

# Uncovering subdominant multipole asymmetries in binary black-hole mergers

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In dynamically formed binaries, the spins of the black holes tend to be misaligned with the system's orbital angular momentum. This causes the spins to precess and leads to an asymmetric emission of gravitational waves. The resulting gravitational-wave multipole asymmetries directly source the recoil of the remnant black hole and are the critical element in fully describing precession. Recoil and precession are of significant astrophysical importance, but multipole asymmetries contribute only minimally to the overall signal strength. Consequently, most current gravitational-wave models either do not incorporate asymmetries at all, or only consider the dominant ones. Here we highlight the importance of subdominant multipole asymmetries for an accurate recoil velocity calculation and discuss their detectability with third generation detectors. Neglecting subdominant asymmetries leads to velocity differences of up to 210 km/s and can, in particular, introduce systematic biases in the inference of masses and the spin geometry. We further discuss universal characteristics of subdominant multipole asymmetries in order to prepare the ground for potential future asymmetry models. In the inspiral regime, the average antisymmetric frequencies can be described by a multiple of the orbital frequency. During ringdown, however, they become equal to their corresponding symmetric frequencies.

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## I. INTRODUCTION

General relativity (GR) predicts that gravitational waves (GWs) have a quadrupolar nature. Accordingly, when decomposing the waveform in a  $s = -2$  spin-weighted spherical harmonic (SWSH) basis, the  $(\ell, m) = (2, \pm 2)$  multipoles dominate the radiation from compact binary coalescences (CBCs). Hence, most of the scientific output from the 218 detected CBC signals by the LIGO-Virgo-Kagra Collaboration [1–4] arises from accurately modeling and measuring these dominant multipoles.

However, GWs exhibit additional structure beyond the leading quadrupole, including higher-order multipoles and asymmetries between the  $+m$  and  $-m$  multipoles. In the cases of GW190412 [5,6], GW190814 [7] and GW241011 [8] evidence for higher-order multipoles has been reported. For GW200129\_065458, the presence of a dominant multipole asymmetry played a key role in the identification of the remnant's recoil, orbital and spin precession [9–12].

In precessing binaries, the precession itself and the largest contribution to the recoil, also known as the “kick,” arise from the components of the binary's spins lying in the orbital plane [13–15]. These spin components lead to an asymmetric emission of gravitational waves [16–18]. In particular, asymmetries in the multipolar structure of the radiation generate a net flux of linear momentum out of the orbital plane, causing the remnant black hole (BH) to recoil in the opposite direction of this flux [19,20].

Observations of precession and kicks with GW detectors are astrophysically highly significant. Spin misalignment can help to reveal formation channels [21–24], while large kick velocities may eject remnants from their host environments. These kicks limit the possibility of remnants undergoing further mergers, impacting the binary black hole (BBH) merger rate [25,26] and the distributions of BH masses and spins [27–29]. In addition, accurate kick estimates, particularly for out-of-plane recoils, are crucial for interpreting potential multimessenger observations of BBH mergers [30], including scenarios such as those proposed in [31].

The GW emission from precessing binaries on quasicircular orbits is fully described by seven intrinsic parameters. These are the mass ratio  $q = m_1/m_2 \geq 1$  and the dimensionless spin vectors  $\vec{\chi}_1$  and  $\vec{\chi}_2$ . In waveform modeling, waveforms are typically scaled by the total mass

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$M = m_1 + m_2$ , while for observations the absolute mass scale remains relevant.

To address the complete two-body problem in GR, one would in principle need to account for orbital eccentricity. In the following, however, we neglect this effect, since eccentricity decreases towards the merger, whereas the asymmetry in the  $\pm m$  multipoles relevant to our study increases. Nevertheless, a recent study emphasizes the importance of accurately understanding multipole asymmetries in both quasicircular and eccentric orbital configurations, since a consistent definition of eccentricity in eccentric and precessing systems is only possible with a correct modeling of the dominant asymmetry [32]. In particular, the frequency difference between the  $(2, 2)$  and  $(2, -2)$  multipoles in the coprecessing frame is the key for their waveform-based definition.

Currently, only a few waveform models incorporate multipole asymmetries. IMRPhenomXO4a [33,34] and IMRPhenomXPNR [35] implement the dominant asymmetry. More recently, asymmetric contributions to the  $\ell = m \leq 4$  multipoles have been incorporated in SEOBNRV5PHM<sub>w/asym</sub> [36]. While these models accurately capture the physics of the asymmetries they include, they do not account for all subdominant contributions. In contrast, the NRSur7dq4 [37] model includes all asymmetries up to  $\ell = 4$ . However, as a data-driven model, it is not explicitly informed by the underlying physical equations, but instead learns an optimal representation directly from the numerical-relativity (NR) data.

The construction of effective one-body (EOB) and phenomenological models relies on a combination of analytical frameworks, most prominently post-Newtonian (PN) expansions and BH perturbation theory formulated in terms of quasinormal modes (QNM), with phenomenological modeling strategies and state-of-the-art NR simulations. Models are often constructed using simplifying frames based on waveform quantities related to orbital parameters. For example, a coprecessing frame [38–40] can be used, in which the coordinates are chosen such that the orbital angular momentum remains aligned with the  $z$  axis at all times. A corotating frame extends this definition by tracking the separation vector between the two BHs along the  $x$  axis. For perturbation theory, the most appropriate frame to work with is the super rest frame [41–44]. This is the center-of-mass frame of the remnant, with the remnant’s spin aligned to the  $z$  axis and supertranslation freedoms fixed.

In Sec. II, we explain the motivation behind investigating the very small effects of subdominant asymmetries. We discuss phenomenological observations in NR data and relate them to concepts in PN and perturbation theory in Sec. III.

## II. IMPORTANCE OF SUBDOMINANT MULTIPOLE ASYMMETRIES

In this Section, we evaluate the importance of subdominant multipole asymmetries in fully characterizing

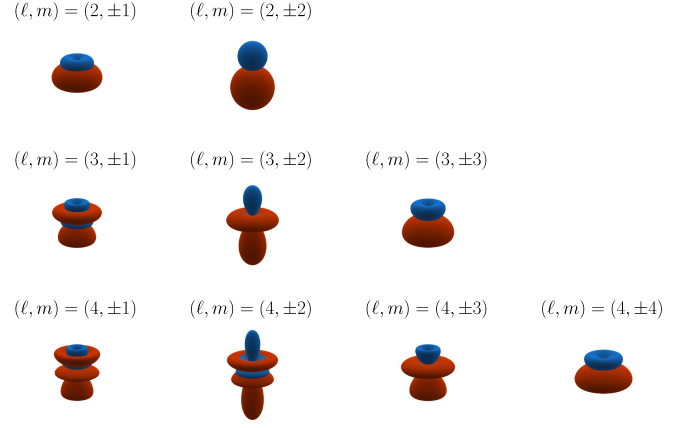


FIG. 1. SWSH surface illustration. The radial distance from the origin encodes the magnitude  $|-^2 Y_{\ell m}|$  scaled with arbitrary amplitudes such that  $|h_{\ell,+m}|/|h_{\ell,-m}| = 2/3$  in order to illustrate an enlarged multipole asymmetry. Blue surfaces correspond to  $m > 0$ , whereas red surfaces denote  $m < 0$ .

out-of-plane kicks and their effect on parameter estimation (PE). To demonstrate the latter, we perform Bayesian inference on synthetic signals injected into a future ground-based detector.

### A. Preliminaries

The waveform of precessing systems is typically expressed as a decomposition in a  $s = -2$  SWSH basis [45,46],

$$h(t, r, \theta, \varphi) = \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} {}^{-2}Y_{\ell m}(\theta, \varphi) h_{\ell m}(t, r), \quad (1)$$

where the multipoles  $h_{\ell m}$  depend on all intrinsic source parameters. Figure 1 illustrates the  ${}^{-2}Y_{\ell m}$  basis and the effect of multipole asymmetries.

For nonprecessing binaries, the dynamics are confined to a fixed orbital plane, which induces a reflection symmetry in the gravitational radiation. As a consequence, the waveform multipoles satisfy

$$h_{\ell,-m} = (-1)^{\ell} h_{\ell,m}^*. \quad (2)$$

Precessing binaries break this symmetry. To describe the multipole asymmetry one defines symmetric (“+”) and antisymmetric (“−”) multipoles as

$$h_{\ell,m}^{\pm} = \frac{1}{2} [h_{\ell,m} \pm (-1)^{\ell} h_{\ell,-m}^*]. \quad (3)$$

We further define the  $+/-$  amplitude  $|h_{\ell,m}^{\pm}| = a_{\ell,m}^{\pm}$ , the phase  $\arg(h_{\ell,m}^{\pm}) = \phi_{\ell,m}^{\pm}$ , and frequency  $\omega_{\ell,m}^{\pm} = \dot{\phi}_{\ell,m}^{\pm}$ . Since the majority of the GW energy is emitted in the  $\ell = 2$ ,  $m = \pm 2$  multipoles, the quantity  $h_{2,2}^{\pm}$  is referred to as the

dominant antisymmetric waveform, whereas all other combinations of  $\ell$  and  $m$  are associated with subdominant multipole asymmetries.

### B. Complete characterization of the out-of-plane kick

A fundamental characteristic of dominant and subdominant multipole asymmetries is that they describe the GW's linear momentum flux above and below the orbital plane of the binary. Due to momentum conservation, the remnant BH receives a kick in the opposite direction to the linear momentum flux given by [46,47]

$$\frac{dP_z}{dt} = \lim_{r \rightarrow \infty} \frac{r^2}{16\pi} \oint d\Omega \cos\theta |\dot{h}|^2. \quad (4)$$

We restrict our analysis to the kick along the  $z$  direction. While an in-plane kick also exists [48,49], it is not affected by multipole asymmetries relative to the orbital plane, when we choose coordinates in a coprecessing frame [38–40].

In this study, we focus on the impact of subdominant multipole asymmetry on the final kick velocity. We start by using a phenomenological tool from Sec. III of our previous study [20], the spin-asymmetry-kick correlation plots. These are distributions of various BBH configurations in the plane spanned by their out-of-plane kick and the maximum value of the antisymmetric amplitude  $\hat{a}^- = \max_i(|\sum_{\ell,m} h_{\ell,m}^-|)$ . The kick magnitudes we will consider may lie within the uncertainty ranges of remnant or waveform models, yet the trends we report are expected to remain robust with increasing model accuracy.

In Fig. 2, we show equal-spin magnitudes with  $q = 3$ , as these configurations exhibit a pronounced effect when the subdominant antisymmetric waveform is set to zero. In addition to observations such as the dependence of the kick range on  $\hat{a}^-$ , which we already found in our previous study, a new observation is that the full range of kick values cannot be recovered when only the dominant multipole asymmetries are included in the kick calculation. For binaries with strong precession and large antisymmetric waveforms, differences of up to 210 km/s are possible. This maximal difference arises from the combined contribution of all subdominant multipole asymmetries, which exhibit the same spin-asymmetry-kick phenomenology but individual contributions to the overall kick amplitude are smaller.

We additionally performed the same spin-asymmetry-kick correlation study as in Fig. 2 with mass ratios of  $q = 1, 2$  and 4. We used the same spin sampling, with equal spin magnitudes of 0.8 uniformly distributed over the sphere. For  $q = 1, 2, 3$ , and 4, the fraction of configurations with kick differences exceeding 100 km/s due to missing subdominant multipole asymmetries is 2.3%, 10.0%, 8.5%, and 5.5%, respectively. This magnitude is astrophysically relevant, especially for systems with kick

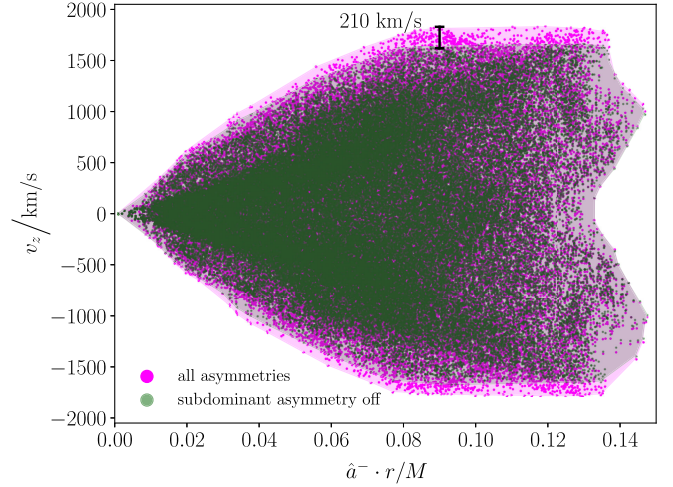


FIG. 2. Spin-asymmetry-kick correlation, showing the difference between including only dominant multipole asymmetries and including all multipole asymmetries in the kick calculation. Configurations with  $q = 3$ , spin magnitude of  $|\vec{\chi}_i| = 0.8$  and varying spin directions were chosen. Waveforms were generated using NRSur7dq4 [37]. Kick computations and coprecessing-frame transformations were performed with SCR1 [50].

velocities comparable to the escape velocities of their host environments, such as young star clusters, globular clusters, or other possible environments [51,52]. In these environments, even small variations in recoil velocity can determine whether the remnant is retained or ejected. Kicks influence the spin distributions of BHs in these systems. Therefore, precise kick estimates are essential for reliably predicting the resulting spin distributions in such environments [29].

As the mass ratio increases, the ratio between the maximal kick difference and the maximal possible kick range, considering all multipoles, also increases, indicating the relevance of subdominant multipole asymmetries for high-mass ratio configurations.

Boyle *et al.* [40] and Ma *et al.* [53] already initiated an analysis of the kick velocity in terms of multipole asymmetries. In our previous work, we derived an algebraic expression for  $dP_z/dt$  that depends explicitly on interactions between the symmetric and antisymmetric parts of the waveform [cf. Eq. (20) in [20] and note that there is the close relation to mirror asymmetries analyzed for instance in Refs. [54,55]]. The dominating part of the kick,

$$\frac{dP_z}{dt} \sim \beta_{\ell^-, m^-, \ell^+, m^+} \cos(\psi_{\ell^-, m^-, \ell^+, m^+}), \quad (5)$$

is a combination of the kick amplitude

$$\beta_{\ell^-, m^-, \ell^+, m^+} = a_{\ell^-, m^-}^- a_{\ell^+, m^+}^+ \omega_{\ell^-, m^-}^- \omega_{\ell^+, m^+}^+, \quad (6)$$

and the phase difference

$$\psi_{\ell^-, m^-, \ell^+, m^+} = \phi_{\ell^-, m^-}^- - \phi_{\ell^+, m^+}^+ \quad (7)$$

In the vast majority of configurations, the  $(\ell^-, m^-, \ell^+, m^+) = (2, 2, 2, 2)$  interactions are the determining factor in the total kick velocity. However, there are two exceptions in which the subdominant interactions could exert a significant influence on the kick. First, the subdominant kick amplitudes are of the same order of magnitude as  $\beta_{2,2,2,2}$ . Second, the cosine of the dominant phase difference at the time of maximal linear momentum emission is close to zero, whereas this is not the case for the subdominant phase differences.

These two cases are illustrated in Fig. 3, and are based on two NR simulations of the latest 3.0.0 version of the Simulating eXtreme Spacetimes (SXS) Collaboration's catalog [56,57]. For SXS:BBH:1941 [37,58], the kick amplitudes associated with the interactions  $(\ell^-, m^-, \ell^+, m^+) = (3, 3, 3, 3)$  and  $(\ell^-, m^-, \ell^+, m^+) = (4, 4, 4, 4)$  are of the same order of magnitude to that of the dominant interaction. The effect is so pronounced that neglecting the subdominant multipole asymmetries in the kick calculation would introduce a larger bias than entirely omitting the dominant asymmetry only, as can be seen in the lower left corner of Fig. 3.

In the case of SXS:BBH:3392 [59], the dominant kick amplitude is large. However, since the cosine of the dominant phase difference is close to zero, the dominant interactions contribute negligibly to the final kick value, albeit it has some influence up to  $t \approx 20M$ . In contrast, the next-to-leading-order phase differences attain values different from zero, such that their associated kick amplitudes contribute significantly to the final kick. Consequently, neglecting the subdominant asymmetries would result in a bias of approximately 50 km/s in the final kick.

### C. Source characterization with next-generation detectors

In addition to the kick velocity, we investigate the role of subdominant multipole asymmetries in estimating the source properties with next-generation ground-based detectors. Subdominant multipole asymmetries contribute minimally to the signal strength. Hence, they will only become relevant for signals with high signal-to-noise ratios. In this Section, we investigate how subdominant asymmetries influence parameter estimation with a next-generation detector such as the Einstein Telescope (ET) [60–62] based on selected examples.

To infer the source properties we use Bayesian inference. This statistical framework estimates a probability distribution for each parameter using Bayes' theorem, defined as

$$p(\vec{\theta}|d, h_m) = \frac{\pi(\vec{\theta}|h_m)\mathcal{L}(d|\vec{\theta}, h_m)}{p(d|h_m)}. \quad (8)$$

As shown by the equation, the posterior probability of a binary signal with parameters  $\vec{\theta}$  given the data  $d$  and a

signal model  $h_m$  is proportional to the prior probability distribution,  $\pi(\vec{\theta}|h_m)$ , and the likelihood of the data given the parameters  $\vec{\theta}$  and a signal model,  $\mathcal{L}(d|\vec{\theta}, h_m)$ . The denominator  $p(d|h_m)$  is known as the evidence and is given by the likelihood marginalized over all parameters. In GW data analysis, it is common to assume stationary Gaussian noise. Under this assumption, the likelihood is proportional to

$$\mathcal{L}(d|\vec{\theta}, h_m) \propto \exp(-\langle d - h_m(\vec{\theta}) | d - h_m(\vec{\theta}) \rangle / 2), \quad (9)$$

where we have introduced the noise-weighted inner product, defined as

$$\langle d|h \rangle = 4\pi \int_0^\infty \frac{\tilde{d}(f)\tilde{h}^*(f)}{S_n(f)} df. \quad (10)$$

$\tilde{d}$  and  $\tilde{h}$  denote the Fourier transforms of  $d(t)$  and  $h(t)$ ,  $*$  indicates complex conjugation and  $S_n$  is the one-sided power spectral density of the detector noise.

In practice, we generate waveforms with limited length. We use  $f_{\min} = 20$  Hz and  $f_{\max} = 2048$  Hz as lower and upper frequency limits, respectively. Assuming the ET-D design power spectral density [61], we perform injections into zero noise, which effectively corresponds to averaging over many noise realizations, in order to isolate waveform modeling biases from statistical noise effects. To inject signals and infer the marginalized posterior distributions, we use BILBY [63] in combination with the nested sampling algorithm DYNesty [64] using  $n_{\text{live}} = 1000$  live points. Posterior samples are post-processed using PESUMMARY [65]. We adopt uniform priors on the component masses,  $m_i \in [1, 1000]M_\odot$ . The spin magnitudes are drawn from a uniform distribution,  $|\vec{\chi}_i| \in [0, 0.99]$ , while the spin directions are assumed to be isotropically distributed. While the extrinsic parameters are not of primary interest in this study, we specify their standard uninformative priors for reproducibility: isotropic sky location  $(\alpha, \delta)$ , isotropic inclination  $\iota$ , uniform polarization and phase angles  $(\psi, \varphi_{\text{ref}})$ , a uniform prior on the luminosity distance in the range  $d_L \in [100, 5000]$  Mpc, and a uniform prior on the geocentric coalescence time  $t_c \in [9.9, 10.1]$  s.

In our injection and recovery study, we simulate signals from BBH mergers. Table I includes all the relevant parameters of our waveform injections with NRSur7dq4. These systems exhibit significant dominant and subdominant multipole asymmetries. This enables us to investigate their influence by recovering with the same waveform model, NRSur7dq4, which incorporates all asymmetries, alongside two artificial waveform models, NRSur7dq4\_dom\_off and NRSur7dq4\_subdom\_off, which can switch off the dominant or subdominant asymmetry, respectively. Note that these models differ from the model NRSur7dq4\_sym used in Ref. [11], in which all multipole asymmetries are set to zero. Motivated by the fact that the

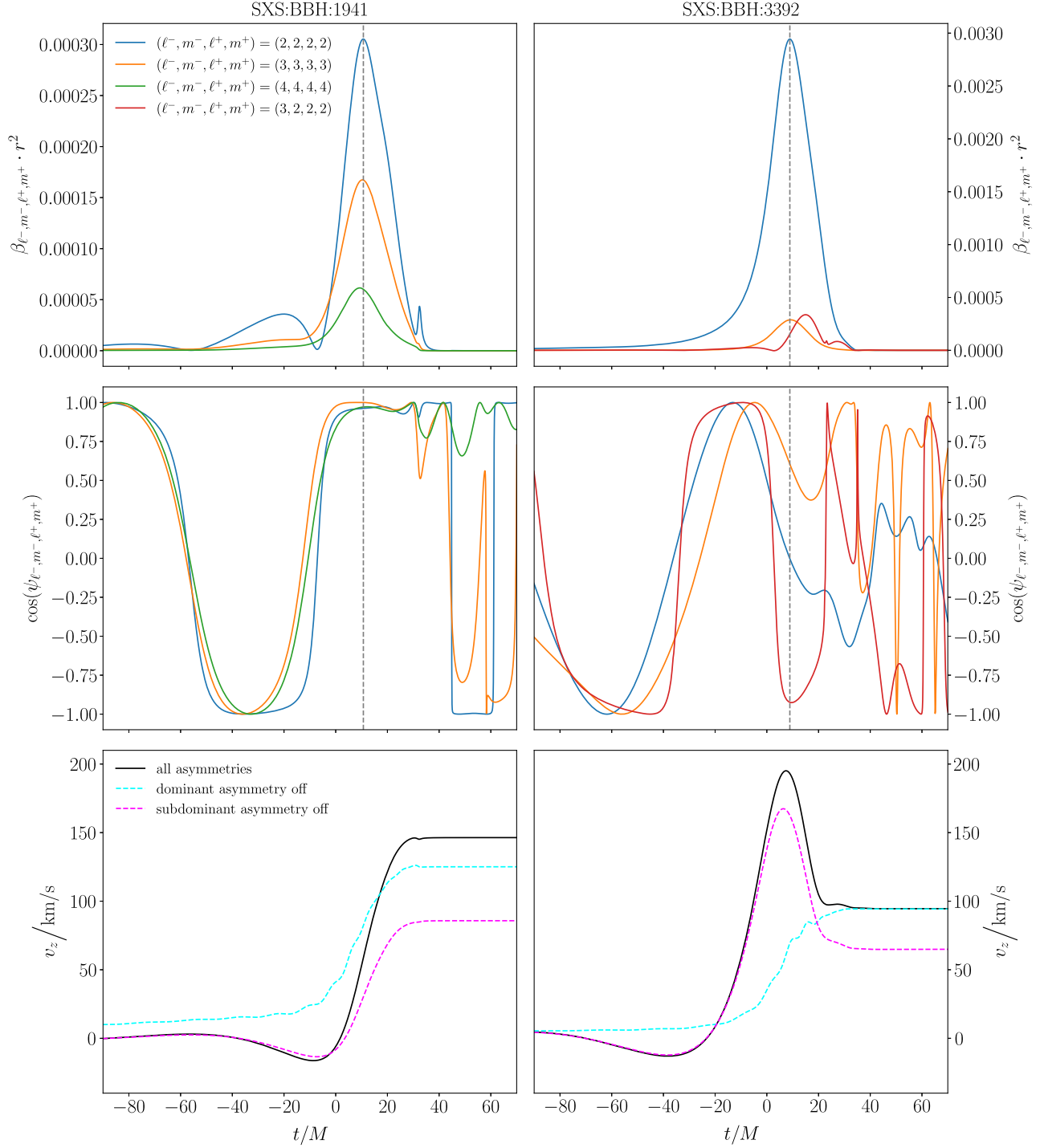


FIG. 3. Influence of subdominant multipole asymmetries on the kick velocity. The individual three kick terms that contribute the most to each of the two highly precessing SXS simulations are shown. The left panel illustrates a case with large subdominant kick amplitudes  $\beta$ , while the right panel depicts a scenario in which the contribution of the dominant multipole asymmetry to the kick is almost negligible. This occurs because, at the time when most of the linear momentum is radiated (dashed gray lines), the dominant antisymmetric and symmetric waveforms are nearly perpendicular, resulting in a cosine value of the phase difference  $\psi$  close to zero. The lower panel provides the cumulative kick velocity calculated with all asymmetries, or when the dominant or subdominant asymmetry is deactivated. This calculation uses the integral of the complete version of Eq. (5).

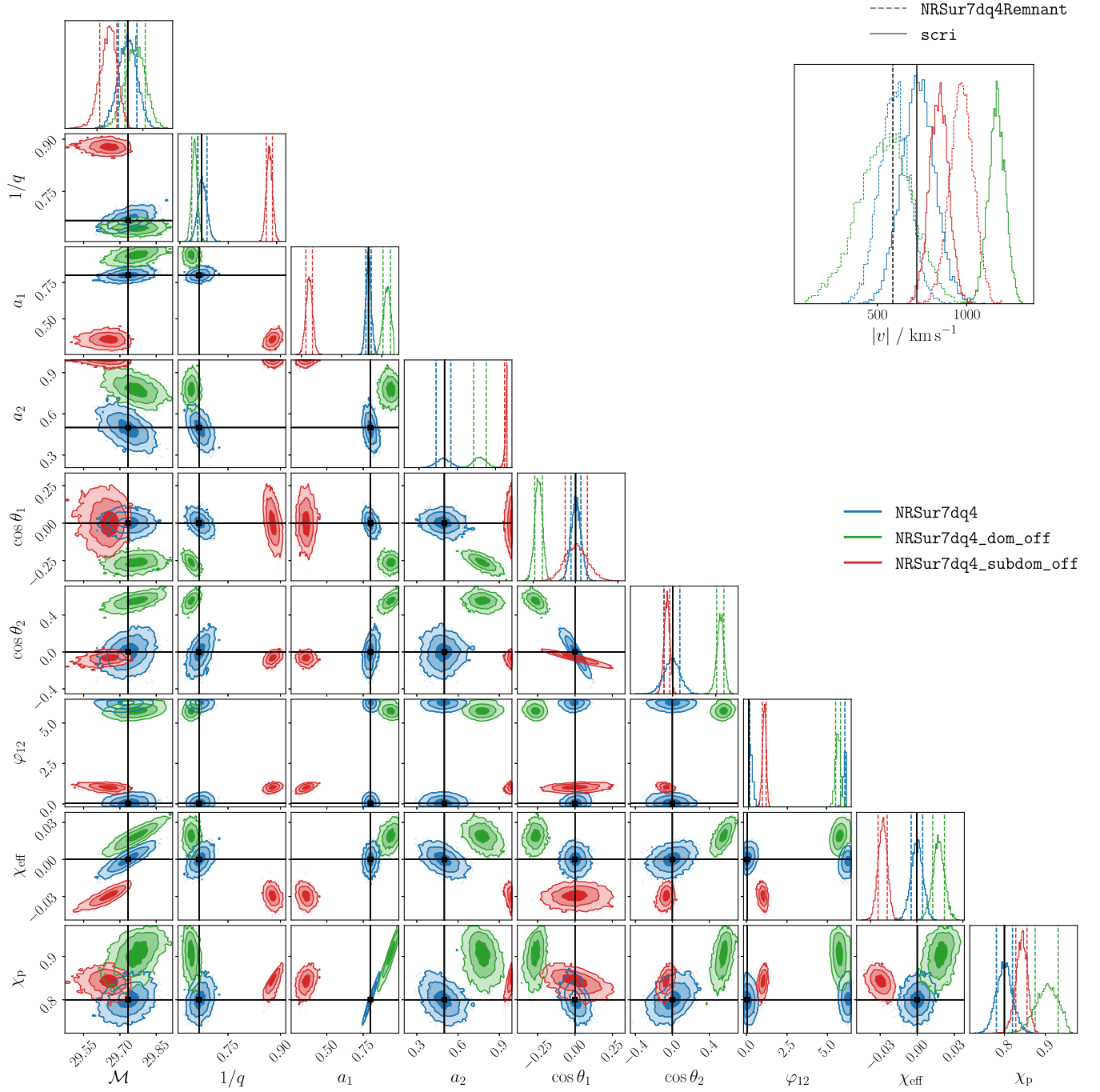


FIG. 4. Recovered posteriors of the chirp mass  $\mathcal{M}$ , the inverted mass ratio  $1/q$ , the spin magnitudes  $a_1$  and  $a_2$ , the spin tilt angles  $\cos \theta_1$  and  $\cos \theta_2$ , the spin azimuthal angle  $\varphi_{12}$ , the effective spin  $\chi_{\text{eff}}$  and the effective spin precession parameter  $\chi_p$ . We use waveform models including all asymmetries, NRSur7dq4, neglecting dominant asymmetries, NRSur7dq4\_dom\_off and neglecting subdominant asymmetries NRSur7dq4\_subdom\_off in the inertial frame. Black lines shows the injected values of a NRSur7dq4 waveform into the ET-D design sensitivity curve. The inset shows the total kick posteriors, which were calculated from the posteriors of the main plot. This calculation was done using NRSur7dq4Remnant and the waveforms directly, together with Eq. (4), which is implemented in SCRI.

kick velocity is usually measured in a fixed inertial reference frame, in Fig. 4 we use the artificial models symmetrized in the inertial frame. In contrast, in Fig. 5 the models are symmetrized in the coprocessing frame,

highlighting the physical impact of neglecting multipole asymmetries.

Figure 4 shows the posterior distributions for several intrinsic parameters described in the caption of that figure.

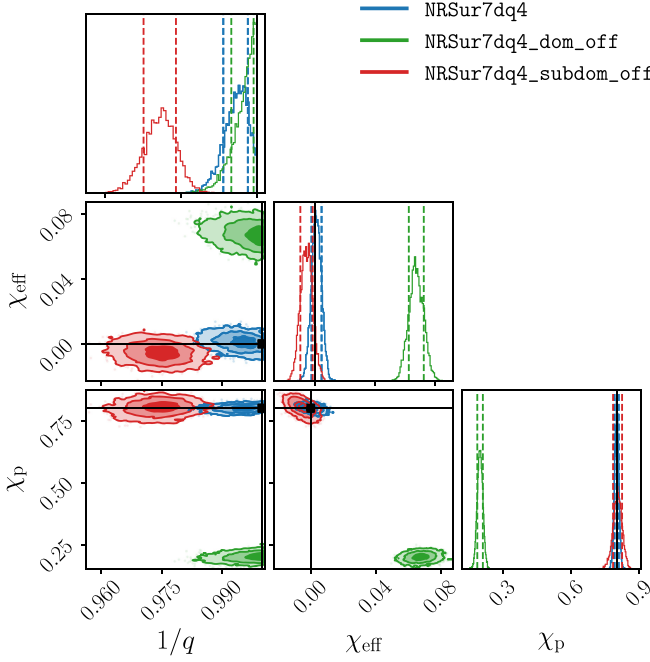


FIG. 5. Systematic biases in selected intrinsic parameters when multipole asymmetries in the coprecessing frame are neglected. The injected superkick signal is observed edge on with ET sensitivity. Even subdominant asymmetries significantly affect the recovery of the inverted mass ratio  $1/q$ .

For this particular injection, we visually identify that for many parameters the bias of neglecting subdominant asymmetries in the inertial frame is larger than the bias of neglecting dominant asymmetries, except for  $\cos\theta_i$  and  $\chi_p$ . This behavior is not expected to be generic for all signals. However, this strongly precessing example demonstrates that such a situation can indeed occur. The reason for this large biases is mode mixing in the  $\ell = 2$

multipoles. For this system, the dominant asymmetries in the coprecessing frame are large. When transforming to the inertial frame, these dominant coprecessing-frame asymmetries leak into the inertial-frame  $(2, \pm 1)$  multipoles. The leakage is strong enough that ignoring subdominant asymmetries in the inertial frame results in larger parameter biases than ignoring dominant asymmetries. Additionally, we assume that the inclination angle of  $\iota = \pi/6$ , in combination with the mass ratio of  $q = 1.5$ , gives subdominant multipoles a relatively large amplitude, and that asymmetries in the subdominant multipoles become important.

Neglecting asymmetries also affects the inferred kick velocities obtained from the recovered intrinsic parameters. The inset of Fig. 4 shows posterior distributions for the kick velocity in the inertial frame, computed from the parameters shown in the main panel using two different methods. NRSur7dq4Remnant [37] is a surrogate model that predicts the kick velocity directly from the binary parameters, while alternatively we use the SCRI implementation of Eq. (4) together with waveforms generated by the three models. Note that agreement between the two methods is not expected. All kick posteriors are biased in both approaches, due to the biased recovered source parameters and to the neglect of asymmetries in the waveform models. In particular, the kick posterior obtained with NRSur7dq4\_dom\_off is shifted toward larger values, since in the inertial frame the effects of the dominant asymmetries are partially encoded in the subdominant multipoles.

In Fig. 5 we inject a superkick configuration with parameters given in Table I and recover it using the three artificial models symmetrized in the coprecessing frame. We decide on superkick system with equal masses and opposite spins aligned in the orbital plane because it exhibits large asymmetries despite not showing any precession in the orbital plane. We adopt an edge-on inclination,  $\iota = \pi/2$ , which reduces the impact of the dominant asymmetries in the coprecessing frame. Nevertheless, their effect on most parameters, exemplified here by the spin parameters, remains larger than that of the subdominant asymmetries. For the mass ratio, however, the bias induced by neglecting the coprecessing subdominant asymmetries is even larger than that obtained when neglecting the dominant asymmetries. We speculate that the edge-on orientation enhances the relative contribution of subdominant multipoles, making asymmetries in these multipoles particularly important for the recovery of the mass ratio. In general we found, that the systematic differences between NRSur7dq4 and NRSur7dq4\_subdom\_off vary with the inclination in a nontrivial way, reflecting the complex structure of the SWSH visualized in Fig. 1. This highlights the need for further investigation to fully quantify and understand the impact of subdominant asymmetries across the parameter space.

TABLE I. Injected parameter values for the analyses shown in Figs. 4 and 5.

| Parameter              | Fig. 4      | Fig. 5       |
|------------------------|-------------|--------------|
| $M$                    | $70M_\odot$ | $90M_\odot$  |
| $q$                    | 1.5         | 1            |
| $\vec{\chi}_1$         | (0.8, 0, 0) | (0.8, 0, 0)  |
| $\vec{\chi}_2$         | (0.5, 0, 0) | (-0.8, 0, 0) |
| $f_{\text{ref}}$       | 20 Hz       | 20 Hz        |
| $d_L$                  | 680 Mpc     | 500 Mpc      |
| $\iota$                | $\pi/6$     | $\pi/2$      |
| $\alpha$               | 0.1 rad     | 0.76 rad     |
| $\delta$               | 0.1 rad     | 1.94 rad     |
| $\varphi_{\text{ref}}$ | 0           | 0            |
| $\psi$                 | $\pi/3$     | 0            |
| $t_c$                  | 10 s        | 10 s         |
| SNR                    | 333.49      | 519.64       |

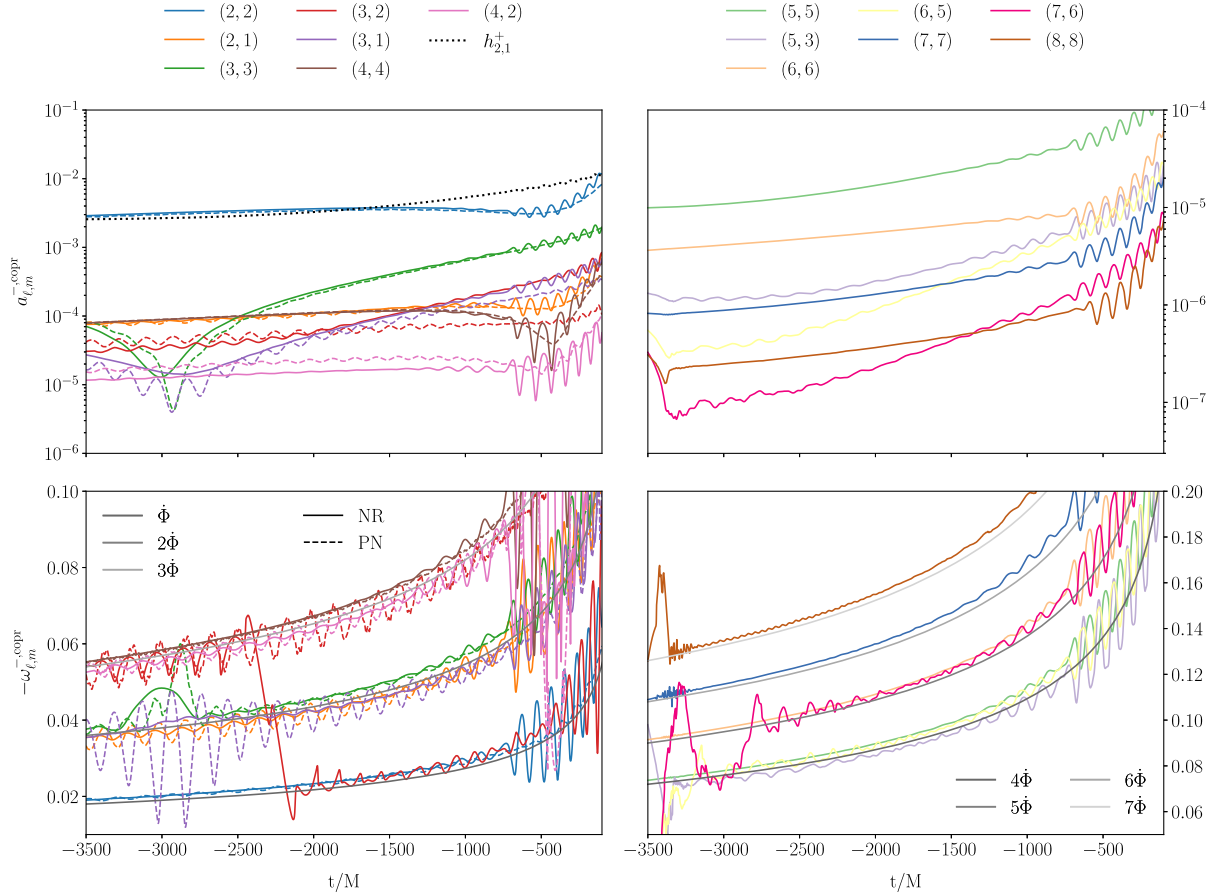


FIG. 6. Antisymmetric amplitudes and frequencies in the inspiral regime, exemplified by SXS:BBH:1207, a  $q = 2$  system with large spin components in the orbital plane. The left panel displays multipoles for which PN results are available, while the right panel shows higher-order antisymmetric multipoles. For reference, the symmetric  $(2, 1)$  amplitude is included as a dotted black line to illustrate the relative order of magnitude. The antisymmetric frequencies scale as  $m \pm 1$  times the orbital frequency  $\Phi$ . The frequency jumping in the  $(3, 2)$  multipole is an exception, as discussed at the end of Sec. III A.

Neglecting subdominant asymmetries leads to comparatively small changes in the waveform and therefore to small mismatches. Standard mismatch-based indistinguishability criteria are known to be conservative in this regime and tend to predict parameter biases at lower signal-to-noise ratios (SNR) than actually required [66], motivating a direct analysis via PE. Here we are in a relatively high SNR regime of  $\rho > 300$ . Since the systematic error is independent of the SNR while the statistical error scales as  $1/\rho$ , the qualitative scaling between statistical and systematic uncertainties remains valid. Consequently, this example serves as an illustration of how, at sufficiently high SNR, waveform systematics induced by missing dominant and subdominant asymmetries can affect parameter estimation. However, because the accuracy requirements typically quoted for NR simulations, and therefore for the surrogate models derived from them, are formally exceeded in this regime [37,67], the absolute numerical value of the inferred systematical biases should not be overinterpreted. This highlights the need for continued improvements in NR simulations to reliably model high-SNR signals.

### III. PHENOMENOLOGY OF SUBDOMINANT MULTIPOLE ASYMMETRIES

This Section develops phenomenological insights into the physics of subdominant multipole asymmetries. We use NR waveforms, PN predictions and analytical QNM models to derive formulas and propose ideas for potential future models of subdominant multipole asymmetries. We base our work on the existing time-domain EOB model for  $\ell = m$  asymmetries [36] and analyze separately the inspiral-plunge and merger-ringdown regime for all subdominant asymmetries.

#### A. Inspiral plunge

The antisymmetric components of the waveform only grow significantly in the merger regime. Consequently, in the inspiral regime, the contributions from antisymmetric multipoles are suppressed by several orders of magnitude compared to the symmetric contributions. Nevertheless, the underlying physics remains of considerable interest in this regime.

TABLE II. Amplitudes  $\kappa_{\ell,m}$  and  $\lambda_{\ell,m}$  according to Eq. (13) for single-in-plane-spin configurations, obtained using the PN expressions in Appendix A of Ref. [40]. If  $\kappa_{\ell,m}$  ( $\lambda_{\ell,m}$ ) dominates, the coprecessing frequency ratio is  $\mathbf{m}_{\ell,m}^- = m + 1$  ( $\mathbf{m}_{\ell,m}^- = m - 1$ ).

| $(\ell, m)$ | $\kappa_{\ell,m}$                                              | $\lambda_{\ell,m}$                                            | $\mathbf{m}_{\ell,m}^-$ |
|-------------|----------------------------------------------------------------|---------------------------------------------------------------|-------------------------|
| (2, 1)      | $\frac{(17+12q)q\chi v^3}{12(1+q)^2}$                          | $\frac{(3+4q)q\chi v^3}{4(1+q)^2}$                            | $m + 1 = 2$             |
| (2, 2)      | 0                                                              | $\frac{iq\chi v^2}{2(1+q)}$                                   | $m - 1 = 1$             |
| (3, 1)      | $\frac{\sqrt{2}q\chi v^3}{3\sqrt{7}(1+q)^2}$                   | 0                                                             | $m + 1 = 2$             |
| (3, 2)      | $-\frac{i\sqrt{5}(-69+40q+56q^2)q\chi v^4}{48\sqrt{7}(1+q)^3}$ | $-\frac{i\sqrt{5}(-35-8q+40q^2)q\chi v^4}{48\sqrt{7}(1+q)^3}$ | $m + 1 = 3$             |
| (3, 3)      | 0                                                              | $-\frac{\sqrt{10}q\chi v^3}{\sqrt{21}(1+q)^2}$                | $m - 1 = 2$             |
| (4, 2)      | $\frac{i9\sqrt{5}q\chi v^4}{112(1+q)^3}$                       | $\frac{i\sqrt{5}q\chi v^4}{336(1+q)^3}$                       | $m + 1 = 3$             |
| (4, 4)      | 0                                                              | $-\frac{i9\sqrt{5}q\chi v^4}{8\sqrt{7}(1+q)^3}$               | $m - 1 = 3$             |

In Fig. 6, we illustrate the characteristics of subdominant and dominant antisymmetric amplitudes, and their associated frequencies in the coprecessing frame for SXS: BBH:1207 [68].

The amplitudes show good agreement with PN predictions [40] implemented in [69], which are available for the (2, 2), (2, 1), (3, 3), (3, 2), (3, 1), (4, 4), (4, 2) antisymmetric multipoles. Minor discrepancies arise, particularly in resolving the small oscillations. However, on average the results show quantitative agreement and remain in the same order of magnitude throughout the inspiral. In Table II, the amplitudes are summarized for the special case of single in-plane-spin configurations. This choice is motivated by the fact that most of the antisymmetric amplitude contribution is expected to arise from the mass ratio and in-plane-spin magnitude.

For the (3, 2) antisymmetric NR amplitude and frequency, we found larger differences to PN in several cases, which will be discussed at the end of this Section. In addition to this multipole, Ref. [36] also reported significant discrepancies for the (2,1) multipole. However, we did not observe this in most of the cases, as the implementation of the antisymmetric PN multipoles [69], which we are using, already accounts for 2PN spin-spin term corrections reported in the documentation. These corrections were not used by Ref. [36].

In the following, our analysis will focus on the frequencies of the antisymmetric multipoles. We observed a universal characteristic that also holds for higher multipoles that are not currently accessible within the PN framework. On average, the antisymmetric frequencies follow integer multiples of the orbital frequency  $\Phi$ , as initially already reported by Boyle *et al.* [40]. In the coprecessing frame it is  $m \pm 1$  times  $\Phi$ , as can be seen from the example in Fig. 6. Note that for  $m = 0$  antisymmetric multipoles, the

frequency is zero by definition. If the metadata of the NR simulations did not provide it, we used the orbital phase derived via [37]

$$\Phi = \frac{1}{4} [\arg(h_{2,-2}^{\text{copr}}) - \arg(h_{2,2}^{\text{copr}})]. \quad (11)$$

To understand when it is  $m + 1$  or  $m - 1$ , we consider the PN expressions given in Appendix A of Ref. [40]. There, the second bracket terms of the Eqs. (A5) correspond to the antisymmetric waveform in the corotating frame.

In order to gain insight into the structure of antisymmetric PN frequencies, we make the following considerations. We only use the leading order expressions and assume a binary with the heavier BH spinning entirely in the orbital plane. Then we can write the symmetric  $\vec{S}$  and antisymmetric  $\vec{\Sigma}$  combinations of the two spins vectors defined in Ref. [40] as

$$S_n = \frac{q^2 M^2}{(1+q)^2} \chi \cos(\varphi), \quad (12a)$$

$$S_\lambda = \frac{q^2 M^2}{(1+q)^2} \chi \sin(\varphi), \quad (12b)$$

$$\Sigma_n = -\frac{qM^2}{1+q} \chi \cos(\varphi), \quad (12c)$$

$$\Sigma_\lambda = -\frac{qM^2}{1+q} \chi \sin(\varphi). \quad (12d)$$

Here  $\varphi(t)$  is the in-plane spin angle in the corotating frame and  $\chi$  the spin magnitude. By inserting Eqs. (12) in Eqs. (A5) of Ref. [40] and using Euler's formula, we can derive a general form of all antisymmetric multipoles,

$$h_{\ell,m}^{\text{coro}} = \kappa_{\ell,m} e^{+i\varphi} + \lambda_{\ell,m} e^{-i\varphi}, \quad (13)$$

where  $(\kappa_{\ell,m}, \lambda_{\ell,m}) \in (\mathbb{R} \times \mathbb{R}) \cup (i\mathbb{R} \times i\mathbb{R})$  are either both purely real or both purely imaginary functions. This means that on average the phase in Eq. (13) rotates with  $+\varphi(t)$  if  $|\kappa_{\ell,m}| > |\lambda_{\ell,m}|$  and with  $-\varphi(t)$  if  $|\kappa_{\ell,m}| < |\lambda_{\ell,m}|$ , see the illustration in Fig. 7. Note that we only have a constant phase shift of  $\pi/2$  when both  $\kappa_{\ell,m}, \lambda_{\ell,m}$  are imaginary. In Tab. II we summarize our findings for all available antisymmetric PN multipoles.

Corotating and coprecessing frame multipoles are related via

$$h_{\ell,m}^{\text{copr}} = h_{\ell,m}^{\text{coro}} \cdot e^{-im\Phi}. \quad (14)$$

The azimuthal angle in the corotating frame can be expressed via [cf. paragraph above Eq. (A3) in Ref. [40]]

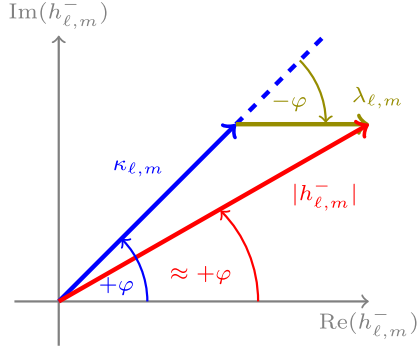


FIG. 7. Illustration of Eq. (13) for the case  $\kappa_{\ell,m}, \lambda_{\ell,m} \in \mathbb{R}^+$  with  $\kappa_{\ell,m} > \lambda_{\ell,m}$ . The resulting sum  $h_{\ell,m}^-$  exhibits an average phase rotation in the positive  $\varphi$  direction.

$$\varphi = \frac{\pi}{2} - \Phi - \alpha, \quad (15)$$

where  $\alpha(t)$  is the precession angle in the coprecessing frame. Thus, in agreement with Ref. [36], we can write the coprecessing frequency as

$$\omega_{\ell,m}^{\text{copr}} \approx \pm \dot{\varphi} - m \dot{\Phi} = \mp \dot{\alpha} - (m \pm 1) \dot{\Phi}. \quad (16)$$

Note that the sign is dependent on the dominance of either  $\kappa_{\ell,m} (m+1)$  or  $\lambda_{\ell,m} (m-1)$ . Assuming that the precession frequency  $\dot{\alpha}$  is small compared to the orbital frequency  $\dot{\Phi}$ , we can give a simple approximation of the antisymmetric frequency ratio

$$\mathbf{m}_{\ell,m}^- = \frac{\omega_{\ell,m}^{\text{copr}}}{-\dot{\Phi}} \approx m \pm 1, \quad (17)$$

as phenomenologically already observed for NR and PN waveforms in Fig. 6.

The simple relation is an universal feature across all multipoles and only weakly depend on the parameter space. To show this, we performed a systematic study of all noncentric and precessing NR waveforms in the SXS catalog. We computed the ratio of the coprecessing anti-symmetric/symmetric frequencies and  $-\dot{\Phi}$ . Then we extracted the median value of this ratio in a window ranging from the reference time to up to 100 M before merger. The distributions across all simulations are shown in Fig. 8. Note that some of the distributions are so narrow that only the median value line is visible.

The well-known symmetric frequency ratio  $\mathbf{m}_{\ell,m}^+ = m$  is correctly recovered by the algorithm. An exception arises for very small subdominant multipoles, such as the (8, 2), (8, 4) or (7, 2) multipoles, where numerical noise could obscure the extraction of a clean frequency ratio.

Nevertheless, the algorithm can be considered reliable for inferring both symmetric and antisymmetric frequency ratios. This is particularly true given that the PN frequency ratios are consistently recovered, except in the case of the

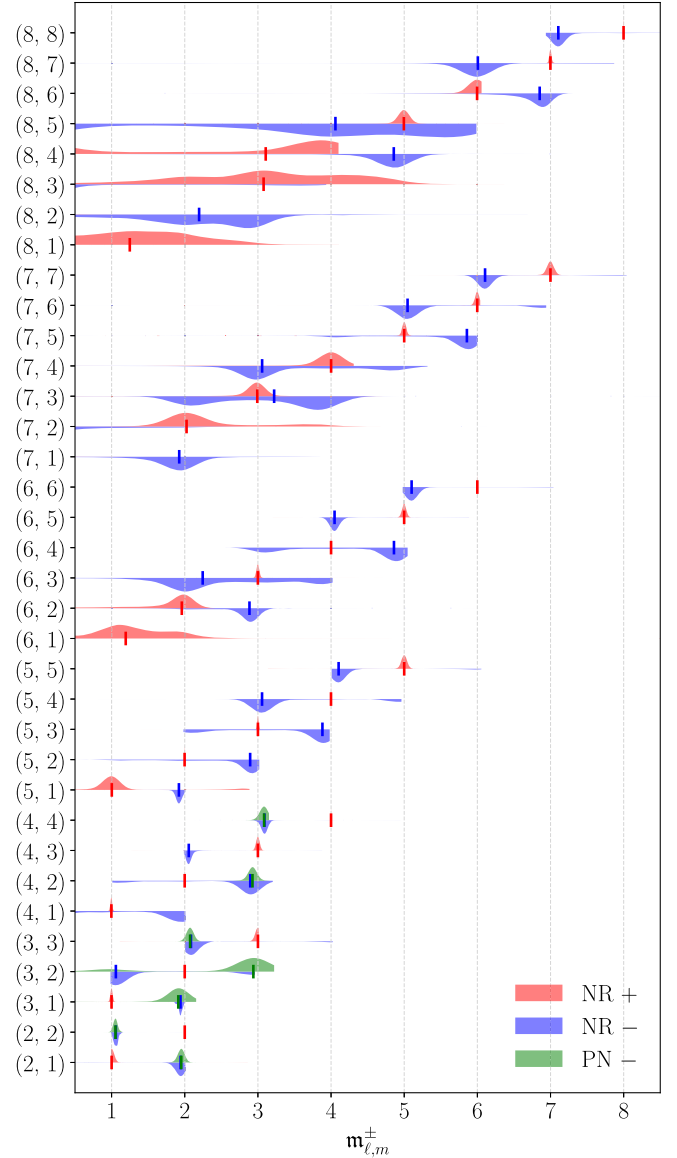


FIG. 8. Frequency ratio distribution across all available multipoles calculated with all quasi-circular precessing simulations of the third SXS catalog. The symmetric frequency ratios in red follow the well-known relation  $\mathbf{m}_{m,\ell}^+ = m$ , while the antisymmetric frequency ratios in blue behave as  $\mathbf{m}_{\ell,m}^- = m \pm 1$ . The antisymmetric PN frequency ratio distributions, which have the same intrinsic parameters as those in the NR catalog, are shown in green. The median values of the distributions are indicated by vertical lines.

(3, 2) ratio. The NR distributions for  $m = \ell$  and  $m = \ell - 1$  peak at  $m - 1$ . The (2, 1) ratio peaks at  $m + 1$ , because  $m - 1$  would otherwise correspond to a zero frequency. For all other values of  $m$ , the antisymmetric ratios are mostly at  $m + 1$ .

Another observation concerns the slight offset of the median values of the distribution from the exact positions at  $m \pm 1$ . Specifically, the median for  $m + 1$  is shifted towards lower values, while the median for  $m - 1$  is shifted

toward higher values. This shift arises from the sign of the precession frequency  $\dot{\alpha}$ , which, according to Eq. (16), is opposite to the sign associated with  $m \pm 1$ .

We do not obtain consistent results for the (3, 2) antisymmetric multipole. The NR frequency occasionally exhibits jumps between one and three times the orbital frequency. The PN frequency remains reliably at three times the orbital phase up to mass ratios of approximately 3.5. For larger mass ratios, however, the frequency is dominated by numerical noise. We examined several hypotheses, such as the possibility that  $a_{3,2}^-$  is very small or that, for different mass ratios, the magnitudes of the parameters  $\kappa_{3,2}$  and  $\lambda_{3,2}$  alternate. None of these provided a satisfactory explanation. Therefore, we suggest that the (3, 2) multipole should be treated with caution.

In summary, we suggest that the frequency of the inspiral-plunge subdominant asymmetry waveform can be easily incorporated into waveform models using Eq. (16) alongside the findings of this study on the  $m \pm 1$  behavior. For instance IMRPhenomXPNR already provides an accurate modeling of the precession  $\dot{\alpha}$  and orbital frequency  $\dot{\Phi}$  needed for Eq. (16). The amplitudes can be found in Table II or its extensions for more complicated spin geometries.

### B. Merger ringdown

The dominant and subdominant antisymmetric amplitude are maximal in the merger regime. As we saw in Sec. II B, the phase difference between the antisymmetric and symmetric waveforms at merger is crucial for the final kick. Unfortunately, we lack an analytical description of this regime, so we are always dependent on calibrating to NR simulations to accurately describe the amplitudes and phases.

However, the radiation emitted by the remnant BH during its ringdown phase is very well understood by perturbation theory. The most general spherical multipole-decomposed analytical QNM model in the super rest frame is given by [44]

$$h_{\ell,m}^{\text{sr}}(t) = \sum_{\ell',n,p} \mathcal{A}_{\ell,m,n,p} e^{-i\omega_{\ell',m,n,p}(t-t_0)} \cdot C_{\ell,\ell',m}(a\omega_{\ell',m,n,p}), \quad (18)$$

where  $n = 0, 1, \dots$  is the overtone index,  $\mathcal{A}_{\ell,m,n,p}$  are the QNM amplitudes,  $\omega_{\ell,m,n,p}$  are the complex QNM frequencies,  $t_0$  is the starting time of the ringdown,  $C_{\ell,\ell',m}$  are the spherical-spheroidal mixing coefficients and  $a$  is the Kerr parameter of the final BH. Ordinary modes have  $p = \text{sgn}[\text{Re}(\omega)] = +1$ , while mirror modes have  $p = -1$ .

In the nonprecessing case where Eq. (2) is valid, the symmetry relations

$$\omega_{\ell,m,n,-p} = -\omega_{\ell,-m,n,p}^*, \quad (19a)$$

$$C_{\ell,\ell',-m}^*(a\omega_{\ell',-m,n,p}) = (-1)^{-\ell-\ell'} \cdot C_{\ell,\ell',m}(a\omega_{\ell',m,n,-p}) \quad (19b)$$

lead to a simple relation for the QNM amplitudes,

$$\mathcal{A}_{\ell,m,n,p} = (-1)^\ell \mathcal{A}_{\ell,-m,n,-p}^*. \quad (20)$$

However, this ringdown amplitude symmetry is broken in the precessing case, while the QNM frequency and mixing coefficient symmetry is preserved. This means that, in the super rest frame, the symmetric and antisymmetric waveform definitions in Eq. (3) lead to the following form in the ringdown regime:

$$h_{\ell,m}^{\text{sr},\pm} = \sum_{\ell',n,p} \mathcal{A}_{\ell',m,n,p}^\pm \cdot e^{-i\omega_{\ell',m,n,p}(t-t_0)} C_{\ell,\ell',m}(a\omega_{\ell',m,n,p}). \quad (21)$$

Here, we defined the antisymmetric (−) and symmetric (+) ringdown amplitudes

$$\mathcal{A}_{\ell,m,n,p}^\pm = \frac{1}{2} [\mathcal{A}_{\ell,m,n,p} \pm (-1)^\ell \mathcal{A}_{\ell,-m,n,-p}^*]. \quad (22)$$

Since the QNM frequencies and mixing coefficients are known from perturbation theory and depend only on the remnant properties, the key to model multipole asymmetries in the ringdown regime lies in accurately capturing the  $+/-$  ringdown amplitudes. These amplitudes represent excitations, which depend on the intrinsic parameters of the binary prior to merger. Consequently, fitting to NR is necessary. Since Eq. (21) is analogous to Eq. (18), the basis for fitting procedures, existing QNM fitting algorithms can be applied directly to the  $+/-$  waveform in the ringdown regime, enabling a consistent and systematic modeling of these waveforms.

This amplitude fitting has, for instance, been performed in Ref. [36] for the dominant antisymmetric multipole. Their merger-ringdown waveform encoded in Eq. (61) is analogous to our Eq. (21) when neglecting mode mixing, i.e.,  $C_{\ell,\ell',m} = \delta_{\ell,\ell'}$ , mirror modes, and when only considering the fundamental  $n = 0$  mode. This means that, in our notation, the key fitting coefficient is given by  $\mathcal{A}_{2,2,0,1}^-$ . In their notation, this is effectively encoded in the coefficient  $c_{i,f}^{22,\text{asym}}$ . Besides determining the antisymmetric ringdown waveform, this coefficient is also used for connecting to the inspiral waveform. This means that, in future subdominant multipole asymmetry models, the antisymmetric ringdown amplitudes can be used to connect different parts of the waveform in a clever way. Connections to the symmetric ringdown amplitudes can also be used. For instance, the authors of Ref. [36] found a simple relationship with the symmetric fitting amplitudes for the antisymmetric

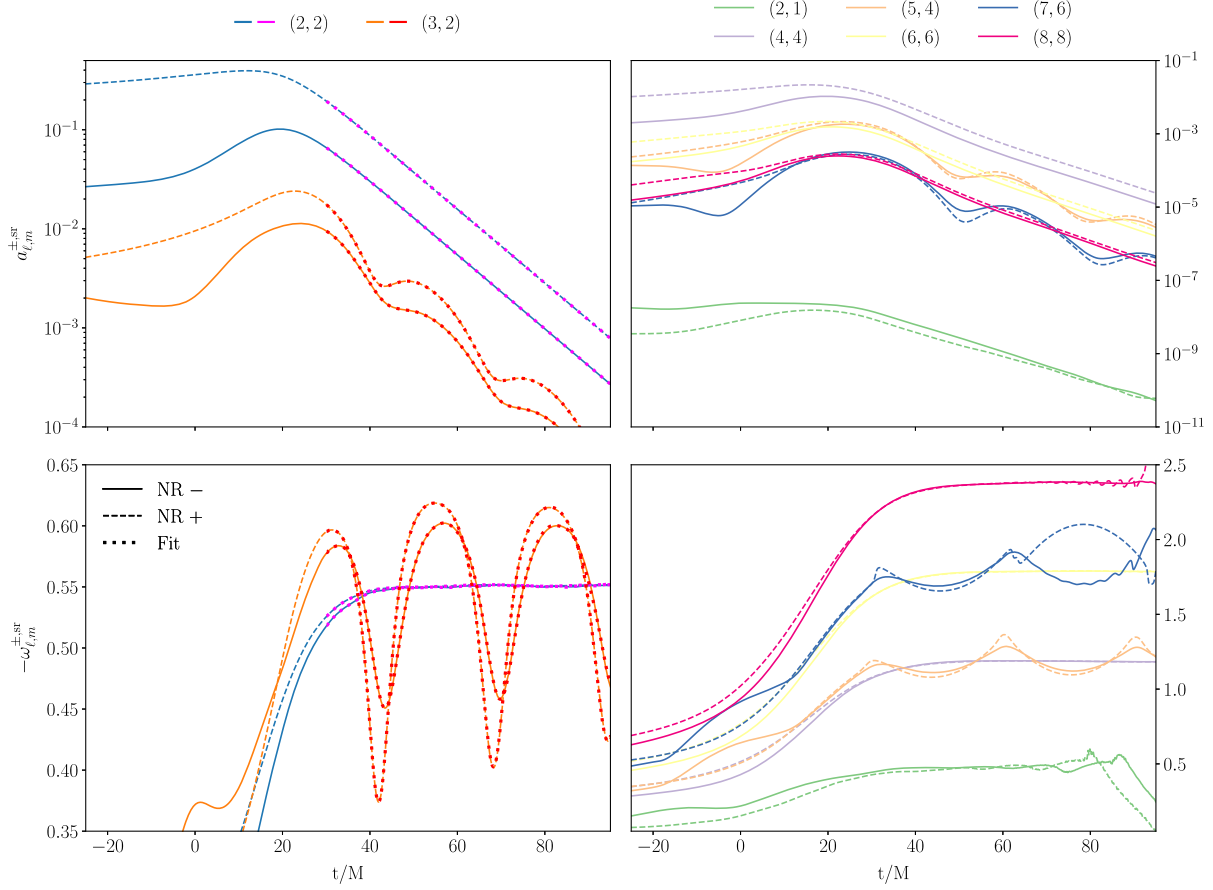


FIG. 9. Super rest frame antisymmetric (solid lines) and symmetric (dashed lines) amplitudes and frequencies in the merger-ringdown regime, shown for the Cauchy-characteristic extracted NR simulation SXS:BBH\_ExtCCE:0009, a  $q = 1$  superkick configuration. The average symmetric and antisymmetric frequencies coincide. The left panel compares the fit model (dotted lines) of Eq. (18) using  $h_{2,2}^{\pm}$  and  $h_{3,2}^{\pm}$ , showing excellent agreement.

multipoles with  $\ell = m$ . This allowed them to avoid fitting the antisymmetric case.

The magnitude of the QNM amplitudes is important for determining the average ringdown frequency of a multipole. In Ref. [70], using an argument analogous to the one we used in the inspiral regime via the dominance of amplitudes in Eq. (13), it was shown that the average ringdown frequency corresponds to the  $p = 1$  QNM frequency if the associated QNM amplitude dominates over that with  $p = -1$ , and vice versa.

This argument can also be applied to Eq. (21). The norm of the  $+/-$  ringdown amplitudes is

$$|\mathcal{A}_{\ell,m,n,p}^{\pm}|^2 = \frac{1}{2} |\mathcal{A}_{\ell,m,n,p}|^2 + \frac{1}{2} |\mathcal{A}_{\ell,-m,n,-p}|^2 \pm (-1)^{\ell} \text{Re}(\mathcal{A}_{\ell,m,n,p} \mathcal{A}_{\ell,-m,n,-p}). \quad (23)$$

Simple complex algebra shows that the last term is always smaller than, or equal to, the two preceding terms. Therefore, the mean value of the ringdown frequency can be approximated by  $\omega_{\ell,m,n,+1}$  if

$$|\mathcal{A}_{\ell,m,n,+1}|^2 + |\mathcal{A}_{\ell,-m,n,-1}|^2 > |\mathcal{A}_{\ell,m,n,-1}|^2 + |\mathcal{A}_{\ell,-m,n,+1}|^2, \quad (24)$$

and by  $\omega_{\ell,m,n,-1}$  in the opposite case. In particular, it also means that antisymmetric and symmetric multipoles share the same average frequency.

We have observed this average frequency characteristic for all sufficiently strong subdominant multipole asymmetries in the super rest frame. An illustrative example for a superkick case is shown in Fig. 9. To generate this plot, we employed the QNMFiTS package of the SXS Collaboration [71]. In this framework, the QNM frequencies and spherical-spheroidal mixing coefficients are provided through the QNM package [72], which implements the Cook-Zalutskiy spectral method [73]. A ringdown starting time of  $t_0 = 30M$  and including mode mixing of the  $(2, \pm 2)$  and  $(3, \pm 2)$  multipoles provided an accurate fit to the  $+/-$  waveform in the ringdown regime.

The fit also reproduces the small oscillations around the average frequency. These arise from the superposition with the nondominant  $p$  contribution in Eq. (21). Moreover, the

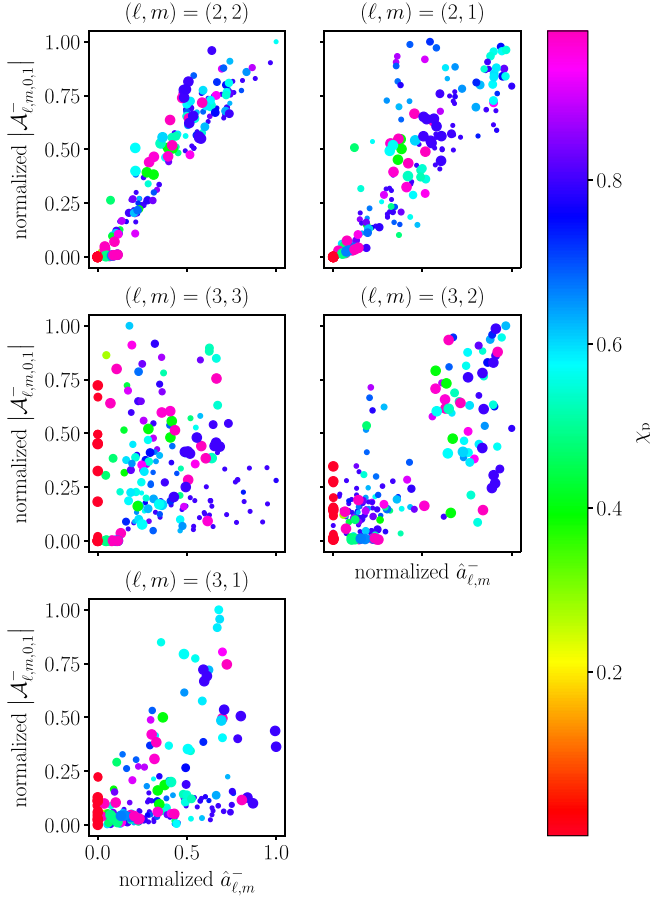


FIG. 10. Relation of antisymmetric ringdown amplitudes defined around  $t \approx 30M$  to intrinsic spin parameters of the system and to the peak of the antisymmetric waveform defined around  $t \approx 10M$ . The marker size is proportional to the mass ratio. The smallest markers correspond to  $q = 1$ , the largest to  $q = 8$ .

small effect of the final mixing term in Eq. (24) is apparent, as the symmetric and antisymmetric oscillations are not perfectly identical.

In addition, Fig. 9 shows the usual  $+/-$  amplitudes. In general, the symmetric amplitude remains larger than the antisymmetric one throughout the merger-ringdown regime. However, for certain subdominant multipoles, the amplitudes can become comparable or, in some cases, the antisymmetric component may even exceed the symmetric one. This amplitude peak structure plays a crucial role in determining the kick. However, it can only be accurately assessed through NR simulations and will be addressed in future work, particularly to analyze the high-asymmetry region on the right-hand side of the spin-asymmetry-kick correlation plots.

Nevertheless, the antisymmetric peak amplitudes in the merger regime can be estimated by analyzing the  $A_{\ell,m,n,p}^\pm$ , which are defined shortly after the merger and are therefore highly correlated with the merger peak amplitudes. There exists a broad scientific community dedicated to the development of highly accurate ringdown fitting algorithms

[74,75] and novel ringdown models [76,77]. Among these outstanding contributions, we have selected a fitting catalog of 250 waveforms in the super rest frame produced by Zhu *et al.* [78,79].

Following the phenomenological approach of Nobili *et al.* [80], who analyzed the dependence of ringdown amplitudes on intrinsic parameters and searched for relations among them, we adopt a similar strategy here. We investigate the correlations between fundamental antisymmetric ringdown amplitudes  $A_{\ell,m,0,1}^-$  around  $t \approx 30M$  calculated from the aforementioned catalog, the system's intrinsic parameters and the amplitude peaks  $\hat{a}_{\ell,m}^-$  around  $t \approx 10$  in the inertial frame. In Fig. 10, the color scale encodes  $\chi_p$ , while the marker size indicates the mass ratio. Note that the horizontal and vertical axis are normalized to its maximum across all systems for each  $(\ell, m)$ .

Several representative correlations can be identified. For all  $(\ell, m)$ , the amplitudes  $A_{\ell,m,0,1}^-$  show an increasing trend with  $\chi_p$ , except for a few very large  $\chi_p$  values. Moreover, for  $\ell = 2$  the peak amplitudes defined near  $t \approx 10M$  and the antisymmetric ringdown amplitudes evaluated at  $t \approx 30M$  are almost proportional. For the  $(3, 1)$  multipole, equal-mass systems tend to correspond to relatively small antisymmetric ringdown amplitudes but comparatively large peak amplitudes.

In summary, fitting the antisymmetric ringdown amplitudes across the parameter space yields the ringdown waveform of multipole asymmetries, which can be incorporated into waveform models via Eq. (21). Additionally, future waveform models may uncover new correlations between the merger and ringdown regimes, as well as between the inspiral and merger regime. Exploiting these correlations could enable the construction of complete waveforms of subdominant asymmetries, without requiring NR simulations across the entire parameter space.

#### IV. CONCLUSIONS

Asymmetries between the subdominant  $+m$  and  $-m$  multipoles are one of the few remaining unmodeled effects in non-eccentric, precessing waveforms. First, we discussed their importance regarding the kick and source characterization with next-generation ground-based detectors. Second, we identified characteristics in the phenomenology of their waveforms in the inspiral-plunge and merger-ringdown regime, that can be used to develop future models. Here we summarize our main results:

- (i) A complete description of the out-of-plane kick requires the inclusion of subdominant asymmetries, as they can make a non-negligible contribution of over 200 km/s to the recoil velocity.
- (ii) Two mechanisms can lead to a kick with significant subdominant asymmetry contribution: A large subdominant kick amplitude  $\beta_{\ell^-, m^-, \ell^+, m^+}$  with  $(\ell^-, m^-, \ell^+, m^+) \neq (2, 2, 2, 2)$ , or approximately

- orthogonal dominant antisymmetric and symmetric waveform contributions at the time of maximal linear momentum emission, so that  $\cos(\psi_{2,2,2,2}) \approx 0$ .
- (iii) For the first time, we performed a high-SNR injection-recovery study using a proposed ET detector sensitivity curve to analyze the influence of subdominant asymmetries. We found that, for certain systems, neglecting subdominant asymmetries can introduce a bias in the recovered posterior that is even larger than the bias introduced by neglecting dominant asymmetries. This bias is nontrivially dependent on the inclination of the system.
  - (iv) In the inspiral regime, the antisymmetric frequencies in the coprocessing frame obey a universal relation to the orbital frequency,  $\omega_{\ell,m}^- \sim (m \pm 1)\dot{\Phi}$ . The sign depends on the specific  $(\ell, m)$  multipole. We determined it using both PN and NR results, which are mutually consistent, except for the (3, 2) case. Because the orbital frequency is typically included in waveform models, implementing this relationship is straightforward.
  - (v) During ringdown, the average  $\omega_{\ell,m}^-$  approaches the corresponding symmetric frequency. Depending on combinations of the  $+m$  and  $-m$  ringdown amplitudes, this frequency can be approximated either by the corresponding ordinary or by the mirror complex frequency.
  - (vi) Existing QNM fitting algorithms can be used to fit antisymmetric ringdown amplitudes. These amplitudes correlate with the system's intrinsic parameters, such as a general increasing trend with  $\chi_p$ . Approximating such correlations across the parameter space can be used for future multipole asymmetry models.

For further research on subdominant asymmetries, more accurate NR simulations are required. In particular, these would help to resolve the inconsistencies in the antisymmetric frequency ratio of the (3, 2) multipole

between PN and NR. Including higher-order PN terms as well as additional multipoles in the PN approach would also be beneficial. The dependence of the kick terms on the in-plane spin angles, as well as the fitting of QNM amplitudes, likewise relies on improved NR data. Regarding our PE study, further investigations could be performed by systematically varying intrinsic parameters, since we only considered a limited set of special cases. Moreover, the inclination dependence of the posteriors should be examined by switching individual multipole asymmetries on and off, rather than deactivating the entire set of subdominant asymmetries.

Software—NUMPY [81], MATPLOTLIB [82], SCIPY [83], LALSUITE [84], SXS [57], SCRI [50], GWSURROGATE [85], PESUMMARY [65], BILBY [63] and QNMFTS [71].

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## DATA AVAILABILITY

The data that support the findings of this article are openly available [86].

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