can be expressed using the shorthand notation introduced in [87],

$$\mathsf{G}^l_X = \int dt \int_{-\infty}^{\infty} \frac{d\tau}{\tau} \mathcal{G}^l_X(t,\tau), \tag{23}$$

which results in the following time-split action:

$$S^{\rm tt}_{\rm time-split} = -\frac{G}{4}\left(\frac{G\mathcal{M}}{c^3}\right)^2 \sum_{l\geq 2} \frac{1}{c^{2l+1}} \left[a_l \beta_l^{(\rm even)} \mathsf{G}^l_I + \frac{b_l}{c^2} \beta_l^{(\rm odd)} \mathsf{G}^l_J \right]. \tag{24}$$

Through relative 1PN order, only the G^2_I, G^3_I, and G^2_J sectors contribute. The mass quadrupole is required through 1PN, whereas the mass octupole and current quadrupole are only required at Newtonian order, as their leading order contributions are suppressed by an additional PN order. The multipole and beta coefficients are given by [9,77,79,114–116]

$$a_2 = \frac{1}{5}, \qquad a_3 = \frac{1}{189}, \qquad b_2 = \frac{16}{45}, \qquad \beta_2 = -\frac{214}{105}, \qquad \beta_3 = -\frac{26}{21}, \tag{25}$$

where $\beta_l^{\rm even} = \beta_l^{\rm odd}$ [20,115,116]. Evaluating these contributions requires only the source mass quadrupole I_{ab}, mass octupole I_{abc}, and current quadrupole J_{ab}. In harmonic coordinates at 1PN order, these moments are given by [1,117]

$$\begin{aligned} I_{ij} &= \mu r_{\langle ij\rangle}\left[1 + \frac{29}{42}\eta^2(1-3\nu)v^2 - \frac{(5-8\nu)}{7}\eta^2\frac{GM}{r}\right] \\ &\quad + \mu\eta^2\frac{1-3\nu}{21}[-12(\mathbf{v}\cdot\mathbf{r})r_{\langle i}v_{j\rangle} + 11 r^2 v_{\langle ij\rangle}], \\ I_{ijk} &= \mu\sqrt{1-4\nu} r_{\langle ijk\rangle}, \\ J_{ij} &= \mu\sqrt{1-4\nu}\eta_{kl\langle i}r_{j\rangle k}v_l, \end{aligned} \tag{26}$$

where

$$\begin{aligned} r_{\langle ijk\rangle} &= r_{ijk} - \frac{3}{5}r^2\delta_{(ij}r_{k)}, \\ \eta_{kl\langle i}r_{j\rangle k}v_l &= (\mathbf{r}\times\mathbf{v})_{\langle i}r_{j\rangle}. \end{aligned} \tag{27}$$

Here, we use standard notation, notably $T^{\langle ij\rangle}$ for the symmetric trace-free part of a tensor, $T^{(ij)} = (T^{ij} + T^{ji})/2$ for the symmetric part, $T_{ij\ldots} = T_i T_j \cdots$ for tensorial products, and η_{ijk} is the Levi-Civita symbol.

From the action,

$$S^{\rm tt}_{\rm time-split} = -\int dt\, H^{\rm tt}(t), \tag{28}$$

the time-split Hamiltonian can be expressed as

$$H^{\rm tt}(t) = H^{\rm tt}_{I_2}(t) + \eta^2[H^{\rm tt}_{I_3}(t) + H^{\rm tt}_{J_2}(t)], \tag{29}$$

where the relative-1PN correction to I_3 and J_2 sectors has been explicitly factored out. The contributions to the tail-of-tail Hamiltonian can be written in a compact form in terms of principal-value (PV) integrals by introducing the following shorthand notation [87]:

$$H[X^{(n,m)}_L(t)] \equiv X^{(n)}_L(t) PV \int_{-\infty}^{\infty} \frac{dt'}{t'-t} X^{(m)}_L(t'). \tag{30}$$

With the convention adopted here, the orbit-averaged bilinear is the negative of a positive-definite spectral norm, as discussed in Sec. V. The contributions to the tail-of-tail Hamiltonian can now be compactly expressed as [87]

$$H^{\rm tt}_{I_2}(t) = -\frac{G}{4}\left(\frac{G\mathcal{M}}{c^3}\right)^2 \frac{1}{c^5}\frac{214}{525} 2H[I_2^{(3,4)}(t)], \tag{31}$$

$$H^{\rm tt}_{J_2}(t) = -\frac{G}{4}\left(\frac{G\mathcal{M}}{c^3}\right)^2 \frac{1}{c^5}\frac{3424}{4725} 2H[J_2^{(3,4)}(t)], \tag{32}$$

$$H^{\rm tt}_{I_3}(t) = -\frac{G}{4}\left(\frac{G\mathcal{M}}{c^3}\right)^2 \frac{1}{c^5}\frac{26}{3969} 2H[I_3^{(4,5)}(t)]. \tag{33}$$

Each contribution is constructed from bilinears in the source multipole moments, which we jointly PN expand and eccentricity expand. We retain the mass quadrupole through relative 1PN order, and all bilinears through $\mathcal{O}(e_t^{12})$. This will be sufficient to determine the $\mathcal{O}(p_r^{12})$ contributions to the EOB Q potential, as shown in Sec. IV.

Schematically, the bilinears of interest have the form

$$I_2^{(3,4)}(u,u') = I_{ij}^{(3)}(t) I_{ij}^{(4)}(t')|_{t=t(u),t'=t(u')}, \tag{34}$$

$$J_2^{(3,4)}(u,u') = J_{ij}^{(3)}(t) J_{ij}^{(4)}(t')|_{t=t(u),t'=t(u')}, \tag{35}$$

$$I_3^{(4,5)}(u,u') = I_{ijk}^{(4)}(t) I_{ijk}^{(5)}(t')|_{t=t(u),t'=t(u')}, \tag{36}$$

which, when PN and eccentricity expanded, lead to a common structure of the form

$$X(u,u') = \sum_{p=0}^{p_{\rm max}} e_t^p [X_{(\eta^0,e_t^p)}(u,u') + \eta^2 X_{(\eta^2,e_t^p)}(u,u')]. \tag{37}$$

The terms up to $\mathcal{O}(e_t^2)$ have already been presented in [87] and are in full agreement with the results presented here. Following [87], we evaluate Eq. (30) after changing the integration variable from $dt'/(t'-t)$ to du', u', and u, leading to a finite set of principal-value integrals. Representative bilinears through $\mathcal{O}(e_t^4)$ are shown in Tables I–III in Appendix A. The complete expressions through $\mathcal{O}(e_t^{12})$ are given in the Supplemental Material [118].

A. Delaunay averaged Hamiltonian

We define the Delaunay average over one radial period by

$$\langle H^{\rm tt}_{X_L}\rangle \equiv \frac{\oint H^{\rm tt}_{X_L}(t)dt}{\oint dt}. \tag{38}$$

The radial period entering the denominator is defined through relative 1PN order by

$$\oint dt = 2\pi a_r^{3/2} + a_r^{1/2}\pi\eta^2(9-\nu), \tag{39}$$

which follows directly from $P = 2\pi/n$ and the 1PN relation in Eq. (10).

Applying the Delaunay averaging procedure to the multipole moment bilinears allows us to determine all the required contributions to construct the averaged tail-of-tail Hamiltonian. For the mass quadrupole I_2, we have two contributions at fractional 1PN order

$$\langle H[I_2^{(3,4)}]\rangle = \langle H[I^{(3,4)}_{2,\eta^0}]\rangle + \eta^2\langle H[I^{(3,4)}_{2,\eta^2}]\rangle, \tag{40}$$

where the two terms are given by

$$2\langle H[I^{(3,4)}_{2,\eta^0}(t)]\rangle = \frac{\pi\nu^2}{a_r^{13/2}}\left[-128 - \frac{4670}{3}e_t^2 - \frac{42955}{6}e_t^4 - \frac{6204647}{288}e_t^6 - \frac{352891481}{6912}e_t^8 - \frac{286907786543}{2764800}e_t^{10} - \frac{6287456255443}{33177600}e_t^{12}\right], \tag{41}$$

$$2\langle H[I^{(3,4)}_{2,\eta^2}(t)]\rangle = \frac{\pi\nu^2}{a_r^{15/2}}\left[\left(\frac{31168}{21} + \frac{256805}{21}e_t^2 + \frac{339379}{12}e_t^4 - \frac{309479}{1728}e_t^6 - \frac{19714929593}{96768}e_t^8 - \frac{1527310235551}{1843200}e_t^{10} - \frac{351712023425809}{154828800}e_t^{12}\right) + \left(-\frac{4288}{7} - \frac{158993}{21}e_t^2 - \frac{146445}{4}e_t^4 - \frac{67251767}{576}e_t^6 - \frac{28402847971}{96768}e_t^8 - \frac{388346663887}{614400}e_t^{10} - \frac{566906059048153}{464486400}e_t^{12}\right)\nu\right]. \tag{42}$$

Similarly, for the current quadrupole J_2 and mass octupole I_3 respectively, we find

$$2\langle H[J_2^{(3,4)}(t)]\rangle = -(1-4\nu)\frac{\pi\nu^2}{a_r^{15/2}}\left[1 + 30e_t^2 + \frac{7321}{32}e_t^4 + \frac{279485}{288}e_t^6 + \frac{73162261}{24576}e_t^8 + \frac{3426964009}{460800}e_t^{10} + \frac{105664935873}{6553600}e_t^{12}\right], \tag{43}$$

$$2\langle H[I_3^{(4,5)}(t)]\rangle = -(1-4\nu)\frac{\pi\nu^2}{a_r^{15/2}}\left[\frac{49209}{5} + \frac{910188}{5}e_t^2 + \frac{367780251}{320}e_t^4 + \frac{849097483}{192}e_t^6 + \frac{104770539213}{8192}e_t^8 + \frac{47178621858259}{1536000}e_t^{10} + \frac{25457292864087389}{393216000}e_t^{12}\right]. \tag{44}$$

Through $\mathcal{O}(e_t^2)$, these expressions agree with the results presented in Ref. [87]. Their overall signs are independently fixed by the spectral norm representation that we introduce in Sec. V. In particular, this identifies an overall sign typo in the I_3 contribution quoted in Ref. [87], noting that this does not affect the final results.

Combining the three multipolar terms, including the 1PN correction to $\mathcal{M}^2$, yields the Delaunay-averaged tail-of-tail Hamiltonian in harmonic coordinates

$$\langle H^{\rm tt}\rangle_{\eta^0} = \frac{\pi\nu^2}{a_r^{13/2}}\left[\frac{6848}{525} + \frac{49969}{315}e_t^2 + \frac{919237}{1260}e_t^4 + \frac{663897229}{302400}e_t^6 + \frac{37759388467}{7257600}e_t^8 + \frac{4385590451443}{414720000}e_t^{10} + \frac{672757819332401}{34836480000}e_t^{12}\right], \tag{45}$$

$$\langle H^{\rm tt}\rangle_{\eta^2} = -\frac{\pi\nu^2}{a_r^{15/2}}\left[\left(\frac{991861}{7350}+\frac{593849}{630}e_t^2+\frac{54097493}{56448}e_t^4-\frac{3935427469}{529200}e_t^6 -\frac{4578610749551}{108380160}e_t^8-\frac{368735424136853}{2709504000}e_t^{10}-\frac{1062619266017932697}{3121348608000}e_t^{12}\right) +\left(\frac{4982}{315}+\frac{1473251}{2450}e_t^2+\frac{1655819311}{352800}e_t^4+\frac{253642801109}{12700800}e_t^6 +\frac{24885308602613}{406425600}e_t^8+\frac{6208604054057939}{40642560000}e_t^{10}+\frac{1290387714679763381}{3901685760000}e_t^{12}\right)\nu\right]. \tag{46}$$

The relative-1PN term receives contributions from the 1PN mass quadrupole, the Newtonian current quadrupole and mass octupole, the 1PN orbital averaging measure, and the ADM mass correction.

We note that the expressions above extend the results of [87] through $\mathcal{O}(e_t^{12})$, while reproducing the previously reported coefficients up to $\mathcal{O}(e_t^2)$. These results provide the harmonic-coordinate input required to determine the corresponding tail-of-tail contributions to the EOB Hamiltonian through $\mathcal{O}(e_t^{12})$ and fractional 1PN order.

IV. DELAUNAY AVERAGED EOB HAMILTONIAN

A. The matching procedure

The harmonic-coordinate results of the previous section can be transcribed into the EOB framework by matching the Delaunay-averaged Hamiltonians [87],

$$\langle H^{\rm tt}_{\rm harmonic}\rangle = \langle H^{\rm tt}_{\rm EOB}\rangle, \tag{47}$$

where

$$\langle H^{\rm tt}_{\rm harmonic}\rangle \equiv \frac{\oint H^{\rm tt}_{\rm harmonic}(t)dt}{\oint dt}, \tag{48}$$

and

$$\langle H^{\rm tt}_{\rm EOB}\rangle \equiv \frac{\oint H^{\rm tt}_{\rm EOB}(t)dt}{\oint dt}. \tag{49}$$

Both sides should be expressed in terms of harmonic QK variables (a_r, e_t), after applying the fractional-1PN map between the EOB and harmonic quasi-Keplerian parameters.

In the DJS gauge [89], the effective Hamiltonian reads

$$\hat{H}_{\rm eff} = \sqrt{A(u,\nu)(1+p_\phi^2u^2+p_r^2A(u,\nu)\bar{D}(u,\nu)+Q)}, \tag{50}$$

where $u = u_{\rm EOB} = 1/r_{\rm EOB}$, p_r is the radial momentum, p_ϕ is the dimensionless canonical angular momentum, and $Q = Q(u, p_r;\nu)$ is a nongeodesic correction.

The connection to the real Hamiltonian follows from the EOB energy map [15]

$$H = Mc^2\sqrt{1+2\nu(\hat{H}_{\rm eff}-1)}. \tag{51}$$

The radial potentials can be expanded as

$$A(u,\nu) = 1-2u+\sum_{n\geq 3}a_n(\nu,\ln u)u^n, \tag{52}$$

$$\bar{D}(u,\nu) = 1+\sum_{n\geq 2}\bar{d}_n(\nu,\ln u)u^n, \tag{53}$$

$$Q(u,p_r;\nu) = \sum_{m\geq 2}p_r^{2m}q_{2m}(u,\nu). \tag{54}$$

At the PN orders explored here, each tail-of-tail contribution contains a leading term entering the real Hamiltonian at 5.5PN and a relative-1PN term entering at 6.5PN.[1] The corrections to the EOB metric potentials A and $\bar{D}$ are given by

$$\delta A = a_{6.5}u^{13/2}+a_{7.5}u^{15/2}, \tag{55}$$

$$\delta\bar{D} = \bar{d}_{5.5}u^{11/2}+\bar{d}_{6.5}u^{13/2}, \tag{56}$$

where the coefficients $\{a_{6.5}, a_{7.5}, \bar{d}_{5.5}, \bar{d}_{6.5}\}$ were determined from the tail-of-tail Hamiltonian in [87]. In contrast, the nongeodesic term admits the expansion

$$Q(u,p_r;\nu) = \sum_{m\geq 2}\sum_n q_{2m,n}(\nu)p_r^{2m}u^n, \tag{57}$$

where the first subscript labels the power of the radial momentum and the second the power of u. Through relative 1PN order, the tail-of-tail contribution in each p_r^{2m} sector contains exactly two powers,

[1]The subscript on a_n, $\bar{d}_n$, and q_n labels the power of u, not the PN order. Thus $a_{6.5}u^{13/2}$ and $\bar{d}_{5.5}u^{11/2}$ both contribute at 5.5PN, while $a_{7.5}u^{15/2}$ and $\bar{d}_{6.5}u^{13/2}$ both contribute at 6.5PN.

$$n \in \left\{ \frac{13}{2} - m, \frac{15}{2} - m \right\}. \tag{58}$$

For bound motion, $p_r^2 \sim \mathcal{O}(u)$, and hence $p_r^{2m} u^{13/2-m} \sim \mathcal{O}(u^{13/2})$ and $p_r^{2m} u^{15/2-m} \sim \mathcal{O}(u^{15/2})$. Since the Newtonian contribution to the Hamiltonian scales as $\mathcal{O}(u)$, these two terms enter the real Hamiltonian at 5.5PN and 6.5PN, respectively, independently of m.

Along the 1PN QK orbit, the radial momentum is $\mathcal{O}(e_t)$, so a term proportional to p_r^{2m} first contributes to the orbital average at $\mathcal{O}(e_t^{2m})$. The eccentricity expansion therefore turns our matching procedure into an iterative process for determining the EOB potentials. The circular sector fixes the A-potential coefficients $a_{6.5}$ and $a_{7.5}$, while the $\mathcal{O}(e_t^2)$ sector fixes the $\bar{D}$-potential coefficients $\bar{d}_{5.5}$ and $\bar{d}_{6.5}$ [87]. Since the nongeodesic potential Q begins at p_r^4 in the DJS gauge [89], it first enters at $\mathcal{O}(e_t^4)$. Extending the expansion through $\mathcal{O}(e_t^{12})$ therefore allows us to determine the tail-of-tail contributions to the Q-potential coefficients up to $\mathcal{O}(p_r^{12})$. Finally, we note that the truncation at $\mathcal{O}(e_t^{12})$ adopted here is purely computational. The procedure outlined here, following [87], can be iterated to arbitrary order in e_t. For example, see Appendix B of Ref. [107], where analogous high-order eccentricity expansions of the nonlocal 5PN tail Hamiltonian were carried out beyond the order used in Refs. [17,18].

At 1PN, we can write the conserved energy $H^{(0)} = E$ and angular momentum $p_\phi^{(0)} = L$ as [87]

$$E = -\frac{1}{2a_r^{\mathrm{e}}} + \eta^2 \frac{3-\nu}{8} \frac{1}{a_r^{\mathrm{e}2}}, \tag{59}$$

$$L = \sqrt{a_r^{\mathrm{e}}(1-e_r^{\mathrm{e}2})} + \eta^2 \frac{\frac{3}{2} + \frac{1}{2} e_r^{\mathrm{e}2}}{\sqrt{a_r^{\mathrm{e}}(1-e_r^{\mathrm{e}2})}}, \tag{60}$$

where $(a_r^{\mathrm{e}}, e_t^{\mathrm{e}}, e_r^{\mathrm{e}})$ are the orbital parameters in EOB coordinates. Following [87], we can use energy conservation to determine $p_r^{(0)}$, finding

$$p_r^{(0)} = p_r^{\mathrm{N}} \left[1 + \eta^2 \left(\frac{1}{r} - \frac{1}{2a_r^{\mathrm{e}}} \right) \right]. \tag{61}$$

Note that there is a typo in Ref. [87], and our expressions differ from [87], who have an additional factor of r multiplying $1/(2a_r^e)$. The expression above follows directly from the 1PN EOB energy-conservation equation and is the form adopted here. The Newtonian prefactor is given by

$$p_r^{\mathrm{N}} = \sqrt{\frac{2}{r} - \frac{r^2 + a_r^{\mathrm{e}2}(1-e_r^{\mathrm{e}2})}{r^2 a_r^{\mathrm{e}}}}. \tag{62}$$

In order to relate our EOB Hamiltonian to the harmonic coordinate expressions in Eqs. (45) and (46), we can make use of the following transformations up to 1PN [87,90,108,119]:

$$e_r^{\mathrm{e}} = e_t^{\mathrm{e}} + \eta^2 \frac{3}{a_r^{\mathrm{e}}} e_t^{\mathrm{e}}, \tag{63}$$

$$a_r^{\mathrm{e}} = a_r + \eta^2, \tag{64}$$

$$e_t^{\mathrm{e}} = e_t - \eta^2 e_t \frac{\nu}{a_r}. \tag{65}$$

B. The tail-of-tail contributions to the EOB potentials

Matching the Delaunay averaged Hamiltonians order-by-order in e_t, we find the following 5.5PN and 6.5PN tail-of-tail corrections to the EOB metric potentials, in agreement with [87]:

$$a_{6.5} = \frac{13696}{525}\nu\pi, \tag{66}$$

$$a_{7.5} = -\nu\pi\left(\frac{512501}{3675} + \frac{10052}{225}\nu \right), \tag{67}$$

$$\bar{d}_{5.5} = \frac{264932}{1575}\nu\pi, \tag{68}$$

$$\bar{d}_{6.5} = -\nu\pi\left(\frac{21288791}{17640} + \frac{893149}{2450}\nu \right). \tag{69}$$

Iterating the matching procedure up to $\mathcal{O}(e_t^{12})$, we find the following tail-of-tail corrections to the Q potential:

$$q_{4,4.5} = \frac{88703}{1890}\nu\pi, \tag{70}$$

$$q_{4,5.5} = -\nu\pi\left(\frac{714117331}{846720} + \frac{224134961}{1058400}\nu \right), \tag{71}$$

$$q_{6,3.5} = -\frac{2723471}{756000}\nu\pi, \tag{72}$$

$$q_{6,4.5} = \nu\pi\left(\frac{1783458013}{56448000} - \frac{1269199661}{127008000}\nu \right), \tag{73}$$

$$q_{8,2.5} = \frac{5994461}{12700800}\nu\pi, \tag{74}$$

$$q_{8,3.5} = \nu\pi\left(\frac{12986592749}{22759833600} - \frac{11266879201}{28449792000}\nu \right), \tag{75}$$

$$q_{10,1.5} = -\frac{1657880149}{26127360000}\nu\pi, \tag{76}$$

$$q_{10,2.5} = \nu\pi\left(\frac{6030897501941}{4551966720000} - \frac{1324939288373}{10241925120000}\nu\right), \tag{77}$$

$$q_{12,0.5} = \frac{23800130803}{3218890752000}\nu\pi, \tag{78}$$

$$q_{12,1.5} = \nu\pi\left(-\frac{1062624721887527}{2403438428160000} + \frac{10982826421327}{163870801920000}\nu\right). \tag{79}$$

As discussed earlier, for each m, the terms with $n = (13/2) - m$ and $n = (15/2) - m$ refer to the 5.5PN and 6.5PN sectors respectively, see the discussion around Eqs. (57) and (58). The $\mathcal{O}(\nu)$ coefficients through $\mathcal{O}(p_r^8)$ are provided in Refs. [86,88]. Our calculation reproduces these terms exactly, and extends the calculation through p_r^{12}. The displayed $\mathcal{O}(\nu^2)$ terms provide qualitatively new 2SF predictions for the nongeodesic potential, extending the 2SF predictions for $a_{7.5}$ and $\bar{d}_{6.5}$ originally derived in [86–88].

V. CROSS-VALIDATION WITH SPECTRAL REPRESENTATIONS

The results of the preceding section were obtained by jointly expanding the multipole bilinears as a PN and eccentricity series before performing the Delaunay average. In this section, we follow a complementary route to the same result. We first decompose the source multipoles into their discrete frequency spectra, and then evaluate the hereditary kernel mode-by-mode. Expanding the resulting spectral sums through $\mathcal{O}(e_t^{12})$ provides a validation of the harmonic-coordinate calculation. This is very closely related to the spectral methods used in [86,90–94,120,121], as the problems are governed by the same source multipoles and orbital frequency spectra. This approach is independent at the level of the hereditary integral and orbital averaging, while using the same source multipoles and QK orbital dynamics, providing a useful cross-validation and self-consistency check of our results.

The spectral representation also highlights two structural features of the calculation. First, the hereditary kernel acts diagonally in frequency space, multiplying each Fourier mode by $i\pi\mathrm{sgn}(\omega)$ [78,92]. The contraction of the STF indices pairs opposite azimuthal modes and the Delaunay average subsequently eliminates all remaining cross-harmonics, collapsing the bilinears to weighted sums of squared Fourier amplitudes. Each averaged hereditary bilinear is thereby given by the negative of a positive-definite spectral norm, fixing the sign of each nonzero-frequency contribution [90,92].

Second, combining the factor $\mathrm{sgn}(\omega)$ with the $2q+1$ time derivatives gives spectral weights proportional to $|\omega|^{2q+1}$. At Newtonian order $\omega = pn$, so the harmonic weights are simply $|p|^{2q+1}$, while at 1PN they become $|p+mk|^{2q+1}$, with $\omega_{pm} = n(p+mk)$ as in Eq. (81). Here, we focus on the Newtonian spectrum. For even spectral weights, Parseval's theorem allows us to relate the harmonic sum to the orbital average of a local-in-time quantity. In particular, we find that $\sum_p p^{2k}|X_L(p)|^2$ is proportional to $\langle (X_L^{(k)})^2\rangle$. Within the QK parametrization, such averages reduce to closed algebraic functions of the eccentricity, such as the Peters-Mathews enhancement function $f(e_t)$ [122]. For an odd weight, however, the identity $|p|^{2q+1} = p^{2q+1}\mathrm{sgn}(p)$ leaves an irreducible factor of $\mathrm{sgn}(p)$ or, equivalently, $\mathrm{sgn}(\omega)$. In the time domain, multiplication by $\mathrm{sgn}(\omega)$ corresponds to a nonlocal principal-value kernel, see Eq. (90) below. It therefore cannot be generated by any finite combination of time derivatives. Consequently, the odd-weight sums cannot be reduced using the same Parseval argument to orbital averages of local quadratic expressions built from finitely many time derivatives of the multipoles. Instead, their corresponding eccentricity enhancement functions are naturally expressed as infinite harmonic sums which can either be evaluated through eccentricity expansions or by direct numerical summation [78,90,91,93].

However, we note that Ref. [93] introduced an elegant resummation scheme for Kapteyn series [123,124], in which the Bessel functions are replaced by their uniform large-p asymptotic expansions and the harmonic sums by integrals, yielding closed-form superasymptotic and hyperasymptotic approximants across the full range of eccentricities. Analogous resummations of the averaged tail-of-tail Hamiltonian could be explored, which we leave to future work.

At Newtonian order, a multipole X_L is periodic in the mean anomaly and admits a Fourier expansion of the form [90,122]

$$X_L(t) = \sum_{p\in\mathbb{Z}} X_L(p)e^{ip\ell}. \tag{80}$$

At 1PN, however, we can no longer ignore the periastron advance and the orbital motion becomes quasiperiodic in the mean anomaly. Writing the multipole as a sum over azimuthal modes m, the periastron advance shifts the pth Fourier harmonic such that [90]

$$X_L(t) = \sum_m \sum_{p\in\mathbb{Z}} X_{L,m}(p)e^{i(p+mk)\ell}, \tag{81}$$

where the corresponding mode frequency is $\omega_{pm} = n(p+mk)$. In the following, we use these representations to construct the Newtonian and fractional-1PN spectral norms and compare their eccentricity expansions directly with the Delaunay-averaged results of the preceding section.

A. Fourier-Bessel amplitudes

Let $F(\ell)$ be a 2π-periodic function of the mean anomaly. Its Fourier expansion can be written as

$$F(\ell) = \sum_{p=-\infty}^{\infty} \mathcal{A}_p[F] e^{ip\ell}, \tag{82}$$

where

$$\begin{aligned}\mathcal{A}_p[F] &\equiv \frac{1}{2\pi}\int_0^{2\pi} d\ell F(\ell) e^{-ip\ell} \\ &= \frac{1}{2\pi}\int_0^{2\pi} du\, \rho(u) F(u) e^{-ip[u - e_t \sin u]}.\end{aligned} \tag{83}$$

In the second line, we used Eq. (1) together with the Jacobian

$$\rho(u) \equiv \frac{d\ell}{du} = 1 - e_t \cos u \tag{84}$$

to express the amplitudes in terms of the eccentric anomaly u. For the orbital functions encountered below, the weighted combination $\rho(u)F(u)$ reduces to a trigonometric polynomial in the eccentric anomaly. If we let $z = e^{iu}$, we see that these polynomials admit a finite Laurent expansion of the form

$$\rho(u)F(u) = \sum_{j=-j_{\max}}^{j_{\max}} c_j z^j. \tag{85}$$

We can use the Laurent expansion to simplify Eq. (83) by making use of the Jacobi-Anger identity [125], which gives us a compact expression for the Fourier amplitudes in terms of the coefficients c_j of the Laurent expansion

$$\mathcal{A}_p[F] = \sum_{j=-j_{\max}}^{j_{\max}} c_j J_{p-j}(pe_t). \tag{86}$$

Amplitudes of this form are closely related to the Hansen coefficients $X_p^{n,m}(e_t)$ familiar from celestial mechanics [126,127]. Although each Fourier amplitude is an exact finite combination of Bessel functions, we shall see that the eccentricity enhancement functions introduced below involve weighted sums of $|\mathcal{A}_p[F]|^2$ over infinitely many harmonics. Because the order and argument of the Bessel functions both grow with p, each such sum is a finite linear combination of Kapteyn series of the second kind [123,124], which converge for $0 \le e_t < 1$.

At Newtonian order, where $e_\phi = e_t$, we will find it useful to introduce

$$\zeta_\pm(u) \equiv \rho(u) e^{\pm i v(u)} = \cos u - e_t \pm is \sin u, \tag{87}$$

where $s \equiv \sqrt{1 - e_t^2}$, so that $\zeta_- = \zeta_+^*$. At 1PN order, the periastron advance is treated separately through the shift to the mode frequencies ω_{pm}. We will denote the radial unit vector by

$$\lambda_i \equiv \frac{r_i}{r}, \tag{88}$$

such that at Newtonian order $\boldsymbol{\lambda} = (\cos v, \sin v, 0)$ and $r_i = a_r \rho \lambda_i$. At 1PN order, the separation direction will be given by $\boldsymbol{\lambda} = (\cos\phi, \sin\phi, 0)$, with $\phi = (1+k)v$.

B. The hereditary kernel as a spectral norm

With an expression for the Fourier amplitudes at hand, the hereditary bilinears can be evaluated mode-by-mode by inserting the Fourier expansion directly into both the time derivatives of the MPM multipoles and the principal-value kernel introduced in Eq. (30). At Newtonian order, the qth time derivative of a multipole is trivially given by

$$X_L^{(q)}(t) = \sum_{p\in\mathbb{Z}} (ipn)^q X_L(p) e^{ip\ell}, \tag{89}$$

and we can make use of the standard principal-value identity [92]

$$PV \int_{-\infty}^{\infty} \frac{dt'}{t'-t} e^{i\omega t'} = i\pi \mathrm{sgn}(\omega) e^{i\omega t}. \tag{90}$$

As discussed above, the time derivatives and the hereditary kernel act diagonally on each Fourier mode, simplifying the resulting calculations.

Substituting these expressions into Eq. (30) results in a double sum over the harmonics p and p'. When we take the Delaunay average, only the terms satisfying $p' = -p$ will survive and the reality of the multipoles enforces $X_L(-p) = X_L^*(p)$. This allows us to derive an equivalent spectral representation of the averaged bilinears, analogous to the result derived in Sec. III. Structurally, we find that the bilinears have the form [90,92]

$$\begin{aligned}\langle H[X_L^{(q,q+1)}]\rangle &= -\pi n^{2q+1} \sum_{p\neq 0} |p|^{2q+1} |X_L(p)|^2 \\ &= -2\pi n^{2q+1} \sum_{p\ge 1} p^{2q+1} |X_L(p)|^2.\end{aligned} \tag{91}$$

Immediately, we can see that the averaged hereditary bilinears are the negative of a positive-definite spectral norm. This provides a convenient check on the overall sign, and allows us to validate the sign of the mass-octupole contribution in Sec. III.

C. Newtonian mass quadrupole

To illustrate this machinery, we derive the Newtonian mass quadrupole. Starting from Eq. (26), we have

$$I_{ij}^{\mathrm{N}} = \mu r_{\langle i} r_{j\rangle} = \mu a_r^2 \rho^2 \lambda_{\langle i}\lambda_{j\rangle}, \tag{92}$$

though we will find it more convenient to work with the dimensionless reduced quadrupole moment [92],

$$\hat{I}_{ij} \equiv \frac{I_{ij}^{\mathrm{N}}}{\mu a_r^2} = \rho^2 \lambda_{\langle i}\lambda_{j\rangle}. \tag{93}$$

When constructing the bilinear in Eq. (34), we end up with a contribution of the form

$$\lambda_{\langle i}\lambda_{j\rangle}\lambda'_{\langle i}\lambda'_{j\rangle} = \frac{1}{6} + \frac{1}{2}\cos\left[2(v - v')\right]. \tag{94}$$

Using $\zeta_\pm = \rho e^{\pm iv}$, we see that the quadrupole term can be expressed as

$$\begin{aligned}\hat{I}_{ij}(u)\hat{I}_{ij}(u') = &\frac{1}{6}\rho^2(u)\rho^2(u') + \frac{1}{4}\zeta_+^2(u)\zeta_-^2(u') \\ &+ \frac{1}{4}\zeta_-^2(u)\zeta_+^2(u'),\end{aligned} \tag{95}$$

where the first term corresponds to a $m = 0$ mode and the next two terms to the $m = \pm 2$ modes.

The problem therefore reduces to finding expressions for the Fourier amplitudes governing these modes, notably

$$D_p = \mathcal{A}_p[\rho^2], \qquad C_p = \mathcal{A}_p[\zeta_+^2], \tag{96}$$

with the ζ_-^2 amplitude trivially following from C_p. For $p \neq 0$, the amplitudes can be expressed in terms of Bessel functions, see Appendix B,

$$D_p = -\frac{2}{p^2} J_p(pe_t), \tag{97}$$

$$C_p = -\frac{2(1 + s^2 - 2ps^3)}{e_t^2 p^2} J_p(pe_t) - \frac{4s(1 - ps)}{e_t p^2} J'_p(pe_t), \tag{98}$$

where a prime denotes differentiation with respect to the argument and the circular limit is understood by continuity. We note that the zero-frequency modes do not contribute as the hereditary bilinear contains time derivatives. The spectral norm for the mass quadrupole bilinear can therefore be written as

$$\hat{I}_{ij}(p)\hat{I}_{ij}^*(p) = \frac{1}{6}|D_p|^2 + \frac{1}{4}|C_p|^2 + \frac{1}{4}|C_{-p}|^2. \tag{99}$$

Evaluating the expression in terms of the Bessel functions, we find that this is directly related to the Peters-Mathews enhancement function [122], such that

$$g(p, e_t) = \frac{p^6}{16}\hat{I}_{ij}(p)\hat{I}_{ij}^*(p), \tag{100}$$

normalized such that $g(2, 0) = 1$. Using Eq. (91), with $q = 3$, we find [86]

$$2\langle H[I_{2,\eta^0}^{(3,4)}]\rangle = -\frac{\pi\nu^2}{a_r^{13/2}}\mathcal{I}_0(e_t), \tag{101}$$

where

$$\begin{aligned}\mathcal{I}_0(e_t) &= 4\sum_{p\geq 1} p^7 \hat{I}_{ij}(p)\hat{I}_{ij}^*(p) \\ &= 128\sum_{p\geq 1}\frac{p}{2} g(p, e_t) \\ &= 128\varphi(e_t),\end{aligned} \tag{102}$$

where $\varphi(e_t)$ is the dimensionless eccentricity factor introduced in [90], itself related to the earlier factor introduced in [78] by a factor of the Peters-Mathews enhancement function $f(e_t)$ [122].

The above result is in full agreement with the earlier calculation presented in Refs. [86,92]. These are closely related to the expressions obtained in [90] for the first-order tail contribution to the averaged gravitational-wave flux and to the expressions in [92] for the averaged second-order tails. This should come as no surprise, since the calculations involve the same underlying source multipoles but with different harmonic weights. Eccentricity expanding the enhancement function in Eq. (101) gives

$$\begin{aligned}\mathcal{I}_0(e_t) = 128 &+ \frac{4670}{3}e_t^2 + \frac{42955}{6}e_t^4 \\ &+ \frac{6204647}{288}e_t^6 + \frac{352891481}{6912}e_t^8 \\ &+ \frac{286907786543}{2764800}e_t^{10} \\ &+ \frac{6287456255443}{33177600}e_t^{12} + \mathcal{O}(e_t^{14}),\end{aligned} \tag{103}$$

in complete agreement with the results obtained in Eq. (41).

D. Current quadrupole and mass octupole

The Newtonian current quadrupole and mass octupole both carry the mass-difference factor,

$$\delta = \sqrt{1 - 4\nu}. \tag{104}$$

We opt to explicitly factor this out of the corresponding enhancement functions by defining the following reduced moments:

$$I_{ijk}^{\mathrm{N}} \equiv \mu\delta a_r^3 \bar{I}_{ijk}, \qquad \bar{I}_{ijk} = \rho^3 \lambda_{\langle i}\lambda_j\lambda_{k\rangle}, \tag{105}$$

$$J_{ij}^{\mathrm{N}} \equiv \mu\delta a_r^3 n \bar{J}_{ij}, \qquad \bar{J}_{ij} = s\rho\,\lambda_{\langle i}\hat{L}_{j\rangle}, \tag{106}$$

where $\hat{L}_i$ denotes the unit vector normal to the orbital plane. From Eq. (91), the current quadrupole and mass octupole contributions are given by

$$2\langle H[J_2^{(3,4)}]\rangle = -(1-4\nu)\frac{\pi\nu^2}{a_r^{15/2}}\mathcal{J}_0(e_t), \tag{107}$$

$$2\langle H[I_3^{(4,5)}]\rangle = -(1-4\nu)\frac{\pi\nu^2}{a_r^{15/2}}\mathcal{O}_0(e_t), \tag{108}$$

where

$$\mathcal{J}_0(e_t) \equiv 4\sum_{p\geq 1} p^7 |\bar{J}_{ij}(p)|^2 = \gamma(e_t), \tag{109}$$

$$\mathcal{O}_0(e_t) \equiv 4\sum_{p\geq 1} p^9 |\bar{I}_{ijk}(p)|^2 = \frac{49209}{5}\beta(e_t). \tag{110}$$

The enhancement functions $\gamma(e_t)$ and $\beta(e_t)$ are exactly those derived in [90]. In particular, $\gamma(e_t)$ is given by

$$\gamma(e_t) = 2s^2\sum_{p=1}^{\infty} p^5\left[J_p'^2(pe_t) + \frac{s^2}{e_t^2}J_p^2(pe_t)\right], \tag{111}$$

and $\beta(e_t)$ by

$$\beta(e_t) = \frac{1}{49209}\sum_{p=1}^{\infty} p^9\left[\frac{5}{2}(|U_p^+|^2 + |U_p^-|^2) + \frac{3}{2}(|V_p^+|^2 + |V_p^-|^2)\right], \tag{112}$$

where we have introduced the Fourier amplitudes

$$U_p^{\pm} \equiv \mathcal{A}_p[\zeta_{\pm}^3], \qquad V_p^{\pm} \equiv \mathcal{A}_p[\rho^2\zeta_{\pm}], \tag{113}$$

whose explicit forms are given in Appendix C. The resulting eccentricity expansions of $\mathcal{J}_0(e_t)$ and $\mathcal{O}_0(e_t)$ can be shown to reduce to

$$\mathcal{J}_0(e_t) = 1 + 30e_t^2 + \frac{7321}{32}e_t^4 + \frac{279485}{288}e_t^6 + \frac{73162261}{24576}e_t^8 + \frac{3426964009}{460800}e_t^{10} + \frac{105664935873}{6553600}e_t^{12} + \mathcal{O}(e_t^{14}), \tag{114}$$

$$\mathcal{O}_0(e_t) = \frac{49209}{5} + \frac{910188}{5}e_t^2 + \frac{367780251}{320}e_t^4 + \frac{849097483}{192}e_t^6 + \frac{104770539213}{8192}e_t^8 + \frac{47178621858259}{1536000}e_t^{10} + \frac{25457292864087389}{393216000}e_t^{12} + \mathcal{O}(e_t^{14}). \tag{115}$$

Inserting these expansions into Eqs. (107) and (108) respectively, we find complete agreement with the earlier results derived in Eqs. (43) and (44).

E. Fractional-1PN mass quadrupole

At fractional 1PN order, the quadrupole amplitudes and their spectral frequencies receive relativistic corrections due to the periastron advance. The 1PN contribution to the Delaunay averaged Hamiltonian can be written as

$$2\langle H[I_{2,\eta^2}^{(3,4)}]\rangle = \frac{\pi\nu^2}{a_r^{15/2}}\mathcal{I}_1(e_t,\nu), \tag{116}$$

where we have introduced the 1PN enhancement function

$$\mathcal{I}_1(e_t,\nu) = A_{I_2}(e_t) + \nu B_{I_2}(e_t). \tag{117}$$

In the hereditary-flux notation of Refs. [90,91], these functions are related by

$$A_{I_2}(e_t) = \frac{13696}{21}\alpha(e_t) + \left(2496 - \frac{1664}{1-e_t^2}\right)\varphi(e_t), \tag{118}$$

$$B_{I_2}(e_t) = -\frac{5696}{21}\theta(e_t) - \frac{1024}{3}\varphi(e_t), \tag{119}$$

where $\alpha(e_t)$ and $\theta(e_t)$ are also standard hereditary enhancement functions introduced in Refs. [90,91]. Eccentricity expanding these functions, we find

$$A_{I_2}(e_t) = \frac{31168}{21} + \frac{256805}{21}e_t^2 + \frac{339379}{12}e_t^4 - \frac{309479}{1728}e_t^6 - \frac{19714929593}{96768}e_t^8 - \frac{1527310235551}{1843200}e_t^{10} - \frac{351712023425809}{154828800}e_t^{12} + \mathcal{O}(e_t^{14}), \tag{120}$$

$$B_{I_2}(e_t) = -\frac{4288}{7} - \frac{158993}{21}e_t^2 - \frac{146445}{4}e_t^4 - \frac{67251767}{576}e_t^6 - \frac{28402847971}{96768}e_t^8 - \frac{388346663887}{614400}e_t^{10} - \frac{566906059048153}{464486400}e_t^{12} + \mathcal{O}(e_t^{14}). \tag{121}$$

When inserted into Eq. (116), we find complete agreement with the result derived in Eq. (42).

VI. REDSHIFT INVARIANTS

The first law of binary mechanics provides a link between the conservative tail-of-tail dynamics derived above and gauge-invariant observables [17,86,88,95,96,128,129], such as the Detweiler-Barack-Sago redshift [130,131]. It is naturally formulated as a variational statement about the ADM mass $\mathcal{M}$ of the system [95,96], which can be related to the binding energy by $\mathcal{M} = M + E_{\rm b}$. Comparison with the redshift invariants provides both a direct test of the small-mass-ratio limit of the averaged Hamiltonian and a natural point of comparison with second-order self-force predictions. Since tail effects are inherently hereditary, we formulate this comparison in terms of the localized Delaunay-averaged tail-of-tail Hamiltonian $\langle H_{\rm tt}\rangle$, for which the first law retains its standard form [97].

For a nonspinning binary on a stable bound orbit, the conservative motion may be parametrized by the radial and azimuthal actions,

$$J_r = \frac{1}{2\pi}\oint p_r\, dr, \qquad J_\phi = L = p_\phi. \tag{122}$$

The fundamental frequencies associated to the angle variables are given by

$$\Omega_r = \frac{2\pi}{T_r}, \qquad \Omega_\phi = \frac{\Phi}{T_r}, \tag{123}$$

where T_r denotes the radial period and Φ the azimuthal phase accumulated during one radial cycle. The averaged Detweiler-Barack-Sago redshift of body a is then defined by [130,132–135]

$$\langle z_a\rangle = \frac{1}{T_r}\int_0^{T_r}\frac{d\tau_a}{dt}\,dt = \frac{\mathcal{T}_a}{T_r}, \tag{124}$$

where $\mathcal{T}_a$ is the proper-time duration of one radial cycle. We also introduce the corresponding inverse redshift invariant,

$$U_a \equiv \frac{1}{\langle z_a\rangle} = \frac{T_r}{\mathcal{T}_a}. \tag{125}$$

To make the connection to GSF results, we need to work in the small-mass-ratio limit. Here, we adopt the conventions of Ref. [129], namely

$$m_1 \ll m_2, \qquad q = \frac{m_1}{m_2}, \qquad M = m_2(1+q), \tag{126}$$

where the larger body sources the background Schwarzschild geometry. For convenience, we will denote the averaged redshift of the smaller body and its inverse by

$$z \equiv \langle z_1\rangle, \qquad U \equiv \frac{1}{z}. \tag{127}$$

For a generic stable bound eccentric orbit, the averaged redshift may be parametrized by the mass ratio and the two dimensionless fundamental frequencies

$$\hat{\Omega}_r \equiv m_2\Omega_r, \qquad \hat{\Omega}_\phi \equiv m_2\Omega_\phi, \tag{128}$$

which are held fixed in the self-force expansions below. For fixed frequencies, we can equivalently describe the system in terms of a reference Schwarzschild geodesic using the Darwin parameters (p, e) [135,136]. At fixed $m_2\Omega_r$ and $m_2\Omega_\phi$, the self-force expansion of z and U take the form

$$z = z_{\rm geo} + q\Delta z^{\rm 1SF} + q^2\Delta z^{\rm 2SF} + \mathcal{O}(q^3), \tag{129}$$

$$U = U_{\rm geo} + q\Delta U^{\rm 1SF} + q^2\Delta U^{\rm 2SF} + \mathcal{O}(q^3), \tag{130}$$

where $U_{\rm geo} = 1/z_{\rm geo}$. Expanding the identity $U = 1/z$ order by order in q yields

$$\Delta U^{\rm 1SF} = -\frac{\Delta z^{\rm 1SF}}{z_{\rm geo}^2}, \tag{131}$$

$$\Delta U^{\rm 2SF} = -\frac{\Delta z^{\rm 2SF}}{z_{\rm geo}^2} + \frac{(\Delta z^{\rm 1SF})^2}{z_{\rm geo}^3}. \tag{132}$$

The remaining task is therefore to evaluate the tail-of-tail contribution to the fixed-frequency mass derivatives at each self-force order.

The first law of binary black hole mechanics can be written in terms of the action variables as [95,96]

$$\delta\mathcal{M} = \Omega_r\delta J_r + \Omega_\phi\delta L + \sum_a \langle z_a\rangle\delta m_a. \tag{133}$$

This variational law relates neighboring binary configurations, connecting variations in the ADM mass, radial action, orbital angular momentum, and component masses through the orbital frequencies and averaged redshifts. The analysis of Ref. [97] shows that the first law retains its standard form for conservative hereditary dynamics, provided one works with a localized Hamiltonian, such as the Delaunay averaged Hamiltonian derived here.

The first law relates the averaged redshift to the variation of the ADM mass with respect to the corresponding mass at fixed actions [85,94–97,137]

$$\langle z_a\rangle = \left.\frac{\partial\mathcal{M}}{\partial m_a}\right|_{J_r,L,m_{b\neq a}}. \tag{134}$$

However, the self-force expansions introduced above are naturally given in terms of fixed fundamental frequencies rather than fixed actions. Following [96], we can replace

the action variables by frequencies via the Legendre transform

$$\mathcal{R}(\Omega_r, \Omega_\phi; m_a) = \mathcal{M} - \Omega_r J_r - \Omega_\phi L, \tag{135}$$

whose variation reads

$$\delta\mathcal{R} = -J_r \delta\Omega_r - L\delta\Omega_\phi + \sum_a \langle z_a \rangle \delta m_a. \tag{136}$$

The redshift then follows as a mass derivative at fixed frequencies,

$$\langle z_a \rangle = \left.\frac{\partial \mathcal{R}}{\partial m_a}\right|_{\Omega_r, \Omega_\phi, m_{b\neq a}}, \tag{137}$$

which is the form we adopt when comparing against GSF results.

Working to linear order in the tail-of-tail interaction, its contribution to $\mathcal{M}$, and hence to $\mathcal{R}$, is given by the Delaunay-averaged Hamiltonian evaluated at the zeroth-order action variables,

$$\mathcal{R}_{\rm tt}(\Omega_r, \Omega_\phi; m_a) = \langle H_{\rm tt} \rangle (J_i^{(0)}(\Omega_r, \Omega_\phi; m_a)). \tag{138}$$

Following the conventions adopted in eccentric self-force calculations [98,129,135,138], we expand in powers of q at fixed m_2-rescaled frequencies $(\hat{\Omega}_r, \hat{\Omega}_\phi)$. Since $m_1 = q m_2$, the mass derivative in Eq. (137) taken at fixed m_2 is equivalent to a derivative with respect to q, and the tail-of-tail contribution to the redshift can be written as

$$\Delta z_{\rm tt} = \left.\frac{\partial}{\partial q}\left(\frac{\mathcal{R}_{\rm tt}}{m_2}\right)\right|_{\hat{\Omega}_r, \hat{\Omega}_\phi, m_2}. \tag{139}$$

Expanding in the mass ratio, the tail-of-tail contributions to the redshift at each self-force order are then defined by

$$\Delta z_{\rm tt} = q \Delta z_{\rm tt}^{\rm 1SF} + q^2 \Delta z_{\rm tt}^{\rm 2SF} + \mathcal{O}(q^3), \tag{140}$$

with the corresponding contributions to the inverse redshift, $\Delta U_{\rm tt}^{\rm 1SF}$ and $\Delta U_{\rm tt}^{\rm 2SF}$, following from Eqs. (131) and (132).

Finally, we note that one could adopt alternative conventions, e.g., Refs. [128,139]. There they choose to expand in powers of ν at fixed M-rescaled frequencies, which necessarily changes the interpretation of the expansion coefficients [129]. Since $M\Omega = (1+q)m_2\Omega$ and $\nu = q/(1+q)^2$, we would expect the two sets of expansion coefficients to differ from first order in the mass ratio onwards.

A. First-order self-force comparison

We begin with the first-order self-force sector, for which independent results are available in the literature, notably Ref. [98]. The averaged tail-of-tail Hamiltonian derived above can be written as

$$\frac{\langle H_{\rm tt} \rangle}{M} = \pi\nu^2 \left[\frac{F_0(e_t)}{a_r^{13/2}} - \frac{F_1(e_t) + \nu G_1(e_t)}{a_r^{15/2}}\right] + \mathcal{O}(a_r^{-17/2}, e_t^{14}), \tag{141}$$

where F_0 is the eccentricity polynomial appearing in the Newtonian-order result in Eq. (45), while F_1 and G_1 are respectively the ν-independent and linear-in-ν parts of the fractional-1PN result in Eq. (46).

As $M = m_2(1+q)$, the tail-of-tail contribution to the frequency-space potential is given by

$$\frac{\mathcal{R}_{\rm tt}}{m_2} = (1+q)\frac{\langle H_{\rm tt} \rangle}{M}. \tag{142}$$

Since $\nu = q/(1+q)^2$, the q-dependent prefactor satisfies

$$(1+q)\nu^2 = q^2 + \mathcal{O}(q^3). \tag{143}$$

Moreover, the term proportional to νG_1 in Eq. (141) first contributes at order q^3 and can be ignored here. The terms relevant to $\mathcal{R}_{\rm tt}$ at 1SF order are therefore

$$\frac{\mathcal{R}_{\rm tt}}{m_2} = q^2 \mathcal{H}_{\rm tt}^{[q^2]}(a_r, e_t) + \mathcal{O}(q^3), \tag{144}$$

where

$$\mathcal{H}_{\rm tt}^{[q^2]}(a_r, e_t) = \pi\left[\frac{F_0(e_t)}{a_r^{13/2}} - \frac{F_1(e_t)}{a_r^{15/2}}\right]. \tag{145}$$

To implement the fixed-frequency derivative, we use the Darwin parameters (p, e) to label the reference Schwarzschild geodesic with the prescribed dimensionless frequencies $(\hat{\Omega}_r, \hat{\Omega}_\phi)$ [136].[2] This follows the convention adopted in eccentric self-force calculations [98,135]. Since the geodesic frequency map between (p, e) and $(\hat{\Omega}_r, \hat{\Omega}_\phi)$ is independent of the mass ratio, derivatives taken at fixed (p, e) coincide with the fixed-frequency derivative of Eq. (139). At fixed (p, e), the QK orbital elements admit expansions of the form

$$a_r = a_{r,\rm geo} + \mathcal{O}(q), \tag{146}$$

$$e_t = e_{t,\rm geo} + \mathcal{O}(q). \tag{147}$$

Because $\mathcal{R}_{\rm tt}$ begins at order q^2, the $\mathcal{O}(q)$ shifts to the orbital parameters first affect the mass derivative at 2SF

[2] In the strong-field regime near the separatrix, the map between (p, e) and $(\hat{\Omega}_r, \hat{\Omega}_\phi)$ can fail to be invertible with physically distinct geodesics sharing frequencies [135,140]. We do not expect this to be a problem for the weak-field expansions considered here.

order. The 1SF result can therefore be directly evaluated along the reference Schwarzschild geodesic.

At the PN accuracy used here, the test-particle relations are given by [135,138,141–144]

$$a_{r,\mathrm{geo}} = \frac{p}{1-e^2} - 1 + \mathcal{O}(p^{-1}), \tag{148}$$

$$e_{t,\mathrm{geo}} = e\left[1 - 3\frac{1-e^2}{p} + \mathcal{O}(p^{-2})\right], \tag{149}$$

$$z_{\mathrm{geo}} = 1 - \frac{3}{2}\frac{1-e^2}{p} + \mathcal{O}(p^{-2}). \tag{150}$$

Taking the derivative in Eq. (139) at fixed (p,e) gives

$$\frac{\partial}{\partial q}\left(\frac{\mathcal{R}_{\mathrm{tt}}}{m_2}\right)\bigg|_{p,e} = 2q\mathcal{H}_{\mathrm{tt}}^{[q^2]}(a_{r,\mathrm{geo}}, e_{t,\mathrm{geo}}) + \mathcal{O}(q^2). \tag{151}$$

Matching to Eq. (140) at order q, and using Eq. (131), we find

$$\Delta z_{\mathrm{tt}}^{\mathrm{1SF}} = 2\mathcal{H}_{\mathrm{tt}}^{[q^2]}(a_{r,\mathrm{geo}}, e_{t,\mathrm{geo}}), \tag{152}$$

$$\Delta U_{\mathrm{tt}}^{\mathrm{1SF}} = -\frac{2\mathcal{H}_{\mathrm{tt}}^{[q^2]}(a_{r,\mathrm{geo}}, e_{t,\mathrm{geo}})}{z_{\mathrm{geo}}^2}. \tag{153}$$

Substituting Eqs. (148)–(150) into this expression, we obtain

$$\Delta U_{\mathrm{tt}}^{\mathrm{1SF}} = \pi\left[\frac{\mathcal{U}_{11/2}^{(1)}(e)}{p^{13/2}} + \frac{\mathcal{U}_{13/2}^{(1)}(e)}{p^{15/2}}\right] + \mathcal{O}(p^{-17/2}, e^{14}), \tag{154}$$

where the subscripts denote the corresponding PN orders and the eccentricity-expanded coefficients are given by

$$\mathcal{U}_{11/2}^{(1)}(e) = -\frac{13696}{525} - \frac{232618}{1575}e^2 + \frac{430889}{3150}e^4 + \frac{18404963}{151200}e^6 - \frac{7527343}{145152}e^8 - \frac{22605883901}{1451520000}e^{10} - \frac{104442581123}{17418240000}e^{12} + \mathcal{O}(e^{14}), \tag{155}$$

$$\mathcal{U}_{13/2}^{(1)}(e) = \frac{81077}{3675} + \frac{2687231}{4410}e^2 + \frac{13695499}{47040}e^4 - \frac{1953554329}{1411200}e^6 + \frac{112283726591}{812851200}e^8 + \frac{7790443135159}{40642560000}e^{10} + \frac{398929659254227}{7803371520000}e^{12} + \mathcal{O}(e^{14}). \tag{156}$$

In the circular limit, these expressions reduce to the known half-integer redshift coefficients [83–85,145]. More generally, the leading-order eccentricity function in Eq. (141) satisfies the exact relation

$$F_0(e) = \frac{6848}{525}\varphi(e), \tag{157}$$

which follows from Eqs. (101) and (102), together with the normalization of the Newtonian mass-quadrupole contribution to the tail-of-tail Hamiltonian. Using the leading-order parts of the geodesic relations in Eqs. (148)–(150), the complete 5.5PN eccentricity dependence can therefore be written as

$$\mathcal{U}_{11/2}^{(1)}(e) = -\frac{13696}{525}(1-e^2)^{13/2}\varphi(e), \tag{158}$$

which agrees exactly, to all orders in eccentricity, with the independent 1SF result of Ref. [98]. Eccentricity expanding this expression reproduces the coefficients displayed above, after accounting for the overall factor of $1/p$ in the PN-series convention of Ref. [98]. At 6.5PN order, the two results agree coefficient by coefficient through $\mathcal{O}(e^{12})$, again after accounting for the overall factor of $1/p$ in [98]. These comparisons provide nontrivial checks of both the leading and fractional-1PN eccentricity dependence of the averaged tail-of-tail Hamiltonian in the small-mass-ratio limit.

B. Eccentric second-order self-force prediction

As discussed in Sec. VI, extending the calculation to 2SF requires the variation of the orbital elements with the mass ratio at fixed dimensionless fundamental frequencies $(\hat{\Omega}_r, \hat{\Omega}_\phi)$ [99,100,146,147]. This is a generic feature of second-order perturbation theory, and also arises in the two-timescale formulation of the self-force problem [148–150]. The results derived here provide weak-field benchmarks for the ongoing 2SF program, which has delivered conservative invariants, fluxes, and waveforms for quasicircular orbits on a Schwarzschild background [102,103,106,151], and for which extensions to eccentric orbits are under active development [152,153], supported by high-order analytic 1SF results for eccentric orbits [98,138,154]. In this context, $\mathcal{R}_{\mathrm{tt}}$ starts at $\mathcal{O}(q^2)$, so its 2SF contribution requires, beyond the explicit mass-ratio dependence of $\langle H_{\mathrm{tt}}\rangle$, only the linear-in-$q$ variation of the orbital elements at fixed frequencies.

For convenience, we introduce the following variables:

$$x \equiv e^2, \qquad w \equiv 1 - e^2 = 1 - x, \qquad x_t \equiv e_t^2, \tag{159}$$

and express the eccentricity polynomials in Eq. (141) as functions of x_t. We write the averaged tail-of-tail Hamiltonian as

$$\frac{\langle H_{\rm tt}\rangle}{M} = \pi\nu^2 \mathcal{B}(q;p,e), \tag{160}$$

where

$$\mathcal{B}(q;p,e) = \frac{F_0(x_t)}{a_r^{13/2}} - \frac{F_1(x_t) + \nu G_1(x_t)}{a_r^{15/2}}. \tag{161}$$

Combining this definition with Eq. (142) gives

$$\frac{\mathcal{R}_{\rm tt}}{m_2} = \pi(1+q)\nu^2 \mathcal{B}(q;p,e). \tag{162}$$

The mass-ratio dependence of $\mathcal{B}$ arises through $\nu(q)$ and the fixed-frequency orbital elements $a_r(q;p,e)$ and $x_t(q;p,e)$.

As before, we use (p,e) to label the reference Schwarzschild geodesic associated with the fixed frequency pair. With this convention, the energy and angular momentum maps can be written as

$$\varepsilon(q) = \varepsilon_{\rm geo} + q\delta_q\varepsilon + \mathcal{O}(q^2), \tag{163}$$

$$j(q) = j_{\rm geo} + q\delta_q j + \mathcal{O}(q^2). \tag{164}$$

Differentiating the QK frequency map [110,111] at fixed $(\hat{\Omega}_r, \hat{\Omega}_\phi)$, following the approach of Refs. [96,135], yields

$$\delta_q\varepsilon \equiv \left.\frac{\partial\varepsilon}{\partial q}\right|_{\hat{\Omega}_r,\hat{\Omega}_\phi,q=0} = \frac{2}{3}\varepsilon_{\rm geo} + \frac{3}{4}\varepsilon_{\rm geo}^2 + \mathcal{O}(\varepsilon_{\rm geo}^3), \tag{165}$$

$$\delta_q j \equiv \left.\frac{\partial j}{\partial q}\right|_{\hat{\Omega}_r,\hat{\Omega}_\phi,q=0} = \frac{2}{3}j_{\rm geo} + \frac{5}{12}(j_{\rm geo} - 6)\varepsilon_{\rm geo} + \mathcal{O}(\varepsilon_{\rm geo}^2). \tag{166}$$

The $\mathcal{O}(\varepsilon_{\rm geo})$ term in Eq. (166) requires the 2PN periastron advance, which was provided in Sec. II.

Combining these shifts with the QK equations of motion and the Schwarzschild relations between (ε, j) and (p,e) gives [135,136,141]

$$a_r(q;p,e) = \frac{p}{w} - 1 - q\left(\frac{2p}{3w} + 1\right) + \mathcal{O}(p^{-1}, qp^{-1}, q^2), \tag{167}$$

$$x_t(q;p,e) = x - \frac{6xw}{p} + qw\left[-\frac{2}{3} + \frac{31-28x}{6p}\right] + \mathcal{O}(p^{-2}, qp^{-2}, q^2), \tag{168}$$

extending Eqs. (148) and (149) to linear order in q.

At fixed (p,e), we expand $\mathcal{B}$ in powers of q as

$$\mathcal{B}(q;p,e) = \mathcal{B}_0(p,e) + q\mathcal{B}_1(p,e) + \mathcal{O}(q^2). \tag{169}$$

Substituting Eqs. (167) and (168) yields

$$\mathcal{B}_0 = \frac{w^{13/2}}{p^{13/2}}F_0 + \frac{w^{15/2}}{p^{15/2}}\left(\frac{13}{2}F_0 - 6xF_0' - F_1\right) + \mathcal{O}(p^{-17/2}), \tag{170}$$

$$\mathcal{B}_1 = \frac{w^{13/2}}{p^{13/2}}\left(\frac{13}{3}F_0 - \frac{2}{3}wF_0'\right) + \frac{w^{15/2}}{p^{15/2}}\left[39F_0 + \frac{5-158x}{6}F_0' + 4xwF_0'' + \frac{2}{3}wF_1' - 5F_1 - G_1\right] + \mathcal{O}(p^{-17/2}), \tag{171}$$

where F_0, F_1, and G_1, and their derivatives, are evaluated at $x = e^2$.

Similar to before, the prefactor in Eq. (162) satisfies $(1+q)\nu^2 = q^2 - 3q^3 + \mathcal{O}(q^4)$. The required expression at $\mathcal{O}(q^3)$ therefore becomes

$$\frac{\mathcal{R}_{\rm tt}}{m_2} = \pi[q^2\mathcal{B}_0 + q^3(\mathcal{B}_1 - 3\mathcal{B}_0)] + \mathcal{O}(q^4). \tag{172}$$

Differentiating Eq. (172) at fixed (p,e), and matching to the expansion in Eq. (140), gives

$$\frac{\partial}{\partial q}\left(\frac{\mathcal{R}_{\rm tt}}{m_2}\right)\bigg|_{p,e} = q\Delta z_{\rm tt}^{\rm 1SF} + q^2\Delta z_{\rm tt}^{\rm 2SF} + \mathcal{O}(q^3). \tag{173}$$

Matching powers of q gives us the 1SF and 2SF contributions

$$\Delta z_{\rm tt}^{\rm 1SF} = 2\pi\mathcal{B}_0, \tag{174}$$

$$\Delta z_{\rm tt}^{\rm 2SF} = 3\pi(\mathcal{B}_1 - 3\mathcal{B}_0), \tag{175}$$

where the 1SF relation agrees with the result obtained in Eq. (152).

The contribution to the 2SF redshift can now be written in terms of the 5.5PN and 6.5PN contributions,

$$\Delta z_{\rm tt}^{\rm 2SF} = \pi\left[\frac{\mathcal{Z}_{11/2}^{(2)}(e)}{p^{13/2}} + \frac{\mathcal{Z}_{13/2}^{(2)}(e)}{p^{15/2}}\right] + \mathcal{O}(p^{-17/2}, e^{12}), \tag{176}$$

where the coefficients are given by

$$\mathcal{Z}^{(2)}_{11/2} = w^{13/2}(4F_0 - 2wF_0'), \tag{177}$$

$$\mathcal{Z}^{(2)}_{13/2} = w^{15/2}\left[\frac{117}{2}F_0 + \frac{5}{2}(1-10x)F_0' + 12xwF_0'' + 2wF_1' - 6F_1 - 3G_1\right]. \tag{178}$$

Because the 2SF expressions involve derivatives of the eccentricity, knowledge of F_i through $\mathcal{O}(e^{12})$ is only sufficient to determine the 2SF result through $\mathcal{O}(e^{10})$.

Finally, we can calculate the 2SF inverse redshift invariant. We let $\Delta z^{1\mathrm{SF}} = \Delta z_0^{1\mathrm{SF}} + \Delta z_{\mathrm{tt}}^{1\mathrm{SF}}$, where $\Delta z_0^{1\mathrm{SF}}$ denotes the tail-free contribution. We can then linearize the cross-term appearing in Eq. (132) in the tail-of-tail interaction, resulting in

$$\Delta U_{\mathrm{tt}}^{2\mathrm{SF}} = -\frac{\Delta z_{\mathrm{tt}}^{2\mathrm{SF}}}{z_{\mathrm{geo}}^2} + \frac{2\Delta z_0^{1\mathrm{SF}}\Delta z_{\mathrm{tt}}^{1\mathrm{SF}}}{z_{\mathrm{geo}}^3}. \tag{179}$$

Since $\Delta z_0^{1\mathrm{SF}} = \mathcal{O}(p^{-1})$ and $\Delta z_{\mathrm{tt}}^{1\mathrm{SF}} = \mathcal{O}(p^{-13/2})$, the cross-term first contributes at $\mathcal{O}(p^{-15/2})$, i.e., at relative 1PN order. As such, we only need the following contributions from the tail-free terms [98,138,141,142]:

$$z_{\mathrm{geo}} = 1 - \frac{3}{2}\frac{w}{p} + \mathcal{O}(p^{-2}), \tag{180}$$

$$\Delta z_0^{1\mathrm{SF}} = \frac{w}{p} + \mathcal{O}(p^{-2}). \tag{181}$$

Substituting these into Eq. (179) gives

$$\Delta U_{\mathrm{tt}}^{2\mathrm{SF}} = \pi\left[\frac{\mathcal{U}^{(2)}_{11/2}(e)}{p^{13/2}} + \frac{\mathcal{U}^{(2)}_{13/2}(e)}{p^{15/2}}\right] + \mathcal{O}(p^{-17/2}, e^{12}). \tag{182}$$

Eccentricity expanding these coefficients leads to

$$\mathcal{U}^{(2)}_{11/2}(e) = \frac{417514}{1575} + \frac{42586}{175}e^2 - \frac{35640523}{50400}e^4 + \frac{804961}{32400}e^6 + \frac{10044408967}{96768000}e^8 + \frac{216499199}{7680000}e^{10} + \mathcal{O}(e^{12}), \tag{183}$$

$$\mathcal{U}^{(2)}_{13/2}(e) = -\frac{9851257}{7350} - \frac{171446197}{39200}e^2 + \frac{7769751313}{1411200}e^4 + \frac{117922725349}{40642560}e^6 - \frac{4876131494921}{2709504000}e^8 - \frac{171112938659393}{433520640000}e^{10} + \mathcal{O}(e^{12}). \tag{184}$$

Equations (183) and (184) give the conservative tail-of-tail contribution to the 2SF inverse redshift through relative 1PN order and $\mathcal{O}(e^{10})$, with terms quadratic in the tail-of-tail interaction entering beyond the PN accuracy considered here. The linearized result therefore gives the complete corresponding 2SF inverse-redshift contributions at 5.5PN and 6.5PN.

We note that the circular-orbit limit of the 2SF results derived above is not guaranteed to coincide with the corresponding quantity computed directly for circular orbits [135,155]. The circular-orbit calculation compares circular configurations at fixed $\hat{\Omega}_\phi$, whereas the eccentric calculation holds both $(\hat{\Omega}_r, \hat{\Omega}_\phi)$ fixed. As such, while taking $e \to 0$ would make the reference geodesic circular, it is not sufficient to ensure that the perturbed binary is circular. To recover the circular result, we instead impose $J_r = 0$ on the perturbed binary while holding $\hat{\Omega}_\phi$ fixed, and reexpand the invariant along this circular sequence [97]. At fixed $\hat{\Omega}_\phi$, the 1SF correction to the periastron advance determines the accompanying shift in $\hat{\Omega}_r$. For circular geodesics, the first law requires [96,97]

$$\left.\frac{\partial z_{\mathrm{geo}}}{\partial \hat{\Omega}_r}\right|_{\hat{\Omega}_\phi} = 0 \tag{185}$$

so that at 1SF the frequency shift enters only through this derivative, leaving the redshift unchanged. At 2SF, however, reexpanding Eq. (129) along the circular sequence generates terms proportional to $\partial\Delta z^{1\mathrm{SF}}/\partial\hat{\Omega}_r$ and $\partial^2 z_{\mathrm{geo}}/\partial\hat{\Omega}_r^2$, with $\hat{\Omega}_\phi$ held fixed. Once these corrections are included, we expect our result to agree with the standard circular limit.

C. Cross-validation with EOB

In this section, we reproduce the redshift coefficients directly from the linearized EOB Hamiltonian, providing a consistency check of both the fixed-frequency expansion and the matching to the tail-of-tail EOB potentials [156]. As discussed above, the fixed-frequency expansion lowers the available eccentricity order by two, so an EOB Hamiltonian through $\mathcal{O}(e_t^4)$ determines the redshift through $\mathcal{O}(e^2)$. Here, we work to $\mathcal{O}(e_t^4)$ for computational convenience, though this comparison can be trivially extended to higher orders.

Our starting point is to linearize the effective Hamiltonian in the tail-of-tail potentials [87], giving

$$\frac{H^{\rm tt}_{\rm EOB}}{M} = \frac{\nu}{2\hat{H}_{\rm eff}\hat{H}}[\delta A(1+p_\phi^2u^2) + p_r^2(2A\bar{D}\delta A + A^2\delta\bar{D}) + A\delta Q], \quad (186)$$

where $u = GM/r_{\rm EOB}$. Through $\mathcal{O}(e_t^4)$, only δA, $\delta\bar{D}$, and the p_r^4 sector of δQ contribute. The required EOB coefficients are therefore $\{a_{6.5}, a_{7.5}, \bar{d}_{5.5}, \bar{d}_{6.5}, q_{4,4.5}, q_{4.5.5}\}$. Evaluating Eq. (186) along the 1PN QK orbit in EOB coordinates, and averaging over one radial period yields

$$\langle H^{\rm tt}_{\rm EOB}\rangle_{\eta^0} = \frac{\pi\nu^2}{a_r^{{\rm e}13/2}}\left[\frac{6848}{525} + \frac{49969}{315}e_t^{\rm e2} + \frac{919237}{1260}e_t^{\rm e4}\right], \quad (187)$$

$$\langle H^{\rm tt}_{\rm EOB}\rangle_{\eta^2} = -\frac{\pi\nu^2}{a_r^{{\rm e}15/2}}\left[\frac{368693}{7350} + \frac{4982}{315}\nu - \left(\frac{3982}{45} - \frac{6263599}{22050}\nu\right)e_t^{\rm e2} - \left(\frac{355973869}{94080} - \frac{208757957}{117600}\nu\right)e_t^{\rm e4}\right], \quad (188)$$

where $a_r^{\rm e} = a_r + \eta^2$ and $e_t^{\rm e} = e_t(1-\eta^2\nu/a_r)$.

The EOB fixed-frequency map can be written in terms of (x, p, w) as

$$a_r^{\rm e}(q;p,e) = \frac{p}{w} - q\left(\frac{2p}{3w}+1\right) + \mathcal{O}(p^{-1}, qp^{-1}, q^2), \quad (189)$$

$$x_t^{\rm e}(q;p,e) = x - \frac{6xw}{p} + qw\left[-\frac{2}{3} + \frac{31-40x}{6p}\right] + \mathcal{O}(p^{-2}, qp^{-2}, q^2). \quad (190)$$

We now iterate the same procedure as above to determine the redshift invariant and its inverse. Following this procedure gives

$$\Delta z_{\rm tt}^{\rm 1SF} = \pi\left(\frac{13696}{525} + \frac{232618}{1575}e^2\right)\frac{1}{p^{13/2}} - \pi\left(\frac{368693}{3675} + \frac{4296083}{4410}e^2\right)\frac{1}{p^{15/2}}, \quad (191)$$

$$\Delta z_{\rm tt}^{\rm 2SF} = -\pi\left(\frac{417514}{1575} + \frac{42586}{175}e^2\right)\frac{1}{p^{13/2}} + \pi\left(\frac{16079941}{7350} + \frac{1605811933}{352800}e^2\right)\frac{1}{p^{15/2}}, \quad (192)$$

and the corresponding inverse-redshift contributions by

$$\Delta U_{\rm tt}^{\rm 1SF} = -\pi\left(\frac{13696}{525} + \frac{232618}{1575}e^2\right)\frac{1}{p^{13/2}} + \pi\left(\frac{81077}{3675} + \frac{2687231}{4410}e^2\right)\frac{1}{p^{15/2}}, \quad (193)$$

$$\Delta U_{\rm tt}^{\rm 2SF} = \pi\left(\frac{417514}{1575} + \frac{42586}{175}e^2\right)\frac{1}{p^{13/2}} - \pi\left(\frac{9851257}{7350} + \frac{171446197}{39200}e^2\right)\frac{1}{p^{15/2}}. \quad (194)$$

These expressions agree with Eqs. (155), (156), (183), and (184) through $\mathcal{O}(e^2)$. This provides a useful consistency check on the fixed-frequency expansion and matching, including the $\mathcal{O}(\nu^2)$ contributions to $a_{7.5}$, $\bar{d}_{6.5}$, and $q_{4.5.5}$.

VII. CONCLUDING REMARKS

We have extended the conservative tail-of-tail dynamics of eccentric, nonspinning binaries through relative 1PN order and $\mathcal{O}(e_t^{12})$. Matching the Delaunay-averaged Hamiltonian to the EOB dynamics determines the 5.5PN and 6.5PN contributions to the nongeodesic Q potential through the p_r^{12} sector, including their complete dependence on the symmetric mass ratio. The terms linear in ν reproduce the available first-order self-force information, while the quadratic terms constitute qualitatively new eccentric 2SF predictions.

These results are supported by several complementary calculations. The time-split derivation of the averaged Hamiltonian agrees with its spectral Fourier-Bessel representation, and the first-law calculation reproduces the known 1SF redshift through $\mathcal{O}(e^{12})$ [98]. At 2SF order, the fixed-frequency expansion yields the tail-of-tail contribution to the inverse redshift at 5.5PN and 6.5PN through $\mathcal{O}(e^{10})$, with a direct EOB calculation providing an internal consistency check through $\mathcal{O}(e^2)$.

The results presented here provide building blocks that form part of the complete description of the dynamics at 5.5PN and 6.5PN accuracy. Their use in EOB or self-force models at the corresponding eccentricity and mass-ratio orders will need the full contributions at the relevant PN orders, which are currently only partially known [17–19,107].

Three extensions of this work are of particular interest. First, direct eccentric 2SF calculations would provide an independent test of the new weak-field predictions and connect them to the strong-field regime. Second, the higher-p_r sectors of the Q potential obtained here could be analytically continued to unbound orbits. Reconstructing the corresponding tail-of-tail scattering angle would allow a direct bound-unbound comparison with the scattering calculation of Ref. [87], and would help clarify the role of the nongeodesic terms in high-PM scattering dynamics. Such predictions could also be incorporated into resummed EOB models of the scattering dynamics [157,158], whose gauge flexibility was recently explored in Ref. [159], and

confronted directly against numerical-relativity scattering simulations [160–165]. Finally, extending the analysis beyond fractional 1PN accuracy would determine the half-integer contributions beyond 6.5PN, at the cost of higher-PN source multipoles and orbital dynamics, and help to further constrain the eccentric EOB potentials.

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DATA AVAILABILITY

The expressions derived in this paper are publicly available [166].

APPENDIX A: MULTIPOLE MOMENT BILINEARS

Tables I–III, we provide the Multipole-moment bilinears up to $\mathcal{O}(e_t^4)$ and fractional 1PN order.

TABLE I. Multipole-moment bilinears for $I_2^{(3,4)}(u,u')$ up to fractional 1PN order and $\mathcal{O}(e_t^4)$. The complete expressions up to $\mathcal{O}(e_t^{12})$ can be found in the Supplemental Material [118].

$$I^{(3,4)}_{2(\eta^0,e_t^0)}(u,u')\frac{64\nu^2}{a_r^{13/2}}[\sin(2(u-u'))]$$

$$I^{(3,4)}_{2(\eta^0,e_t^1)}(u,u')\frac{4\nu^2}{a_r^{13/2}}[38\sin(3u-2u')+13\sin(2u-u')+65\sin(2u-3u')+10\sin(u-2u')]$$

$$\begin{aligned}I^{(3,4)}_{2(\eta^0,e_t^2)}(u,u')\frac{\nu^2}{6a_r^{13/2}}[&-120\sin(2u')+1320\sin(4u-2u')+741\sin(3u-u')+3705\sin(3(u-u'))\\&+2280\sin(2(u-u'))+3420\sin(2u-4u')+197\sin(u-u')+975\sin(u-3u')+156\sin(2u)+2\sin(u+u')]\end{aligned}$$

$$\begin{aligned}I^{(3,4)}_{2(\eta^0,e_t^3)}(u,u')\frac{\nu^2}{12a_r^{13/2}}[&-195\sin(u')-975\sin(3u')+3000\sin(5u-2u')+2145\sin(4u-u')+10725\sin(4u-3u')\\&+10206\sin(3u-2u')+16245\sin(3u-4u')+3282\sin(2u-u')+14340\sin(2u-3u')+10920\sin(2u-5u')\\&+2864\sin(u-2u')+4275\sin(u-4u')+199\sin(u)+741\sin(3u)-106\sin(u+2u')+102\sin(2u+u')]\end{aligned}$$

$$\begin{aligned}I^{(3,4)}_{2(\eta^0,e_t^4)}(u,u')\frac{\nu^2}{24a_r^{13/2}}[&-2970\sin(2u')-4275\sin(4u')+5880\sin(6u-2u')+4875\sin(5u-u')+24375\sin(5u-3u')\\&+29790\sin(4u-2u')+47025\sin(4(u-u'))+14559\sin(3u-u')+63045\sin(3(u-u'))\\&+51870\sin(3u-5u')+29562\sin(2(u-u'))+59100\sin(2u-4u')+28560\sin(2u-6u')+4137\sin(u-u')\\&+17975\sin(u-3u')+13650\sin(u-5u')+3384\sin(2u)+2145\sin(4u)-33\sin(u+u')-925\sin(u+3u')\\&-102\sin(2u+2u')+468\sin(3u+u')]\end{aligned}$$

$$I^{(3,4)}_{2(\eta^2,e_t^0)}(u,u')\frac{32\nu^2}{21a_r^{15/2}}[252(u-u')\cos(2(u-u'))-(487-201\nu)\sin(2(u-u'))]$$

$$\begin{aligned}I^{(3,4)}_{2(\eta^2,e_t^1)}(u,u')\frac{2\nu^2}{21a_r^{15/2}}[&9576(u-u')\cos(3u-2u')+3276(u-u')\cos(2u-u')+16380(u-u')\cos(2u-3u')\\&+2520(u-u')\cos(u-2u')+2(3657\nu-6805)\sin(3u-2u')+7(393\nu-919)\sin(2u-u')\\&+7(1797\nu-3815)\sin(2u-3u')+2(1143\nu-3167)\sin(u-2u')]\end{aligned}$$

$$\begin{aligned}I^{(3,4)}_{2(\eta^2,e_t^2)}(u,u')\frac{\nu^2}{84a_r^{15/2}}[&10080(u-u')\cos(2u')+110880(u-u')\cos(4u-2u')+62244(u-u')\cos(3u-u')\\&+311220(u-u')\cos(3(u-u'))+223776(u-u')\cos(2(u-u'))+287280(u-u')\cos(2u-4u')\\&+16380(u-u')\cos(u-u')+81900(u-u')\cos(u-3u')+13104(u-u')\cos(2u)-8(1143\nu-2915)\sin(2u')\\&+40(2001\nu-2641)\sin(4u-2u')-(90403-50163\nu)\sin(3u-u')\\&-(348275-228471\nu)\sin(3(u-u'))+24(7235\nu-13971)\sin(2(u-u'))\\&+4(52599\nu-93385)\sin(2u-4u')-(42359-15595\nu)\sin(u-u')\\&+5(14373\nu-36221)\sin(u-3u')+28(393\nu-739)\sin(2u)+2(23\nu-339)\sin(u+u')]\end{aligned}$$

$$\begin{aligned}I^{(3,4)}_{2(\eta^2,e_t^3)}(u,u')\frac{\nu^2}{168a_r^{15/2}}[&16380(u-u')\cos(u')+81900(u-u')\cos(3u')+252000(u-u')\cos(5u-2u')+180180(u-u')\cos(4u-u')\\&+900900(u-u')\cos(4u-3u')+1010520(u-u')\cos(3u-2u')+1364580(u-u')\cos(3u-4u')\\&+327600(u-u')\cos(2u-u')+1466640(u-u')\cos(2u-3u')+917280(u-u')\cos(2u-5u')\\&+279720(u-u')\cos(u-2u')+359100(u-u')\cos(u-4u')+16380(u-u')\cos(u)+62244(u-u')\cos(3u)\\&+10080(u-u')\cos(u+2u')+8064(u-u')\cos(2u+u')+(38405-15549\nu)\sin(u')\\&-5(14373\nu-32945)\sin(3u')+8(21165\nu-16127)\sin(5u-2u')+35(3933\nu-5065)\sin(4u-u')\\&+35(17817\nu-16745)\sin(4u-3u')+2(373053\nu-498431)\sin(3u-2u')\\&+133(7167\nu-8095)\sin(3u-4u')+14(18713\nu-34367)\sin(2u-u')\\&+4(271311\nu-426346)\sin(2u-3u')+56(11316\nu-15805)\sin(2u-5u')\end{aligned}$$

(Table continued)

TABLE I. *(Continued)*

	$+12(20178\nu-47039)\sin(u-2u')+5(60465\nu-135109)\sin(u-4u')$
	$+(15641\nu-36737)\sin(u)+(50163\nu-66463)\sin(3u)$
	$-2(3219\nu-2599)\sin(u+2u')+2(3881\nu-5097)\sin(2u+u')]$
$I^{(3,4)}_{2(\eta^2,e_t^4)}(u,u')$	$\frac{\nu^2}{336a_r^{15/2}}[16380(u-u')+289800(u-u')\cos(2u')+359100(u-u')\cos(4u')+493920(u-u')\cos(6u-2u')$
	$+409500(u-u')\cos(5u-u')+2047500(u-u')\cos(5u-3u')+2945880(u-u')\cos(4u-2u')$
	$+3950100(u-u')\cos(4(u-u'))+1470924(u-u')\cos(3u-u')+6540660(u-u')\cos(3(u-u'))$
	$+4357080(u-u')\cos(3u-5u')+3374784(u-u')\cos(2(u-u'))+6113520(u-u')\cos(2u-4u')$
	$+2399040(u-u')\cos(2u-6u')+409500(u-u')\cos(u-u')+1833300(u-u')\cos(u-3u')$
	$+1146600(u-u')\cos(u-5u')+335664(u-u')\cos(2u)+180180(u-u')\cos(4u)+26460(u-u')\cos(u+u')$
	$+81900(u-u')\cos(u+3u')+8064(u-u')\cos(2u+2u')+38304(u-u')\cos(3u+u')$
	$-2(124287\nu-261901)\sin(2u')-25(12093\nu-24149)\sin(4u')+56(5415\nu-797)\sin(6u-2u')$
	$-(222401-292395\nu)\sin(5u-u')+5(262995\nu-85901)\sin(5u-3u')$
	$+2(1030845\nu-799819)\sin(4u-2u')+65(39903\nu-19475)\sin(4(u-u'))$
	$-(1396449-1118377\nu)\sin(3u-u')-(4241879-4584735\nu)\sin(3(u-u'))$
	$+14(204474\nu-141175)\sin(3u-5u')+154(16471\nu-25199)\sin(2(u-u'))$
	$+20(215991\nu-259772)\sin(2u-4u')+56(27561\nu-27016)\sin(2u-6u')$
	$-(815827-363595\nu)\sin(u-u')+5(302157\nu-607141)\sin(u-3u')$
	$+70(13110\nu-25321)\sin(u-5u')+4(67436\nu-98893)\sin(2u)+35(3933\nu-3085)\sin(4u)$
	$+(-59\nu-14041)\sin(u+u')-35(1467\nu-1117)\sin(u+3u')$
	$+14(329\nu-2801)\sin(2u+2u')+2(17597\nu-11622)\sin(3u+u')]$

TABLE II. Multipole-moment bilinears for $J_2^{(3,4)}(u,u')$ up to fractional 1PN order and $\mathcal{O}(e_t^4)$. The complete expressions up to $\mathcal{O}(e_t^{12})$ can be found in the Supplemental Material [118].

$J^{(3,4)}_{2(\eta^0,e_t^0)}(u,u')$	$\frac{\nu^2}{2a_r^{15/2}}(1-4\nu)[\sin(u-u')]$
$J^{(3,4)}_{2(\eta^0,e_t^1)}(u,u')$	$\frac{\nu^2}{4a_r^{15/2}}(1-4\nu)[-\sin(u')+7\sin(2u-u')+15\sin(u-2u')+\sin(u)]$
$J^{(3,4)}_{2(\eta^0,e_t^2)}(u,u')$	$\frac{\nu^2}{8a_r^{15/2}}(1-4\nu)[-15\sin(2u')+25\sin(3u-u')+105\sin(2(u-u'))+38\sin(u-u')+90\sin(u-3u')+7\sin(2u)$ $+\sin(u'+u)]$
$J^{(3,4)}_{2(\eta^0,e_t^3)}(u,u')$	$\frac{\nu^2}{16a_r^{15/2}}(1-4\nu)[-37\sin(u')-90\sin(3u')+65\sin(4u-u')+375\sin(3u-2u')+243\sin(2u-u')$ $+630\sin(2u-3u')+390\sin(u-2u')+350\sin(u-4u')+39\sin(u)+25\sin(3u)+5\sin(u+2u')$ $+9\sin(2u+u')]$
$J^{(3,4)}_{2(\eta^0,e_t^4)}(u,u')$	$\frac{\nu^2}{32a_r^{15/2}}(1-4\nu)[-385\sin(2u')-350\sin(4u')+140\sin(5u-u')+975\sin(4u-2u')+860\sin(3u-u')$ $+2250\sin(3(u-u'))+2385\sin(2(u-u'))+2450\sin(2u-4u')+702\sin(u-u')+2120\sin(u-3u')$ $+1050\sin(u-5u')+252\sin(2u)+65\sin(4u)+36\sin(u+u')+20\sin(u+3u')+65\sin(2u+2u')$ $+30\sin(3u+u')]$

TABLE III. Multipole-moment bilinears for $I_3^{(4,5)}(u,u')$ up to fractional 1PN order and $\mathcal{O}(e_t^4)$. The complete expressions up to $\mathcal{O}(e_t^{12})$ can be found in the Supplemental Material [118].

$I^{(4,5)}_{3(\eta^0,e_t^0)}(u,u')$	$\frac{3\nu^2}{20a_r^{15/2}}(1-4\nu)[32805\sin(3(u-u'))+\sin(u-u')]$
$I^{(4,5)}_{3(\eta^0,e_t^1)}(u,u')$	$\frac{3\nu^2}{40a_r^{15/2}}(1-4\nu)[-\sin(u')+212625\sin(4u-3u')+59535\sin(3u-2u')+316305\sin(3u-4u')-17\sin(2u-u')$ $+40095\sin(2u-3u')-33\sin(u-2u')+\sin(u)]$
$I^{(4,5)}_{3(\eta^0,e_t^2)}(u,u')$	$\frac{3\nu^2}{80a_r^{15/2}}(1-4\nu)[33\sin(2u')+801900\sin(5u-3u')+385875\sin(4u-2u')+2050125\sin(4(u-u'))$ $+73621\sin(3u-u')+932310\sin(3(u-u'))+1652805\sin(3u-5u')+73326\sin(2(u-u'))$ $+386595\sin(2u-4u')-82\sin(u-u')+42225\sin(u-3u')-17\sin(2u)-11\sin(u+u')]$

(Table continued)

TABLE III. *(Continued)*

$I^{(4,5)}_{3(\eta^0,e_t^3)}(u,u')$	$\frac{3\nu^2}{160a_r^{15/2}}(1-4\nu)[71\sin(u')-42225\sin(3u')+2296350\sin(6u-3u')+1455300\sin(5u-2u')$ $+7731900\sin(5u-4u')+477485\sin(4u-u')+5533350\sin(4u-3u')+10712625\sin(4u-5u')$ $+1457607\sin(3u-2u')+6714915\sin(3u-4u')+6245505\sin(3u-6u')+90879\sin(2u-u')$ $+1137435\sin(2u-3u')+2020095\sin(2u-5u')+77283\sin(u-2u')+408665\sin(u-4u')$ $-93\sin(u)+73621\sin(3u)+161\sin(u+2u')+69\sin(2u+u')]$
$I^{(4,5)}_{3(\eta^0,e_t^4)}(u,u')$	$\frac{3\nu^2}{320a_r^{15/2}}(1-4\nu)[-77122\sin(2u')-408665\sin(4u')+5528250\sin(7u-3u')+4167450\sin(6u-2u')$ $+22141350\sin(6u-4u')+1801190\sin(5u-u')+20727000\sin(5u-3u')$ $+40401900\sin(5(u-u'))+8512695\sin(4u-2u')+38610975\sin(4(u-u'))$ $+40480125\sin(4u-6u')+1780532\sin(3u-u')+13860930\sin(3(u-u'))$ $+32193420\sin(3u-5u')+19144755\sin(3u-7u')+1783074\sin(2(u-u'))$ $+8161250\sin(2u-4u')+7633395\sin(2u-6u')+95742\sin(u-u')+1233860\sin(u-3u')$ $+2138115\sin(u-5u')+90948\sin(2u)+477485\sin(4u)+324\sin(u+u')-40855\sin(u+3u')$ $+1157\sin(2u+2u')+59949\sin(3u+u')]$

APPENDIX B: FOURIER-BESSEL AMPLITUDES: A WORKED EXAMPLE

Here we schematically outline the derivation of $D_p = \mathcal{A}_p[\rho^2]$, where $\rho = 1 - e_t \cos u$. As was detailed in Sec. V, the Fourier-Bessel amplitude can be written as

$$D_p = \mathcal{A}_p[\rho^2] = \frac{1}{2\pi}\int_0^{2\pi} du(1-e_t\cos u)\rho^2 e^{-ip(u-e_t\sin u)}, \tag{B1}$$

which can also be expressed in terms of a Laurent expansion $\rho(u)F(u) = \sum_j c_j z^j$, with $z = e^{iu}$. Here, the weight function $\rho(u)F(u)$ is trivially given by $\rho^3 = (1-e_t\cos u)^3$. Writing $\cos u = (z+z^{-1})/2$, the weight function can be expanded as

$$\rho^3 = \left[1 - \frac{e_t}{2}(z+z^{-1})\right]^3, \tag{B2}$$

$$= c_0 + \sum_{j=1}^{3} c_j (z^j + z^{-j}), \tag{B3}$$

where the expansion terminates at $|j| = 3$ due to the trigonometric polynomial nature of the expressions. The resulting coefficients can be matched, leading to

$$c_0 = 1 + \frac{3}{2}e_t^2,$$
$$c_1 = -\frac{3}{2}e_t\left(1+\frac{e_t^2}{4}\right),$$
$$c_2 = \frac{3}{4}e_t^2,$$
$$c_3 = -\frac{e_t^3}{8}.$$

This allows us to express D_p as

$$D_p = c_0 J_p + c_1(J_{p-1}+J_{p+1}) + c_2(J_{p-2}+J_{p+2}) + c_3(J_{p-3}+J_{p+3}), \tag{B4}$$

where $J_m = J_m(pe_t)$. Using the standard Bessel recurrence relations,

$$J_{n-1}(pe_t) + J_{n+1}(pe_t) = \frac{2n}{pe_t}J_n(pe_t), \tag{B5}$$

$$J_{n-1}(pe_t) - J_{n+1}(pe_t) = 2J_n'(pe_t), \tag{B6}$$

where prime denotes differentiation with respect to the argument, we find that, for $p \neq 0$, D_p reduces to

$$D_p = -\frac{2}{p^2}J_p(pe_t), \tag{B7}$$

in agreement with Eq. (97). The derivation of the other coefficients follows a similar pattern.

APPENDIX C: FOURIER-BESSEL AMPLITUDES: $U_p^{\pm}$ AND $V_p^{\pm}$

The Fourier amplitudes $U_p^{\pm}$ and $V_p^{\pm}$ introduced in Eq. (113) are given by

$$U_p^{\pm} = \frac{3}{e_t^3 p^3}[\pm s A_p^{\pm} J_p(pe_t) + e_t B_p^{\pm} J_p'(pe_t)], \tag{C1}$$

$$V_p^{\pm} = \frac{6 \mp 5ps}{p^3}[J_p'(pe_t) \pm \frac{s}{e_t}J_p(pe_t)], \tag{C2}$$

where $A_p^{\pm}$ and $B_p^{\pm}$ are polynomials in p and s,

$$A_p^{\pm} \equiv 4p^2s^4 + 2s^2 + 6 \mp ps(7+5s^2), \tag{C3}$$

$$B_p^{\pm} \equiv 4p^2s^4 + 6s^2 + 2 \mp ps(5+7s^2). \tag{C4}$$

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