

Top-quark decay at next-to-next-to-next-to-leading order in QCD

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We present the first complete QCD corrections to the inclusive decay width Γ_t , the W -helicity fractions $f_{L,R,0}$ and semi-inclusive distributions for the top-quark decay process $t \rightarrow b + W^+ + X_{\text{QCD}}$ at next-to-next-to-next-to-leading order (NNNLO) in the strong coupling constant α_s . In particular, the pure NNNLO QCD correction decreases Γ_t by about 0.8% relative to the previous next-to-next-to-leading order (NNLO) result at the scale of the top-quark pole mass, exceeding the uncertainty estimated by the conventional scale-variation prescription. After taking into account errors from various sources, we get $\Gamma_t = 1.3148_{-0.005}^{+0.003} \times |V_{tb}|^2 + 0.027(m_t - 172.69)$ GeV, whose uncertainty meets the request of future colliders. On the other hand, the NNNLO QCD effects on $f_{L,R,0}$ are found to be much smaller, at the level of one one per mille for the dominant f_0 , making them precision observables for the top-quark decay process.

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I. INTRODUCTION

The top-quark t , the heaviest fundamental particle discovered to-date, plays a special role both in precision test of the Standard Model (SM), especially of the electroweak sector, and in searches for new physics. The properties of its production and decay have been actively studied at the Tevatron and the LHC, and will also be among the core physics programs at future high-energy lepton colliders [1–4].

The large t -quark mass m_t , currently measured to be 172.69 ± 0.30 GeV [5] usually interpreted as the pole mass, not only sets a sufficiently large scale to apply perturbative quantum chromodynamics (QCD), but also makes for a unique feature relevant for QCD phenomenology: the t -quark decays almost exclusively to W and bottom-quark b before it hadronizes. The current most precise measurement for the t -quark decay width Γ_t comes from the CMS [6] which gives $1.36 \pm 0.02(\text{stat.})_{-0.11}^{+0.14}(\text{syst.})$ GeV. The anticipated experimental uncertainties in the measurement of Γ_t at the future hadron and lepton colliders can be reduced to

about 20–26 MeV [2,7–9]. To fully take advantage of the data, the theoretical error should be ideally smaller than about one third of the experimental one, say less than 7 MeV. Although QCD corrections to t -quark decay have been calculated perturbatively up to next-to-next-to-leading order (NNLO) in the strong coupling α_s , e.g., in Refs. [10–23], we will see below that the theoretical error at NNLO cannot meet this request. In addition, there are concerns about the convergence of the perturbative series related to the fact that both the pole mass m_t and the decay width Γ_t are sensitive to the infrared-renormalon issue (see, e.g., Refs. [24–29]). It is thus desirable to explicitly determine the complete QCD corrections at next-to-next-to-next-to-leading order (NNNLO) in α_s , which we accomplish for the first time in this paper.

One particularly interesting phenomenon in t -quark decay is that the produced W boson is polarized even if the t -quark is unpolarized, due to the chiral structure of the weak interaction. The current measurements [30] of the fractions of longitudinal (f_0), left-handed (f_L), and right-handed (f_R) polarization states of W boson by ATLAS are $f_0 = 0.684 \pm 0.005(\text{stat.}) \pm 0.014(\text{syst.})$, $f_L = 0.318 \pm 0.003(\text{stat.}) \pm 0.008(\text{syst.})$, and $f_R = -0.002 \pm 0.002(\text{stat.}) \pm 0.014(\text{syst.})$. The current uncertainties in f_0 and f_L are dominated by the $t\bar{t}$ -production modeling, jet energy scale and also the experimental uncertainties in m_t , while for f_R the uncertainties in α_s and b -quark mass m_b matter too. All these experimental uncertainties are expected to improve significantly at future lepton colliders [1–4,31,32], and the experimental uncertainties in $f_{L,0}$ could therefore be comparable or

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smaller than the theoretical uncertainties based on NNLO QCD calculations [18–20]. It is thus also desirable to improve the accuracy of the theoretical predictions for these precision observables at higher orders in QCD.

In this Letter we formulate an efficient method to tackle this problem, which not only allows us to obtain high-precision results at one order higher in QCD but also provides us with needed for a future fully differential calculation of any infrared-safe observables for t -quark decay process at NNNLO later on.

II. THE METHOD

Aiming to obtain NNNLO QCD corrections to the semi-inclusive distributions related to the W produced in $t \rightarrow b + W^+ + X_{\text{QCD}}$, from which inclusive observables such as Γ_t and $f_{\text{L,R},0}$ can also be constructed, we write Γ_t in terms of the semi-inclusive hadronic tensor $\mathcal{W}_{tb}^{\mu\nu}$ integrated over the W momentum k as follows,

$$\Gamma_t = \frac{1}{2m_t} \int \frac{d^{d-1}k}{(2\pi)^{d-1}2E} \mathcal{W}_{tb}^{\mu\nu} \sum_{\lambda} \epsilon_{\mu}^*(k, \lambda) \epsilon_{\nu}(k, \lambda), \quad (1)$$

with fixed t -quark momentum p . For an on-shell W with mass m_W , the polarization-sum $\sum_{\lambda} \epsilon_{\mu}^*(k, \lambda) \epsilon_{\nu}(k, \lambda)$ can be reduced to $(-g^{\mu\nu} + k^{\mu}k^{\nu}/m_W^2)$. For W with a definite helicity λ , the corresponding projectors can be found in Ref. [33] and see also Ref. [34]. Anticipating potential infrared-soft and/or collinear (IR) divergence in the phase-space integration over k in the region where E reaches its maximum, we introduce the dimensional regularization (DR) with spacetime dimension denoted by $d \equiv 4 - 2\epsilon$.

$\mathcal{W}_{tb}^{\mu\nu}$ can be decomposed into five linearly independent Lorentz-tensor structures

$$\mathcal{W}_{tb}^{\mu\nu}(p, k) = W_1 g^{\mu\nu} + W_2 p^{\mu} p^{\nu} + W_3 k^{\mu} k^{\nu} + W_4 (p^{\mu} k^{\nu} + k^{\mu} p^{\nu}) + W_5 i \epsilon^{\mu\nu\rho\sigma} p_{\rho} k_{\sigma}, \quad (2)$$

where the Levi-Civita tensor $\epsilon^{\mu\nu\rho\sigma}$ appears due to the chiral structure of the weak tbW -vertex. Since the γ_5 from the tbW -vertex always appears on an open fermion chain in the contributing QCD amplitudes, γ_5 can be treated fully anticommutatively [35–38] in a straightforward manner.

Each form factor W_i is a function of m_t , m_W , and the W -energy E , as well as m_b , and receives both virtual and real-radiation QCD corrections. Loop integrals are reduced to relatively simpler master integrals using integration-by-parts (IBP) identities [39], employing the recently released package `Blade` [40], which equipped with the strategy of block-triangular form [41,42] and utilizes the `FiniteFlow` [43] package (for other reduction packages on the market, see Refs. [44–56]). The resulting master integrals are computed using the differential equation method [57,58] based on series expansion (See, e.g., Refs. [59–64]). To fix the

boundary conditions for these differential equations, we employ the auxiliary mass flow method [65–68] implemented in `AMFlow` [69]. The phase-space integrals over the momenta of final-state particles, except with k kept fixed, are treated in the same manner as loop integrals by means of the reverse unitarity [70–72]. To give an idea of the complexity encountered in our calculation, the number of integrals in the $\mathcal{O}(\alpha_s^3)$ corrections to $\mathcal{W}_{tb}^{\mu\nu}$ is about 7×10^4 , which are reduced to 2988 master integrals to be solved using the aforementioned method. Armed with all these highly efficient techniques, we are able to construct a piecewise series expansion representation (PSE) for each master integral, deeply expanded to about 200 orders in E , using the differential equation solver in `AMFlow`. Consequently, a high-precision result for $\mathcal{W}_{tb}^{\mu\nu}$ is obtained in the form of deeply expanded PSE.

A tricky point arises when completing the phase-space integration of $\mathcal{W}_{tb}^{\mu\nu}$ over k to calculate physical observables: W_i in (2) contain IR divergence in the region with E reaching its maximum, and the integration over k shall be done with a proper regularization which we take to be DR. The phase-space integration of $\mathcal{W}_{tb}^{\mu\nu}$ over the IR-dangerous regions is done with ϵ assigned with the values, $10^{-3} + n \times 10^{-4}$ for $n = 0, 1, \dots, 15$, to regularize the potential IR divergences [68,69]. The fit to the ϵ -dependence is done only at the very end for the final finite physical objects of interest, which can be the inclusive Γ_t and $f_{\text{L,R},0}$ as well as IR-safe distributions. This treatment saves us from the need to explicitly perform Laurent expansions in ϵ at intermediate stages, which significantly reduces the computational time for our problem. This method is also expected to work with physical jet-based observables where the dimensionally regularized phase-space integration can be carried out using numerical integration methods such as the Monte Carlo technique, with certain phase-space cuts imposed. (The numerical values for ϵ should then be adapted.) We validated the above method using the simplest inclusive observable of the process, namely Γ_t , by comparing the numerical results constructed from $\mathcal{W}_{tb}^{\mu\nu}$, to be presented below, against those calculated by applying the optical theorem (where the integration over k can be done using the reversed unitarity [70–72]). We find a perfect agreement between the two methods within our high numerical precision.

III. THE INCLUSIVE DECAY WIDTH

We are now ready to present our numerical results, beginning with Γ_t . The QCD effects on Γ_t in the SM can be parametrized as

$$\Gamma_t = \Gamma_0 \left[\mathbf{c}_0 + \frac{\alpha_s}{\pi} \mathbf{c}_1 + \left(\frac{\alpha_s}{\pi} \right)^2 \mathbf{c}_2 + \left(\frac{\alpha_s}{\pi} \right)^3 \mathbf{c}_3 + \mathcal{O}(\alpha_s^4) \right], \quad (3)$$

where we introduce a prefactor $\Gamma_0 \equiv \frac{G_F m_W^2 m_t |V_{tb}|^2}{12\sqrt{2}}$ with G_F the Fermi constant and the CKM matrix element V_{tb} taken to be 1 in the following numerical results. The perturbative coefficients $\mathbf{c}_i$ are functions of the kinematic ratio m_W^2/m_t^2 in the limit $m_b = 0$, and the renormalization scale μ . Below we present the numerical results for the QCD corrections computed using a 5-flavor Lagrangian where m_t is renormalized in the on-shell scheme and α_s is $\overline{\text{MS}}$ -renormalized with 5 massless quark flavors.¹

At the scale $\mu = m_t/2$, close to the kinetic energy $m_t - m_W - m_b$ of the final state, we obtain

$$\begin{aligned} \mathbf{c}_0 &= 1.93851, & \mathbf{c}_1 &= -4.85519, \\ \mathbf{c}_2 &= -21.2260, & \mathbf{c}_3 &= -174.265, \end{aligned} \quad (4)$$

using the SM input parameters $m_t = 172.69$ GeV and $m_W = 80.377$ GeV. In (4) we have truncated our internal high-precision results for the perturbative coefficients $\mathbf{c}_i$ to the first six digits for display purpose (Results with higher precision can be obtained from the Supplemental Material [73] that provides machine-readable expressions and routines for evaluating them.) Up to NNNLO our result (4) agrees with those in Refs. [16,17,20,23]. With $G_F = 1.166379 \times 10^{-5}$ GeV⁻² and $\alpha_s(m_t/2) \approx 0.1189$ at scale $\mu = m_t/2$, obtained by solving the renormalization-group equation for the running α_s at four-loop order [74–77] with input $\alpha_s(m_Z) = 0.1179$ at the Z-pole mass $m_Z = 91.1876$ GeV, we further obtain

$$\begin{aligned} \Gamma_t &= 1.48642 - 0.140877 - 0.023306 - 0.007240 \text{ GeV} \\ &= 1.31500 \text{ GeV}, \end{aligned} \quad (5)$$

where the first line provides the decomposition of the total NNNLO result according to the α_s order. Therefore the QCD corrections continue to decrease the Born-level result for Γ_t up to $\mathcal{O}(\alpha_s^3)$.

Results for Γ_t at other scales can be readily derived from (5) using the renormalization-group equation method, which are provided as a function in the Supplemental Material [73] where the value of m_t can also be changed. In Fig. 1 we plot the scale dependence of our results for Γ_t for $\mu/m_t \in [0.1, 1]$ up to NNNLO. It can be seen that the scale dependence has been further reduced at NNNLO, as expected. However, due to the turning point of the NNLO green curve (at about $\mu/m_t = 0.14$) in Fig. 1, its scale variation never covers the NNNLO result at any scales less than $\mu/m_t = 0.6$, including the central value $\mu/m_t = 0.5$ chosen in this paper. This clearly demonstrates that the NNLO results can

¹The contributions with closed top-loop diagrams not included, which start from two-loop order. This part amounts to about 0.2% of the pure $\mathcal{O}(\alpha_s^2)$ correction and thus contributes less than 10^{-4} of the total result.

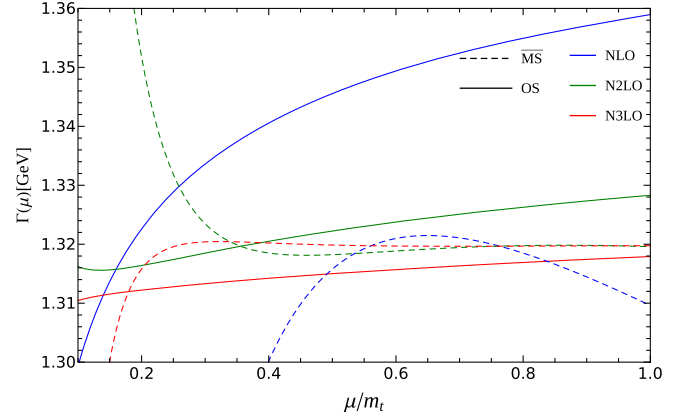


FIG. 1. The scale dependence of the fixed-order results for Γ_t in $\mu/m_t \in [0.1, 1]$ in both the on-shell and $\overline{\text{MS}}$ mass scheme.

underestimate the theoretical uncertainty if one studies only the μ dependence, and thus determining the $\mathcal{O}(\alpha_s^3)$ corrections explicitly is very important. We note that at the benchmark scale $\mu = m_t$ typically chosen in literature, the pure $\mathcal{O}(\alpha_s^3)$ correction decreases Γ_t by about 0.8% relative to the previous NNLO result, roughly 10 MeV, which significantly exceeds the usual NNLO scale uncertainty.

Despite an overall good convergence in (5), by examining the successive ratios $\mathbf{c}_{i+1}/\mathbf{c}_i$ starting from $i = 1$, we begin to see hints that the convergence of the perturbative series deteriorates as the perturbative order goes higher. This is expected due to the well-known infrared-renormalon sensitivity of the on-shell (pole) mass definition for heavy quarks (see, e.g., Refs. [24–29]), which can be avoided by rewriting the results using the renormalon-free $\overline{\text{MS}}$ -mass $\bar{m}_t$. Using the three-loop conversion formula from the pole mass m_t to $\bar{m}_t$ [78], we get $\bar{m}_t(\bar{m}_t) = 163.094$ GeV. Then we obtain the corresponding decay rate $\bar{\Gamma}_t(\mu = \bar{m}_t) = 1.2125 + 0.09830 + 0.00845 + 0.00034 = 1.31959$ GeV, the convergence rate of which is clearly improved, as shown in Fig. 1.

IV. THE W-HELICITY FRACTIONS AND ANGULAR ASYMMETRIES

As the first example beyond the inclusive Γ_t at NNNLO, we consider the W -helicity fractions $f_{L,R,0}$ resulting from a t -quark decay, which are unequal due to the chiral structure of the tbW -vertex in the SM. To present our results for QCD corrections to $f_{L,R,0}$, following their definitions, we introduce further $f_\lambda^{[n]}$ accurate to the n th order in α_s as in Ref. [18], $f_\lambda^{[n]} = \frac{\sum_{i=0}^n \Gamma_\lambda^{[i]}}{\sum_{i=0}^n \Gamma_t^{[i]}}$ where $\Gamma_t^{[n]} = \sum_{\lambda=0,L,R} \Gamma_\lambda^{[n]}$ is the total top-decay width at $\mathcal{O}(\alpha_s^n)$, expressed as the sum over $\Gamma_\lambda^{[n]}$ with a polarized W defined by $\Gamma_\lambda^{[n]} = \frac{1}{2m_t} \int \frac{d^{d-1}k}{(2\pi)^{d-1}2E} \mathcal{W}_{tb}^{\mu\nu} \varepsilon_\mu^*(k, \lambda) \varepsilon_\nu(k, \lambda)$ at $\mathcal{O}(\alpha_s^n)$ with the projector for $\varepsilon_\mu^*(k, \lambda) \varepsilon_\nu(k, \lambda)$ taken from Ref. [33].

Choosing the default parameters used above, we obtain the following numerical results for the NNNLO QCD corrections to the unexpanded ratios:

$$\begin{aligned}
 f_0^{[3]} &= 0.697706 - 0.008401 - 0.001954 - 0.000613 \\
 &= 0.686737, \\
 f_L^{[3]} &= 0.302294 + 0.007254 + 0.001799 + 0.000586 \\
 &= 0.311933, \\
 f_R^{[3]} &= 0. + 0.001147 + 0.000155 + 0.000027 \\
 &= 0.001330,
 \end{aligned} \tag{6}$$

at $\mu = m_t/2$, where the numbers in the first equality for each $f_\lambda^{[3]}$ give the difference relative to the proceeding perturbative order. In perturbation theory, it is conventional to further expand the ratios f_λ in α_s and truncate to the third order, resulting

$$\begin{aligned}
 f_0^{[3],\text{exp}} &= 0.697706 - 0.007605 - 0.002328 - 0.000832 \\
 &= 0.686942, \\
 f_L^{[3],\text{exp}} &= 0.302294 + 0.0065665 + 0.002109 + 0.000777 \\
 &= 0.311747, \\
 f_R^{[3],\text{exp}} &= 0. + 0.001038 + 0.000217 + 0.000055 \\
 &= 0.001312,
 \end{aligned} \tag{7}$$

The differences relative to the original unexpanded results (6) are much smaller than the QCD scale uncertainty of $\Gamma_t^{[3]}$ (except for the tiny f_R). When evaluated at $\mu = m_t$, our results up to NNLO are in good agreement with the numbers given in Ref. [18]. Defined as ratios, the scale uncertainties of $f_{L,R,0}$ are naturally quite small.

The three $f_{L,R,0}$ with numerical results in (6) govern the $\cos\theta^*$ angular distribution through [79]

$$\begin{aligned}
 \frac{1}{\Gamma_t} \frac{d\Gamma_t}{d\cos\theta^*} &= \frac{3}{4} (\sin^2\theta^*) f_0 + \frac{3}{8} (1 - \cos\theta^*)^2 f_L \\
 &\quad + \frac{3}{8} (1 + \cos\theta^*)^2 f_R,
 \end{aligned}$$

where θ^* is the angle between the momentum of the charged lepton from the W -decay in W -rest frame and the W -momentum in t -rest frame. For reducing measurement uncertainties when extracting results on $f_{L,R,0}$, Ref. [80] suggested a generalized angular asymmetry based on this $\cos\theta^*$ distribution,

$$\mathcal{A}_z = \frac{N(\cos\theta^* > z) - N(\cos\theta^* < z)}{N(\cos\theta^* > z) + N(\cos\theta^* < z)},$$

where $N(\cos\theta^* > z) = \int_z^1 \frac{d\Gamma_t}{d\cos\theta^*} d\cos\theta^*$, proportional to the number of charged leptons in t -quark decay observed

with $\cos\theta^*$ larger than a given value z [and similarly for $N(\cos\theta^* < z)$]. The forward-backward asymmetry of the charged lepton in $\cos\theta^*$ distribution corresponds to the special case of $\mathcal{A}_z$ at $z = 0$, which equals to $\frac{3}{4}(f_L - f_R)$. Our results (6) can be readily used to determine the QCD corrections to these quantities up to NNNLO.

V. THE W -ENERGY DISTRIBUTION

Having obtained the results for $\mathcal{W}_{tb}^{\mu\nu}$, we are able to calculate the NNNLO QCD corrections to the W -energy distribution in t -quark decay in the t -quark rest frame, shown in Fig. 2. The W -energy distribution increases as E becomes larger, and becomes singular in the limit of E reaching its maximum $E_{\max}$ where the QCD radiations have to be either soft and/or collinear to the b -quark. The fixed-order prediction breaks down in this region, and we take an average over a 1 GeV-bin at the rightmost end of the distribution, where the QCD corrections up to $\mathcal{O}(\alpha_s^3)$ decrease the Born-level result in a way similar to the Γ_t case.

From the lower panel of Fig. 2, one can see that the QCD corrections are positive and quite sizable, in particular, the pure $\mathcal{O}(\alpha_s^3)$ correction modifies the lowest-order result by about 7 ~ 14% for $E \in [94, 104]$ GeV. The scale variation of the next-to-leading (NLO) curve is solely determined by the change in α_s , and thus the blue band is independent of E . Starting from NNLO, the appearance of large logarithm $\ln((E_{\max} - E)/E)$ leads to the increase of corrections as E approaches $E_{\max}$. At NNNLO, the change in the scale-variation band in this region becomes more visible due to poor convergence of the perturbative series contaminated by the aforementioned large logarithmic structure.

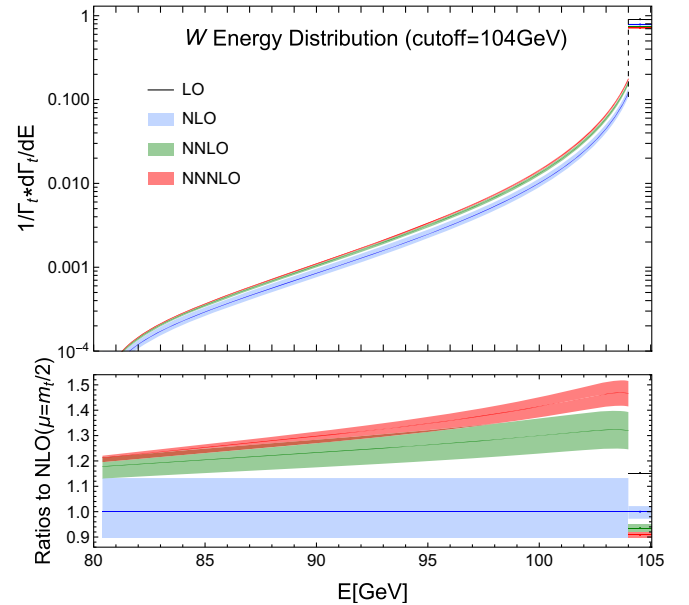


FIG. 2. The W -energy distribution in t -quark decay observed in the t -quark rest frame.

VI. MISCELLANEOUS EFFECTS

With the expression for Γ_t in the form of a deeply expanded PSE with respect to m_W^2/m_t^2 in hand, valid in the whole physical region, we can investigate the so-called off-shell- W effect up to $\mathcal{O}(\alpha_s^3)$. The standard definition of $\tilde{\Gamma}_t$ including off-shell- W effect (See, e.g., Ref. [81]) is obtained by replacing the W propagator with a Breit-Wigner distribution, where the W -momentum square is essentially the invariant-mass of the lepton pair produced from W -decay. $\tilde{\Gamma}_t$ admits a perturbative parametrization similar to (3) but with the i th order coefficient denoted as $\tilde{c}_i$ for distinction. With W total decay width $\Gamma_W = 2.085$ GeV, $\delta_i \equiv (\tilde{c}_i - c_i)/c_i$ are found to be quite small and decrease, albeit very slowly, as the α_s -order increases: δ_i takes -1.54% , -1.53% , -1.39% , -1.23% for $i = 0, 1, 2, 3$, respectively.

For the finite b -quark mass effect, we denote the i th order coefficient for Γ_t with a nonzero m_b as $c_i^{m_b}$. With $m_b = 4.78$ GeV we find that $(c_1^{m_b} - c_1)/c_1 \approx (c_2^{m_b} - c_2)/c_2 \approx -1.47\%$. This strongly indicates that the small nonzero m_b effect at $\mathcal{O}(\alpha_s^3)$ may follow a similarly small ratio, well below the sub-per-mille level for the total Γ_t .

Taking into account the aforementioned finite m_b and off-shell W effects, as well as the NLO electroweak corrections [82–86] which we rederived and included through a multiplicative K -factor,² our final result for Γ_t reads

$$\Gamma_t = 1.3148_{-0.005}^{+0.003} \times |V_{tb}|^2 + 0.027(m_t - 172.69) \text{ GeV}, \quad (8)$$

where the second term parametrizes the main source of uncertainty in Γ_t originating from the experimental uncertainty in the input t -quark mass, while V_{tb} has been restored explicitly for completeness. The numbers in the superscript and subscript in (8) correspond to a conservative estimate of the QCD scale uncertainty obtained by varying μ/m_t in $[0.1, 1]$.

Being ratios, the m_b effects on $f_{L,R,0}$ are even smaller, below one per mille, except for the small f_R whose Born-level expression vanishes at $m_b = 0$. After incorporating the NLO electroweak correction [85] our final results for $f_{L,R,0}$ read

$$\begin{aligned} f_0^{[3]} &= 0.686_{-0.003}^{+0.002}, & f_L^{[3]} &= 0.312_{-0.002}^{+0.001}, \\ f_R^{[3]} &= 0.00157_{-0.00002}^{+0.00002}. \end{aligned} \quad (9)$$

²To be more specific, the central value in (8) was obtained by multiplying our full NNNLO QCD result for Γ_t , with the finite m_b and off-shell W effects, by our recalculated NLO electroweak K -factor 1.0168 (See also [82–86]). If these NLO electroweak corrections are included in an additive manner, the central value then reads $\Gamma_t = 1.3180$ GeV, leading to a difference still compatible with the perturbative uncertainty given in (8).

Since the scale uncertainties of $f_{L,R,0}$ are naturally quite small because they are ratios, the errors for the above results, given in numbers in superscript and subscript are obtained by combining the errors of the following main sources: the conservative QCD-scale-uncertainty error for Γ_t determined in (8) and the errors induced by the input t -quark mass 172.69 ± 0.30 GeV and b -quark mass $4.78_{-0.03}^{+0.02}$ GeV, as well as that of $\alpha_s(m_Z) = 0.1179 \pm 0.0009$ [5] in the case of $f_R^{[3]}$.

Our results for Γ_t and $f_{L,R,0}$ in (8) and (9) respectively are thus sufficient to meet the anticipated precision requirements for the theoretical predictions for these observables to be measured at the future hadron and lepton colliders [2,7–9], as quoted in the Introduction.

VII. CONCLUSION

We have presented the first complete NNNLO QCD corrections to the t -decay width Γ_t , W -helicity fractions $f_{L,R,0}$ as well as the related asymmetries and the W -energy distribution in this paper, in which the off-shell- W and finite m_b effects are also taken into account. At $\mu = m_t/2$, our best prediction for Γ_t reads $\Gamma_t = 1.3148$ GeV with a conservative QCD scale-uncertainty of $[+3, -5]$ MeV. The QCD effects in $f_{L,R,0}$ are found to be much smaller and the corresponding theoretical uncertainties are therefore also quite small. For the W -energy distribution, we find quite sizable QCD corrections. Moreover, the treatment of the phase-space integration over W -momentum in the present computation may be adapted and further optimized to become useful for the fully differential calculation of any infrared-safe observables in t -quark decay process at NNNLO in QCD. These results represent the most precise theoretical predictions to date and will be useful for the purpose of testing the SM and probing new physics.

The calculations accomplished in this paper demonstrate the efficiency of our approach formulated. It can be readily applied to the decay of polarized t -quarks at NNNLO, where phenomenologically more interesting observables can be constructed, as well as to the mixed QCD-electroweak corrections needed to achieve a reliable per mille level phenomenological analysis for this process. Last but not least, our approach and results can also be applied to studies of other heavy-to-light quark decays, in particular the B-meson semileptonic decays. For example, the lepton-pair invariant-mass spectrum in this process can be readily evaluated using the functions provided in the associated Supplemental Material [73].

Note added. While the work in this paper was being finalized, a preprint [87] appeared in which the three-loop QCD correction to heavy-light form factors was obtained in the color-planar limit. Parallel to the calculations done in this paper, the leading-color part of the t -quark inclusive

decay width Γ_t was computed independently in [88], and a perfect agreement was found for this contribution.

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DATA AVAILABILITY

The *Mathematica* files containing machine-readable expressions and routines for evaluating the results reported in this article are available in the Supplemental Material [73].

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