

Larger physical volume and bounds on the number of matter fields in noncompact gauge theories on a lattice

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The work was motivated by the numerical result that, in a pure SU(2) gauge theory, the ratio R of the effective noncompact and compact lattice spacing is larger than 1 and increases with decreasing gauge coupling, with the expectation that it should further increase extending the parameter space. This means that with a noncompact regularization, at a given number of lattice sites and comparable scaling, one can obtain a larger physical volume whose importance for the control of size effects has long been known. We confirm qualitatively results and expectation by a perturbative evaluation of the effective lattice spacing in an expansion in the Plank constant of noncompact pure SU(2) and Abelian gauge theories, but we find in addition that R reaches the maximum value of $\sqrt{2}$. Including matter fields, we find that R increases (still up to $\sqrt{2}$) or decreases depending on the difference between the number of scalar and spinor degrees of freedom, and there are bounds on such a difference.

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I. INTRODUCTION

In recent years, there has been intense activity on multiflavor gauge theories in different contexts: asymptotic safety [1] with large number of colors and matter fields in four space-time dimensions and condensed matter and critical phenomena in three space-time dimensions with Abelian [2–4] or SU(2) [5] gauge fields, especially in connection with a possible symmetry enlargement [6] at a second-order phase transition.

Although the above themes constitute in perspective one of our motivations, the present work started in a different context, which turned out to be related to the above, motivated by the results of a numerical simulation of pure SU(2) gauge theory on a lattice: Di Carlo and Scimia [7] found that, in the parameter space accessible to them, the effective lattice spacing in the noncompact regularization [8] is up to 20% larger than in Wilson's [9]. Moreover, they advanced the idea that the ratio of such spacings, called R , should further increase by extending the parameter space. The need of a large physical volume was recognized long ago [10], and for recent references see [11]. But a larger physical volume at fixed number of sites and comparable scaling can be obtained with a larger effective spacing [12].

We will not tackle this issue in the present work, but the computations presented are a necessary preliminary step.

So we wanted to see the point at which the DiCarlo Scimia [7] expectation would be confirmed and we came to the conclusion, by a perturbative expansion to first order in the Plank constant, *that R does indeed continue to grow but only up to $\sqrt{2}$, which puts a bound on the parameter space.*

In the course of such an investigation, it is natural to wonder how our results would change by inclusion of matter fields, which is our additional motivation related to the subjects quoted at the beginning, and we found by the same perturbative expansion that R increases (still up to $\sqrt{2}$) or decreases depending on the difference between the number of scalar and spinor degrees of freedom, and there are bounds on such a difference. Because we restrict ourselves to Abelian and SU(2) gauge fields, we cannot compare our results with the Veneziano limit [13] and then with asymptotic safety [1], in which both numbers of colors and matter fields are let to grow, but we hope to come back to this question in a future work.

Let us define what we mean by noncompact gauge theories on a lattice. In general, such a name is given to regularizations in which the kinetic part of the Lagrangian is written in terms of the (noncompact) gauge fields while the coupling with matter fields is written in terms of the unitary Wilson variables. Hoping that by our premise we can avoid any confusion, we call instead such a formulation (which does not exist for non-Abelian theories) half compact, to distinguish it from the totally noncompact one that we are going to use in which the entire Lagrangian is written in terms of noncompact fields. Such a

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regularization [8] was constructed with the idea of being closer to the continuum since the gauge fields live in the algebra of the gauge group, at variance with Wilson's [9] in which they are elements of the group. Exact gauge invariance at finite lattice spacing is enforced by means of auxiliary fields which decouple in the continuum limit. In the SU(2) case, an evidence for confinement was found [14] comparable to that of Wilson (at the time in which the comparison was made), while perturbative studies of the scaling properties showed that it has the same fixed point as Wilson's [15,16]. The fixed point is the same but the renormalization trajectory is different, as also shown by the numerical simulation [7].

As already said, at present we cannot compare with the Veneziano limit [13] on the number of colors and matter fields, because our results are restricted to $N_c = 2$. In general, in noncompact gauge theories there are colored auxiliary fields which have the same coupling as the physical ones, and one neutral auxiliary field which has a different coupling [8,17]. The case of SU(2) is simpler because there are no colored auxiliary fields [18] and the coupling of the neutral field has been shown to be a function of the gauge coupling [15,16]. We presume that this will remain true also for $N_c > 2$, but their relationship, which should involve the number of matter fields, is not presently known. Apart from this, extension of our results to $N_c > 2$ should not present any difficulty, and we plan to do it in a future work.

Our results are also valid for Abelian theories; in four dimensions, these have been shown after several studies [19] to be trivial, at least coupled to scalars or to fermions with four flavors in the chiral limit [20], but in three dimensions they are truly interacting theories. We work in Euclidean space, setting $c = 1$.

The work is organized in the following way. In Sec. II, we describe for the convenience of the reader and to fix the notation the known formulation of the noncompact regularization of SU(2) and Abelian gauge theories in Cartesian and polar form. In Sec. III, we derive the expression of the effective lattice spacing in four dimensions to first order in the Plank constant, compare our perturbative values with the numerical results [7], and derive the bounds mentioned above on the number of matter fields. In Sec. IV, we briefly discuss the change to three dimensions and in Sec. V, we present our conclusions.

II. NONCOMPACT GAUGE THEORIES ON A LATTICE

In this section, we report the formulation of the noncompact regularization of the SU(2) gauge theory on a lattice. Section II A is devoted to its formulation in terms of Cartesian fields: This is close to the continuum and shows the role of the auxiliary field in ensuring exact gauge invariance at finite lattice spacing. In Sec. II B, the interaction with matter fields is presented and, in Sec. II C, the

formulation in terms of polar fields is given: This is close to the Wilson formulation and more convenient for the perturbative calculations of the work. The case of Abelian fields can be simply derived from the non-Abelian case, and its explicit treatment can be found in Ref. [21].

A. Cartesian fields

The basic ingredient of the noncompact regularization of SU(2) gauge theories is the quaternion

$$D_\mu(x) = V_\mu(x) + iA_{\mu a}(x)T_a, \quad (1)$$

where V_μ is a real auxiliary field, $A_{\mu a}(x)$ are the gauge fields, and the $T_a = \sigma_a/2$ are the generators of the gauge group, σ_a being the Pauli matrices. We assume a convention of summation over repeated indices.

In terms of this quaternion, one can define the quantities

$$F_{\mu\nu}(x) = \frac{1}{i} [D_\mu(x)D_\nu(x+\mu) - D_\nu(x)D_\mu(x+\nu)], \quad (2)$$

$$\begin{aligned} \mathcal{L}_{\text{YM}}(x) &= \frac{\beta}{8} \sum_{\mu\nu} \text{Tr}[F_{\mu\nu}^\dagger(x)F_{\mu\nu}(x)] \\ &= \frac{\beta}{4} \sum_{\mu\nu} \text{Tr}[D_\mu^\dagger(x+\nu)D_\nu^\dagger(x)D_\nu(x)D_\mu(x+\nu) \\ &\quad - D_\nu^\dagger(x+\mu)D_\mu^\dagger(x)D_\nu(x)D_\mu(x+\nu)], \end{aligned} \quad (3)$$

where Tr is the trace over colors.

Under the gauge transformations

$$g(x) = \exp(-i\theta_a(x)T_a), \quad (4)$$

$D_\mu(x)$ and $F_{\mu\nu}$ transform *homogeneously*:

$$\begin{aligned} D'_\mu(x) &= g(x)D_\mu(x)g^\dagger(x+\mu), \\ F'_{\mu\nu}(x) &= g(x)F_{\mu\nu}(x)g^\dagger(x+\mu+\nu), \end{aligned} \quad (5)$$

while $\mathcal{L}_{\text{YM}}(x)$ is invariant. For small $\theta_a(x)$

$$\begin{aligned} A'_{\mu a} &= A_{\mu a} - \frac{1}{2}\epsilon_{abc}A_{\mu b}[\theta_c(x+\mu) - \theta_c] \\ &\quad + V_\mu[\theta_a(x+\mu) - \theta_a], \\ V'_\mu &= V_\mu - \frac{1}{4}A_{\mu a}[\theta_a(x+\mu) - \theta_a]. \end{aligned} \quad (6)$$

What has been presented so far is a matrix model characterized by two related features: the fields of the quaternion, unlike physical gauge fields on a lattice, transform homogeneously and instead of discrete space-time there appears only a four-dimensional grid of points: there is no spacetime scale and then no derivative.

One can make contact with physical gauge theories in two steps. Firstly, one can introduce a bare lattice spacing

“a” by means of the gauge-invariant potential

$$\mathcal{L}_c = \frac{1}{2}\gamma^2 \sum_{\mu} \left[D_{\mu}^{\dagger} D_{\mu} - \frac{1}{a^2} \right]^2, \quad (7)$$

and define the Lagrangian

$$\mathcal{L} = \mathcal{L}_{\text{YM}} + \mathcal{L}_c. \quad (8)$$

There is no space-time scale, and the minimum of the potential does not occur at $V_{\mu} = 0$, but at

$$\bar{V}_{\mu} = \pm \frac{1}{a}. \quad (9)$$

In nonperturbative calculations, one has to account for both minima, what can be naturally realized in the polar representation [15]. In a perturbative calculation, however, one can expand about one of the minima (jumps between them are energetically suppressed) introducing the fluctuations of the auxiliary field

$$V_{\mu} = \bar{V}_{\mu} - W_{\mu}, \quad (10)$$

so that

$$D_{\mu}(x) = \frac{1}{a} - W_{\mu}(x) + iA_{\mu a}(x)T_a. \quad (11)$$

This is the second step towards a contact with physical gauge fields, as we are going to show. The above change of variables does not break gauge invariance, but it changes the homogeneous gauge field transformations into nonhomogeneous ones. For small $\theta_a(x)$, such transformations can be found in Ref. [18] (where the group generators have a different normalization):

$$\begin{aligned} A'_{\mu a} &= A_{\mu a} + \nabla_{\mu}^+ \theta_a - \epsilon_{abc} A_{\mu b} \theta_c \\ &\quad - a \left(W_{\mu} \nabla_{\mu}^+ \theta_a + \frac{1}{2} \epsilon_{abc} A_{\mu b} \nabla^+ \theta_c \right), \\ W'_{\mu} &= W_{\mu} + \frac{1}{4} a A_{\mu a} \nabla_{\mu}^+ \theta_a, \end{aligned} \quad (12)$$

where

$$\nabla_{\mu}^{\pm} f = \pm \frac{1}{a} [f(x \pm \mu) - f(x)] \quad (13)$$

is the ordinary forward-leftward derivative on the lattice. Notice that in the formal continuous limit, $a \rightarrow 0$, A_{μ} transforms as the physical vector potential while the auxiliary field W_{μ} becomes invariant and decouples as confirmed by the expression of the potential

$$\begin{aligned} \mathcal{L}_c &= \frac{1}{2}\gamma^2 \sum_{\mu} \left\{ D_{\mu}^{\dagger} D_{\mu} - \frac{1}{a^2} \right\}^2, \\ &= \frac{1}{2}\gamma^2 \sum_{\mu} \left\{ -\frac{2}{a^2} W_{\mu} + W_{\mu}^2 + \frac{1}{4} A_{\mu a} A_{\mu a} \right\}^2, \end{aligned} \quad (14)$$

which gives to the auxiliary field W_{μ} the square mass $4\gamma^2/a^2$.

An arbitrary gauge-invariant function $\mathcal{P}$ of the gauge fields can be added for particular purposes, but only when such a function is a polynomial is the theory strictly local. In conclusion, the total gauge fields Lagrangian is

$$\mathcal{L}_g = \mathcal{L}_{\text{YM}} + \mathcal{L}_c + \mathcal{P}. \quad (15)$$

The couplings β and γ are not independent from each other. Firstly, stability of the classical action requires [15]

$$\gamma^2 > \frac{3}{2}\beta = \frac{6}{g^2}, \quad (16)$$

where g is the gauge coupling constant. Secondly, perturbation theory to one loop gives [15]

$$\gamma = \gamma_1 \beta + \gamma_2, \quad (17)$$

where γ_1, γ_2 are independent of β , γ_1 is arbitrary and γ_2 can be, but it has not been evaluated. This result has been confirmed by a calculation in the Hamiltonian formalism [16].

B. Matter fields Lagrangians

Let us now write down the interactions with matter fields. For a field ϕ transforming as

$$\phi'(x) = g(x)\phi(x), \quad (18)$$

one can define right/left covariant derivatives

$$\begin{aligned} (\mathcal{D}_{\mu}^+ \phi)(x) &= D_{\mu}(x)\phi(x + \mu) - \frac{1}{a}\phi(x), \\ (\mathcal{D}_{\mu}^- \phi)(x) &= \frac{1}{a}\phi(x) - D_{\mu}^{\dagger}(x - \mu)\phi(x - \mu), \end{aligned} \quad (19)$$

and the symmetric covariant derivative

$$(\mathcal{D}_{\mu} \phi)(x) = \frac{1}{2} [D_{\mu}(x)\phi(x + \mu) - D_{\mu}^{\dagger}(x - \mu)\phi(x - \mu)]. \quad (20)$$

They all transform according to Eq. (18). The Lagrangian of a scalar field ϕ with such a transformation is

$$\mathcal{L}_\phi = \frac{1}{2} \sum_\mu |\mathcal{D}_\mu \phi|^2 + \frac{1}{2} m_\phi^2 |\phi|^2 + \mathcal{V}(|\phi|), \quad (21)$$

where $\mathcal{V}(|\phi|)$ is a potential. We remind our convention of summation on repeated indices and we use the notation

$$|\phi|^2 = (\phi^f)^* \phi^f. \quad (22)$$

The Lagrangian of a spinor field ψ transforming according to Eq. (18) is

$$\mathcal{L}_\psi = \frac{i}{2} \sum_\mu \bar{\psi} \gamma_\mu \mathcal{D}_\mu \psi + m_\psi \bar{\psi} \psi. \quad (23)$$

Such a Lagrangian is usually called naive because it is plagued by the so-called doubling problem, which will be ignored in the present work.

C. Polar fields

We still denote by D_μ the covariant derivative in terms of polar fields

$$D_\mu(\rho_\mu, \alpha_\mu) = \frac{1}{a} \rho_\mu U_\mu, \quad (24)$$

where the polar radius ρ_μ is the auxiliary field now gauge-invariant and

$$U_\mu = \exp(ia g \alpha_{\mu a} T_a) \quad (25)$$

are the Wilson variables parameterized in terms of the angular fields $\alpha_\mu(x)$. The gauge coupling constant has been inserted in the definition of Wilson's variables in order to get rid of it in the perturbative kinetic energy of the α -fields in Eq. (36). The polar fields are related to the Cartesian ones according to

$$\begin{aligned} \rho_\mu &= a \left[\left(W_\mu - \frac{1}{a} \right)^2 + A_\mu^2 \right]^{\frac{1}{2}}, \\ U_\mu &= \frac{a}{\rho_\mu} \left(\frac{1}{a} - W_\mu + i A_\mu \right). \end{aligned} \quad (26)$$

Notice that ρ_μ is not a quaternion but it is a real function of the lattice sites. The Yang-Mills Lagrangian density in polar form is

$$\begin{aligned} \mathcal{L}_{\text{YM}}(x) &= \frac{\beta}{8} \sum_{\mu, \nu} \{ \rho_\nu^2(x + \mu) \rho_\mu^2(x) \\ &\quad - \rho_{P\mu\nu}(x) \text{Tr}(U_{P\mu\nu}(x) - a^{-4}) \}, \end{aligned} \quad (27)$$

where

$$\begin{aligned} \rho_{P\mu\nu}(x) &= \rho_\mu(x) \rho_\nu(x + \mu) \rho_\nu(x) \rho_\mu(x + \nu), \\ U_{P\mu\nu}(x) &= \frac{1}{a^4} [1 - U_\mu^\dagger(x + \nu) U_\nu^\dagger(x) U_\mu(x) U_\nu(x + \mu)], \end{aligned} \quad (28)$$

while

$$\mathcal{L}_c = \frac{1}{2} \gamma^2 \frac{1}{a^4} \sum_\mu (\rho_\mu^2 - 1)^2. \quad (29)$$

Notice that in the limit $\gamma \rightarrow \infty$, the compact regularization is recovered. The gauge fields partition function in polar coordinates is

$$T_g = \int_0^\infty d\rho_\mu \rho_\mu^3 \int dU_\mu \exp \left[-a^4 \sum_x (\mathcal{L}_{\text{YM}} + \mathcal{L}_c + \mathcal{P}) \right], \quad (30)$$

where dU_μ is the Haar measure. The measure ρ_μ^3 can be eliminated by assuming the arbitrary function

$$\mathcal{P} = \frac{\hbar}{a^4} \sum_\mu \ln \rho_\mu^2. \quad (31)$$

We note that since it is not a polynomial when expressed in terms of the Cartesian fields, strict locality is lost. The partition function can now be rewritten in the form

$$T_g = \frac{1}{2} \int_0^\infty d\rho_\mu^2 \int dU_\mu \exp \left(-a^4 \sum_x \mathcal{L}_{\text{YM}} + \mathcal{L}_c \right). \quad (32)$$

III. EFFECTIVE SPACING FOR SU(2) GAUGE THEORIES

In this section, we work in four space-time dimensions and therefore we do not discuss Abelian theories that in such a case are trivial. As we are going to show, the ratio R between lattice spacing in the noncompact and compact regularization is determined by the value of the polar field ρ_μ at the minimum of the effective potential that we will evaluate to first order in the Plank constant $\hbar$.

For perturbative calculations, the Yang-Mills Lagrangian density can be rewritten, ignoring irrelevant terms [15], as

$$\begin{aligned} \mathcal{L}_{\text{YM}}(x) &= \frac{1}{2} \beta a^4 \sum_{\mu, \nu} \rho_\nu(x) \rho_\mu(x + \nu) \rho_\mu(x) \rho_\nu(x + \mu) \\ &\quad \times \text{Tr} U_{P\mu\nu}(x). \end{aligned} \quad (33)$$

It is convenient to perform the change of variables

$$\rho_\mu^2 = 1 + \frac{a^2}{\gamma} t_\mu, \quad (34)$$

which makes the potential $\mathcal{L}_c$ quadratic

$$\mathcal{L}_c = \frac{1}{2} \sum_{\mu} t_{\mu}^2. \quad (35)$$

Notice that γ appears now as an inverse coupling constant in the Yang-Mills Lagrangian of Eq. (33).

The total gauge fields Lagrangian density in the quadratic approximation is

$$\begin{aligned} \mathcal{L}_g = & \frac{1}{2} \sum_{\mu} t_{\mu}^2 + \frac{1}{2} \sum_{\mu\nu} \rho_{\mu}^2 \rho_{\nu}^2 (\nabla_{\mu}^+ \alpha_{\nu} - \nabla_{\nu}^+ \alpha_{\mu})^2 \\ & + \frac{1}{\xi} \left(\sum_{\mu} \rho_{\mu} \nabla_{\mu}^+ \alpha_{\mu} \right)^2 + \bar{\eta} \nabla_{\mu}^- \rho_{\mu} \nabla_{\mu}^+ \eta, \end{aligned} \quad (36)$$

where the last terms are gauge fixing and ghost contributions [15]. The gauge coupling constant disappears at this order because of the definition (25) of Wilson's variables.

We evaluate the effective potential following the functional method of Jackiw [22]. We firstly perform the shift, equal for simplicity in all directions

$$t_{\mu} \rightarrow \hat{t} + t_{\mu}, \quad (37)$$

and accordingly we rewrite ρ_{μ} as

$$\rho_{\mu}^2 = \hat{\rho}^2 \left(1 + \frac{1}{\gamma} \hat{a}^2 t_{\mu} \right), \quad (38)$$

where

$$\hat{\rho}^2 = 1 + \frac{1}{\gamma} a^2 \hat{t}, \quad \hat{a} = \frac{a}{\hat{\rho}}. \quad (39)$$

Now the covariant derivative reads

$$D_{\mu} = \frac{1}{\hat{a}} \sqrt{1 + \frac{1}{\gamma} \hat{a}^2 t_{\mu}} U_{\mu}, \quad (40)$$

showing that $\hat{a}$ is the effective lattice spacing, namely the physical spacing, and the ratio of the spacings is

$$R = \frac{1}{\hat{\rho}}. \quad (41)$$

A. Evaluation of the effective potential

For the evaluation of the effective potential, it is convenient to perform the further change of variables

$$t_{\mu} = \frac{1}{a^2} s_{\mu}, \quad s_{\mu} \rightarrow \hat{s} + s_{\mu}. \quad (42)$$

Then, following Jackiw [22], we define the shifted Lagrangian

$$\begin{aligned} \hat{\mathcal{L}}_g = & \frac{1}{2} \sum_{\mu} (s_{\mu}^2 + \hat{s}_{\mu}^2) + \frac{1}{2} \hat{\rho}^4 \sum_{\mu\nu} (\nabla_{\mu}^+ \alpha_{\nu} - \nabla_{\nu}^+ \alpha_{\mu})^2 \\ & + \frac{1}{\xi} \hat{\rho}^2 \left(\sum_{\mu} \nabla_{\mu}^+ \alpha_{\mu} \right)^2 + \hat{\rho} \bar{\eta} \sum_{\mu} \nabla_{\mu}^- \nabla_{\mu}^+ \eta, \end{aligned} \quad (43)$$

obtained by retaining only quadratic terms in all the field fluctuations and neglecting the linear ones. The partition function can now be written as

$$T_g = \frac{1}{2} \int_{-\gamma}^{\infty} ds_{\mu} \int dU_{\mu} \exp \left(-a^4 \sum_x \hat{\mathcal{L}}_g \right), \quad (44)$$

where dU_{μ} is the Haar measure. It follows that the classical gauge fields effective potential is

$$\mathcal{V}_g^{(0)} = 2\hat{s}^2. \quad (45)$$

It is what remains of the potential (14) after all our manipulations. Its minimum occurs at $\hat{s} = 0$, namely at the bare lattice spacing. We then evaluate the partition function to order $\hbar$. Integration over s_{μ} can be ignored because it does not change the dependence on $\hat{s}$ and then on $\hat{\rho}$ of the effective potential.

After Fourier transformation, we get the action

$$a^4 \sum_x \hat{\mathcal{L}}_g = \frac{1}{N^4} \sum_p \tilde{\alpha}_{\mu c}^*(p) M_{\mu\nu}(p) \alpha_{\nu c}(p), \quad (46)$$

where N is the number of lattice spacings in one direction and the matrix

$$M_{\mu\nu}(p) = \delta_{\mu\nu} p^2 \hat{\rho}^4 - \left(\hat{\rho}^4 - \frac{1}{\xi} \hat{\rho}^2 \right) p_{\mu} p_{\nu} \quad (47)$$

has determinant

$$\det M(p) = \frac{1}{\xi} (p^2)^4 (\hat{\rho}^2)^7. \quad (48)$$

In the same way, one obtains the ghost contribution for each momentum and color

$$\hat{\rho}^{2\frac{1}{2}}, \quad (49)$$

therefore ignoring irrelevant factors

$$T_g = (\hat{\rho}^2)^{-9N^4}. \quad (50)$$

Following Jackiw [22], the contribution of order $\hbar$ to the effective potential can be obtained by dividing the effective action by the space-time volume

$$\mathcal{V}_g^{(1)} = 9 \frac{\hbar}{a^4} \ln \hat{\rho}^2, \quad (51)$$

so that the total effective potential turns out to be

$$\mathcal{V}_g = 2 \frac{\gamma^2}{a^4} \left[(\hat{\rho}^2 - 1)^2 + \frac{1}{2} \sigma_g \ln \hat{\rho}^2 \right], \quad (52)$$

where

$$\sigma_g = 9 \frac{\hbar}{\gamma^2}. \quad (53)$$

At first sight it might be surprising that the gauge coupling constant does not appear, but we should remember that at variance with standard cases, here the shifted field is the auxiliary field, whose coupling constant is indeed $1/\gamma$. We should also remember that according to Eq. (17), γ is a function of β .

The coupling constant $\frac{1}{\xi}$ of the gauge fixing disappeared, but obviously this does not mean *per se* that the result is gauge independent. There are several investigations with conclusions that physical results derived from the effective potential are gauge independent [23]. The ratio of lattice spacings is not a physical quantity, but in the following subsection we will extend our analysis, including the coupling with matter fields, and then we will derive bounds on their number which are physical results. Because our procedure is not standard, however, it is not obvious to us that the quoted results [23] are valid also in the present case. We leave such a question for a future investigation.

The effective potential diverges to $-\infty$ for $\hat{\rho}^2 \rightarrow 0$, but the gauge fields Lagrangian is bounded from below [15], so that such negative divergence is not a problem of the regularization but of the perturbative expansion that cannot be trusted in this region. We do not think that it should cast doubts on the validity of the correction evaluated at finite lattice spacing, and therefore we think it can be ignored in the present context. All of the above refers to a pure SU(2) theory, and the situation can be changed by matter fields, as shown in Sec. III C.

The first derivative of the effective potential

$$\frac{\partial \mathcal{V}_g}{\partial \hat{\rho}^2} = 4 \frac{\gamma^2}{a^4} \frac{1}{\hat{\rho}^2} \left[(\hat{\rho}^2)^2 - \hat{\rho}^2 + \frac{1}{4} \sigma_g \right] \quad (54)$$

vanishes at

$$\hat{\rho}_\pm^2 = \frac{1}{2} [1 \pm \sqrt{1 - \sigma_g}], \quad (55)$$

so that the effective potential takes a maximum at $\hat{\rho}_-$ and a local minimum at $\hat{\rho}_+$ with the following exceptions. For $\sigma_g = 1$, $\hat{\rho}_\pm = 1/\sqrt{2}$ is neither a maximum nor a minimum but an inflection. For $\sigma_g = 0$, the Wilson limit, the minimum occurs at $\hat{\rho} = 1$, namely $\hat{a} = a$.

Otherwise, the roots are real under the condition on γ

$$\frac{\gamma^2}{\hbar} > 9 \quad (56)$$

to be compared with the condition (16). Under such a condition, since the theory is defined at the minimum of the effective potential,

$$1 < R = \frac{\hat{a}}{a} = \frac{1}{\hat{\rho}_+} < \sqrt{2}. \quad (57)$$

B. Comparison with the numerical simulation of Di Carlo and Scimia

Di Carlo and Scimia [7] made a plot of the ratio of two hadron masses as a function of $1/\gamma$ and β . The resulting surface has a valley starting at the Wilson point $1/\gamma = 0$ that they regarded as a scaling window and called the scaling valley. *The bottom of this valley at given $1/\gamma$ determines this parameter as a function of β , a function which was found consistent with the perturbative one (17).* This allowed them to make a plot (in their Fig. 7) of the lattice spacing $\hat{a}_{\text{num}}$, which is a function of both γ and β , as a function of γ only, which is directly comparable to our result which comes out naturally as a function of γ .

Their plot is done in the interval $0.0 \leq 1/\gamma \leq 0.18$. We now compare some values of R extracted from the plot, which we call R_{num} , with ours evaluated according to Eq. (57):

$\frac{1}{\gamma} = 0.00$	$R = 1$	$R_{\text{num}} = 1.00$
$\frac{1}{\gamma} = 0.06$	$R = 1.00$	$R_{\text{num}} = 1.02$
$\frac{1}{\gamma} = 0.12$	$R = 1.02$	$R_{\text{num}} = 1.07$
$\frac{1}{\gamma} = 0.18$	$R = 1.04$	$R_{\text{num}} = 1.15$
$\frac{1}{\gamma} = 0.24$	$R = 1.09$	
$\frac{1}{\gamma} = 0.30$	$R = 1.18$	
$\frac{1}{\gamma} \rightarrow \frac{1}{3}$	$R \rightarrow \sqrt{2}$.	

(58)

We see that the agreement is only qualitative, there being a factor four of discrepancy. However, we confirm the expectation of Di Carlo and Scimia that R should have increased, extending the range of γ , which was the original motivation of the present work, but we find the additional result that in pure SU(2) γ is bound according to Eq. (56).

C. Inclusion of matter fields

Inclusion of matter fields is straightforward. From Eqs. (21) and (23), using the definition (24) of covariant derivative and retaining only quadratic fluctuations, we get the matter fields shifted Lagrangians

$$\begin{aligned}\hat{\mathcal{L}}_\phi &= \frac{1}{2}\hat{\rho}^2 \sum_\mu |\nabla_\mu \phi|^2 + \frac{1}{2}m_\phi^2 |\phi|^2, \\ \hat{\mathcal{L}}_\psi &= \frac{i}{2}\hat{\rho} \sum_\mu \bar{\psi} \gamma_\mu \nabla_\mu \psi + m_\psi \bar{\psi} \psi.\end{aligned}\quad (59)$$

As for the pure SU(2) theory and for the same reason, self interactions and Yukawa couplings would not appear in the effective potential, so we omit them.

Assuming matter fields to be massless for simplicity, to first order in the Plank mass they give the effective potential the contribution

$$\mathcal{V}_{\text{mat}} = \frac{1}{2} \frac{\hbar}{a^4} (\nu_s - \nu_f) \ln \hat{\rho}^2, \quad (60)$$

where ν_s, ν_f are the numbers of scalars and fermion degrees of freedom, respectively: ν_s is the color times flavor dimension of the scalar representations, ν_f the color times flavor times Dirac matrices dimension of the fermion representation. The latter is 4 in four space-time dimensions, but 4 or 2 in three space-time dimensions depending on whether parity is conserved or not.

In conclusion, now the total effective potential reads

$$\mathcal{V} = 2 \frac{\gamma^2}{a^4} \left[(\hat{\rho}^2 - 1)^2 + \frac{1}{2} \sigma \ln \hat{\rho}^2 \right], \quad (61)$$

where

$$\sigma = \frac{1}{2} (18 + \nu_s - \nu_f) \frac{\hbar}{\gamma^2}. \quad (62)$$

If $\sigma > 0$, the equations derived for pure gauge theories remain valid after replacing σ_g by σ :

$$0 < 18 + \nu_s - \nu_f < 2 \frac{\gamma^2}{\hbar} = \frac{2}{\hbar} (\gamma_1 \beta + \gamma_2)^2. \quad (63)$$

The first inequality is a bound on the number of degrees of freedom of all the fields, independent of the coupling constants, while the second one, which replaces (56), involves the parameter γ and then the inverse coupling β . It can be regarded as a further bound on the number of fields at given γ , or as a condition on this parameter for given number of fields.

A completely different situation occurs if $\sigma < 0$:

$$18 + \nu_s - \nu_f < 0. \quad (64)$$

In such a case, the effective potential for $\hat{\rho} \rightarrow 0$ diverges to $+\infty$, $\hat{\rho}_-$ becomes imaginary and must be discarded, and $\hat{\rho}_+ > 1$ becomes an absolute minimum so that $R < 1$, namely the effective lattice spacing, is smaller than the bare one.

IV. THREE DIMENSIONS

All the above concerns the theory in four space-time dimensions. In order to evaluate the effective potential in three dimensions, it is necessary to change the function $\mathcal{P}$ (31), chosen to get rid of the measure of integration on the radial field in four dimensions, which now must be

$$\mathcal{P} = \frac{1}{a^3} \hbar \sum_\mu \ln \rho_\mu. \quad (65)$$

Then for SU(2) gauge theories, we look at the expression of the determinant of the gauge fields matrix (48) for each momentum and color: reducing the space-time dimensions by one, it becomes

$$\det M(p) = \frac{1}{\xi} G(p^2)^3 (\hat{\rho}^2)^5, \quad (66)$$

so that all the results for three dimensions can be obtained by replacing σ_g according to

$$\sigma_g^{(3)} = 6 \frac{\hbar}{\gamma^2}. \quad (67)$$

Concerning Abelian gauge theories, we must remember that while in four dimensions, they have been shown to be trivial [19,20], in three dimensions they are truly interacting and largely used in condensed matter physics [2–4], as noted in the Introduction.

Gauge transformations as well as Lagrangian density can be derived immediately and can be found in Ref. [21]. Then, in the evaluation of the shifted Lagrangian, one has to omit the ghost contribution; and in the evaluation of the effective potential, one must omit the factor due to colors, with the result

$$(\sigma_g^{(3)})_{\text{Abelian}} = \frac{5}{2} \frac{\hbar}{\gamma^2}. \quad (68)$$

V. CONCLUSIONS

We confirmed qualitatively to order $\hbar$ of perturbation theory the numerical results [7] that R , the ratio between lattice spacings in the noncompact and compact regularizations of the pure SU(2) gauge theory, is larger than 1. We confirmed also the expectation that R should further increase with decreasing γ , but we found that γ has a lower bound, and for this value, R reaches its maximum $R = \sqrt{2}$.

Of course, quantitatively we should trust the numerical simulations more than the perturbative results. Therefore, if the factor 4 between R_{num} and R persists up to the minimum value of γ , we might expect the maximum of R_{num} to be of the order of 5.

We extended the investigation including massless matter fields and showed that the ratio R increases (still up to $\sqrt{2}$) or decreases depending on the difference between the number of scalar and spinor degrees of freedom, and there are bounds on such a difference.

These bounds might be relevant to the other theme which occurred as a second motivation of the work, namely ultraviolet fixed points and the related Veneziano limit [13], but in order to proceed in this direction, two further steps are needed. The first is the extension to large number of colors. In $SU(N_c > 2)$, in addition to the neutral auxiliary field, there are colored auxiliary fields with the same coupling constant as the physical ones [8,17] which will contribute to the effective potential. The second step is the extension of the relationship (17) to large number of colors and matter fields. We hope to come back to these issues in a future work. The above results are also valid in

three dimensions, where they are relevant to condensed matter physics.

Finally, a comment about Abelian gauge theories. One might wonder whether in four dimensions, due to the absence of the relationship (17) between β and γ , which remain two independent couplings, the noncompact regularization might define a theory different from the standard one, with a continuum nontrivial limit. This seems, however, very unlikely. Indeed, the noncompact theory is renormalizable by power counting and integration on the auxiliary field [21], although in a perturbative regime, it yields only irrelevant contributions.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the author.

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