

Klein bottle cosmology

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We explore a higher-dimensional universe that is a product of Minkowski space and the nonorientable Klein bottle. The topology explicitly breaks important symmetries, such as translational invariance and $(5 + 1)$ -dimensional CP invariance. Somewhat surprisingly, the $(3 + 1)$ -dimensional cp of the Minkowski space can also be broken by the Klein bottle, both explicitly and in the presence of a brane. The topology enforces a background of fermion correlations that amounts to a condensate wall localized in the Klein bottle. The wall acts as an order parameter for the broken symmetries. If a brane passes through the wall, brane fermions that couple to the condensate are produced as quantified by the Bogoliubov coefficients for a time-dependent mass. The scenario meets the conditions, including cp violation, to potentially generate the matter-antimatter asymmetry of the universe.

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I. EXTRA DIMENSIONS IN COSMOLOGY

The Klein bottle [1] is an intriguing space to explore nonorientable extra dimensions, as detailed in a preceding paper [2]. Notably, the Klein bottle induces condensate walls from bulk fields, a feature not seen in the torus, the orientable counterpart [2–4].

In particular, a free, massless bulk fermion—the minimal fermion with no new parameters introduced—contributes a condensate wall with peaks around special axes: the flip axis at the origin of the twisted identification and the axis diametrically opposite the flip axis. Depending on the spin of the bulk field, the energy density of the wall may contribute to the cosmological constant, a possible signal of hidden dimensions [2,5].

As well as generating condensates from bulk fields, the Klein bottle topology explicitly breaks symmetries on the covering space [2]. Although the local geometry is flat and homogeneous, the topology explicitly breaks translational invariance, leading to a condensate wall that varies across the Klein bottle. The discrete symmetries C , P , and CP of the Dirac equation can also be broken. A fermion transported once around the Möbius direction of the Klein bottle flips chirality. There are various boundary conditions that can be imposed on a fermion under this reflection. Generally, they are either periodic or antiperiodic on the covering torus [6]. Each of these boundary conditions explicitly breaks a different set of symmetries. The symmetry breaking is again manifest in the condensate. Some boundary conditions will explicitly break charge symmetry and/or parity (and/or chirality). We review the explicitly broken symmetries found in [2] and discuss the additional possibility of spontaneous symmetry breaking.

Perhaps most intriguing, $(5 + 1)$ -dimensional CP is explicitly broken, with various physical implications, including a possible source for the matter asymmetry in the universe. Curiously, $(3 + 1)$ -dimensional cp is also broken, which is significant if the condensate couples to a brane. For antiperiodic boundary conditions, $(3 + 1)$ -dimensional cp is explicitly broken while for periodic

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boundary conditions cp is broken by the location of the brane.

In this sequel to [2,7–9], we explore the implications for the early universe of the broken symmetries in the Klein bottle. In particular, we imagine a $(3+1)$ -D brane boosted along the Klein bottle and calculate particle production of $(3+1)$ -D fermions as the brane passes through the condensate. The process we consider can lead to baryogenesis via leptogenesis. As a byproduct, we find a general expression for the Bogoliubov coefficients that quantify the particle production for a brane fermion with a time-dependent Dirac mass.

Further cosmological implications of the Klein bottle, which we leave for study elsewhere, include dark matter candidates that receive their mass from the condensate, dark energy contributions from the same, and a condensate-induced potential that can stabilize the volume of the Klein bottle as well as bring the brane to rest.

II. THE KLEIN BOTTLE

A. Covering space and Γ matrices

We work in $(5+1)$ dimensions composed of a $(3+1)$ Minkowski spacetime times a $2D$ Klein bottle, $\mathcal{M} \times \mathcal{K}$. We adopt a mostly minus metric $(+, -, -, -, -, -)$ for ease of comparison with canonical particle physics convention. The Dirac equation in $(5+1)$ dimensions is $(i\not{D} - m\mathbb{1}_8)\psi = 0$ with positive and negative frequency modes ψ on the covering space.

We build 8×8 Γ^M matrices and a chirality operator $\bar{\Gamma} = \Gamma^0 \Gamma^1 \Gamma^2 \Gamma^3 \Gamma^4 \Gamma^5$, using the notation $M = 0, 5$ and $J = 1, 5$. The Γ 's obey the Clifford algebra

$$\{\Gamma^M, \Gamma^N\} = 2\eta^{MN}\mathbb{1}_8, \quad \eta^{MN} = \text{diag}(1, -1, -1, -1, -1, -1). \quad (1)$$

It follows that $(\Gamma^0)^2 = \mathbb{1}_8$ and $\Gamma^0 = \Gamma^{0\dagger}$ while $(\Gamma^J)^2 = -\mathbb{1}_8$ and $\Gamma^J = -\Gamma^{J\dagger}$. Also $\bar{\Gamma} = \bar{\Gamma}^\dagger$ and $\bar{\Gamma}^2 = \mathbb{1}_8$. We will make use of $\{\Gamma^N, \bar{\Gamma}\} = 0$. The explicit representation is

$$\begin{aligned} \Gamma^\mu &= \gamma^\mu \otimes \sigma^3 = \begin{pmatrix} \gamma^\mu & 0 \\ 0 & -\gamma^\mu \end{pmatrix}, \\ \Gamma^4 &= \mathbb{1}_4 \otimes i\sigma^1 = i \begin{pmatrix} 0 & \mathbb{1}_4 \\ \mathbb{1}_4 & 0 \end{pmatrix}, \\ \Gamma^5 &= \mathbb{1}_4 \otimes i\sigma^2 = \begin{pmatrix} 0 & \mathbb{1}_4 \\ -\mathbb{1}_4 & 0 \end{pmatrix}, \\ \bar{\Gamma} &= -\bar{\gamma} \otimes \sigma^3 = \begin{pmatrix} -\bar{\gamma} & 0 \\ 0 & \bar{\gamma} \end{pmatrix}. \end{aligned} \quad (2)$$

These are built with the $(3+1)$ γ matrices which obey the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu,\nu}\mathbb{1}_4$ with $(\gamma^0)^2 = \mathbb{1}_4$ and

$(\gamma^j)^2 = -\mathbb{1}_4$. With this convention $\gamma^0 = \gamma^{0\dagger}$ and $\gamma^j = -\gamma^{j\dagger}$. In the chiral basis

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \quad \text{with } \sigma^\mu = (1_2, \sigma^i) \text{ and } \bar{\sigma}^\mu = (1_2, -\sigma^i). \quad (3)$$

Just to have on hand for convenience, the Hermitian Pauli matrices are

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (4)$$

for which $\sigma^{i\dagger} = \sigma^i$, $(\sigma^i)^2 = 1_2$, $\{\sigma^i, \sigma^j\} = 2\delta^{ij}\mathbb{1}_2$, and $[\sigma^i, \sigma^j] = 2i\epsilon_{ijk}\sigma^k$. The chiral γ will be called $\bar{\gamma}$ instead of the conventional γ^5 to avoid confusion with the fifth dimension. We have $\bar{\gamma} = i\gamma^0\gamma^1\gamma^2\gamma^3$ which is $\bar{\gamma} = \text{diag}(-1, -1, 1, 1)$. Also $\bar{\gamma} = \bar{\gamma}^\dagger$ and $(\bar{\gamma})^2 = \mathbb{1}_4$ and $\bar{\gamma}$ anti-commutes with the others $\{\gamma^\mu, \bar{\gamma}\} = 0$.

B. Klein bottle boundary conditions

We build a Klein bottle by identifying the sides of a rectangle with a twist on traversing the x_5 direction

$$\begin{aligned} (x^\mu, x^4, x^5) &\approx (x^\mu, x^4 + 2\pi r_4, x^5), \\ (x^\mu, x^4, x^5) &\approx (x^\mu, -x^4, x^5 + 2\pi r_5). \end{aligned} \quad (5)$$

We introduce the notation for convenience,

$$\tilde{x} = (x^\mu, -x^4, x^5 + 2\pi r_5). \quad (6)$$

The Klein bottle is nonorientable. To accommodate spinors in the Klein bottle, begin with the periodic modes on the covering torus [10] of size $(2\pi r_4, 2\pi(2r_5))$:

$$\psi(x) = \psi(x^\mu, x^4 + 2\pi r_4, x^5), \quad (7)$$

$$\psi(x) = \psi(x^\mu, x^4, x^5 + 2(2\pi r_5)), \quad (8)$$

of the form $\psi \sim e^{\mp ik \cdot x}$ (on-shell) with

$$k_4 = \frac{n_4}{r_4}, \quad k_5 = \frac{n_5}{2r_5}, \quad n_4, n_5 \in \mathbb{Z}. \quad (9)$$

To determine which of these spinors descend to the Klein bottle, we need to specify the boundary conditions imposed in the twisted direction, of which there are a number of possibilities. Consider, for example, the reflection of a spinor upon being transported around x_5 . We denote the

reflection operation by R_4 and denote modes on the Klein bottle by Ψ to distinguish from modes ψ on the covering torus. The reflection matrix is

$$R_4 = \Gamma^4 \bar{\Gamma}, \quad (10)$$

for which $R_4 = R_4^\dagger$. There is a notable phase ambiguity. The fermion can be periodic under reflection, which we will call R_4^+ boundary conditions,

$$R_4^+ : \Psi(x) = R_4 \Psi(\tilde{x}), \quad (11)$$

using the notation of Eq. (6). Periodicity is ensured since $R_4^2 = \mathbb{1}_8$, which on traversing x_5 twice gives $\Psi(x) = \Psi(x + 4\pi r_5 \delta_5^M)$. The fermions can also be antiperiodic, which we call R_4^- boundary conditions,

$$R_4^- : \Psi(x) = i R_4 \Psi(\tilde{x}). \quad (12)$$

Antiperiodicity is assured since $(i R_4)^2 = -\mathbb{1}_8$, which on traversing x_5 twice gives $\Psi(x) = -\Psi(x + 4\pi r_5 \delta_5^M)$. For a more thorough discussion of general phases see [2]. We will be slightly cavalier with the notation and use R_4 to stand for both the operation and the matrix representation.

An additional possibility is that orientation reversal is accompanied by charge conjugation $\Psi^C = C\Psi^*$,

$$CR_4^+ : \Psi(x) = CR_4 \Psi^*(\tilde{x}). \quad (13)$$

(Note, we take C first then R_4 , so $CR_4 = R_4 \circ C$.) The charge conjugation matrix and CR_4 are

$$C = -i\Gamma^2 \Gamma^4 \bar{\Gamma} = -i\Gamma^2 R_4, \quad (14)$$

$$CR_4 = -i\Gamma^2. \quad (15)$$

There is again a phase ambiguity, and indeed we use a different phase than in our previous paper [2], which can be consulted for a more general discussion of phases. With these definitions, $C^2 = \mathbb{1}$ and $C = C^\dagger$. We take note that for any choice of phase

$$(\Psi^C)^C = -\Psi, \quad (16)$$

and the CR_4^+ boundary conditions are always periodic on the covering torus.

The boundary conditions are satisfied by mode solutions of the form, due to the (anti)periodicity,

$$R_4^+ : \Psi(x) = \frac{1}{\sqrt{2}}(\psi(x) + R_4 \psi(\tilde{x})),$$

$$R_4^- : \Psi(x) = \frac{1}{\sqrt{2}}(\psi(x) + i R_4 \psi(\tilde{x})),$$

$$CR_4^+ : \Psi(x) = \frac{1}{\sqrt{2}}(\psi(x) + CR_4 \psi^*(\tilde{x})). \quad (17)$$

The full Kaluza-Klein (KK) spectrum of solutions was presented in [2].

A note on chiral fermions. Since R_4 does not commute with $\bar{\Gamma}$, we are unable to construct chiral fermions. Indeed, the situation is a bit worse than the usual obstruction present in compactifications of higher-dimensional theories. For instance, the left- or right-handed projection of a mode on the Klein bottle is not itself a solution for the reasons described above. Consequently, bulk fermions cannot couple to a chiral gauge theory and must be singlets under the standard model gauge interactions. This would seem to preclude the three large dimensions from possessing nonorientable topology if the standard model gauge interactions are exact. To have chiral fermions, we therefore assume they are confined to a $(3+1)$ -D brane that lives in the $\mathcal{M} \times \mathcal{K}$ spacetime.

C. Explicitly broken discrete symmetries

The fermion boundary conditions break some of the discrete symmetries we had on the covering space as summarized in Table I and detailed in [2]. The massive Dirac equation is symmetric under C and P as well as under reflection in any given direction, $R_J = \Gamma^J \bar{\Gamma}$. We can ask if these transformations map solutions out of the solution space.

For instance, with R_4^+ boundary condition,

$$R_4^+ : \Psi(x) = R_4 \Psi(\tilde{x}), \quad (18)$$

a Klein bottle mode in the solution space is mapped under parity P to a new mode

TABLE I. The boundary conditions explicitly break some symmetries on the covering space. A+ indicates the symmetry is preserved, a – that the symmetry is broken. We note that $P_{123} = iR_1 R_2 R_3$ and so $P = \Gamma^0 = R_1 R_2 R_3 R_4 R_5 = -iP_{123} R_4 R_5$. In the final row, this phase difference accounts for the breaking of P but not $P_{123} R_4 R_5$. In general, the final row is very dependent on phase choices in the definition of C . The results for arbitrary phase are summarized in [2].

	P_{123}	R_4	R_5	P	C	$e^{ia\bar{\Gamma}}$
$R_4^+ : \Psi(x) = R_4 \Psi(\tilde{x})$	–	+	–	+	–	–
$R_4^- : \Psi(x) = i R_4 \Psi(\tilde{x})$	–	+	+	–	+	–
$CR_4^+ : \Psi(x) = CR_4 \Psi^*(\tilde{x})$	–	–	+	–	–	+

$$\begin{aligned}
\Psi'(x) &= P\Psi(t, -x^J) = \Gamma^0\Psi(t, -x^J) \\
&= \Gamma^0 R_4 \Psi(t, -x^j, x^4, -x^5 + 2\pi r_5) \\
&= R_4 \Gamma^0 \Psi(t, -x^j, x^4, -x^5 + 2\pi r_5) \\
&= R_4 \Psi'(\tilde{x}),
\end{aligned} \tag{19}$$

from which we conclude that the parity transformed field obeys the boundary condition and therefore is also a solution.

However, for the C transformed field,

$$\begin{aligned}
\Psi'(x) &= C\Psi^*(x) = C R_4^* \Psi^*(\tilde{x}) \\
&= -R_4 C \Psi^*(\tilde{x}) \\
&= -R_4 \Psi'(\tilde{x}) \\
&\neq R_4 \Psi'(\tilde{x}),
\end{aligned} \tag{20}$$

from which we conclude that the C transformed field does not obey the boundary condition and therefore is not a solution. The R_4^+ boundary condition preserves P but explicitly breaks C as well as CP .

Following the same logic for R_4^- , we find P is explicitly broken while C is preserved. Again, the important combination CP is broken [2].

For massless fermions, the Dirac equation is also chirally symmetric, under transformations of the field by $\bar{\Gamma}$, $\Psi' = e^{i\alpha\bar{\Gamma}}\Psi$. We can ask if chiral symmetry for massless fermions is preserved. (More accurately, chiral rotations when broken are broken to the Z_2 subgroup $\alpha \in \{0, \pi\}$ for $R_4^\pm$.) The results for the various boundary conditions and symmetries are tabulated in Table I.

In Sec. IV C, we will be interested in coupling to a $(3+1)$ -D brane, so we also tabulate the outcomes for a $(3+1)$ -dimensional parity, $P_{123} = iR_1 R_2 R_3$. The phase is chosen so that $P_{123}^2 = \mathbb{1}_8$.

III. THE CONDENSATE WALL

In the torus, the two-point correlator encodes the free propagation of a fermion through space. However, for the Klein bottle, the vacuum supports a self-consistent background of fermionic condensates in the absence of an external source. The correlator thereby acquires an additional, position-dependent expectation value even at coincident points. This nonzero expectation value is the fermion condensate, which we calculate here.

To keep the scope manageable, we will work with R_4^+ boundary conditions. (The correlators were calculated for all three boundary conditions using different methods in [2].) To isolate the condensate, we express the correlator in terms of the free propagator and the condensate wall:

$$\langle \Psi(x) \bar{\Psi}(x') \rangle = S_{T^2}(x, x') + i\bar{\Gamma} W(x, x'), \tag{21}$$

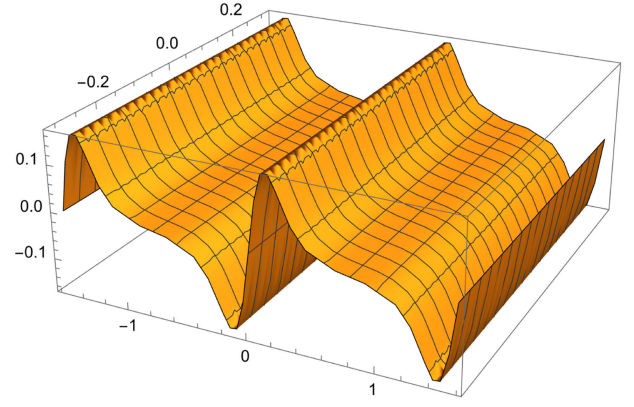


FIG. 1. The condensate wall as a function of x_4 and x_5 .

where our notation exploits the fact that the free propagator is the same as the free propagator on the torus and the wall W is the unique contribution to the condensate. This decomposition and the Γ dependence are a bit of foreshadowing, as is the depiction in Fig. 1.

In addition to $\tilde{x}$ defined in Eq. (6), we introduce the notation

$$\begin{aligned}
\tilde{\tilde{x}} &= (x^\mu, -x^4, x^5 - 2\pi r_5), \\
\tilde{\tilde{k}} &= (k^\mu, -k^4, k^5),
\end{aligned} \tag{22}$$

for which we have the useful combinations

$$\begin{aligned}
k \cdot (\tilde{x} - \tilde{x}') &= \tilde{\tilde{k}} \cdot (x - x'), \\
k \cdot (\tilde{\tilde{x}} - \tilde{x}') &= \tilde{\tilde{k}} \cdot (x - \tilde{\tilde{x}}').
\end{aligned} \tag{23}$$

We will consider R_4^+ boundary conditions for which the modes can be decomposed as

$$\begin{aligned}
\Psi(x) &= \frac{1}{\sqrt{2}} (\psi(x) + R_4 \psi(\tilde{x})), \\
\bar{\Psi}(x') &= \frac{1}{\sqrt{2}} (\bar{\psi}(x') + \bar{\psi}(\tilde{x}') R_4^\dagger).
\end{aligned} \tag{24}$$

The condensate in terms of modes on the covering torus is then

$$\begin{aligned}
2\langle \Psi(x) \bar{\Psi}(x') \rangle &= \langle \psi(x) \bar{\psi}(x') \rangle + R_4 \langle \psi(\tilde{x}) \bar{\psi}(\tilde{x}') \rangle R_4^\dagger \\
&\quad + \langle \psi(x) \bar{\psi}(\tilde{x}') \rangle R_4^\dagger + R_4 \langle \psi(\tilde{x}) \bar{\psi}(x') \rangle.
\end{aligned}$$

The first term is the propagator on the covering torus. The second term is

$$\begin{aligned}
R_4 \langle \psi(\tilde{x}) \bar{\psi}(\tilde{x}') \rangle R_4^\dagger &= \sum_{w_4} \sum_{w_5} \int \frac{d^6 k}{(2\pi)^6} \frac{i R_4(\not{k} + m) R_4^\dagger}{k^2 - m^2 + i\epsilon} e^{-ik \cdot (\tilde{x} - \tilde{x}')} e^{-ik_4 2\pi w_4 r_4} e^{-ik_5 4\pi w_5 r_5} \\
&= \sum_{w_4} \sum_{w_5} \int \frac{d^6 k}{(2\pi)^6} \frac{i(\not{k} + m)}{k^2 - m^2 + i\epsilon} e^{-i\tilde{k} \cdot (x - x')} e^{-ik_4 2\pi w_4 r_4} e^{-ik_5 4\pi w_5 r_5} \\
&= \sum_{w_4} \sum_{w_5} \int \frac{d^6 k}{(2\pi)^6} \frac{i(\not{k} + m)}{k^2 - m^2 + i\epsilon} e^{-ik \cdot (x - x')} e^{-ik_4 2\pi w_4 r_4} e^{-ik_5 4\pi w_5 r_5},
\end{aligned}$$

having used $R_4 \not{k} R_4^\dagger = \tilde{k}$ and $R_4 R_4^\dagger = \mathbb{1}_8$. In the last step we redefine $k_4 \rightarrow -k_4$ and $w_4 \rightarrow -w_4$. This conspires with the first term to give the free propagator for $\langle \Psi(x) \bar{\Psi}(x') \rangle$, which matches that on the covering torus, $S_{T^2}(x, x')$. We hereafter drop this toroidal contribution, which will vanish on taking the trace in the bilinear built from the condensate, and focus on the new contribution from the Klein bottle. For the two last terms in $2\langle \Psi \bar{\Psi} \rangle$, we have $\langle \psi(x) \bar{\psi}(\tilde{x}') \rangle R_4^\dagger + R_4 \langle \psi(\tilde{x}) \bar{\psi}(x') \rangle$

$$\begin{aligned}
&= \sum_{w_4} \sum_{w_5} \left[\int \frac{d^6 k}{(2\pi)^6} \frac{i}{k^2 - m^2 + i\epsilon} ((\not{k} + m) e^{-ik \cdot (x - \tilde{x}')} R_4^\dagger + R_4 (\not{k} + m) e^{-ik \cdot (\tilde{x} - x')}) \right] e^{-ik_4 2\pi w_4 r_4} e^{-ik_5 4\pi w_5 r_5} \\
&= \sum_{w_4} \sum_{w_5} \left[\int \frac{d^6 k}{(2\pi)^6} \frac{i}{k^2 - m^2 + i\epsilon} ((\not{k} + m) e^{-ik \cdot (x - \tilde{x}')} + (\tilde{k} + m) e^{-i\tilde{k} \cdot (x - \tilde{x}')} R_4) \right] e^{-ik_4 2\pi w_4 r_4} e^{-ik_5 4\pi w_5 r_5} \\
&= \sum_{w_4} \sum_{w_5} \left[\int \frac{d^6 k}{(2\pi)^6} \frac{i}{k^2 - m^2 + i\epsilon} ((\not{k} + m) e^{-ik \cdot (x - \tilde{x}')} + (\not{k} + m) e^{-ik \cdot (x - \tilde{x}')} R_4) \right] e^{-ik_4 2\pi w_4 r_4} e^{-ik_5 4\pi w_5 r_5} \\
&= (i\not{p} + m) R_4 \sum_{w_4} \sum_{w_5} \left[\int \frac{d^6 k}{(2\pi)^6} \frac{i}{k^2 - m^2 + i\epsilon} (e^{-ik \cdot (x - \tilde{x}')} + e^{-ik \cdot (x - \tilde{x}')} R_4) \right] e^{-ik_4 2\pi w_4 r_4} e^{-ik_5 4\pi w_5 r_5}
\end{aligned}$$

where $R_4^\dagger = R_4$ and we have used Eq. (23). Since the sums are over all w_5 , we can shift to $w_5 \rightarrow -w_5$ with no affect on the second term in parentheses. Then, to consolidate the expression, we write

$$2\langle \Psi(x) \bar{\Psi}(x') \rangle = (i\not{p} + m) R_4 \sum_{w_4} \sum_{w_5} (D_F(x - \tilde{X}') + D_F(x - \tilde{\tilde{X}}')), \quad (25)$$

where D_F is the $(5+1)$ -D scalar propagator and we define

$$\begin{aligned}
\tilde{X}' &= (x'^\mu, -x'^4 + 2\pi w_4 r_4, x'^5 + 2\pi(2w_5 + 1)r_5), \\
\tilde{\tilde{X}}' &= (x'^\mu, -x'^4 + 2\pi w_4 r_4, x'^5 - 2\pi(2w_5 + 1)r_5).
\end{aligned} \quad (26)$$

In the coincident limit $x \rightarrow x'$, only the $\partial_{4,5}$ terms survive, giving

$$\begin{aligned}
2\langle \Psi(x) \bar{\Psi}(x') \rangle &= \sum_{w_4} \sum_{w_5} (i\Gamma^4 \partial_4 + i\Gamma^5 \partial_5 + m) R_4 (D_F(x - \tilde{X}') + D_F(x - \tilde{\tilde{X}}')) \\
&= i \sum_{w_4} \sum_{w_5} \left(\Gamma^4 \frac{(x_4 - \tilde{X}'_4)}{r} + \Gamma^5 \frac{(x_5 - \tilde{X}'_5)}{r} \right) R_4 \partial_r D_F(x - \tilde{X}') \\
&\quad \times i \sum_{w_4} \sum_{w_5} \left(\Gamma^4 \frac{(x_4 - \tilde{\tilde{X}}'_4)}{r} + \Gamma^5 \frac{(x_5 - \tilde{\tilde{X}}'_5)}{r} \right) R_4 \partial_r D_F(x - \tilde{\tilde{X}}') \\
&\quad + \sum_{w_4} \sum_{w_5} m R_4 (D_F(x - \tilde{X}') + D_F(x - \tilde{\tilde{X}}')).
\end{aligned} \quad (27)$$

In the limit of $x' \rightarrow x$, $(x_5 - \tilde{\tilde{X}}'_5) = 2\pi(w_5 + 1)r_5 = -(x_5 - \tilde{X}'_5)$ and the terms proportional to Γ^5 cancel. Going over to the massless case to make this simpler,

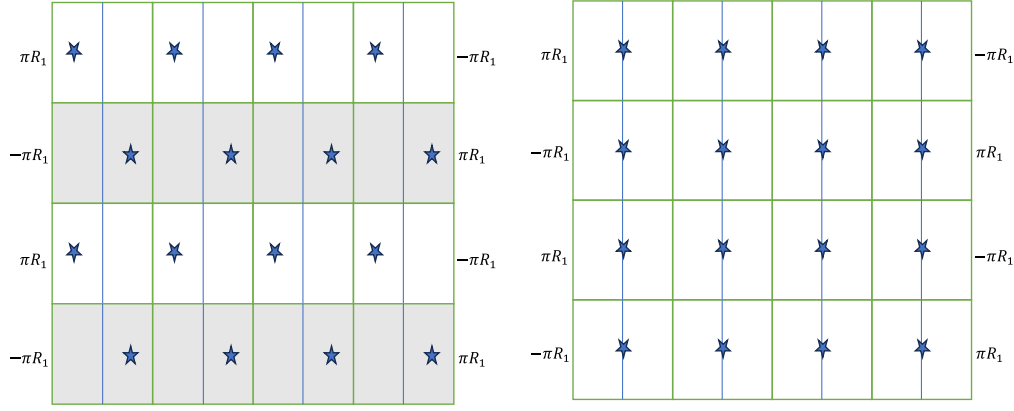


FIG. 2. Klein bottle tilings of the plane. The horizontal axis is x_4 and the vertical axis is x_5 . The flip axis is indicated by the faint vertical blue line bisecting the tiles. Left: if a star is arbitrarily off the flip axis, there are a series of images in a T^2 of size $2\pi(r_4, 2r_5)$ and there is a second series of images in all the shaded cells at $-x_4$ as well as all of its images in a T^2 of size $2\pi(r_4, r_5)$. Right: for a star located on the flip axis, all images look like those in a T^2 of size $2\pi(r_4, r_5)$. The same would be true for the images of a star located at the identified points $x_4 = \pm\pi r_4$. We find the condensate vanishes exactly on, and is centered around, these two special axes: the flip axis at $x_4 = 0$ and the identified axes $x_4 = \pm\pi r_4$.

$$D_F(x - \tilde{X}') = \int \frac{d^6 k}{(2\pi)^6} \frac{i}{k^2 + i\epsilon} e^{-ik \cdot (x - \tilde{X}')} \\ = \frac{1}{4\pi^3} \left(\frac{1}{(x - \tilde{X}') \cdot (x - \tilde{X}')} \right)^2. \quad (28)$$

In the limit $x' \rightarrow x$, $D_F(x - \tilde{X}') = D_F(x - \tilde{\tilde{X}}') = D_F(r)$:

$$D_F(r) = \frac{1}{4\pi^3} \frac{1}{r^4}, \\ r^2 = (2x_4 - 2\pi w_4 r_4)^2 + (2\pi(w_5 + 1)r_5)^2. \quad (29)$$

We then have

$$\langle \Psi \bar{\Psi}(x) \rangle = -\frac{i}{\pi^3} \Gamma^4 R_4 \sum_{w_4} \sum_{w_5} \frac{(2x_4 - 2\pi w_4 r_4)}{r^6}. \quad (30)$$

We consolidate these results as

$$R_4^+ : (m=0) \quad \langle \Psi \bar{\Psi}(x) \rangle = -iW(x_4) \Gamma^4 R_4 = iW(x_4) \bar{\Gamma} \quad (31)$$

with the wall coefficient defined as

$$W(x_4) = \frac{1}{\pi^3} \sum_{w_4} \sum_{w_5} \frac{(2x_4 - 2\pi w_4 r_4)}{((2x_4 - 2\pi w_4 r_4)^2 + (2\pi(2w_5 + 1)r_5)^2)^3}. \quad (32)$$

The condensate is forced to zero at $x_4 = 0$ because the sum is odd in w_4 . The condensate is also forced to zero on the edges, $x_4 = \pm\pi r_4$, as can be seen by shifting to $w'_4 = w_4 \mp 1$, respectively, so that the sum will be odd and exactly zero on those axes.

As shown in Fig. 1, there are two double-bumped walls around these axes, $x_4 = 0$ and $x_4 = \pm\pi r_4$, which are special locations where the infinite images are those of a simple torus and translational symmetry is restored.¹ Figure 2 depicts the pattern of images for the Klein bottle and also shows that the pattern of images reduces to those for a torus on these special axes. The modes $w_4 = -1, 0, 1$ capture the dominant features of both hills. Including both walls to lowest order gives

$$W(x_4) \approx \frac{1}{\pi^3} \left[\frac{(2x_4)}{((2x_4)^2 + (2\pi r_5)^2)^3} + \frac{(2x_4 \mp 2\pi r_4)}{((2x_4 \mp 2\pi r_4)^2 + (2\pi r_5)^2)^3} \right] \quad (34)$$

(which is odd and vanishes on integration over the Klein bottle). The first term has a maximum at $x_{4max} = \frac{\pi}{\sqrt{5}} r_5$, so we will take $r_4 > \frac{\pi}{\sqrt{5}} r_5$ for the sake of argument. In Fig. 1, we show the wall for $r_4 = 4r_5$. As $r_4 \gg r_5$, the distance between the peaks at the origin and those on the $x_4 = \pm\pi r_4$ edges becomes longer and flatter.

The pseudoscalar bilinear with a nonzero vev for the condensate from the R_4^+ boundary conditions is

$$\langle \bar{\Psi} i \bar{\Gamma} \Psi \rangle = -i \text{Tr}(\bar{\Gamma} \langle \Psi \bar{\Psi} \rangle) = W(x_4) \text{Tr}(\bar{\Gamma} \bar{\Gamma}) = 8W(x_4) \quad (35)$$

¹In other words, translations in the x^4 direction

$$\Psi'(x) = \Psi(x^\mu, x^4 + a, x^5) \quad (33)$$

are broken to the Z_2 subgroup $a \in \{0, \pi r_4\}$.

TABLE II. The first three columns are the boundary condition, the corresponding condensate, and its explicit value. The condensate has the same symmetries tabulated in Table I.

B.C.	Condensate	Value	Bilinear	$\langle \text{Bilinear} \rangle$
R_4^+	$\langle \Psi \bar{\Psi} \rangle$	$-iW(x_4)\Gamma^4 R_4$	$\bar{\Psi} i \Gamma \Psi$	$8W(x_4)$
CR_4^+	$\langle \Psi \bar{\Psi}^C \rangle$	$-iW(x_4)\Gamma^4 R_4$	$\bar{\Psi}^C i \Gamma \Psi$	$8W(x_4)$

TABLE III. Transformation of the R_4^+ bilinear under various charge-parity transformations. These are $CP = C\Gamma^0$, $CP_{123} = CiR_1R_2R_3$, $CP_{1234} = CP_{123}R_4$, and $CP_{1235} = CP_{123}R_5$. The symbol $\times$ indicates an explicit breaking of the symmetry by the boundary conditions.

B.C.	Bilinear	CP	CP_{123}	CP_{1234}	CP_{1235}
R_4^+	$\bar{\Psi} i \Gamma \Psi$	$\times$	$\bar{\Psi} i \Gamma \Psi _{-\vec{x}}$	$-\bar{\Psi} i \Gamma \Psi _{-\vec{x}, -x^4}$	$\times$

TABLE IV. $(3+1)$ -dimensional cp transformation of brane bilinears in f .

$\bar{f}f$	$\bar{f}i\gamma f$	$(\bar{f}^c f + \bar{f}f^c)$	$(\bar{f}^c f - \bar{f}f^c)$
+	-	+	-

as summarized in Table II. As expected, the condensate respects the same symmetries that the boundary conditions do in Table I. All other attempts to construct scalars or pseudoscalars would vanish on taking the trace. All toroidal contributions also vanish on taking the trace.

Similarly, from CR_4^+ boundary conditions, a pseudoscalar bilinear with a nonvanishing vev, $\langle \bar{\Psi}^C i \Gamma \Psi \rangle = 8W(x_4)$, can be constructed from the condensate $\langle \Psi \bar{\Psi}^C \rangle$ with $\bar{\Psi}^C = \Psi^T C^\dagger \Gamma^0$. As above, all other attempts to construct scalars or pseudoscalars would vanish on taking the trace.

The bilinear and condensate wall for R_4^- boundary conditions are found in [2]. It is interesting that the periodic

boundary conditions lead to a wall that is antisymmetric in x_4 . Conversely, the antiperiodic boundary conditions lead to a wall that is symmetric in x_4 .

IV. LEPTOGENESIS AND BARYOGENESIS

The condensates can be put to good use to explain the excess of matter over antimatter. By introducing a 3D brane moving inside the Klein bottle and coupling a brane fermion to the bulk fermion, we can meet Sakharov's three conditions for successful baryogenesis [11]: (i) Baryon number violation, (ii) cp violation, and (iii) out-of-equilibrium dynamics.

As the brane moves through the condensate walls, the mass term varies sharply, which leads to the production of brane fermions as we show explicitly in Sec. IV B. The production mechanism is a strictly out-of-equilibrium condition. Additionally, there can be lepton number violating mass terms so that leptogenesis can accompany the bursts of particle production. The standard model on the brane then provides the baryon number violation through $B-L$ conserving processes. The location of the brane breaks translation symmetry, which in turn leads to spontaneous breaking of certain cp symmetries, as we discuss below.

A. Brane fermion masses

We consider brane-bulk couplings in which the bulk condensate provides a position-dependent mass to brane fermions. The standard definitions for charge conjugation c and parity p in $3+1$ dimensions are

$$\begin{aligned} c: f(x) &\rightarrow -i\gamma^2 f^*(x), \\ p: f(x) &\rightarrow \gamma^0 f(t, -x^1, -x^2, -x^3). \end{aligned} \quad (36)$$

The properties of the bilinear $\bar{\Psi} i \Gamma \Psi$ are summarized in Table III. The properties of various bilinears in f are summarized in Table IV. Finally, the properties of various mass terms are summarized in Table V.

TABLE V. Spontaneous $(3+1)$ -dimensional cp violation. Various Hermitian interactions between a brane fermion and a bulk fermion subject to R_4^+ boundary conditions on the Klein bottle are shown. Each interaction leads to a mass term for the brane fermion through the condensate. Both g and $m_f = 8W(x_4)$ are real. The columns tabulate lepton number violation; the interaction's transformation under $cp + CP_{123}$; the interaction's transformation under $cp + CP_{1234}$; the mass term from the vev of the bulk bilinear; and finally the subsequent cp character of the mass term. A + indicates the symmetry is preserved and a - that the symmetry is broken. The second and fourth rows have a $cp + CP$ invariance that is spontaneously broken by the brane.

Interaction	ΔL	$cp + CP_{123}$	$cp + CP_{1234}$	Mass term	cp
$-g\bar{f}f(\bar{\Psi} i \Gamma \Psi)$	No	+	-	$-\bar{f}f m_f$	+
$-g\bar{f}i\gamma f(\bar{\Psi} i \Gamma \Psi)$	No	-	+	$-\bar{f}i\gamma f m_f$	-
$-\frac{g}{2}(\bar{f}^c f + \bar{f}f^c)(\bar{\Psi} i \Gamma \Psi)$	Yes	+	-	$-\frac{1}{2}(\bar{f}^c f + \bar{f}f^c)m_f$	+
$-\frac{ig}{2}(\bar{f}^c f - \bar{f}f^c)(\bar{\Psi} i \Gamma \Psi)$	Yes	-	+	$-\frac{1}{2}(\bar{f}^c f - \bar{f}f^c)im_f$	-

Of these, the most interesting is the Majorana interaction

$$\mathcal{L}_{\text{Maj}} = -\frac{ig}{2}(\bar{f}^c f - \bar{f} f^c)(\bar{\Psi} i \bar{\Gamma} \Psi), \quad (37)$$

where g is real. The term allows for lepton number violation, which is needed for baryogenesis. The interaction is cp even under $f \rightarrow f^{cp}$ when combined with the $CP_{1234} = CP_{123} R_4$ symmetry.

This symmetry is spontaneously broken by the location of the brane, as is evident from the mass term generated by the vev of the bilinear,

$$\mathcal{L}_{\text{Maj}} \rightarrow \frac{1}{2}(\bar{f}^c f - \bar{f} f^c) i m_f, \quad (38)$$

which is odd under $f \rightarrow f^{cp}|_{-\vec{x}}$. The real parameter

$$m_f = g \langle \bar{\Psi} \bar{\Gamma} \Psi \rangle = -ig \text{Tr}(\bar{\Gamma} \langle \Psi \bar{\Psi} \rangle) = 8gW(x_4) \quad (39)$$

leads to the imaginary mass $i m_f$. If the brane is at $x_4 = 0$ (or the identified edges $x_4 = \pm \pi r_4$), then the mass vanishes and the symmetry is restored. If the brane is not located on these two special axes, then the mass term appears to a brane observer to be cp violating. This mass term now introduces two of the key elements needed for leptogenesis, lepton number violation, and cp violation.

B. Particle production from brane motion through the wall

If the brane is boosted in the 4-direction with position $x_{b4} = v_4 t$, then as the brane moves through the condensate wall there will be bursts of particle production on the sharp slopes near $x_4 \sim 0$ and $x_4 \sim \pm \pi r_4$. Although above we consider a variety of masses, here we will just treat f as a solution to the ordinary Dirac equation with a Dirac mass in order to streamline the calculation for the particle production. We will simplify by imposing $r_4 \gg r_5$ and just consider scattering through the wall around the origin for which

$$m_f = \frac{8g}{\pi^3} \left(\frac{2v_4 t}{r^6} \right) \quad (40)$$

with $r^2 = (2x_4)^2 + (2\pi r_5)^2$ from Eq. (29) with $w_4 = w_5 = 0$. We expect a maximum of m_f at $x_4 = \pi r_5 / \sqrt{5}$, as shown in Fig. 3.

To begin, we expand in the Fourier modes f_k :

$$f(x) = \int \frac{d^3 k}{(2\pi)^3} e^{i\mathbf{k} \cdot \mathbf{x}} \sum_s (a_{\mathbf{k},s} f_{\mathbf{k},s}^u + b_{-\mathbf{k},s}^\dagger f_{-\mathbf{k},s}^v), \quad (41)$$

dropping the cumbersome s notation hereafter. The time-dependent spinor modes $f_{\pm \mathbf{k}}^{u,v}(t)$ are solutions to the Dirac equation with a time-dependent mass. Expressing the Dirac

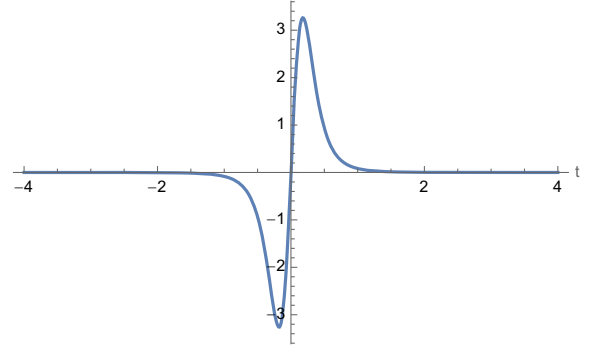


FIG. 3. m_f versus t with $g = 1/2$, $v_4 = 1/2$, and $2\pi r_5 = 0.4$.

equation as a Schrödinger equation for the modes gives

$$i \dot{f}_{\mathbf{k}}(t) = H(t) f_{\mathbf{k}}(t). \quad (42)$$

Both $f_{\mathbf{k}}^u$ and $f_{-\mathbf{k}}^v$ satisfy this Schrödinger equation with the Hermitian Hamiltonian

$$H(t) = \gamma^0 (\boldsymbol{\gamma} \cdot \mathbf{k} + m_f(t) \mathbb{1}_4) = \begin{pmatrix} -\mathbf{k} \cdot \boldsymbol{\sigma} & m_f(t) \mathbb{1}_2 \\ m_f(t) \mathbb{1}_2 & \mathbf{k} \cdot \boldsymbol{\sigma} \end{pmatrix},$$

$$H(t)^2 = \begin{pmatrix} \omega_{\mathbf{k}}^2 \mathbb{1}_2 & 0 \\ 0 & \omega_{\mathbf{k}}^2 \mathbb{1}_2 \end{pmatrix}, \quad \omega_{\mathbf{k}}^2 = \mathbf{k}^2 + m_f(t)^2, \quad (43)$$

and instantaneous eigenvectors

$$H(t) u_{\mathbf{k}} = \omega_{\mathbf{k}}(t) u_{\mathbf{k}}, \quad H(t) v_{-\mathbf{k}} = -\omega_{\mathbf{k}}(t) v_{-\mathbf{k}},$$

$$u_{\mathbf{k}}^\dagger H(t) = \omega_{\mathbf{k}}(t) u_{\mathbf{k}}^\dagger, \quad v_{-\mathbf{k}}^\dagger H(t) = -\omega_{\mathbf{k}}(t) v_{-\mathbf{k}}^\dagger. \quad (44)$$

However, the instantaneous eigenvectors are not necessarily solutions to the time-dependent Dirac equation. Using orthogonality and normalizing [12]

$$u_{\mathbf{k}}^\dagger u_{\mathbf{k}} = 1, \quad v_{-\mathbf{k}}^\dagger v_{-\mathbf{k}} = 1, \quad \bar{u}_{\mathbf{k}} v_{\mathbf{k}} = 0, \quad u_{\mathbf{k}}^\dagger v_{-\mathbf{k}} = 0, \quad (45)$$

although generally, $u_{\mathbf{k}}^\dagger v_{\mathbf{k}} \neq 0$. This normalization is consistent with (41).

We expand mode solutions to the time-dependent Dirac equation (42) in terms of the instantaneous eigenvectors and the Bogoliubov coefficients α_k, β_k [13]:

$$f_{\mathbf{k},s}^u = \alpha_{\mathbf{k}} u_{\mathbf{k}} + \beta_{\mathbf{k}} v_{-\mathbf{k}}. \quad (46)$$

There are two sets of Bogoliubov coefficients, one for $s = 1/2$ and one for $s = -1/2$, though we will continue to suppress the s index. Feeding this into (42) gives

$$i \dot{\alpha}_{\mathbf{k}} u_{\mathbf{k}} + i \dot{\beta}_{\mathbf{k}} v_{-\mathbf{k}} + i \alpha_{\mathbf{k}} \dot{u}_{\mathbf{k}} + i \beta_{\mathbf{k}} \dot{v}_{-\mathbf{k}} = \alpha_{\mathbf{k}} \omega_{\mathbf{k}} u_{\mathbf{k}} - \beta_{\mathbf{k}} \omega_{\mathbf{k}} v_{-\mathbf{k}}. \quad (47)$$

Using orthogonality with $u_{\mathbf{k}}^\dagger$ isolates $\dot{\alpha}_{\mathbf{k}}$ while orthogonality with $v_{-\mathbf{k}}^\dagger$ isolates $\dot{\beta}_{\mathbf{k}}$:

$$\begin{aligned} i\dot{\alpha}_{\mathbf{k}} + i\alpha_{\mathbf{k}}u_{\mathbf{k}}^\dagger\dot{u}_{\mathbf{k}} + i\beta_{\mathbf{k}}u_{\mathbf{k}}^\dagger\dot{v}_{-\mathbf{k}} &= \alpha_{\mathbf{k}}\omega_{\mathbf{k}}, \\ i\dot{\beta}_{\mathbf{k}} + i\alpha_{\mathbf{k}}v_{-\mathbf{k}}^\dagger\dot{u}_{\mathbf{k}} + i\beta_{\mathbf{k}}v_{-\mathbf{k}}^\dagger\dot{v}_{-\mathbf{k}} &= -\beta_{\mathbf{k}}\omega_{\mathbf{k}}. \end{aligned} \quad (48)$$

To find $v_{-\mathbf{k}}^\dagger\dot{u}_{\mathbf{k}}$ and similar combinations we multiply the time derivative of $Hu_{\mathbf{k}} = \omega_{\mathbf{k}}u_{\mathbf{k}}$ with $v_{-\mathbf{k}}^\dagger$, which gives

$$\begin{aligned} v_{-\mathbf{k}}^\dagger\dot{H}(t)u_{\mathbf{k}} + v_{-\mathbf{k}}^\dagger H(t)\dot{u}_{\mathbf{k}} &= \dot{\omega}_{\mathbf{k}}v_{-\mathbf{k}}^\dagger u_{\mathbf{k}} + \omega_{\mathbf{k}}v_{-\mathbf{k}}^\dagger\dot{u}_{\mathbf{k}}, \\ v_{-\mathbf{k}}^\dagger\dot{H}(t)u_{\mathbf{k}} - \omega_{\mathbf{k}}v_{-\mathbf{k}}^\dagger\dot{u}_{\mathbf{k}} &= \omega_{\mathbf{k}}v_{-\mathbf{k}}^\dagger\dot{u}_{\mathbf{k}}, \end{aligned} \quad (49)$$

having used $v_{-\mathbf{k}}^\dagger H(t) = (H(t)v_{-\mathbf{k}})^\dagger = -\omega_{\mathbf{k}}v_{-\mathbf{k}}^\dagger$ for real $\omega_{\mathbf{k}}$ from Eq. (44). Solving, we have

$$v_{-\mathbf{k}}^\dagger\dot{u}_{\mathbf{k}} = \frac{1}{2\omega_{\mathbf{k}}}v_{-\mathbf{k}}^\dagger\dot{H}(t)u_{\mathbf{k}}. \quad (50)$$

We can push this a bit further with an operator related to the helicity operator

$$h_{\mathbf{k}} = \mathbf{k} \cdot \mathbf{S} = \frac{1}{2} \begin{pmatrix} \mathbf{k} \cdot \boldsymbol{\sigma} & 0 \\ 0 & \mathbf{k} \cdot \boldsymbol{\sigma} \end{pmatrix}. \quad (51)$$

Reverting the s index, we note that

$$h_{\mathbf{k}}u_{\mathbf{k},s} = sku_{\mathbf{k},s}, \quad h_{\mathbf{k}}v_{\mathbf{k},s} = -skv_{\mathbf{k},s}, \quad (52)$$

where $s = \pm \frac{1}{2}$ and by k we mean the magnitude of the 3-momentum, $k = |\mathbf{k}|$. We will use this and reexpress

$$H(t) = 2\bar{\gamma}h_{\mathbf{k}} + \gamma^0 m_f. \quad (53)$$

We simplify the following:

$$\begin{aligned} v_{-\mathbf{k}}^\dagger\dot{H}(t)u_{\mathbf{k}} &= \dot{m}_f v_{-\mathbf{k}}^\dagger\gamma^0 u_{\mathbf{k}} = \frac{\dot{m}_f}{\omega_{\mathbf{k}}}v_{-\mathbf{k}}^\dagger\gamma^0 Hu_{\mathbf{k}} \\ &= \frac{\dot{m}_f}{\omega_{\mathbf{k}}}v_{-\mathbf{k},s}^\dagger\gamma^0 2\bar{\gamma}h_{\mathbf{k}}u_{\mathbf{k},s} = \frac{2sk\dot{m}_f}{\omega_{\mathbf{k}}}v_{-\mathbf{k},s}^\dagger\gamma^0\bar{\gamma}u_{\mathbf{k},s} \\ &= \frac{2sk\dot{m}_f}{\omega_{\mathbf{k}}}v_{-\mathbf{k},s}^\dagger v_{-\mathbf{k},s} = \pm \frac{k\dot{m}_f}{\omega_{\mathbf{k}}}. \end{aligned} \quad (54)$$

We used $2s = \pm 1$ and

$$v_{-\mathbf{k},s} = \gamma^0\bar{\gamma}u_{\mathbf{k},s}, \quad (55)$$

which is consistent with Eq. (44). Again, the upper sign corresponds to $s = 1/2$ while the lower sign corresponds to $s = -1/2$.

The combinations we need are

$$\begin{aligned} v_{-\mathbf{k}}^\dagger\dot{u}_{\mathbf{k}} &= \frac{1}{2\omega_{\mathbf{k}}}v_{-\mathbf{k}}^\dagger\dot{H}(t)u_{\mathbf{k}} = \pm \frac{k\dot{m}_f}{2\omega_{\mathbf{k}}^2}, \\ u_{\mathbf{k}}^\dagger\dot{v}_{-\mathbf{k}} &= -\frac{1}{2\omega_{\mathbf{k}}}u_{\mathbf{k}}^\dagger\dot{H}(t)v_{-\mathbf{k}} = \mp \frac{k\dot{m}_f}{2\omega_{\mathbf{k}}^2}, \\ v_{-\mathbf{k}}^\dagger\dot{v}_{-\mathbf{k}} &= u_{\mathbf{k}}^\dagger\dot{u}_{\mathbf{k}} = 0. \end{aligned} \quad (56)$$

The final line amounts to a gauge choice for an arbitrary phase. Using this in (48)

$$\begin{aligned} \dot{\alpha}_{\mathbf{k}} &= -i\omega_{\mathbf{k}}\alpha_{\mathbf{k}} \pm \beta_{\mathbf{k}}k\frac{\dot{m}_f}{2\omega_{\mathbf{k}}^2}, \\ \dot{\beta}_{\mathbf{k}} &= i\omega_{\mathbf{k}}\beta_{\mathbf{k}} \mp \alpha_{\mathbf{k}}k\frac{\dot{m}_f}{2\omega_{\mathbf{k}}^2}. \end{aligned} \quad (57)$$

These are often easier to integrate numerically but we could also shift the Bogoliubov coefficients to separate out the fast oscillations,

$$\begin{aligned} \alpha_{\mathbf{k}} &= \tilde{\alpha}_{\mathbf{k}} e^{-i\int^t \omega(t')dt'}, \\ \beta_{\mathbf{k}} &= \tilde{\beta}_{\mathbf{k}} e^{i\int^t \omega(t')dt'}. \end{aligned} \quad (58)$$

Using $\dot{m}_f/\omega_{\mathbf{k}} = \dot{\omega}_{\mathbf{k}}/m_f$, we have the general result for the evolution of the Bogoliubov coefficients for a fermion with a time-dependent mass:

$$\begin{aligned} \dot{\tilde{\alpha}}_{\mathbf{k}} &= \pm \tilde{\beta}_{\mathbf{k}} \left(\frac{k}{m_f} \right) \frac{\dot{\omega}_{\mathbf{k}}}{2\omega_{\mathbf{k}}} e^{2i\int^t \omega(t')dt'}, \\ \dot{\tilde{\beta}}_{\mathbf{k}} &= \mp \tilde{\alpha}_{\mathbf{k}} \left(\frac{k}{m_f} \right) \frac{\dot{\omega}_{\mathbf{k}}}{2\omega_{\mathbf{k}}} e^{-2i\int^t \omega(t')dt'}, \end{aligned} \quad (59)$$

with $|\alpha_{\mathbf{k}}|^2 + |\beta_{\mathbf{k}}|^2 = 1$. Again, the upper sign is for $s = 1/2$ while the lower sign is for $s = -1/2$.

As compared to the more familiar scalar equations for which $|\alpha_{\mathbf{k}}|^2 - |\beta_{\mathbf{k}}|^2 = 1$,

$$\begin{aligned} \dot{\tilde{\alpha}}_{\mathbf{k}} &= \tilde{\beta}_{\mathbf{k}} \frac{\dot{\omega}_{\mathbf{k}}}{2\omega_{\mathbf{k}}} e^{2i\int^t \omega(t')dt'}, \\ \dot{\tilde{\beta}}_{\mathbf{k}} &= \tilde{\alpha}_{\mathbf{k}} \frac{\dot{\omega}_{\mathbf{k}}}{2\omega_{\mathbf{k}}} e^{-2i\int^t \omega(t')dt'}, \end{aligned} \quad (60)$$

the fermion equations have a relative sign difference and a factor of k/m_f .

In the far past, the instantaneous eigenmodes are solutions to the massless Dirac equation and correspond to a well-defined vacuum. In the far future, the instantaneous eigenmodes are again solutions to the massless Dirac equation and the Bogoliubov coefficients account for the mismatch in early- and late-time modes. The number density of particles created is given by $n_{\mathbf{k}} = |\beta_{\mathbf{k}}|^2$ at late times. We show in Fig. 4 the numerically integrated

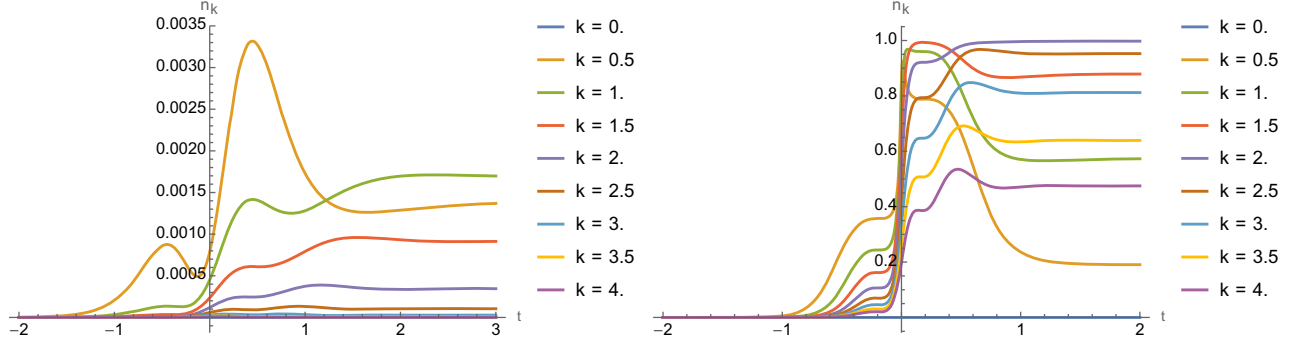


FIG. 4. The particle number density for a given spin as a function of time and for a range of modes. The parameters are $g = 1/2$, $v_4 = 1/2$ with $2\pi r_5 = 1$ on the left and $2\pi r_5 = 0.4$ on the right.

$n_{\mathbf{k}} = |\beta_{\mathbf{k}}|^2$, which we expect to always be less than 1 due to fermi statistics. Notice that there is a large spike around $t = 0$ when $\dot{m}_f/\omega_{\mathbf{k}}^2$ is a maximum. This suggests a non-adiabatic burst of particle production. We should get this burst twice per orbit as the brane hits two walls for N orbits. The accumulated phase could delay the peak to just after $t = 0$ if the peak is not sharp.

The energy density is

$$\rho = g_s \int d^3k \omega_{\mathbf{k}} n_{\mathbf{k}}, \quad (61)$$

where $g_2 = 2$ accounts for the two spin states. As we have assumed that the Klein bottle geometry has been stabilized, the only source for this energy production is a reduction in the brane's kinetic energy, which will decrease with each new instance of particle production. Eventually the brane will come to rest and particle production will end. The final mass of the brane fermion will depend on the resting place of the braneworld in the Klein bottle.

C. Implications for cosmology

We have seen that there are bursts of particle production as the brane passes through the condensate wall. To place this in the larger cosmological context, we consider the conditions needed to create the universe's observed matter asymmetry. The cp violating phases in the standard model Lagrangian are too small to meet the demands of baryogenesis, which requires an asymmetry today of

$$\eta = \frac{n_B - n_{\bar{B}}}{s} \sim 10^{-1} \epsilon \sim 8.6 \times 10^{-11}, \quad (62)$$

where the numerator is the difference between the baryon number density and the antibaryon number density and the denominator is the entropy density of the universe [14]. The cp violation is quantified in the parameter ϵ , which can be related to elements in the fermion mass matrices, for instance.

The Klein bottle condensate can contribute new sources of cp violation and thereby revive baryogenesis. We have

set up all of the ingredients for leptogenesis, which customarily generates a lepton asymmetry as a byproduct of out-of-equilibrium decays of a heavy right-handed neutrino. The condensate potentially rewrites leptogenesis by providing a spontaneously cp violating Majorana mass matrix for multiple right-handed neutrinos,

$$-\frac{1}{2} \bar{\mathbf{f}}^c M \mathbf{f} (\bar{\Psi} i \bar{\Gamma} \Psi) + \text{H.c.} \quad (63)$$

where M is a matrix built out of complex couplings to the condensate $M_{ij} = y_{ij} \langle \Psi \bar{i} \bar{\Gamma} \Psi \rangle$ for multiple bulk fermions. The Majorana mass matrix above is a module in leptogenesis that could also include Dirac mass terms as well as other non- cp violating mass terms. It is worth noting that the condensate can also introduce cp violation in the decays of the heavy neutrino that lead to leptogenesis. The standard model can then do the rest, transforming lepton number violation into baryon number violation with $B - L$ conserving sphaleron transitions active at $T > 100$ GeV.

Typical mass ranges for leptogenesis can be as high as

$$M_L \sim 10^9 - 10^{14} \text{ GeV}. \quad (64)$$

As the brane sweeps through the wall, there would be bursts of production of heavy right-handed neutrinos, whose mass peaks at around

$$M_L \sim g \frac{1}{r_5} \quad (65)$$

with the units of $[g] \sim [r_5]^4$, which translates to a size for the space in the range

$$r_5 \sim 10^{-23} - 10^{-28} \text{ cm}. \quad (66)$$

If the brane comes to rest near the bottom of the wall, then these same neutrinos contribute to dark matter, which in a CDM WIMP scenario typically have masses in the range

$$M \sim 1 \text{ GeV} - 10 \text{ TeV} \quad (67)$$

if the brane resting place is very near the origin of the wall, $x_{4b} \sim 10^{-6} - 10^{-14} r_5$. Equivalently, the brane could come to rest between walls, roughly $x_{4b} \sim 10^2 - 10^3 r_5$. Our ambitions do not extend to naturalness, technical or otherwise.

We mention that through the dependence on the brane location, the magnitude of the cp violation varies as the brane moves and the velocity changes. The cp violation is also dynamical on cosmological timescales if the size of the internal dimensions is dynamical, with the maximum of the wall function scaling as r_5^{-5} . That flexibility allows for a landscape of possibilities that would impact the early universe yet evade detection in the low-energy physics of today. It would be intriguing to apply this novel topological mechanism for cp violation to other important quandaries, such as the strong cp problem.

While we leave the complex details of this novel baryogenesis proposal for further study, we conclude by emphasizing the key ingredients of the scenario:

- (1) nonequilibrium particle production due to brane motion through the wall,

- (2) spontaneous cp violation in the neutrino Majorana mass matrix,
- (3) early universe, heavy, fermion for leptogenesis,
- (4) lighter, but still heavy, neutrinos as CDM candidates.

The braneworld proves valuable since the scenario meets all three of Sakharov's conditions for baryogenesis. However, if baryogenesis can transpire in the full higher-dimensional spacetime, we could also exploit the explicit breaking of $(5 + 1)$ -dimensional CP in the Klein bottle.

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DATA AVAILABILITY

The data are not publicly available. The data are available from the authors upon reasonable request.

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