

Topological susceptibility and excess kurtosis in SU(3) Yang-Mills theory

Stephan Dürr^{1,2} and Gianluca Fuwa¹

¹Department of Physics, *University of Wuppertal*, 42119 Wuppertal, Germany

²Jülich Supercomputing Centre, *Forschungszentrum Jülich*, 52425 Jülich, Germany



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We present a high-precision study of the topological susceptibility in SU(3) pure gauge theory in four space-time dimensions. The result is based on ensembles at seven lattice spacings and in seven physical volumes to facilitate a controlled continuum and infinite-volume extrapolation. We use a gluonic topological charge measurement, with gradient flow smoothing in the operator. Two complementary smoothing strategies are used (one keeps the flow time fixed in lattice units, one in physical units). Our data support the idea that both strategies yield a universal continuum limit; we find $\chi_{\text{top}}^{1/4} r_0 = 0.4775(14)(11)$ or $\chi_{\text{top}}^{1/4} = 198.1(0.7)(2.7)$ MeV. Our Appendix data suggest that the excess kurtosis $\langle q^4 \rangle / \langle q^2 \rangle^2 - 3$ decreases $\propto L^{-2}$ for large box sizes L .

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I. INTRODUCTION

Yang-Mills (YM) theories in four space-time dimensions dynamically generate a scale by a process called “dimensional transmutation” [1]. This scale reflects itself in any dimensionful quantity, for instance the topological susceptibility $\chi_{\text{top}} = \lim_{V \rightarrow \infty} \langle q^2 \rangle / V$. Here q is the (global) topological charge of the gauge background (see below) and V the volume of the four-dimensional Euclidean box.

The topological susceptibility has the dimension MeV^4 . Our goal is to calculate, from first principles, the ratio between $\chi_{\text{top}}^{1/4}$ and another dimensionful quantity for $N_c = 3$ colors (the result has no free parameters). In this work we use the Sommer radius r_0 [2] to set the scale, since this facilitates comparison with previous works (see below). In practical terms this means that we shall calculate the dimensionless quantity $\chi_{\text{top}}^{1/4} r_0$ for a number of lattice spacings a/r_0 (again in dimensionless units) so that we can extrapolate the results with $(a/r_0)^2 \rightarrow 0$ to the continuum.

The motivation to study the topological susceptibility in QCD-like theories is twofold. On the one hand, χ_{top} serves as a vacuum diagnostics tool. In YM theory it depends on N_c , while in QCD it depends on N_c and the N_f individual quark masses (see Refs. [3,4] for details). On the other hand, the YM susceptibility appears in the Witten-Veneziano formula [5,6]

$$\chi_{\text{top}}^{\text{YM}} \doteq \frac{F^2}{2N_f} (M_{\eta'}^2 + M_{\eta}^2 - 2M_K^2) \quad (1)$$

which is supposed to hold¹ at leading order in the $1/N_c$ expansion. Interestingly, the right-hand side refers to full QCD quantities² only, so the relation links two distinct theories.

In the continuum the topological susceptibility in a finite Euclidean volume V may be defined as

$$\begin{aligned} \chi_{\text{top}} &= \int \langle q(x) q(0) \rangle d^4x \\ &= \lim_{p^2 \rightarrow 0} \frac{1}{V} \int \langle q(x) q(y) \rangle e^{ip(x-y)} d^4x d^4y, \end{aligned} \quad (2)$$

where $q(x)$ is the topological charge density. In this approach two limits are involved, zero virtuality ($p^2 \rightarrow 0$) and infinite volume ($V \rightarrow \infty$) in toroidal geometry. Alternatively, one may use the definition

$$\chi_{\text{top}} = \frac{\langle q^2 \rangle}{V} \quad \text{with} \quad q = \int q(x) d^4x \quad (3)$$

the global topological charge $q \in \mathbb{Z}$. Again a limit $V \rightarrow \infty$ is required, but now one is restricted to $p^2 = 0$. The two

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¹For an explicit check on the lattice see Ref. [7] and references therein.

²We use the Bern normalization $F_\pi = f_\pi / \sqrt{2}$ of the pion decay constant, where $F_\pi^{\text{phys}} = 92.4(3)$ MeV in QCD with physical quark masses and $F = 86.2(5)$ MeV in the two-flavor chiral limit [8].

approaches are equivalent (up to a possible contact term [9]).

On the lattice one may start from definitions analogous to (2) or (3), but the renormalization details are different. On a smoothed copy ($t > 0$ in the terminology of Sec. II) of the gauge field

$$\begin{aligned} q_{\text{nai}}(x) &= \frac{1}{32\pi^2} \epsilon_{\mu\nu\rho\sigma} \text{Tr}[F_{\mu\nu}(x)F_{\rho\sigma}(x)] \\ &= \frac{1}{4\pi^2} \text{Tr}[F_{12}(x)F_{34}(x) - F_{13}(x)F_{24}(x) \\ &\quad + F_{14}(x)F_{23}(x)] \end{aligned} \quad (4)$$

is a (t -dependent) gluonic definition of the field-strength tensor $F_{\mu\nu}(x) = F_{\mu\nu}(x)^a T^a$ with $T^a = \frac{1}{2}\lambda^a$ and thus of the *local* topological charge density $q(x)$. In this case the topological susceptibility

$$\begin{aligned} \chi_{\text{top}} &= Z_q^2(\beta, t) \chi_{\text{nai}} + M(\beta, t) \\ \text{with } \chi_{\text{nai}} &= \frac{a^4}{N} \sum_{x, y \in \Lambda} q_{\text{nai}}(x) q_{\text{nai}}(y) \end{aligned} \quad (5)$$

akin to (2) renormalizes both multiplicatively and additively [10–12]. Here $N = (L/a)^4$ is the number of lattice sites, while $V = L^4$ is the box volume in physical units. Alternatively

$$q_{\text{ren}} = \text{round}(Z_q(\beta, t) q_{\text{nai}}) \quad \text{with} \quad q_{\text{nai}} = a^4 \sum_{x \in \Lambda} q_{\text{nai}}(x) \quad (6)$$

is a *global* topological charge wherein q_{nai} at $t > 0$ is only multiplicatively³ renormalized. Based on q_{ren} defined in (6) on smoothed gauge fields, one may proceed to define the topological susceptibility

$$\chi_{\text{top}} = \frac{\langle q_{\text{ren}}^2 \rangle}{V} \quad (7)$$

akin to (3), and no further renormalization is needed in this last step [13–21]. Put differently, the relative weights $\mathcal{Z}_q(V)/\mathcal{Z}(V)$ of the topological sectors in the $\theta = 0$ YM partition function

$$\mathcal{Z}(V) = \sum_{q=-\infty}^{\infty} \mathcal{Z}_q(V) = \sum_{q=-\infty}^{\infty} \int e^{-S_{\text{gluc}}[U]} [\text{DU}]_{q,V} \quad (8)$$

at $t > 0$ qualify as observables, and $\langle q^2 \rangle$ is defined as the second moment of the partition function. We will also

³An additive renormalization of q_{nai} is excluded by the CP symmetry of the lattice theory.

address the higher moment combination $\langle q^4 \rangle - 3\langle q^2 \rangle^2$, known as “excess kurtosis.”

The remainder of this article is organized as follows. In Sec. II we specify how we smooth the gauge configurations to define a variety of $q_{\text{nai}} \in \mathbb{R}$ and $q_{\text{ren}} \in \mathbb{Z}$ which is less susceptible to UV noise. In Sec. III we discuss how we choose the bare parameters to generate a number of ensembles with a joint physical volume and decreasing lattice spacings, and we give details of how we compute the Z_q factors for the global topological charge (6). In Sec. IV our analysis is presented which yields the continuum limit of the topological susceptibility in a fixed physical volume, together with a robust estimate of the theoretical uncertainty involved. In Sec. V the same type of analysis is repeated for the excess kurtosis of the global charge distribution. In Sec. VI we use another set of simulations to study the infinite-volume behavior of the quantities studied in the previous two sections, again with a careful estimate of the theoretical uncertainty involved. In Sec. VII our result is compared to the literature, and Sec. VIII gives a summary and discusses some prospects for future research.

II. SMOOTHING VIA STOUT SMEARING OR WILSON FLOW

In contemporary lattice field theory two closely related smoothing schemes are used, stout smearing [22] and gradient flow [23–25]. We use the output $V_\mu(x)$ of either one to define a clover operator

$$\begin{aligned} C_{\mu\nu}(x) &= V_\mu(x) V_\nu(x + \hat{\mu}) V_\mu^\dagger(x + \hat{\nu}) V_\nu^\dagger(x) \\ &\quad + V_\nu(x) V_\mu^\dagger(x - \hat{\mu} + \hat{\nu}) V_\nu^\dagger(x - \hat{\mu}) V_\mu(x - \hat{\mu}) \\ &\quad + V_\mu^\dagger(x - \hat{\mu}) V_\nu^\dagger(x - \hat{\mu} - \hat{\nu}) V_\mu(x - \hat{\mu} - \hat{\nu}) V_\nu(x - \hat{\nu}) \\ &\quad + V_\nu^\dagger(x - \hat{\nu}) V_\mu(x - \hat{\nu}) V_\nu(x + \hat{\mu} - \hat{\nu}) V_\mu^\dagger(x) \end{aligned} \quad (9)$$

for a given lattice site $x \in \Lambda$. Here $\hat{\mu}$ denotes a times the unit vector in the direction μ . This $C_{\mu\nu}(x)$ is identified with $4I_{N_c}$ plus $4i$ times the field-strength operator. Unlike $F_{\mu\nu}(x)$ in the continuum⁴ the latter is not exactly Hermitian (in color space) and exactly traceless. Therefore we define

$$\begin{aligned} F_{\mu\nu}(x) &= P_{\text{TH}} \left[\frac{1}{4i} C_{\mu\nu}(x) \right] \\ \text{with } P_{\text{TH}}[M] &= \frac{1}{2} (M + M^\dagger) - \frac{1}{2N_c} \text{Tr}(M + M^\dagger) I_{N_c} \end{aligned} \quad (10)$$

as the traceless Hermitian part of (9) divided by $4i$.

⁴We like observables like the field strength $F_{\mu\nu}(x)$ or the gauge potential $A_\mu(x)$ in $U_\mu(x) = P\{\exp[ig \int_x^{x+\hat{\mu}} A(s) ds]\}$ to be Hermitian quantities, as is common practice in quantum mechanics.

Our smeared field $V_\mu(x)$ emerges from the unsmeared $U_\mu(x)$ through n steps of stout⁵ smearing

$$\begin{aligned} V_\mu(x) &= V_\mu^{(n)}(x), & V_\mu^{(n)}(x) &= e^{i\rho Q_\mu^{(n-1)}(x)} V_\mu^{(n-1)}(x), \\ V_\mu^{(0)}(x) &= U_\mu(x), \end{aligned} \quad (11)$$

where the stout parameter should be chosen in the interval $0 < \rho < 0.125$ in 4D [26]. The operator

$$Q_\mu^{(n-1)}(x) = P_{\text{TH}} \left[\frac{1}{i} S_\mu^{(n-1)}(x) V^{(n-1)\dagger}(x) \right] \quad (12)$$

contains the product of $V_\mu(x)$ and $S_\mu^\dagger(x)$ which also appears in the Wilson gauge action, with

$$\begin{aligned} S_\mu^{(k)}(x) &= \sum_{\nu \neq \mu} \{ V_\nu^{(k)}(x) V_\mu^{(k)}(x + \hat{\nu}) V_\nu^{(k)\dagger}(x + \hat{\mu}) \\ &\quad + V_\nu^{(k)\dagger}(x - \hat{\nu}) V_\mu^{(k)}(x - \hat{\nu}) V_\nu^{(k)}(x + \hat{\mu} - \hat{\nu}) \} \end{aligned} \quad (13)$$

being the staple around the link $V_\mu^{(k)}(x)$, pointing in the same direction as the link itself.

The main advantage of stout smearing is that one stays in the gauge group, hence no “backprojection” to $SU(3)$ is needed. This is the technical basis of the suggestion made in Refs. [23–25] to consider the limit $\rho \rightarrow 0$ and $n \rightarrow \infty$ where the product $n \times \rho = t/a^2$ is kept constant. The quantity t/a^2 is called the “flow time in lattice units” and has the meaning of a cumulative sum of the ρ parameters used in all steps. Hence, to reach $t/a^2 = 0.84$ one may factor the sum as 7×0.12 or 14×0.06 or 28×0.03 , and so on. In this sequence the step-size error in the flow time decreases (ideal Wilson flow means zero step size). For some applications (e.g. measuring t_0 or w_0 [24,25,27]) it is important to keep the flow time discretization effect small. For other applications (e.g. the one we have in mind) it is less important (we shall come back to this point in Sec. III). The conceptual link between stout smearing and (ideal) Wilson flow has been discussed in Refs. [25,28,29].

In today’s lattice literature the main difference between “stout smearing” and “gradient flow” is the quantity which is held fixed in the continuum limit $a \rightarrow 0$. The terminology stout smearing usually implies that ρ and n , and hence the flow time t/a^2 in *lattice units* is kept constant at all β . With this strategy the naive topological charge density (4) is an ultralocal operator (fixed footprint in lattice units) at all β and becomes pointlike in the limit $a \rightarrow 0$. The terminology gradient flow is usually chosen when the flow time $t/r_0^2 = t/a^2 \times (a/r_0)^2$ in *physical units* is held fixed at all β . In our approximation (via stout smearings with ρ fixed) the

number of steps is then bound to increase like $(r_0/a)^2$ toward the continuum. As a result, $q_{\text{nai}}(x)$ is “regularized” over a distance $\sqrt{8t}$ (which is a fixed distance in r_0 units) and no longer ultralocal [24]. Hence, in this strategy a second regulator is in place, which persists in the continuum limit $a \rightarrow 0$; see e.g. Ref. [29] for a discussion.

In Refs. [30,31] it is stated that for the topological susceptibility χ_{top} the continuum limit can be taken at a fixed flow time in physical units, too, not just at a fixed flow time in lattice units (as was traditionally done). In other words, for this observable the additional flow regulator does not introduce any systematic bias (the continuum extrapolated value $\chi_{\text{top}} r_0^4$ at fixed t/r_0^2 would be the same as, say, with seven stout steps), provided $\sqrt{t}$ is in a range where it smoothes out short-range fluctuations but leaves long-range fluctuations (say, for $r > r_0$) unaffected. In this article we aim for a numerical comparison of the two strategies.

III. LATTICE SETUP AND RENORMALIZATION FACTORS

Our goal is to set up a series of lattice simulations for χ_{top} with a fixed physical volume V . We choose $V = (2.4783r_0)^4$ which is about 51% larger than the volume $V = (2.2356r_0)^4$ of Ref. [16]. We use the Wilson gauge action, and Eq. (14) of Ref. [16] parametrizes r_0/a as a function of β . Today, t_0 [24,25] and w_0 [27] are valuable alternatives to set the scale, but previous results refer to r_0 , and we see no reason to believe that the continuum limit would be different⁶ at all.

Details of our plan are presented in Table I. We use three smoothing strategies, named “7 stout” (fixed $t/a^2 = 0.84$), “0.21 fm,” and “0.30 fm” (fixed t/r_0^2), respectively. To get statistically independent continuum extrapolated values, we shall generate $7 \times 3 = 21$ independent⁷ ensembles. The flow time discretization effects in the latter two strategies are expected⁸ to be small.

Table II lists the ensembles generated, along with the number of measurements made and the number of update packages between adjacent measurements. An update package consists of a heat bath sweep [32–35] followed by four overrelaxation sweeps [36–38]. Occasionally, a P transformation [39] of the gauge configuration is applied (this preserves the action and flips the sign of q_{ren}). The last

⁶The quantity t_0 has the dimension of an area (or inverse mass squared); hence, the ratio $\sqrt{t_0}/r_0$ is a real number which assumes a universal value in the continuum limit $a \rightarrow 0$.

⁷The original plan was to generate only five lattice spacings. At $\beta = 6.0314, 6.1912$ a slight increase of ρ from 0.06 to ~ 0.061 or from 0.12 to ~ 0.122 was needed to fit in these lattices.

⁸Comparing the top-left and bottom-left panels in Fig. 13 of Ref. [19] we see no hint for any flow time discretization effects in $\chi_{\text{top}}^{1/4} r_0$ at flow times and step sizes similar to ours.

⁵Alternatively, one may remove the i and $1/i$ in (11) and (12) and replace P_{TH} by the traceless anti-Hermitian projector.

TABLE I. Overview of the box sizes and couplings selected for the continuum scaling analysis. The volume $V = (2.4783r_0)^4$ is fixed in physical units, based on r_0/a as given in Eq. (14) of Ref. [16]. The “7 stout” smoothing strategy keeps the flow time in lattice units fixed at $t/a^2 = 7 \times 0.12 = 0.84$, tantamount to $\sqrt{8t} \simeq 2.59a \rightarrow 0$ in physical units for $\beta \rightarrow \infty$. The “flow 0.21 fm” strategy sets the flow time to $t/a^2 = (N/4)^2 \times 0.06$, tantamount to $\sqrt{8t} = 0.429r_0 \simeq 0.21$ fm. The “flow 0.30 fm” strategy sets the flow time to $t/a^2 = (N/4)^2 \times 0.12$, tantamount to $\sqrt{8t} = 0.607r_0 \simeq 0.30$ fm.

L/a	β	r_0/a	a [fm]	7 stout	flow 0.21 fm	flow 0.30 fm
12	5.9421	4.842	0.098	7×0.12	$9 \times 0.06 = 0.54$	$9 \times 0.12 = 1.08$
14	6.0314	5.649	0.084	7×0.12	$12 \times 0.06125 = 0.735$	$12 \times 0.1225 = 1.47$
16	6.1142	6.456	0.074	7×0.12	$16 \times 0.06 = 0.96$	$16 \times 0.12 = 1.92$
18	6.1912	7.263	0.066	7×0.12	$20 \times 0.06075 = 1.215$	$20 \times 0.1215 = 2.43$
20	6.2629	8.070	0.059	7×0.12	$25 \times 0.06 = 1.5$	$25 \times 0.12 = 3.00$
24	6.3929	9.684	0.049	7×0.12	$36 \times 0.06 = 2.16$	$36 \times 0.12 = 4.32$
28	6.5079	11.30	0.042	7×0.12	$49 \times 0.06 = 2.94$	$49 \times 0.12 = 5.88$

TABLE II. Details of the ensembles used in the continuum extrapolation. Columns three to five contain the number of measurements and the number of update packages between adjacent measurements in the format $n_{\text{meas}}[n_{\text{sepa}}]$, a separate stream was generated for each smoothing strategy. The last three columns contain $\tau_{\text{int}}(q_{\text{ren}}^2)$, where q_{ren} uses the smoothing strategy listed in the column head.

L/a	β	7 stout	flow 0.21 fm	flow 0.30 fm	7 stout	flow 0.21 fm	flow 0.30 fm
12	5.9421	100000[10]	100000[10]	100000[10]	0.650(10)	0.640(10)	0.660(10)
14	6.0314	209519[10]	200000[10]	200000[10]	1.120(20)	1.130(20)	1.150(20)
16	6.1142	40851[64]	50654[64]	49617[64]	0.570(10)	0.560(10)	0.550(10)
18	6.1912	310191[81]	54760[81]	54577[81]	0.730(20)	0.710(20)	0.730(20)
20	6.2629	78749[100]	83474[100]	106925[100]	1.020(2)	1.060(30)	1.030(20)
24	6.3929	93292[100]	56238[144]	69276[144]	3.08(10)	2.209(90)	2.280(80)
28	6.5079	68641[196]	126804[196]	87726[196]	4.81(24)	5.04(26)	5.34(28)

three columns give $\tau_{\text{int}}(q_{\text{ren}}^2)$, where q_{ren} uses the smoothing strategy listed in the column head.

The original (unsmeared) plaquettes are displayed in the first panel of Fig. 1. In log-log representation versus a/r_0 (or our a/L) they appear almost linear. A marked difference between the 7 stout smoothing strategy on the one hand and the flow 0.21 fm and flow 0.30 fm strategies on the other hand is illustrated in the second panel. With a fixed flow time in lattice units (7 stout) the slope in the log-log

representation is small, while with a fixed flow time in physical units (flow 0.21 fm and flow 0.30 fm) it is much steeper. This is unsurprising, since with the latter two strategies the number of stout steps in Table I proliferates $\propto (r_0/a)^2$ toward the continuum.

Finally, we need a nonperturbative determination of the renormalization factors $Z_q(\beta, t)$. Following Refs. [15,16,40,41] we calculate, for each ensemble and smoothing strategy, the quantity

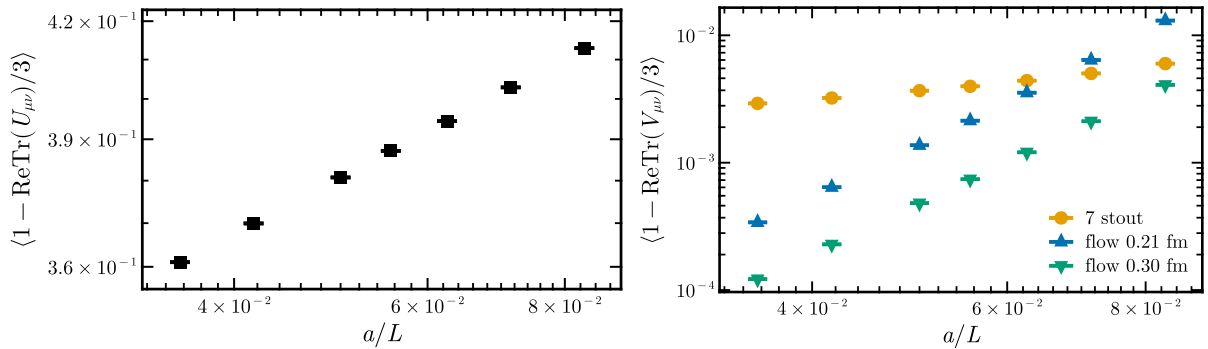


FIG. 1. $\langle 1 - \text{ReTr}(U_{\mu\nu})/3 \rangle$ of the ensembles used in the continuum extrapolation, unsmeared (left) and with one of the three smoothing strategies (right).

TABLE III. The multiplicative renormalization factor Z_q of the gluonic topological charge, determined for each smoothing strategy at various lattice spacings in a fixed physical volume $V = (2.4783r_0)^4$.

L/a	β	7 stout	flow 0.21 fm	flow 0.30 fm
12	5.9421	1.2757(18)	1.3782(57)	1.2337(20)
14	6.0314	1.22881(57)	1.2522(12)	1.15821(41)
16	6.1142	1.1974(10)	1.17807(80)	1.11444(49)
18	6.1912	1.17499(62)	1.13280(43)	1.08579(26)
20	6.2629	1.15682(53)	1.10237(47)	1.06672(17)
24	6.3929	1.13312(31)	1.06700(16)	1.04426(11)
28	6.5079	1.11818(37)	1.04740(10)	1.03149(12)

$$\chi_{\min}^2 = \min_{1 \leq Z \leq 2} \sum_{i=1}^{n_{\text{conf}}} (Zq_{\text{nai}}^{(i)} - \text{round}(Zq_{\text{nai}}^{(i)}))^2 \quad (14)$$

and the Z which realizes the minimum is (for the given β and smoothing recipe) the global topological charge renormalization factor Z_q in Eq. (6). The results are tabulated in Table III and displayed in Fig. 2. For each smoothing strategy the Z_q factor decreases toward the continuum. Plotting Z_q as a function of a^2 (left panel) the function seems to pass through 1 at $a = 0$ for the physical flow time strategies (flow 0.21 fm and flow 0.30 fm), even though these were unconstrained fits. If the flow time is fixed in lattice units (7 stout), the extrapolation in $(a/r_0)^2$ misses 1 in the continuum by many sigmas. The theoretically better motivated extrapolation in $g_0^2 = 6/\beta$ (right panel) is more cumbersome to fit. Eventually, we were successful with the Ansatz $Z_q = (1 + a_1/\beta + a_2/\beta^2)/(1 + b_1/\beta + b_2/\beta^2)$. With d.o.f. = $7 - 4 = 3$ it yields the P values 0.171, 0.896 and 0.350 for the 7 stout, flow 0.21 fm and flow 0.30 fm strategies, respectively.

IV. CONTINUUM ANALYSIS FOR THE TOPOLOGICAL SUSCEPTIBILITY

With these preparatory steps completed, we are in a position to present the analysis for the continuum extrapolation of the topological susceptibility.

Our $7 \times 3 = 21$ ensembles with a joint physical volume are used to measure the naive topological charge (4) and, based on the Z_q listed in Table III, the renormalized charge (6). In either case the second moment of the distribution is determined, and the resulting $\langle q_{\text{nai}}^2 \rangle$ (first three columns) and $\langle q_{\text{ren}}^2 \rangle$ (last three columns) are listed in Table IV. Given (7) and $V = (2.4783r_0)^4$, these numbers must be divided by 2.4783^4 to obtain $\chi_{\text{top}} r_0^4$. All that remains to be done is to get rid of the discretization effects by means of a continuum extrapolation with $O(a^2)$ cutoff effects.

First we discuss this extrapolation for the data based on q_{ren}^2 (last three columns of Table IV), since this is the standard procedure [13–21]. We obtain good fits with a “const + linear” fit (in a^2) that excludes the coarsest ($\beta = 5.9421$) lattice spacing, and with a “const + linear + quadratic” fit that includes all lattice spacings. These fits are shown in Fig. 3. The extrapolated values at $a = 0$ for the three smoothing strategies are statistically independent (as are the ensembles). With the linear Ansatz they agree very closely, and with the quadratic Ansatz (in a^2) there is a visible spread, but they still agree within statistical errors. Extrapolating $\chi_{\text{top}} r_0^4$ is not the only possibility, also $\chi_{\text{top}}^{1/4} r_0$ or other powers are permissible. We add the latter option to control the pertinent systematics; the results are shown in Fig. 4. Again const + linear without the coarsest lattice spacing and const + linear + quadratic with all data included are found to yield acceptable fits.

We have, for each one of the three smoothing strategies, two continuum extrapolations of the topological susceptibility and two of its fourth root. These results are shown in Table V. For a given smoothing strategy the four entries

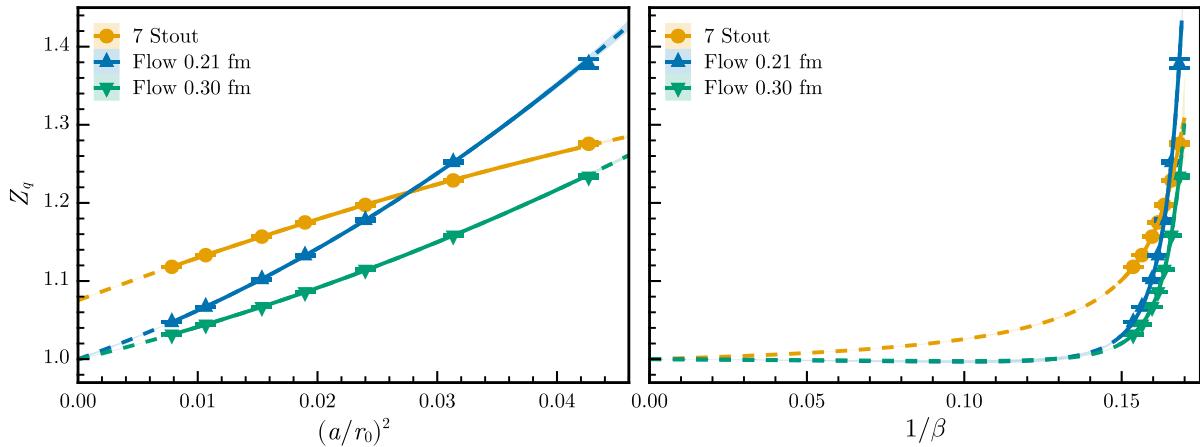


FIG. 2. The Z_q factors involved, with quadratic fits in $(a/r_0)^2$ (left) and rational fits in g_0^2 (right).

TABLE IV. Ensemble average and statistical error of $\langle q_{\text{nai}}^2 \rangle$ (columns three to five) and $\langle q_{\text{ren}}^2 \rangle$ (last three columns), as determined for each smoothing strategy and lattice spacing in the fixed physical volume $V = (2.4783r_0)^4$. These results reflect 7×3 different ensembles.

L/a	β	7 stout	flow 0.21 fm	flow 0.30 fm	7 stout	flow 0.21 fm	flow 0.30 fm
12	5.9421	1.4653(78)	1.2158(64)	1.5970(85)	2.453(12)	2.387(12)	2.486(13)
14	6.0314	1.5362(74)	1.4734(73)	1.7419(88)	2.369(11)	2.364(12)	2.365(12)
16	6.1142	1.554(12)	1.633(11)	1.806(11)	2.268(17)	2.294(14)	2.254(14)
18	6.1912	1.5888(58)	1.715(13)	1.871(14)	2.2154(81)	2.214(16)	2.212(17)
20	6.2629	1.619(12)	1.781(13)	1.897(12)	2.185(16)	2.171(16)	2.162(14)
24	6.3929	1.617(19)	1.834(24)	1.899(22)	2.088(25)	2.090(27)	2.072(24)
28	6.5079	1.627(28)	1.847(24)	1.926(32)	2.046(35)	2.027(26)	2.050(34)

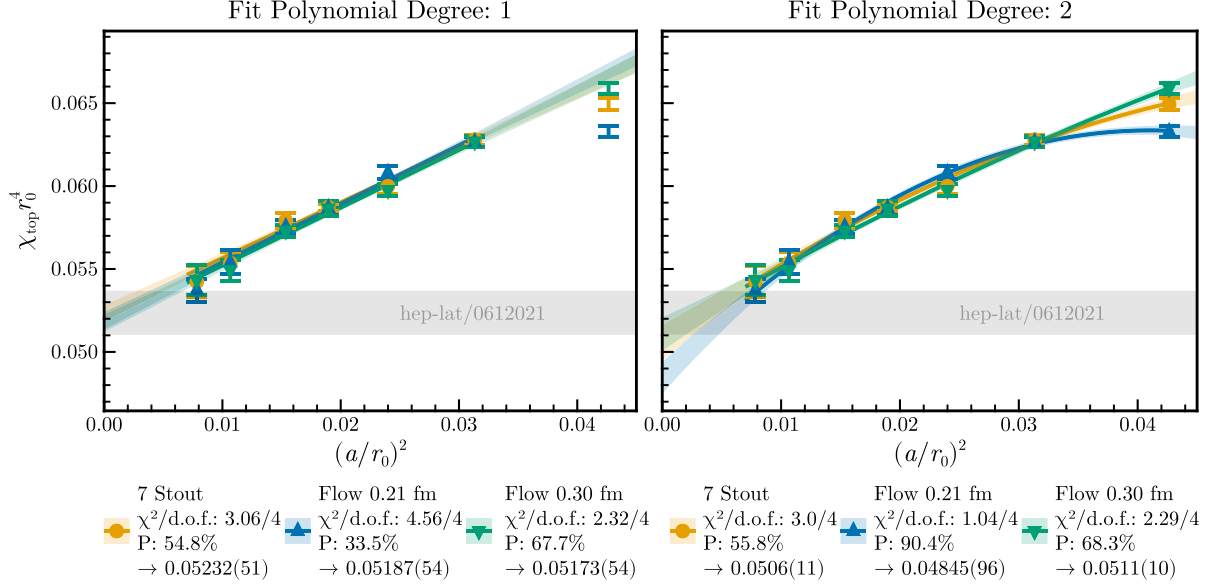


FIG. 3. Continuum extrapolation of the topological susceptibility with an *Ansatz* linear in a^2 based on the six finest spacings (left) and with an *Ansatz* quadratic in a^2 based on all seven spacings (right).

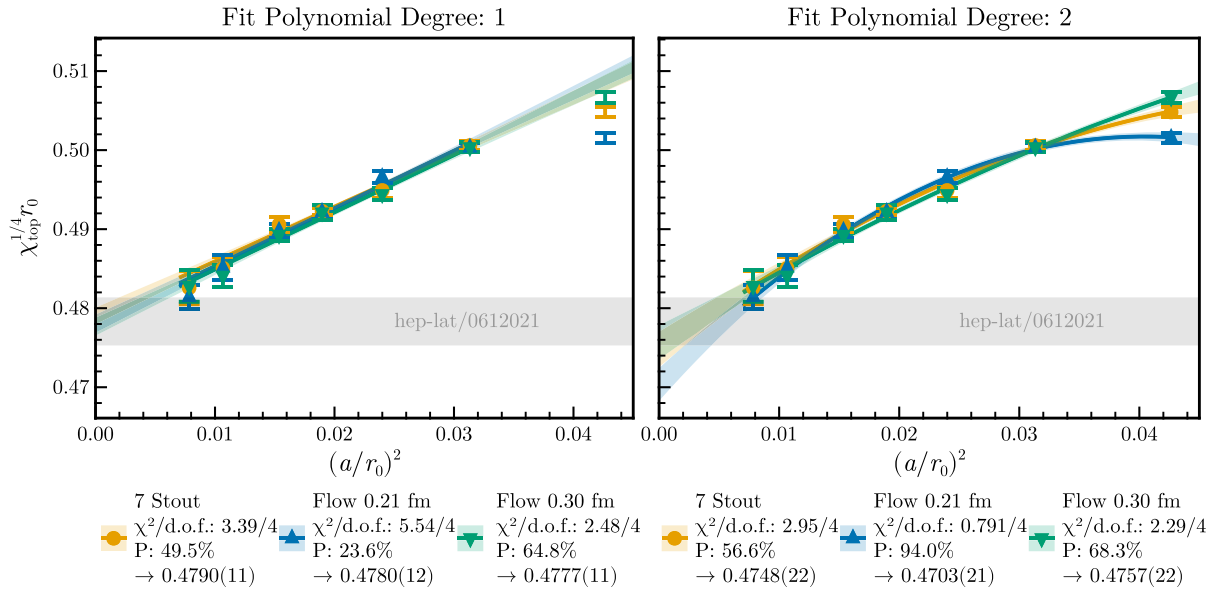


FIG. 4. The same as Fig. 3 but with the quantity $\chi_{\text{top}}^{1/4} r_0$ on the ordinate (and six freshly created fits).

TABLE V. Continuum value of $\chi_{\text{top}} r_0^4$ and its fourth root, in a fixed physical volume $V = (2.4783 r_0)^4$, extracted with one of two fitting *Ansätze* and one of three smearing strategies. The statistical errors are highly correlated along each line but fully uncorrelated along each column.

	$[\chi_{\text{top}} r_0^4]_{\text{lin}}$	$[\chi_{\text{top}} r_0^4]_{\text{quad}}$	$[\chi_{\text{top}}^{1/4} r_0]_{\text{lin}}$	$[\chi_{\text{top}}^{1/4} r_0]_{\text{quad}}$
7 stout	0.05232(51)	0.0506(11)	0.4790(11)	0.4748(22)
flow 0.21 fm	0.05187(54)	0.04845(96)	0.4780(12)	0.4703(21)
flow 0.30 fm	0.05173(54)	0.0511(10)	0.4777(11)	0.4757(22)

describe a joint continuum limit (their spread indicates a systematic uncertainty). It is, *a priori*, not clear that the “ultralocal” strategy (7 stout) and either one of the “fixed physical flow time” strategies (flow 0.21 fm and flow 0.30 fm) would yield the same continuum limit. As mentioned in Sec. III, the latter two strategies introduce a second regulator which persists in the continuum limit. In Refs. [30,31] it is stated that with a reasonable choice of $\sqrt{8t}$ the second regulator leaves a negligible imprint on $\chi_{\text{top}}^{1/4} r_0$ at $a \rightarrow 0$. We are thus left with the task to condense, for each smoothing strategy, the four entries in Table V into a single number (with statistical and systematic uncertainties) to see whether our data support this statement.

We proceed in two steps (first we combine the two fit *Ansätze*, then the two ordering options of the root and the extrapolation). For the first line of Table V this means that we combine $0.05230(57)_{\text{stat}}$ and $0.0506(11)_{\text{stat}}$ to become $0.05194(68)_{\text{stat}}$, where the average of the statistical errors was done with the same weights that were used for the average of the central values, since these errors are highly correlated. In addition, the combined value needs to be attributed a systematic uncertainty to reflect the spread between the two numbers from which it was derived. For combining two central values $c^{(1)}, c^{(2)}$ with statistical uncertainties $\sigma_{\text{stat}}^{(1)}, \sigma_{\text{stat}}^{(2)}$ Refs. [42,43] recommend using

$$\sigma_{\text{syst}} = |c^{(1)} - c^{(2)}| \text{erf} \left(\frac{|c^{(1)} - c^{(2)}|}{\sqrt{2} \max(\sigma_{\text{stat}}^{(1)}, \sigma_{\text{stat}}^{(2)})} \right) \quad (15)$$

since this is the difference between the extrapolations with the two fitting *Ansätze*, multiplied by the probability that this difference is due to a statistical fluctuation. For the 7 stout strategy this yields $\chi_{\text{top}} r_0^4 = 0.05202(61)_{\text{stat}}(152)_{\text{syst}}$ in the continuum and $\chi_{\text{top}}^{1/4} r_0 = 0.4782(13)_{\text{stat}}(40)_{\text{syst}}$. Combining the latter two values by the same recipe yields

$0.4779(14)_{\text{stat}}(37)_{\text{syst}}(01)_{\text{syst}}$ in the continuum. This number is stored in the last column of Table VI, along with the flow 0.21 fm and flow 0.30 fm counterparts. These three figures are consistent within (overall) errors, in line with Refs. [30,31].

This nice agreement among the smoothing strategies at the level of the continuum result contrasts with a technical difference that was alluded to in Fig. 2 already. Based on the extrapolation in the left panel, both $Z_q^{0.21 \text{ fm}}$ and $Z_q^{0.30 \text{ fm}}$ seem to tend to 1 in the limit $a \rightarrow 0$, while $Z_q^{7 \text{ stout}}$ does not. This suggests that one might obtain the correct continuum limit for $\chi_{\text{top}} r_0^4$ based on $\langle q_{\text{nai}}^2 \rangle$, that is *without* the Z_q -factor inherent in $\langle q_{\text{ren}}^2 \rangle$, but only for the fixed-flow-time strategies. Such an extrapolation of $\chi_{\text{top}} r_0^4$, based on the naive topological charge (4), is shown with dashed lines in Fig. 5. The standard extrapolation, based on the renormalized topological charge (6), is shown with full lines. We observe that the fixed-flow-time strategies work with both $\langle q_{\text{ren}}^2 \rangle$ and $\langle q_{\text{nai}}^2 \rangle$. By contrast, the 7 stout strategy works only with $\langle q_{\text{ren}}^2 \rangle$, and the difference between the correct continuum limit (with Z_q) and the incorrect one (without), roughly $0.052 - 0.044 \simeq 0.008$, reflects the additive renormalization constant M in (5) which is due whenever χ_{top} is calculated from a naive charge at zero flow time. Near the continuum the flow 0.21 fm and flow 0.30 fm curves in Fig. 5 are found to be flatter *without* the Z_q factors, and this leads to *smaller* statistical errors in the continuum. This suggests that our final results in Table VI come with conservatively assessed systematics.

To summarize, we note that the three smoothing strategies yield perfectly consistent results, in line with Refs. [30,31]. In view of this, we select the “0.30 fm” quote in Table VI as our final result (or average over the three strategies; it hardly makes any difference). As we shall see in Sec. VI, this final result of the continuum extrapolation is subject to a tiny finite-volume shift.

TABLE VI. Results of the continuum extrapolation of $\chi_{\text{top}} r_0^4$ (left column) or $\chi_{\text{top}}^{1/4} r_0$ (middle column).

	$\chi_{\text{top}} r_0^4$	$\chi_{\text{top}}^{1/4} r_0$	Combined
7 stout	$0.05202(61)(152) = [0.4776(14)(35)]^4$	$0.4782(13)(40)$	$0.4779(14)(37)(01)$
flow 0.21 fm	$0.05105(64)(342) = [0.4753(15)(80)]^4$	$0.4761(14)(77)$	$0.4759(15)(79)(01)$
flow 0.30 fm	$0.05159(64)(030) = [0.4766(15)(07)]^4$	$0.4773(13)(13)$	$0.4769(14)(10)(03)$

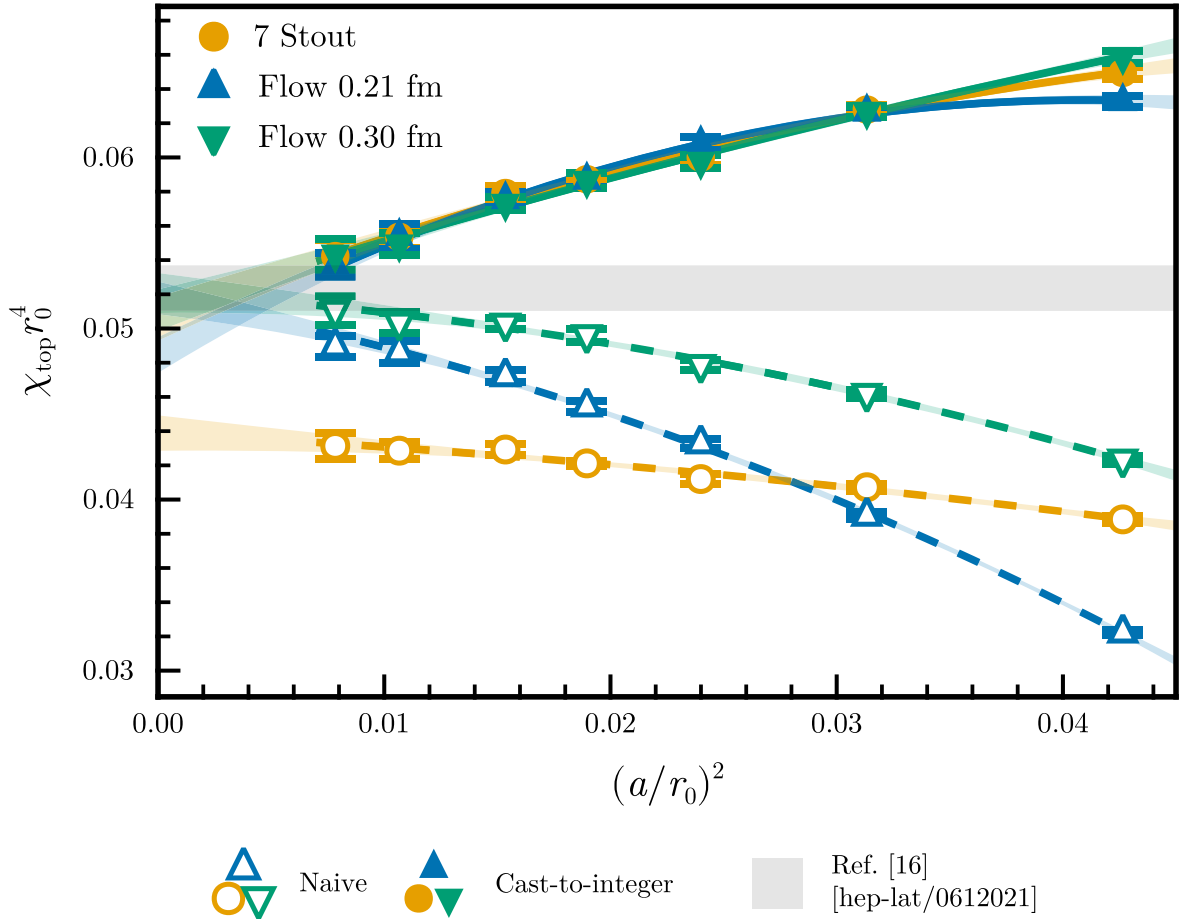


FIG. 5. The standard quadratic-fit extrapolation as shown in the right panel of Fig. 3 (full lines connect $\langle q_{\text{ren}}^2 \rangle / V$) is compared to another approach where the Z_q factor and the cast-to-integer operation are omitted (dashed lines connect $\langle q_{\text{nai}}^2 \rangle / V$). In both cases the data for all three smearing strategies are shown. The most precise determination to date, Ref. [16], is included for comparison.

V. CONTINUUM ANALYSIS FOR THE TOPOLOGICAL EXCESS KURTOSIS

The measured distribution of q_{ren} is, for each β , precise enough that we may attempt to determine a fourth-order cumulant (with a subsequent continuum limit). For a random variable X with mean μ and variance σ^2 , the “standard kurtosis” is defined as $\langle Y^4 \rangle$, where $Y = (X - \mu)/\sigma$. If X is Gaussian, then $\langle Y^2 \rangle = 1$ and $\langle Y^4 \rangle = 3$. Therefore one defines the “excess kurtosis” of X as $\langle Y^4 \rangle / \langle Y^2 \rangle^2 - 3$ or $\langle Y^4 \rangle / \langle Y^2 \rangle - 3 \langle Y^2 \rangle$ or $\langle Y^4 \rangle - 3 \langle Y^2 \rangle^2$; each one of these quantities is zero for a Gaussian X .

We measured these four varieties of the excess kurtosis of q_{ren} for all our ensembles; the results are given in Table VII. The 7 stout, flow 0.21 fm and flow 0.30 fm results are given as a comma separated list in each cell. These results are also displayed in Fig. 6; the data seem to approach a well-defined continuum limit. Upon dropping the coarsest (and for the flow 0.30 fm strategy also the finest) lattice spacing, we get good fits with a single constant for each smearing strategy. The continuum limit in the fixed physical volume $V = (2.4783r_0)^4$ is found to be

$$\langle q_{\text{ren}}^4 \rangle / \langle q_{\text{ren}}^2 \rangle^2 - 3 = 0.1150(76)_{7 \text{ stout}}, 0.1232(91)_{\text{flow}0.21 \text{ fm}}, 0.1235(90)_{\text{flow}0.30 \text{ fm}}, \quad (16)$$

$$\langle q_{\text{ren}}^4 \rangle / \langle q_{\text{ren}}^2 \rangle - 3 \langle q_{\text{ren}}^2 \rangle = 0.259(17)_{7 \text{ stout}}, 0.278(21)_{\text{flow}0.21 \text{ fm}}, 0.276(20)_{\text{flow}0.30 \text{ fm}}, \quad (17)$$

$$\langle q_{\text{ren}}^4 \rangle - 3 \langle q_{\text{ren}}^2 \rangle^2 = 0.578(39)_{7 \text{ stout}}, 0.622(46)_{\text{flow}0.21 \text{ fm}}, 0.614(45)_{\text{flow}0.30 \text{ fm}} \quad (18)$$

TABLE VII. Ensemble average and statistical error of the excess kurtosis varieties on all ensembles in a physical volume $V = (2.4783r_0)^4$. Results for the 7 stout, flow 0.21 fm and flow 0.30 fm smoothing strategies are comma separated in each cell. The pertinent β are the same as in Table IV.

L/a	$\langle q_{\text{ren}}^4 \rangle / \langle q_{\text{ren}}^2 \rangle^2 - 3$	$\langle q_{\text{ren}}^4 \rangle / \langle q_{\text{ren}}^2 \rangle - 3 \langle q_{\text{ren}}^2 \rangle$	$\langle q_{\text{ren}}^4 \rangle - 3 \langle q_{\text{ren}}^2 \rangle^2$
12	0.135(20), 0.088(18), 0.129(18)	0.331(48), 0.209(43), 0.320(45)	0.81(12), 0.50(10), 0.80(11)
14	0.105(14), 0.111(15), 0.131(15)	0.249(33), 0.262(34), 0.310(35)	0.590(78), 0.620(82), 0.733(84)
16	0.098(29), 0.138(21), 0.116(20)	0.221(65), 0.317(49), 0.261(44)	0.50(15), 0.73(11), 0.587(100)
18	0.120(12), 0.093(24), 0.130(25)	0.265(26), 0.205(52), 0.287(56)	0.586(57), 0.45(12), 0.64(12)
20	0.127(22), 0.148(24), 0.131(19)	0.277(48), 0.320(52), 0.283(41)	0.61(10), 0.69(11), 0.612(90)
24	0.131(34), 0.136(35), 0.075(32)	0.273(71), 0.283(73), 0.155(67)	0.57(15), 0.59(15), 0.32(14)
28	0.112(47), 0.157(36), 0.249(48)	0.229(96), 0.318(74), 0.51(10)	0.47(20), 0.64(15), 1.03(21)

and we recall that these figures (or the panels in Fig. 6) are vertically correlated but not horizontally.

Regardless which definition of the excess kurtosis is chosen, it is striking to the eye that the horizontal bands of the three smoothing strategies in Fig. 6 agree within errors. This is consistent with what we saw for the second moment

(compare Table VI). Note that the statement in Refs. [30,31] that the universal cutoff $\mu = (8t)^{-1/2}$ of the gradient-flow smoothing strategy has no visible impact on $\langle q^2 \rangle$ applies to $\langle q^4 \rangle$, too. In fact, any moment of the global topological charge distribution (8), and thus the distribution itself, is supposed to enjoy a negligible

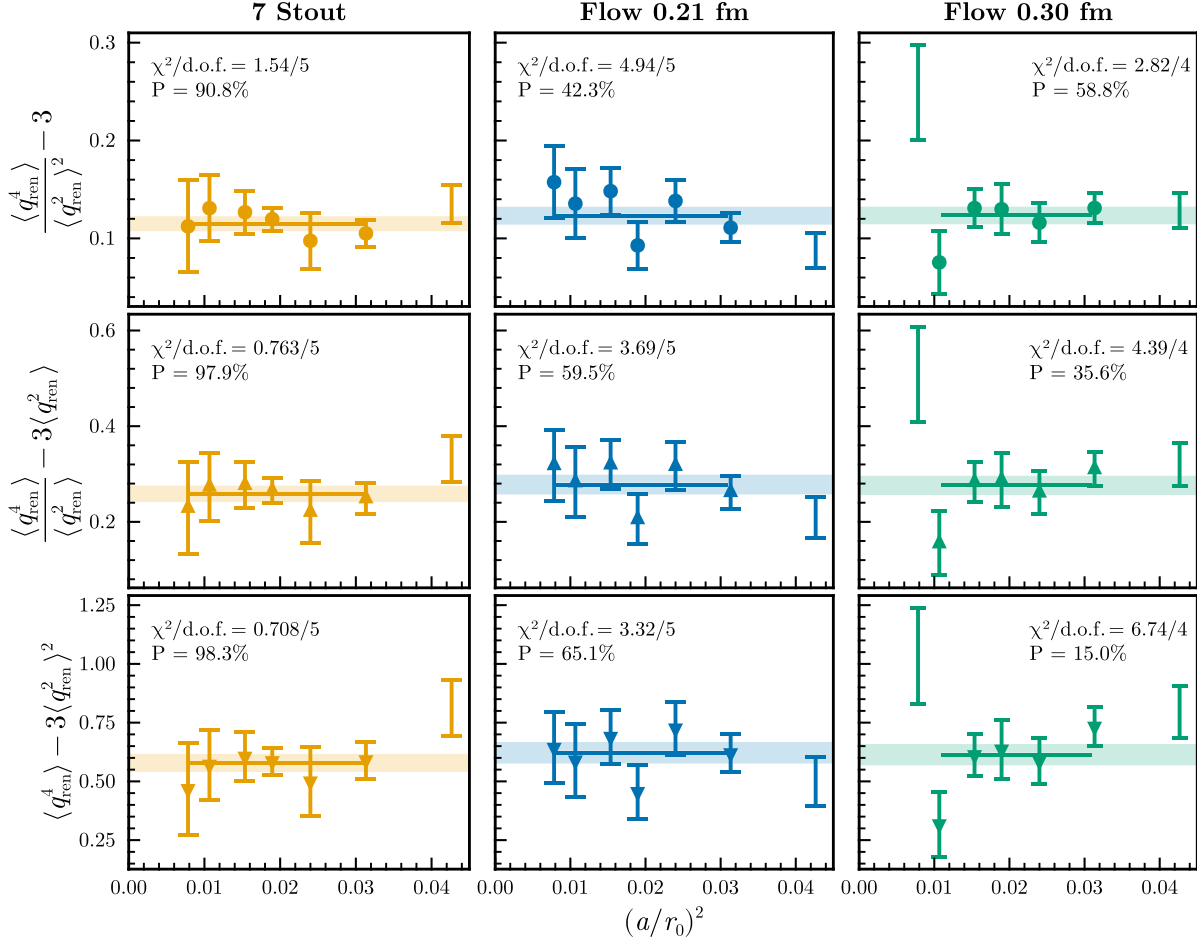


FIG. 6. Continuum extrapolation of three varieties of the excess kurtosis (top, middle, bottom) for our three smearing strategies (7 stout, left; flow 0.21 fm, middle; flow 0.30 fm, right).

systematic bias, provided $(8t)^{-1/2} \ll 4\pi F_\pi$ holds true [24,25]. Our flow-scale choices $\mu = (0.21 \text{ fm})^{-1} \simeq 940 \text{ MeV}$ and $\mu = (0.30 \text{ fm})^{-1} \simeq 660 \text{ MeV}$ seem to satisfy this criterion.

Last but not least we remind the reader that the final results of both this section and the previous one are subject to potential finite-volume artifacts, which we shall address next.

VI. INFINITE-VOLUME EXTRAPOLATIONS

To get rid of potential finite-volume effects in the final results of Secs. IV and V we need data with different box sizes L . In a gapped theory these finite-volume effects scale asymptotically with L like [44]

$$\chi_{\text{top}}(L) = \chi_{\text{top}}(\infty) \cdot [1 + \text{const} \cdot e^{-M_G L}], \quad (19)$$

where M_G is, in our case, the mass of the lightest glueball that couples to $q(x)$.

We select our intermediate coupling ($\beta = 6.1912$, 18^4 lattice) in the continuum series and augment it with smaller or larger boxes as indicated in Table VIII. We do fewer updates between adjacent measurements [to establish $\tau_{\text{int}}(q_{\text{ren}}^2) \simeq 1$] and (over)compensate this by a larger number of measurements. Since potential finite-volume effects originate in the IR physics of the original (unsmeared) gauge configuration, it is sufficient to do this for one β and one smoothing strategy (we select 7 stout). We expect that the factor Z_q in the definition (6) of q_{ren} is essentially independent of L/r_0 , unless the box volume is so small that deconfinement effects are present. Indeed, the last column of Table VIII confirms this expectation, but the errors grow with the volume. Therefore we follow Ref. [16] and use, in the analysis below, the value $Z_q = 1.17499(62)$ of the 18^4 box for all volumes.

With these preparatory steps taken, we determine the second and fourth moments⁹ of the q_{ren} distribution on the ensembles mentioned. For the latter moment we measure the same three varieties of the excess kurtosis that were studied in Sec. V. The results are listed in Table IX.

The first data column of this table is easy to interpret. The quantity $\langle q_{\text{ren}}^2 \rangle$ grows linearly with the box volume, so $\chi_{\text{top}} = \langle q_{\text{ren}}^2 \rangle / V$ assumes a finite value in the limit $V \rightarrow \infty$, in agreement with theory [3]. In Fig. 7 we illustrate this by plotting the topological susceptibility versus $1/L$ and $1/V$. The standard volume used in Sec. IV (indicated by a dashed vertical line) is almost large enough to avoid finite-size effects (within the statistical precision that we have). Still, fitting the *Ansatz* (19) to our data with

⁹We checked that the first and third moments $\langle q_{\text{ren}} \rangle$ and $\langle q_{\text{ren}}^3 \rangle$ are zero within errors.

$L/a \geq 14$ suggests¹⁰ that there is a factor 1.0048(08) between $\chi_{\text{top}}(L \simeq 2.48r_0)$ and $\chi_{\text{top}}(L = \infty)$. Multiplying the last entry in Table VI with a quarter root of this factor one finds

$$\begin{aligned} [\chi_{\text{top}}^{1/4} r_0]_{0.30 \text{ fm}} &= 0.4775(14)_{\text{stat}}(11)_{\text{syst}} \\ &= 0.4775(18)_{\text{tot}} \end{aligned} \quad (20)$$

for the topological susceptibility with all systematics included (the previous uncertainty and the one from the finite-volume correction factor were added in quadrature). In physical units we obtain

$$\chi_{\text{top}}^{1/4} = \frac{0.4775(18)}{0.4757(64) \text{ fm}} = 198.1(0.7)(2.7) \text{ MeV}, \quad (21)$$

where the first parentheses reflect the total uncertainty of our calculation (with statistics and systematics added in quadrature) and the second one the uncertainty of r_0 from Ref. [45]. The central value and the 0.4% uncertainty of $\chi_{\text{top}}^{1/4} r_0$ in (20) will be compared to the literature in Sec. VIII.

The last three columns of Table IX are not so easy to interpret. The $\langle q_{\text{ren}}^4 \rangle / \langle q_{\text{ren}}^2 \rangle^2 - 3$ variety of the excess kurtosis definitely decreases with the box volume, the $\langle q_{\text{ren}}^4 \rangle / \langle q_{\text{ren}}^2 \rangle - 3 \langle q_{\text{ren}}^2 \rangle$ variety seems somewhat undecided, and the $\langle q_{\text{ren}}^4 \rangle - 3 \langle q_{\text{ren}}^2 \rangle^2$ variety likely increases toward the largest volumes. Hence, we include a factor $(L/r_0)^4$ in the first case, a factor 1 in the second case, and a factor $(r_0/L)^4$ in the third case to plot them in Fig. 8 (left panels). It is striking to see that—with these volume factors included—the data for the three varieties resemble each other, except for a change in the vertical scale. For each variety we get a reasonable fit to a constant, if we include the four largest box volumes, and the resulting value is consistent with other works with similar volumes [30,41].

However, from Fig. 8 (left) it is not obvious that the large-volume asymptotic of these quantities is a finite constant. In the Appendix we will present an auxiliary study (at coarser lattice spacings) that suggests that one must include a factor $(L/r_0)^2$, $(r_0/L)^2$ and $(r_0/L)^6$, respectively, in order to obtain a finite value in the infinite-volume limit. The right panels of Fig. 8 give a preview how this hypothesis works on the present data. Again, the three excess kurtosis varieties are found to look similar to each other, except for a change in the overall scale. Each panel includes a fit of the largest four volumes to a constant, which (by the P value) is worse than the respective fit in the left panel.

¹⁰As a side result, this fit yields $M_G a = 0.903(38)$ for the lightest 0^{-+} glueball mass. Our result $M_G r_0 = 6.56(28)$ or $M_G = 2720(120)(40) \text{ MeV}$ is compatible with $M_G r_0 = 5.276(45) \times 1.160(6) = 6.12(6)$ in Ref. [20].

TABLE VIII. Details of the ensembles used in the infinite-volume extrapolation. The format is the same as in Tables II and III, except that this time we restrict ourselves to the 7 stout strategy.

L/a	β	$n_{\text{meas}}[n_{\text{sepa}}]$	$\tau_{\text{int}}(q_{\text{ren}}^2)$	Z_q (7 stout)
12	6.1912	200000[50]	1.117(17)	1.18218(64)
14	6.1912	200000[50]	1.133(18)	1.17604(35)
16	6.1912	212789[50]	1.036(19)	1.17378(52)
18	6.1912	310191[81]	0.730(20)	1.17499(62)
20	6.1912	204997[50]	1.010(27)	1.17282(39)
24	6.1912	324894[50]	1.045(37)	1.17292(58)
28	6.1912	246512[50]	1.056(51)	1.1717(16)

TABLE IX. Ensemble average and statistical error of the squared topological charge and the three kurtosis varieties for the ensembles listed in Table VIII. Throughout, the 7 stout strategy is used.

L/a	β	$\langle q_{\text{ren}}^2 \rangle$	$\langle q_{\text{ren}}^4 \rangle$	$\langle q_{\text{ren}}^4 \rangle / \langle q_{\text{ren}}^2 \rangle^2 - 3$	$\langle q_{\text{ren}}^4 \rangle / \langle q_{\text{ren}}^2 \rangle - 3 \langle q_{\text{ren}}^2 \rangle$	$\langle q_{\text{ren}}^4 \rangle - 3 \langle q_{\text{ren}}^2 \rangle^2$
12	6.1912	0.1911(17)	0.2923(49)	4.998(79)	0.956(14)	0.1827(34)
14	6.1912	0.6689(38)	1.744(22)	0.897(26)	0.600(18)	0.401(12)
16	6.1912	1.3482(63)	5.884(61)	0.238(15)	0.320(20)	0.432(27)
18	6.1912	2.2154(81)	15.31(13)	0.120(12)	0.265(26)	0.586(57)
20	6.1912	3.394(15)	35.17(35)	0.053(13)	0.181(43)	0.61(14)
24	6.1912	7.019(25)	150.0(1.2)	0.045(11)	0.318(74)	2.24(52)
28	6.1912	13.050(53)	511.6(4.5)	0.005(11)	0.06(14)	0.8(1.8)

For now we can only conclude that the data in Table IX are insufficient to decide on the scaling exponent α in the asymptotic large-volume behavior $\propto (L/r_0)^\alpha$ of the quantities $\langle q_{\text{ren}}^4 \rangle / \langle q_{\text{ren}}^2 \rangle^2 - 3$, $\langle q_{\text{ren}}^4 \rangle / \langle q_{\text{ren}}^2 \rangle - 3 \langle q_{\text{ren}}^2 \rangle$, and $\langle q_{\text{ren}}^4 \rangle - 3 \langle q_{\text{ren}}^2 \rangle^2$. This is why we decided to generate another dataset to investigate the situation in more detail

(see the Appendix). As we shall see, these new data suggest the scaling exponents $\alpha = -2$, 2 and 6 for these excess kurtosis varieties. In retrospect we will say that the volumes considered in the present section are large enough to reach definite conclusions for the large-volume behavior of $\langle q_{\text{ren}}^2 \rangle$ but not for the three excess kurtosis varieties.

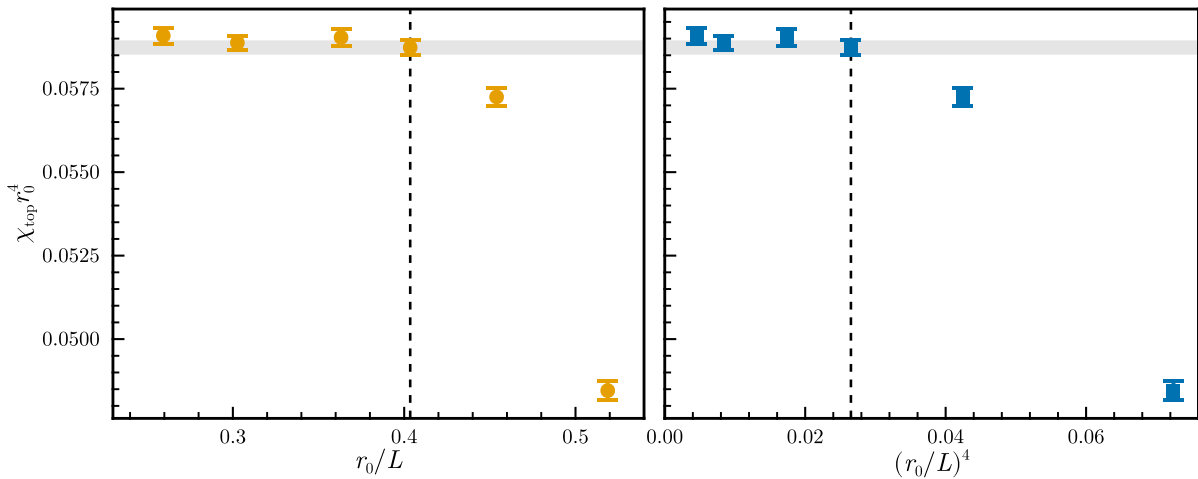


FIG. 7. Volume scaling of the topological susceptibility plotted versus $1/L$ (left) and $1/V$ (right). The standard inverse size or volume used in Sec. IV is marked with a dashed vertical line. The runs of Ref. [16] were performed with a box size $\sim 10\%$ smaller, which coincides roughly with the fifth data point. Our seventh data point is out of scale (both horizontally and vertically).

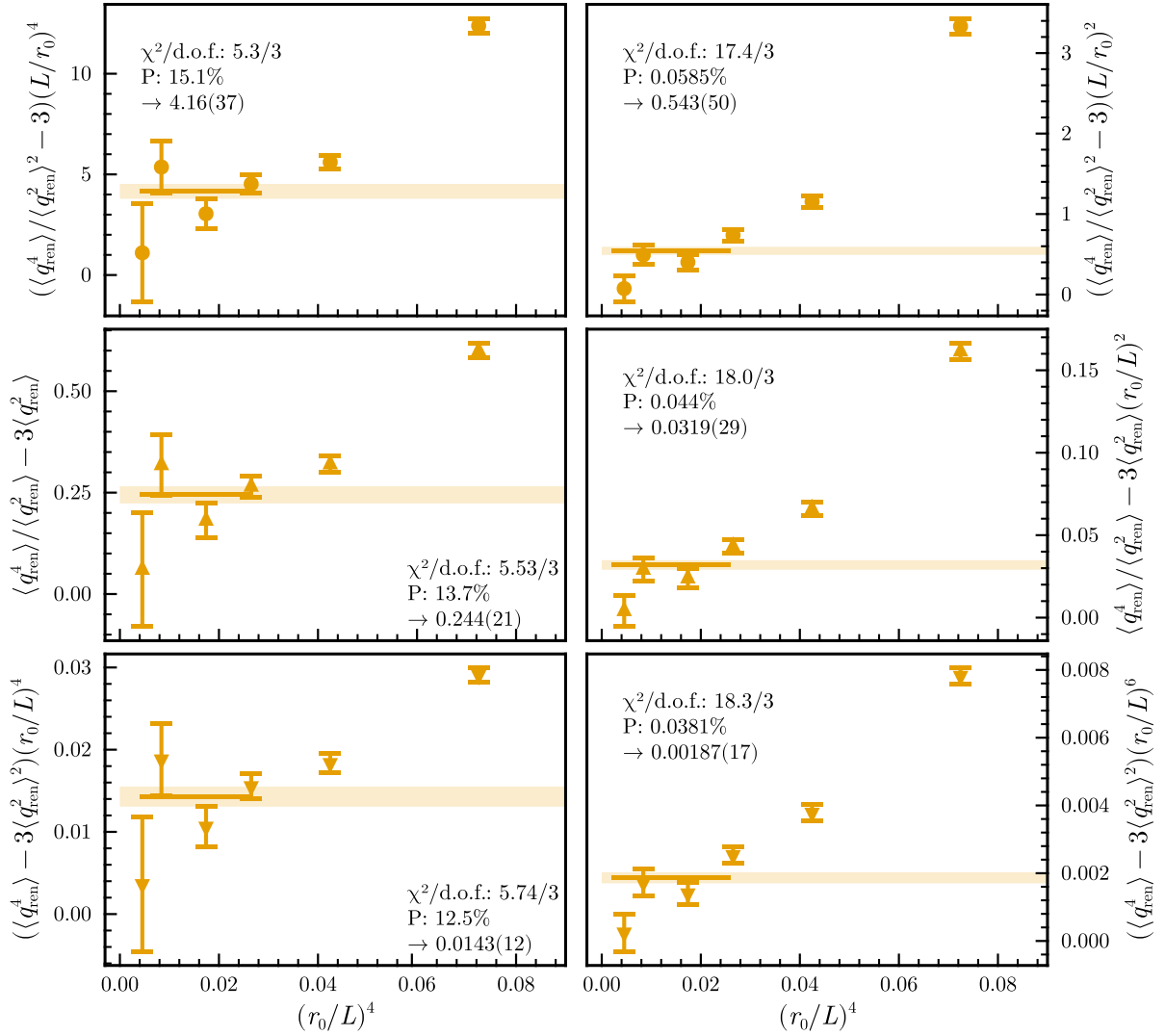


FIG. 8. Large-volume behavior of the kurtosis varieties (7 stout smoothing). In the left panel one variety is multiplied with $(L/r_0)^4$, one unchanged, one divided by $(L/r_0)^4$ (top to bottom). In the right panel the factors are $(L/r_0)^2$, $(r_0/L)^2$ and $(r_0/L)^6$. The seventh data point is out of scale.

VII. DISCUSSION

Our final result for $\chi_{\text{top}}^{1/4} r_0$ was given in Eqs. (20) and (21). In Table X this quantity and its fourth power are compared to results in the literature, under the proviso that they come with a continuum extrapolation. Except for Refs. [20,21] all these papers effectively compute $\chi_{\text{top}}^{1/4} r_0$ or $\chi_{\text{top}} r_0^4$. This is why we ignore their final quote in physical units (if given) and convert, in the last column, all results with the same factor to MeV units. This factor $r_0^{-1} = 414.8(5.6)$ MeV is taken from Ref. [45]; its error bar is reflected by the last parentheses (which dominates the final uncertainty).

Considering the $\chi_{\text{top}}^{1/4} r_0$ column, there is some tension (2.9 and 2.4 combined standard deviations) with Refs. [21,47], reasonable agreement (1.2, 1.2, 1.6 and 1.4 combined sigmas) with Refs. [20,30,46,48], and perfect

agreement (0.2, 0.9, 0.5, 0.3 and 0.7 sigmas) with Refs. [31,41,7,16,49]. This is what one expects if every paper assesses its uncertainty correctly; see Fig. 9 for an overview. We think that we follow this tradition; we have seven lattice spacings available, down to $a \simeq 0.042$ fm, and we use a large variety of fit functions and cuts to determine our systematic uncertainty.

Regarding the excess kurtosis, the situation is different. The continuum limit in a fixed volume is benevolent (see Fig. 6), but the large-volume scaling behavior is not. We agree with Refs. [30,41] on the value in a volume $V \simeq (2.5r_0)^4$, but we cannot confirm their statement that $\langle q^4 \rangle / \langle q^2 \rangle - 3\langle q^2 \rangle$ has a finite value in the limit $L \rightarrow \infty$. Our data in Sec. VI are inconclusive on this point, and the auxiliary data in the Appendix suggest that the latter quantity scales like L^2 .

TABLE X. Summary of χ_{top} values in SU(3) gauge theory with continuum extrapolation, in units of r_0^{-4} . The results $\chi_{\text{top}}^{1/4}/\sqrt{\sigma} = 0.4187(53)$ [20] and $\chi_{\text{top}}^{1/4}/\sqrt{\sigma} = 0.4246(36)$ [21] are converted via $r_0\sqrt{\sigma} = 1.160(6)$. The values in the $\chi_{\text{top}}^{1/4}r_0$ column are converted to MeV by means of $r_0 = 0.4757(64)$ fm [45].

		$\chi_{\text{top}}^{1/4}r_0$	$\chi_{\text{top}}r_0^4$	$\chi_{\text{top}}^{1/4}$ [MeV]
This work	(2025)	0.4775(18)	0.05199(78)	198.1(0.7)(2.7)
Ref. [31]	(2023)	0.4794(86)	0.0528(38)	198.9(3.6)(2.7)
Ref. [21]	(2021)	0.4926(48)	0.0589(23)	204.3(2.0)(2.8)
Ref. [20]	(2020)	0.4857(67)	0.0557(31)	201.5(2.8)(2.7)
Ref. [41]	(2015)	0.4708(72)	0.0491(30)	195.3(3.0)(2.7)
Ref. [30]	(2015)	0.4829(40)	0.0544(18)	200.3(1.7)(2.7)
Ref. [7]	(2015)	0.470(14)	0.049(6)	195.0(5.8)(2.6)
Ref. [46]	(2010)	0.498(13)	0.0615(64)	206.6(5.4)(2.8)
Ref. [16]	(2006)	0.4784(21)	0.05236(94)	198.4(0.9)(2.7)
Ref. [47]	(2004)	0.4928(62)	0.059(3)	204.4(2.6)(2.8)
Ref. [48]	(2003)	0.4928(104)	0.059(5)	204.4(4.3)(2.8)
Ref. [49]	(2003)	0.4821(67)	0.054(3)	200.0(3.9)(2.7)

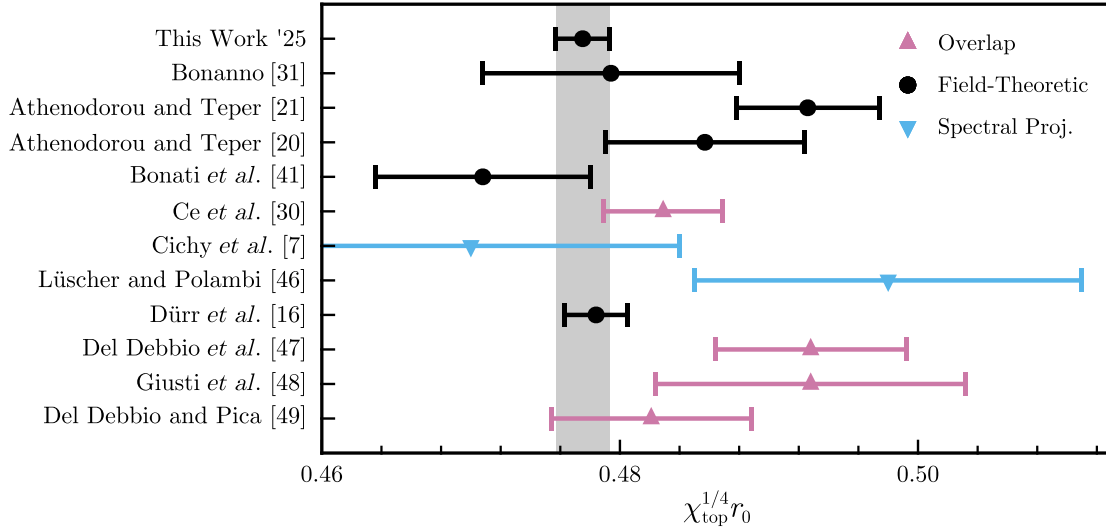


FIG. 9. Graphical display of the results for $\chi_{\text{top}}^{1/4}r_0$ as listed in the first column of Table X.

VIII. CONCLUSIONS

In many respects the present study is an update to Ref. [16] which, for many years, was the most precise determination of $\chi_{\text{top}}r_0^4$ in SU(3) pure gauge theory. The present work increases the precision but only marginally (the overall uncertainty decreases from 1.8% to 1.5%). Our study uses exactly the same methodology as Refs. [13–21] (and likely many more), with a smoothed-link gluonic definition of the global topological charge, and a careful continuum extrapolation to get rid of the $O(a^2)$ lattice artifacts. In addition, an extensive finite-volume scaling study was performed.

The point where we differ from previous investigations is that we compare different smoothing strategies in the definition of the topological charge operator. Our 7 stout strategy implements the traditional choice where the

smoothing radius $\sqrt{8t}$ is held fixed in lattice units (in our case at $2.59a$). By contrast, our 0.21 and 0.30 fm strategies increase, on the way to the continuum, the smoothing radius $\sqrt{8t}$ in lattice units so that it stays fixed in physical units (at the distance indicated by the name). In line with theoretical arguments [30,31] our data support the idea that there is a joint continuum limit under the auspices of either strategy.

Evidently, the research reported here is a snapshot in the broader context of lattice gluodynamics. It would be interesting to follow Refs. [14,15,21,50] and vary the number of colors beyond $N_c = 3$. Also quantities with nonzero virtuality would be interesting, such as $\chi'_{\text{top}}(p^2)|_{p^2=0}$ [15,31,41,50] or the glueball mass extracted from the long-distance behavior of $q(x)q(0)$. On the technical level, perhaps an improved field-strength definition as advocated in Ref. [51] might be a useful addition.

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DATA AVAILABILITY

Most of the data on which the conclusions of this article rely are contained in the tables and figures of the article. Primary data are available as Supplemental Material to this article at [52].

APPENDIX: LARGE-VOLUME SCALING TEST WITH ANOTHER DATASET

In an attempt to improve on the large-volume scaling tests presented in Sec. VI, we decided to generate another series of lattices which would reach toward larger physical volumes. For the intermediate coupling $\beta = 6.1912$, as used in that section, we exhausted our computational resources. Hence, another attempt must use a lower β (stronger coupling).

We select $\beta = 5.9421$ for which there is a series of 100 000 lattices in a 12^4 box available. We complement this with $L/a = 10, 14, 16, 18, 20, 22$, thus spanning the range $2.07 \leq L/r_0 \leq 4.54$, as opposed to $1.65 \leq L/r_0 \leq 3.86$ in Sec. VI. The details of these “postproduction ensembles” are listed in Table XI. A technical point is that the statistical precision of Z_q , as listed in this table, degrades for large volumes. In line with Ref. [16] and our procedure in Sec. VI we use, in the analysis below, the value $Z_q = 1.2757(18)$ of the 12^4 box for all volumes.

On these ensembles we measure the topological susceptibility $\langle q_{\text{ren}}^2 \rangle / V$ and the same excess kurtosis varieties as in Sec. VI. Again we limit ourselves to the 7 stout smoothing strategy. The results are listed in Table XII and shown in Fig. 10. For comparison the old data ($\beta = 6.1912$) are included. For the topological susceptibility some mild discretization effects are visible, while for the fourth-order cumulants they seem to be negligible at the current level of statistical precision.

We apply power-law fits $c(L/r_0)^\alpha$ to the observables $\langle q_{\text{ren}}^4 \rangle / \langle q_{\text{ren}}^2 \rangle^2 - 3$, $\langle q_{\text{ren}}^4 \rangle / \langle q_{\text{ren}}^2 \rangle - 3\langle q_{\text{ren}}^2 \rangle$ and $\langle q_{\text{ren}}^4 \rangle - 3\langle q_{\text{ren}}^2 \rangle^2$. With the five largest volumes included, we find $\alpha = -1.85(53)$, $2.13(52)$ and $6.02(52)$, respectively, with $P \simeq 0.1$ (in all three cases). And the prefactors c are $0.36(23)$, $0.024(15)$ and $0.0018(11)$, respectively, for these excess kurtosis varieties.

Note that the fitted power α of the first excess kurtosis variety is negative by almost 4σ . Hence, if $\langle q_{\text{ren}}^4 \rangle / \langle q_{\text{ren}}^2 \rangle^2 - 3$ is used to quantify the deviation from a normal distribution, our fit suggests that any non-Gaussian shape of the topological charge histogram is a finite-volume effect.

TABLE XI. Details of the postproduction ensembles; the format is the same as in Table VIII.

L/a	β	$n_{\text{meas}}[n_{\text{sepa}}]$	$\tau_{\text{int}}(q_{\text{ren}}^2)$	Z_q (7 stout)
10	5.9421	281979[10]	0.7039(76)	1.28139(96)
12	5.9421	100000[10]	0.653(11)	1.2757(18)
14	5.9421	535206[10]	0.6533(48)	1.2757(79)
16	5.9421	846551[10]	0.6485(67)	1.279(59)
18	5.9421	420704[10]	0.6481(76)	1.13(32)
20	5.9421	707643[10]	0.6465(90)	1.201(12)
22	5.9421	809898[10]	0.6369(93)	1.27(62)

In addition, it is worth pointing out that the fitted powers α are deceptively close¹¹ to the integer values -2 , $+2$ and $+6$, respectively. In view of the well-known volume scaling law [3]

$$\langle q_{\text{ren}}^2 \rangle \propto L^4, \quad (\text{A1})$$

for the second cumulant, the fits suggest the scaling laws

$$\begin{aligned} \langle q_{\text{ren}}^4 \rangle / \langle q_{\text{ren}}^2 \rangle^2 - 3 &\propto L^{-2}, \\ \langle q_{\text{ren}}^4 \rangle / \langle q_{\text{ren}}^2 \rangle - 3\langle q_{\text{ren}}^2 \rangle &\propto L^2, \\ \langle q_{\text{ren}}^4 \rangle - 3\langle q_{\text{ren}}^2 \rangle^2 &\propto L^6 \end{aligned} \quad (\text{A2})$$

for the three excess kurtosis varieties. Hence the first relation stipulates that $\langle q_{\text{ren}}^4 \rangle / \langle q_{\text{ren}}^2 \rangle^2$ tends, in the infinite-volume limit, to the value 3 in such a way that the difference is asymptotically suppressed by two powers of L . Similarly, the second relation says that $\langle q_{\text{ren}}^4 \rangle / \langle q_{\text{ren}}^2 \rangle$ and $3\langle q_{\text{ren}}^2 \rangle$ both grow like L^4 , but the difference grows only like L^2 . And the third relation suggests that $\langle q_{\text{ren}}^4 \rangle$ and $3\langle q_{\text{ren}}^2 \rangle^2$ individually grow like L^8 , while the difference stays behind by two powers of L .

In the event the conjectures (A2) turn out to be correct, the quantities

$$\begin{aligned} [\langle q_{\text{ren}}^4 \rangle / \langle q_{\text{ren}}^2 \rangle^2 - 3] \frac{L^2}{r_0^2}, \quad & [\langle q_{\text{ren}}^4 \rangle / \langle q_{\text{ren}}^2 \rangle - 3\langle q_{\text{ren}}^2 \rangle] \frac{r_0^2}{L^2}, \\ [\langle q_{\text{ren}}^4 \rangle - 3\langle q_{\text{ren}}^2 \rangle^2] \frac{r_0^6}{L^6} \end{aligned} \quad (\text{A3})$$

assume universal (finite) values in the combined $a \rightarrow 0$, $L \rightarrow \infty$ limit. Obviously, in these equations r_0 [2] may be replaced by another suitable distance, e.g. $t_0^{1/2}$ [24,25] or w_0 [27]. Our results for the respective constants c were mentioned in the text ahead of Eq. (A1).

¹¹These findings are hard to reconcile with the standard view that $\langle q^4 \rangle / \langle q^2 \rangle^2 - 3\langle q^2 \rangle$ tends to a constant value in the $V \rightarrow \infty$ limit [15,30,41,50,53]. Interestingly, our value (17) is in good agreement with the values 0.233(45) and 0.259(19) obtained in Refs. [30,41] in a similar volume $V \simeq (2.5r_0)^4$. To the best of our knowledge, the volumes considered in this appendix are larger than those of other works, but there is no guarantee that we see the true asymptotic behavior.

TABLE XII. Ensemble average and statistical error of the squared topological charge and the three kurtosis definitions for the ensembles listed in Table XI. Throughout, the 7 stout strategy is used.

L/a	β	$\langle q_{\text{ren}}^2 \rangle$	$\langle q_{\text{ren}}^4 \rangle$	$\langle q_{\text{ren}}^4 \rangle / \langle q_{\text{ren}}^2 \rangle^2 - 3$	$\langle q_{\text{ren}}^4 \rangle / \langle q_{\text{ren}}^2 \rangle - 3 \langle q_{\text{ren}}^2 \rangle$	$\langle q_{\text{ren}}^4 \rangle - 3 \langle q_{\text{ren}}^2 \rangle^2$
10	5.9421	1.1095(38)	4.167(34)	0.385(14)	0.427(15)	0.474(17)
12	5.9421	2.450(13)	18.81(23)	0.135(20)	0.331(48)	0.81(12)
14	5.9421	4.546(10)	63.13(32)	0.0550(74)	0.250(34)	1.14(15)
16	5.9421	7.671(13)	178.06(70)	0.0261(55)	0.201(42)	1.54(33)
18	5.9421	12.257(31)	456.7(2.7)	0.0408(84)	0.50(10)	6.1(1.3)
20	5.9421	18.557(36)	1041.1(4.5)	0.0221(61)	0.41(11)	7.6(2.1)
22	5.9421	27.251(49)	2244.6(9.1)	0.0237(57)	0.65(15)	17.5(4.2)

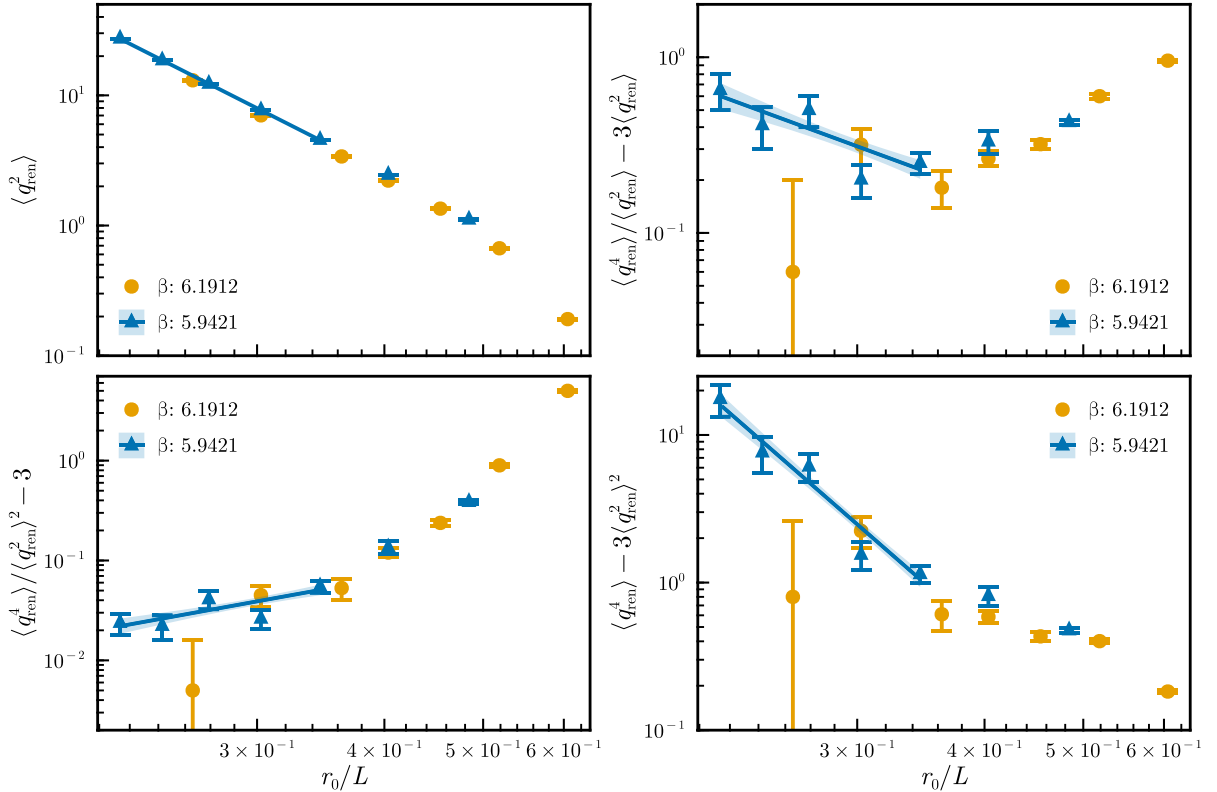


FIG. 10. Large-volume scaling of the topological susceptibility (top left) and of the three kurtosis varieties (remaining panels) with the 7 stout strategy. The new data ($\beta = 5.9421$, blue triangles) show mild cutoff effects relative to the old ones ($\beta = 6.1912$, orange circles), but they extend to larger box sizes. The power-law fits based on the five largest volumes suggest the scaling laws (A2).

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- [1] S. Coleman, *Aspects of Symmetry: Selected Erice Lectures* (Cambridge University Press, Cambridge, England, 1985).
 - [2] R. Sommer, A new way to set the energy scale in lattice gauge theories and its applications to the static force and α_s in SU(2) Yang-Mills theory, *Nucl. Phys.* **B411**, 839 (1994).
 - [3] H. Leutwyler and A. V. Smilga, Spectrum of Dirac operator and role of winding number in QCD, *Phys. Rev. D* **46**, 5607 (1992).
 - [4] S. Durr, Topological susceptibility in full QCD: Lattice results versus the prediction from the QCD partition function with granularity, *Nucl. Phys.* **B611**, 281 (2001).

- [5] E. Witten, Current algebra theorems for the U(1) Goldstone boson, *Nucl. Phys.* **B156**, 269 (1979).
- [6] G. Veneziano, U(1) without instantons, *Nucl. Phys.* **B159**, 213 (1979).
- [7] K. Cichy, E. Garcia-Ramos, K. Jansen, K. Ottnad, and C. Urbach (ETM Collaboration), Non-perturbative test of the Witten-Veneziano formula from lattice QCD, *J. High Energy Phys.* **09** (2015) 020.
- [8] Y. Aoki *et al.* (Flavour Lattice Averaging Group (FLAG)), FLAG review 2021, *Eur. Phys. J. C* **82**, 869 (2022).
- [9] E. Seiler and I. O. Stamatescu, Some remarks on the Witten-Veneziano formula for the eta-prime mass, MPI-PAE-PTH-10-87.
- [10] M. Campostrini, A. Di Giacomo, H. Panagopoulos, and E. Vicari, Topological charge, renormalization and cooling on the lattice, *Nucl. Phys.* **B329**, 683 (1990).
- [11] A. Di Giacomo and E. Vicari, Renormalization and topological susceptibility on the lattice, *Phys. Lett. B* **275**, 429 (1992).
- [12] B. Alles, M. D’Elia, A. Di Giacomo, and R. Kirchner, A critical comparison of different definitions of topological charge on the lattice, *Phys. Rev. D* **58**, 114506 (1998).
- [13] J. Hoek, M. Teper, and J. Waterhouse, Topological fluctuations and susceptibility in SU(3) lattice gauge theory, *Nucl. Phys.* **B288**, 589 (1987).
- [14] B. Lucini and M. Teper, SU(N) gauge theories in four-dimensions: Exploring the approach to $N = \infty$, *J. High Energy Phys.* **06** (2001) 050.
- [15] L. Del Debbio, H. Panagopoulos, and E. Vicari, theta dependence of SU(N) gauge theories, *J. High Energy Phys.* **08** (2002) 044.
- [16] S. Durr, Z. Fodor, C. Hoelbling, and T. Kurth, Precision study of the SU(3) topological susceptibility in the continuum, *J. High Energy Phys.* **04** (2007) 055.
- [17] S. Borsanyi, M. Dierigl, Z. Fodor, S. D. Katz, S. W. Mages, D. Nogradi, J. Redondo, A. Ringwald, and K. K. Szabo, Axion cosmology, lattice QCD and the dilute instanton gas, *Phys. Lett. B* **752**, 175 (2016).
- [18] C. Bonati and M. D’Elia, Comparison of the gradient flow with cooling in SU(3) pure gauge theory, *Phys. Rev. D* **89**, 105005 (2014).
- [19] C. Alexandrou, A. Athenodorou, K. Cichy, A. Dromard, E. Garcia-Ramos, K. Jansen, U. Wenger, and F. Zimmermann, Comparison of topological charge definitions in lattice QCD, *Eur. Phys. J. C* **80**, 424 (2020).
- [20] A. Athenodorou and M. Teper, The glueball spectrum of SU(3) gauge theory in 3 + 1 dimensions, *J. High Energy Phys.* **11** (2020) 172.
- [21] A. Athenodorou and M. Teper, SU(N) gauge theories in 3 + 1 dimensions: Glueball spectrum, string tensions and topology, *J. High Energy Phys.* **12** (2021) 082.
- [22] C. Morningstar and M. J. Peardon, Analytic smearing of SU(3) link variables in lattice QCD, *Phys. Rev. D* **69**, 054501 (2004).
- [23] R. Narayanan and H. Neuberger, Infinite N phase transitions in continuum Wilson loop operators, *J. High Energy Phys.* **03** (2006) 064.
- [24] M. Luscher, Properties and uses of the Wilson flow in lattice QCD, *J. High Energy Phys.* **08** (2010) 071; **03** (2014) 92.
- [25] M. Luscher and P. Weisz, Perturbative analysis of the gradient flow in non-Abelian gauge theories, *J. High Energy Phys.* **02** (2011) 051.
- [26] S. Capitani, S. Durr, and C. Hoelbling, Rationale for UV-filtered clover fermions, *J. High Energy Phys.* **11** (2006) 028.
- [27] S. Borsanyi *et al.* (BMW Collaboration), High-precision scale setting in lattice QCD, *J. High Energy Phys.* **09** (2012) 010.
- [28] M. Nagatsuka, K. Sakai, and S. Sasaki, Equivalence between the Wilson flow and stout-link smearing, *Phys. Rev. D* **108**, 094506 (2023).
- [29] M. Ammer and S. Durr, Stout smearing and Wilson flow in lattice perturbation theory, *Phys. Rev. D* **110**, 054504 (2024).
- [30] M. Ce, C. Consonni, G. P. Engel, and L. Giusti, Non-Gaussianities in the topological charge distribution of the SU(3) Yang–Mills theory, *Phys. Rev. D* **92**, 074502 (2015).
- [31] C. Bonanno, The topological susceptibility slope χ' of the pure-gauge SU(3) Yang–Mills theory, *J. High Energy Phys.* **01** (2024) 116.
- [32] M. Creutz, Monte Carlo study of quantized SU(2) gauge theory, *Phys. Rev. D* **21**, 2308 (1980).
- [33] N. Cabibbo and E. Marinari, A new method for updating SU(N) matrices in computer simulations of gauge theories, *Phys. Lett.* **119B**, 387 (1982).
- [34] K. Fabricius and O. Haan, Heat bath method for the twisted Eguchi-Kawai model, *Phys. Lett.* **143B**, 459 (1984).
- [35] A. D. Kennedy and B. J. Pendleton, Improved heat bath method for Monte Carlo calculations in lattice gauge theories, *Phys. Lett.* **156B**, 393 (1985).
- [36] S. L. Adler, An overrelaxation method for the Monte Carlo evaluation of the partition function for multiquadratic actions, *Phys. Rev. D* **23**, 2901 (1981).
- [37] M. Creutz, Overrelaxation and Monte Carlo simulation, *Phys. Rev. D* **36**, 515 (1987).
- [38] F. R. Brown and T. J. Woch, Overrelaxed heat bath and Metropolis algorithms for accelerating pure gauge Monte Carlo calculations, *Phys. Rev. Lett.* **58**, 2394 (1987).
- [39] D. B. Leinweber, A. G. Williams, J. B. Zhang, and F. X. Lee, Topological charge barrier in the Markov chain of QCD, *Phys. Lett. B* **585**, 187 (2004).
- [40] S. Durr, Gauge action improvement and smearing, *Comput. Phys. Commun.* **172**, 163 (2005).
- [41] C. Bonati, M. D’Elia, and A. Scapellato, θ dependence in SU(3) Yang–Mills theory from analytic continuation, *Phys. Rev. D* **93**, 025028 (2016).
- [42] C. Alexandrou *et al.* (ETM Collaboration), Probing the energy-smear R ratio using lattice QCD, *Phys. Rev. Lett.* **130**, 241901 (2023).
- [43] C. Bonanno, F. D’Angelo, and M. D’Elia, The chiral condensate of $N_f = 2 + 1$ QCD from the spectrum of the staggered Dirac operator, *J. High Energy Phys.* **11** (2023) 013.

- [44] M. Luscher, Volume dependence of the energy spectrum in massive quantum field theories. I. Stable particle states, *Commun. Math. Phys.* **104**, 177 (1986).
- [45] T. M. B. Asmussen, R. Hollwieser, F. Knechtli, and T. Korzec, The determination of potential scales in $2 + 1$ flavor QCD, *Eur. Phys. J. C* **85**, 673 (2025).
- [46] M. Luscher and F. Palombi, Universality of the topological susceptibility in the SU(3) gauge theory, *J. High Energy Phys.* **09** (2010) 110.
- [47] L. Del Debbio, L. Giusti, and C. Pica, Topological susceptibility in the SU(3) gauge theory, *Phys. Rev. Lett.* **94**, 032003 (2005).
- [48] L. Giusti, M. Luscher, P. Weisz, and H. Wittig, Lattice QCD in the epsilon regime and random matrix theory, *J. High Energy Phys.* **11** (2003) 023.
- [49] L. Del Debbio and C. Pica, Topological susceptibility from the overlap, *J. High Energy Phys.* **02** (2004) 003.
- [50] E. Vicari and H. Panagopoulos, Theta dependence of SU(N) gauge theories in the presence of a topological term, *Phys. Rep.* **470**, 93 (2009).
- [51] S. O. Bilson-Thompson, D. B. Leinweber, and A. G. Williams, Highly improved lattice field strength tensor, *Ann. Phys. (Amsterdam)* **304**, 1 (2003).
- [52] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/dw3m-vcdm> for raw topological charge measurements.
- [53] C. Bonanno, C. Bonati, and M. D'Elia, Strong CP problem, theta term and QCD topological properties, [arXiv:2510.03059](https://arxiv.org/abs/2510.03059).