Universal content of the proper time flow in scalar and Yang-Mills theories

Gabriele Giacometti [*]
Dipartimento di Fisica e Astronomia "Ettore Majorana",
Università di Catania, Via S. Sofia 64, 95123, Catania, Italy
and INFN, Sezione di Catania, Via Santa Sofia 64, 95123 Catania, Italy

Daniele Rizzo [†]
National Centre for Nuclear Research, Pasteura 7, 02-093 Warsaw, Poland

Dario Zappalà [‡]
INFN, Sezione di Catania, Via Santa Sofia 64, I-95123 Catania, Italy
and Centro Siciliano di Fisica Nucleare e Struttura della Materia, Catania, Italy

(Received 10 October 2025; accepted 30 January 2026; published 23 February 2026)

We investigate the perturbative structure of the proper time renormalization group flow in scalar and Yang-Mills theories. Although the proper time flow does not belong to the class of exact functional renormalization group equations, we show that it correctly reproduces the universal coefficients of the β functions at one and two loops. For the O(N) scalar theory, we derive the one- and two-loop contributions to the running quartic coupling and also confirm the expected anomalous dimension. For the SU(N) Yang-Mills theory, using the background field method, we compute the gauge coupling renormalization recovering the correct two-loop β function without generating any gauge-symmetry–violating terms. These results highlight that, despite its limitations for reconstructing the full effective action, the proper time flow retains the essential universal content of renormalization, accounting for its reliability in diverse applications ranging from statistical models to quantum gravity.

DOI: 10.1103/dsgp-mlkp

I. INTRODUCTION

The functional renormalization group (FRG) approach in statistical mechanics and quantum field theory inspired by the pioneering work of Wilson [1] has achieved a widespread success in the last three decades (see, e.g., [2,3]), providing a powerful tool to study the general renormalization properties of various field theories and to analyze in detail their evolution when the observational (renormalization) scale moves from high [ultraviolet (UV)] to low [infrared (IR)] energy ranges.

The content of the FRG is enclosed in a functional differential flow equation that describes the evolution of the average effective action between the classical action, taken as the UV (in the limit of the renormalization scale $k \to \infty$) boundary condition of the equation, and the full effective action (the Legendre transform of the generator of the connected Green's functions), which, in principle, is the solution of the flow equation in the extreme IR region, i.e., when $k \to 0$. The details of the intermediate steps (finite k) of the flow equation are model dependent, in the sense that they depend on the particular choice of the k dependent regulator function that selects the set of modes that are integrated out along the flow, respecting the constraint of recovering, in all cases, the full effective action for $k \to 0$ [4–6]. Because of the latter property, the generation of the perturbative diagrammatic loop expansion is naturally produced by the FRG [7,8], although, on the practical side, replicating perturbative results beyond one loop could require a nontrivial effort, as in the simple case of the computation of the two-loop β function of the scalar field [9]. The problem of recovering the perturbative scheme from a different approach to the flow equation, derived by the original idea of generating a flow equation for a Wilsonian action by means of a progressive integration of modes [1,10], was also addressed [11–16], and, although the first perturbative order (one loop) is totally under control, it is evident that the nature of the sharp regulator acting on momenta, characteristic of this method, brings in serious flaws when it comes to two-loop computations.

[*]Contact author: gabriele.giacometti@phd.unict.it
[†]Contact author: daniele.rizzo@ncbj.gov.pl
[‡]Contact author: dario.zappala@ct.infn.it

Published by the American Physical Society under the terms of the Creative Commons Attribution 4.0 International license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Funded by SCOAP³.

The difficulties associated with a sharp momentum cutoff can be somehow circumvented by using the proper time (PT) formulation of the renormalization group flow where the blocking procedure is performed on a suitably introduced variable, the proper time [13,17–20]. The corresponding flow equation turns out to be particularly accurate in various contexts, such as the determination of critical indices in second order phase transition [19,21,22], or the determination of ground state and energy gap of a quantomechanical double well potential [23]. It has been largely employed in the study of the gravitational theory [24–29], and it has also been recently related to the essential renormalization group [30].

Despite this versatility, it was shown that the PT flow does not belong to the class of exact flows for the effective average action [7,8,20], because it is not capable of reproducing the correct loop expansion of the effective action; rather, it can be regarded as a special case of background field flow [8,31]. Along this line, the missing elements of the PT flow to be assimilated to an exact flow are computed in [32].

In this paper, we reconsider this old issue and compute the coefficient of the first two orders of the perturbative series of the β function of the O(N) scalar theory and of the SU(N) Yang-Mills theory from the PT flow, being motivated not only by its reliability in many applications, but also by the study in [33,34], where the PT renormalization group equation is regarded as a Wilsonian flow, where, at each step of the integration, a new local action is generated, in contrast with the exact flow that deals with the evolution of the full generator of one-particle irreducible diagrams. In this sense, we can still expect to get significant results at least for the determination of the one- and two-loop β function. In fact, this is a reduced goal with respect to the reconstruction of the full diagrammatic structure of the two-loop effective action which is only achieved within the class of exact FRG flows, but it still gives access to some universal quantities that are characteristic of each specific theory. In addition, the computation of the Yang-Mills β function allows us to verify that the PT does not generate any gauge-violating term (at least in this particular case), thus respecting the gauge symmetry of the theory. Then, in the next section the basic details of the calculation are summarized, while Secs. III and IV are devoted respectively to the O(N) theory and the SU(N) Yang-Mills theory. The conclusions are reported in Sec. V.

II. PERTURBATIVE EXPANSION AND THE PT FLOW

We start from the regularized PT representation of the inverse and the logarithm of an operator X,

$$X^{-1} = \int_{1/\Lambda^2}^{1/k^2} ds e^{-sX} \tag{1}$$

and

$$\log[X] = -\int_{1/\Lambda^2}^{1/k^2} \frac{ds}{s} e^{-sX}, \tag{2}$$

where the integral is performed on the PT variable s and the infinite upper limit and zero lower limit of the integrals, which define the unregularized original expressions, here are, respectively, replaced with $1/k^2$ and $1/\Lambda^2$, and k and Λ are two scales that protect computations from possible divergences (IR in the limit $k \to 0$ and UV in the limit $\Lambda \to \infty$) related to the spectrum of X. In particular, Eq. (2) allows us to regularize the one-loop effective action and also obtain a renormalization group (RG) equation for the Wilsonian effective action S_k, indicated as the PT RG flow equation,

$$k\partial_k S_k = -\frac{k\partial_k}{2} \int_{1/\Lambda^2}^{1/k^2} \frac{ds}{s} \mathrm{TR}[e^{-sS_k''}], \tag{3}$$

where S_k'' is the second functional derivative of the Wilsonian action $S_k[\phi]$ with respect to its degrees of freedom (the field ϕ), and the trace TR is performed on all free indices of S_k''. In particular, in addition to all internal indices associated with ϕ, the trace includes the integration over the space and momentum variables appearing in S_k''.

In order to recover the perturbative series in powers of the coupling, one can simply replace the running operator S_k'' in the right-hand side of Eq. (3), with its value at a fixed order of the series, thus obtaining the contribution to S_k to the next order. To one loop, this procedure is straightforward. In fact, it is sufficient to replace S_k'' with the second functional derivative of the bare action which, up to UV counterterms (that are not relevant for our purposes), coincides with the tree action S_0. With this replacement, the integration of Eq. (3) in the limit $1/k^2 \to \infty$ and $1/\Lambda^2 \to 0$ reproduces the well known one-loop correction

$$S_1 = -\frac{1}{2} \int_0^\infty \frac{ds}{s} \mathrm{TR}[e^{-sS_0''}] = \frac{1}{2} \mathrm{TR} \log[S_0''] \tag{4}$$

and the full one-loop action is $S_{1L} = S_0 + S_1$. The one-loop β function is easily derived by following the same procedure. In fact, after replacing S_k'' with S_0'' in Eq. (3), it is sufficient to pick the coupling of interest on both sides of the equation with a suitable projector, to obtain the one-loop flow of the selected coupling.

Unfortunately, the computation of the two-loop correction to the effective action, S_2, from the straightforward extension of Eq. (4),

$$S_1 + S_2 = -\frac{1}{2} \int_{1/\Lambda^2}^{1/k^2} \frac{ds}{s} \mathrm{TR}[e^{-s(S_0''+S_1'')}], \tag{5}$$

presents some inconvenience. This is due to the structure of the PT flow equation that does not possess the structure of the exact flow equation [7,8,32], which implies the

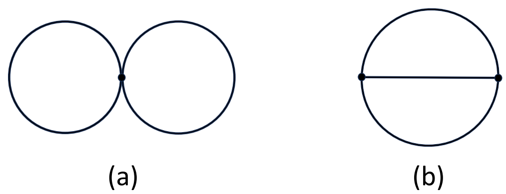

FIG. 1. Two-loop diagrams contributing to the effective action. Diagram (a) is order λ and diagram (b) is order λ^2.

impossibility of reconstructing the complete perturbative structure of the effective action. So, for instance, the direct computation of the two-loop effective action from the PT flow yields a wrong weight for the $O(\lambda^2)$ two-loop vacuum diagram (b) in Fig. 1, as shown in [8]. Nevertheless, we notice that in Eq. (5), it is still possible to expand at first order the exponential in the right-hand side,

$$S_1 + S_2 = -\frac{1}{2}\int_{1/\Lambda^2}^{1/k^2} \frac{ds}{s}\mathrm{TR}[e^{-sS_0''}(1 - sS_1'')], \quad (6)$$

so that the zeroth order of the expansion gives back the one-loop action S_1, while the first order produces the two-loop action S_2. But in practice, this formal structure requires the full knowledge of the second functional derivative of the one-loop action S_1'', including its full momentum dependence.

Therefore, treating the exponential in Eq. (4) as a simple function of S_0'', the second functional derivative of the one-loop action

$$(S_1)'' = \left(-\frac{1}{2}\int_{1/\Lambda^2}^{s}\frac{dt}{t}\mathrm{TR}[e^{-tS_0''}]\right)'' = -\frac{1}{2}\int_{1/\Lambda^2}^{s}\frac{dt}{t}\mathrm{TR}[e^{-tS_0''}(-tS_0^{(4)} + t^2 S_0^{(3)}S_0^{(3)})], \quad (7)$$

does not actually lead to the complete result for S_1''. In the latter equation, $S_0^{(3)}$ and $S_0^{(4)}$ indicate, respectively, three and four functional derivatives of S_0 with respect to the field, and the proper time variable is indicated as t to distinguish it from s, which instead refers to the proper time variable of the "external" loop in Eq. (6). Nevertheless, it must be remarked that Eq. (7) still provides the correct result for constant (x-independent) fields, i.e., it can still be retained to determine S_1'' at zero external momentum.

The computation of the non-zero external momentum component of S_1'' requires instead the use of the following expansion, [35],

$$\mathrm{TR}[e^{-(A+B)}] = \mathrm{TR}\Bigg[e^{-A} - Be^{-A} + \frac{(-1)^2}{2}\int_0^1 du_1 (Be^{-(1-u_1)A}Be^{-u_1 A}) + \cdots + \frac{(-1)^{n+1}}{n+1}\int_0^1 du_1 \ldots \int_0^1 du_n (u_1^{n-1}u_2^{n-2}\ldots u_{n-1}^1 u_n^0) \times Be^{-(1-u_1)A}Be^{-u_1(1-u_2)A}Be^{-u_1u_2(1-u_3)A}\ldots Be^{-u_1u_2\ldots u_{n-1}u_nA} + \cdots\Bigg] \quad (8)$$

where A depends on some momentum, while B contains space-dependent fields. Therefore, the reconstruction of the full momentum dependence of S_1'', for instance to correctly recover the diagrams in Fig. 1, does require computing even an infinite sum of terms in the series in Eq. (8). This is extremely challenging and beyond the scope of the present paper.

Instead, we recall the indications of [33,34], which suggest that the running action entering the PT flow equation is a Wilsonian local action. This, in turn, means that the one-loop S_1'' in the right-hand side of Eq. (6) is the second derivative of a local object, whose only dependence on the square momentum comes from the double functional derivative of its kinetic part. Therefore, in our computation, we shall treat S_1 as a momentum-independent quantity, except in the presence of an external momentum associated with the two functional derivatives of S_1'' in Eq. (6). This will be the case of the computation of the two-point function at two loop where, with the help of the expansion in Eq. (8), we shall retain only the quadratic contributions in the external momentum.

III. O(N) SCALAR THEORY

We start with the action of the O(N) scalar theory (the index j runs from 1 to N)

$$S_0 = \int d^4x \left[\frac{1}{2}\partial\phi_j\partial\phi_j + \frac{m^2}{2}\phi_j\phi_j + \frac{\lambda}{4!}(\phi_j\phi_j)^2\right] \quad (9)$$

and we evaluate the β function of the renormalized coupling $\lambda_R = Z^{-2}\lambda$ (the factor Z comes from the field renormalization $\phi_R = Z^{1/2}\phi$),

$$\beta_{\lambda_R} = k\partial_k(Z^{-2}\lambda) = -2Z^{-3}\lambda k\partial_k(Z) + Z^{-2}k\partial_k\lambda, \quad (10)$$

by computing the k derivative of Z and λ, order by order, directly from the PT flow.

By expanding Z in powers of the coupling, $Z = Z_0 + Z_1 + Z_2 + \cdots$, with normalization $Z_0 = 1$, it is known that $Z_1 = 0$ if the effective action does not contain vertices with odd powers of the field, and therefore, Z_1 does not contribute to Eq. (10). As sketched below Eq. (4), the flow of the coupling λ at one-loop order is obtained by replacing S_k'' with S_0'' in the right-hand side of Eq. (3), and then extracting the coefficient of the fourth power of ϕ on both sides of the equation,

$$[k\partial_k\lambda]_{1L} = -\frac{k\partial_k}{2}\mathrm{P}_{\phi^4}\left\{\int_{1/\Lambda^2}^{1/k^2}\frac{ds}{s}\int\frac{d^4p}{(2\pi)^4}\mathrm{Tr}[e^{-s[(p^2+m^2+\frac{\lambda}{6}\phi_j\phi_j)\delta_{ab}+\frac{\lambda}{3}\phi_a\phi_b]}]\right\}\bigg|_{\phi=0}, \tag{11}$$

where we inserted the explicit form of S_0'' in the exponential and split the full trace of Eq. (3) into the product of the integral over the four-momentum p times the trace, Tr, over the internal indices associated with the N field components and, finally, we introduced the projector $\mathrm{P}_{\phi^4} = (d^4/(d\phi_i)^4)$ (i.e., the fourth derivative with respect to the arbitrary field component ϕ_i) to single out the evolution of the coupling λ. The projector P_{ϕ^4} can be safely used here, as no nonlocal effects are present in the one-loop calculation and one can reduce Eq. (11) to the case of x-independent fields ϕ, with no loss of generality.

By using the following standard result $\int[d^4p/(2\pi)^4]e^{-sp^2} = 1/(16\pi^2 s^2)$, one easily finds

$$\begin{aligned}[k\partial_k\lambda]_{1L} &= -\frac{\lambda^2}{(16\pi^2)}\frac{(N+8)}{6}(k\partial_k)\int_{1/\Lambda^2}^{1/k^2}\frac{ds}{s}e^{-sm^2} \\ &= \frac{\lambda^2}{(16\pi^2)}\frac{(N+8)}{3}e^{-\frac{m^2}{k^2}}.\end{aligned} \tag{12}$$

We observe that the PT regulator automatically leads to a mass dependence, while the universal features of the β function at one and two loop order are recovered by means of mass independent renormalization schemes. Clearly, in Eq. (12), at one loop order, it is easy to consider the massless case, $m^2 = 0$, or, more generally to restrict the analysis to the regime of the running scale k: $m^2/k^2 \ll 1$, where we can approximate $e^{-(m^2/k^2)} \simeq 1$. In this regime, from Eq. (12) we recover the universal one-loop β function,

$$\beta_{\lambda_R}^{1L} = \frac{\lambda^2}{(16\pi^2)}\frac{(N+8)}{3}. \tag{13}$$

However, for the scalar theory there is no symmetry protecting the constraint $m = 0$ and it is known that the value of the mass is modified by potentially large quantum corrections. In fact, with the same procedure followed to compute the β function in Eq. (11) and, by integrating the PT variable s up to $1/k^2 = \infty$, we find the one loop corrected square mass m_c^2 (here we need to introduce the small scale ε to avoid an infrared logarithmic divergence),

$$m_c^2 = m^2 + \frac{\lambda}{(16\pi^2)}\frac{(N+2)}{6}\left[\Lambda^2 - m^2\log\left(\frac{\Lambda^2}{\varepsilon^2}\right)\right] \tag{14}$$

and, in order to keep m_c^2 finite in the limit $\Lambda \to \infty$, we have to rewrite m^2 as

$$m^2 = m_c^2 + \delta_c, \tag{15}$$

where the counterterm δ_c is automatically defined through Eq. (14).

Since we are interested in reproducing a perturbative result, expressible as a power series in λ, we have to perform our calculation of the β function in the following regime of the physical mass m_c and the scales k and Λ,

$$\frac{\lambda\Lambda^2}{16\pi^2} \ll m_c^2 \ll k^2 < \Lambda^2, \tag{16}$$

that refines the condition under which Eq. (13) was established. According to Eq. (16), δ_c is higher order in λ with respect to m_c^2 and, by retaining the leading contribution, it reads,

$$\delta_c = -\frac{\lambda}{(16\pi^2)}\frac{(N+2)}{6}\Lambda^2. \tag{17}$$

Then, when inserting Eq. (15) into Eq. (12), we find that δ_c produces a $O(\lambda^3)$ contribution to the β function that can be neglected at one loop but, as discussed below, it is essential in cancelling a UV divergent term at two loop order. At the same time, the universal one loop β function in Eq. (13) is recovered because, in the regime displayed in Eq. (16), m_c^2/k^2 is negligible.

Let us now consider the two-loop order. As discussed above, for the renormalization of λ the x dependence of the fields at one-loop order can be neglected. Then, Eq. (6) gives

$$\begin{aligned}&[k\partial_k\lambda]_{2L} \\ &= -\frac{k\partial_k}{2}\mathrm{P}_{\phi^4}\left\{\int_{1/\Lambda^2}^{1/k^2}\frac{ds}{s}\int\frac{d^4p_e}{(2\pi)^4}\mathrm{Tr}[e^{-sS_0''}(-sS_1'')]\right\}\bigg|_{\phi=0}.\end{aligned} \tag{18}$$

The use of the projector P_{ϕ^4} is allowed because the fields are treated as x-independent variables. Then, S_1'' is computed according to Eq. (7),

$$(S_1)'' = -\frac{1}{2}\int_{\Lambda^{-2}}^{s}\frac{dt}{t}\int\frac{d^4p_i}{(2\pi)^4}\mathrm{Tr}[e^{-tS_0''}(-tS_0'' + t^2 S_0''' S_0''')], \tag{19}$$

where p_i is the momentum associated with the "internal loop" of S_1 [to be distinguished from p_e, which is instead associated with the "external loop" in Eq. (18)].

After replacing Eq. (19) in Eq. (18), by applying the projector P_{ϕ^4} and integrating on the momentum variables p_i and p_e and by retaining mass m^2 effects only at lowest order in the coupling, $O(\lambda^0)$, we find

$$[k\partial_k\lambda]_{2L} = \frac{\lambda^3}{(16\pi^2)^2} k\partial_k \left\{ \int_{1/\Lambda^2}^{1/k^2} \frac{ds}{4s^2} e^{-sm_c^2} \int_{1/\Lambda^2}^{s} \frac{dt}{t^3} e^{-tm_c^2} \times \left[\frac{t^3(N^2+30N+104)}{9} + \frac{st^2(2N^2+28N+128)}{9} + \frac{s^2t(N^2+10N+16)}{9} \right] \right\}. \tag{20}$$

For the determination of the two-loop β function from Eq. (20), we observe that, after integrating in the PT variable t, the last term in square brackets yields a contribution proportional to Λ^2 that is precisely cancelled by the counterterm δ_c in Eq. (17) which, as observed before, when inserted in Eq. (12) gives a two-loop, $O(\lambda^3)$, term. In addition, the terms proportional to (st^2) in square brackets of Eq. (20) can be discarded in our computation because, after the integration in t and s, they produce a square logarithm, $[\log^2(\Lambda/k)]$, which is reabsorbed in the one loop renormalization of the coupling constant. Then, by sticking to the scale hierarchy in Eq. (16), we neglect m_c^2 in Eq. (20), as we did before for the one loop β function, and we get

$$[k\partial_k\lambda]_{2L} = -\frac{\lambda^3}{(16\pi^2)^2}\frac{(10N+44)}{9}. \tag{21}$$

Finally, we evaluate the wave-function renormalization Z, which enters the β function computation as indicated in Eq. (10). Z renormalizes the derivative term of Eq. (9) and it can be computed from the derivative of the two-point function with respect to the square external momentum q^2,

$$Z = \left(\frac{\partial}{\partial q^2} \left[\frac{\delta^2 S}{\delta\tilde{\phi}_i(q)\delta\tilde{\phi}_i(-q)} \right]_{\phi=0} \right)_{q^2=0}. \tag{22}$$

Note that the index i in Eq. (22) is not summed, as it simply indicates a generic component of the field, and Z does not depend on its particular value. Moreover, as mentioned at the beginning of this section, the first nontrivial contribution comes from S_2, since the two-point function shows explicit dependence on the external momentum q starting from two-loop order, which is essential to extract a nonzero contribution in Eq. (22).

Then, in order to select the linear dependence in q^2 of S_2'' we need to use the expansion in Eq. (8) to properly evaluate the two functional derivatives of S_1'', before inserting it into Eq. (6). Moreover, since in Eq. (22) one has to set $\phi = 0$, it is evident that only the term proportional to B^2 is actually relevant in the present computation. Therefore, we find

$$Z_2 = \left[\frac{\partial}{\partial q^2} \left(-\int_{1/\Lambda^2}^{1/k^2} \frac{ds}{4} \int \frac{d^4p_e}{(2\pi)^4} \int_{1/\Lambda^2}^{s} \frac{dt}{t} \int \frac{d^4p_i}{(2\pi)^4} \times \int_0^1 du \frac{\lambda^2(2N+4)}{3} e^{-sp_e^2} e^{-tp_i^2(1-u)} e^{-su(p_i+q)^2} \right) \right]_{q^2=0} = \frac{\lambda^2}{(16\pi^2)^2}\frac{(N+2)}{36}\log\frac{\Lambda^2}{k^2}, \tag{23}$$

where, consistently with the previous computation, we neglect the mass m_c^2 by assuming the scale hierarchy in Eq. (16) is respected.

By plugging the results of Eqs. (21) and (23) into Eq. (10), we read the correct two-loop β function for the O(N) model,

$$\beta_{\lambda_R}^{2L} = \frac{\lambda^3}{(16\pi^2)^2}\left(\frac{(N+2)}{9} - \frac{(10N+44)}{9} \right) = -\frac{\lambda^3}{(16\pi^2)^2}\frac{3N+14}{3}. \tag{24}$$

Moreover, since the two-loop anomalous dimension η is related to Z by the definition $Z = 1 - \eta\log(k/\Lambda)$, we find from Eq. (23)

$$\eta = \frac{\lambda^2}{(16\pi^2)^2}\frac{N+2}{18}. \tag{25}$$

Equations (13), (24), and (25) are the expected universal renormalization properties of the O(N) scalar theory at two-loop level, directly derived from the PT flow equations.

Before concluding this section we add a couple of comments. First, in the above computation we made use of the PT flow in Eq. (3), which can be regarded as the limiting case of the class of PT flows, defined by a "smooth" regulator parametrized by the real number $n > 1$ (usually chosen as an integer), whose flow after some manipulations reads

$$k\partial_k S_k = \mathrm{TR}\left(1 + \frac{S_k''}{nk^2} \right)^{-(n+1)}, \tag{26}$$

and, in the limit $n \to \infty$, Eq. (26) reproduces the flow in Eq. (3). Then, it is easy to realize that the one-loop β function extracted from Eq. (26) coincides with Eq. (13). However, the two-loop β function and anomalous dimension from

Eq. (26) contain additional $O(1/n)$ terms, with respect to Eqs. (24) and (25), that vanish only in the limit $n \to \infty$. Therefore, at two-loop order, these universal properties are only recovered by the PT flow in Eq. (3).

Second, we remark that the two-loop β-function computation of the O(N) theory by means of the exponential PT flow, was already studied in [36] where, instead of working with the full action S_k flow as in the present case, the PT flow of the two-, four-, and six-point vertices is analyzed and the nesting of the one loop diagrams into the vertex flow equations is used to evaluate two-loop diagrams. Actually, the two-loop β function found in [36] shows some spurious dependence on ratios of various momenta and of the running scale k, which disappear in the limit $k \to 0$ and, consequently, the correct two-loop result was obtained by taking this limit. Yet, this limit has no physical meaning and the correct β function should not show such spurious dependence.

However, it is not difficult to show that in the calculation performed in [36] the two-loop β function can also be recovered by keeping k finite and setting instead to zero all the external momenta of the one-loop diagrams, when they are inserted in the flow equations to compute the two-loop diagrams [except for the anomalous dimension computation where the dependence on the external momentum q of the two-point function has to be retained at least to order q^2, to find a nonvanishing contribution to Z according to its definition in Eq. (22)]. The latter procedure shares the same spirit of the computation discussed in this paper, where only the local sector (with constant fields) of S_1 is used to evaluate S_2, again expect for the calculation of the wave function renormalization where $O(q^2)$ terms are maintained. In this sense, our present findings on the two-loop β function fully agree with those of [36].

IV. SU(N) YANG-MILLS THEORY

Let us now consider the SU(N) Yang-Mills theory, defined by the antisymmetric tensor, built by means of the gluon field $\hat{A}^a_\mu(x)$ $(a = 1, 2, \ldots, (N^2 - 1))$,

$$F^a_{\mu\nu}(\hat{A}) = \partial_\mu \hat{A}^a_\nu(x) - \partial_\nu \hat{A}^a_\mu(x) + g f^{abc} \hat{A}^b_\mu(x) \hat{A}^c_\nu(x), \qquad (27)$$

where f^{abc} are the totally antisymmetric structure constants of the SU(N) group.

Since the use of the background field method [37] is particularly suitable to construct the Wilsonian flow [33], we split the field into background and fluctuations $\hat{A}^a_\mu = A^a_\mu + Q^a_\mu$ and perform the functional integration over the fluctuations Q^a_μ. To this purpose, the background gauge-fixing term is introduced as the covariant derivative, depending on the background field only, applied to the fluctuations

$$D^{ac}_\mu(A) Q^c_\mu = [\partial_\mu \delta^{ac} + g f^{abc} A^b_\mu(x)] Q^c_\mu(x), \qquad (28)$$

so that the full gauge action, result of the sum of the Yang-Mills and gauge-fixing part, is

$$\begin{aligned} S_A &= S_{\mathrm{YM}} + S_{gf} \\ &= \int d^4x' \left\{ \frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} + \frac{1}{2} (D^{ab}_\mu(A) Q^b_\mu)(D^{ac}_\nu(A) Q^c_\nu) \right\}, \end{aligned} \qquad (29)$$

where $F^a_{\mu\nu}$ is a function of the full field $\hat{A}^a_\mu(x)$. The corresponding ghost action is

$$S_G = \int d^4x' \{ -\bar{\theta}_a (D^{ab}_\mu(A) D^{bc}_\mu(\hat{A})) \theta_c \}, \qquad (30)$$

and the ghost field is also to be split into background and fluctuations $\theta_a = \chi_a + c_a$, in order to integrate out the fluctuations c_a.

It is well known that the integration over the gluon and ghost fluctuations produces, after turning off the ghost background $\chi_a = 0$, an effective action that depends on the background A^a_μ and is invariant under gauge transformations of the background [37]. However, preserving gauge invariance means that the integration of the quantum fluctuations must dress the tree level Yang-Mills Lagrangian density by a multiplicative factor, Z_A,

$$\frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} \longrightarrow \frac{Z_A}{4} F^a_{\mu\nu} F^a_{\mu\nu} \qquad (31)$$

where Z_A is regarded as the background field renormalization constant: $A_R = \sqrt{Z_A} A$. Then, since the internal structure of $F^a_{\mu\nu}(A)$, defined in Eq. (27) must also be preserved, one immediately realizes that the quantum corrections to the coupling g must be inversely related to those of the background field,

$$g_R = \frac{g}{\sqrt{Z_A}}, \qquad (32)$$

where it is understood that g is kept constant, i.e., independent of the renormalization scale, while all the effects of the renormalization are included in the parameter Z_A, in contrast with the scalar case where both coupling and field are affected by quantum corrections in an uncorrelated way.

To determine the loop corrections within the PT flow framework, it is then sufficient to extract Z_A from the analogous versions for the Yang-Mills case of Eqs. (4) and (5). The computation of Z_A results in a double expansion in powers of g^2 and of $\log(k^2/\Lambda^2)$, as we discard terms that vanish in the limit $\Lambda \to \infty$,

$$Z_A = 1 + \left[\frac{g^2N}{16\pi^2}\beta_1 + \left(\frac{g^2N}{16\pi^2}\right)^2\beta_2 + O(g^6)\right]\log\frac{k^2}{\Lambda^2} + O\left(\log^2\frac{k^2}{\Lambda^2}\right). \tag{33}$$

Then, we get β_{g_R} from Eq. (32),

$$\beta_{g_R} = k\partial_k g_R = -\frac{g}{2Z_A^{\frac{3}{2}}}k\partial_k(Z_A) = -g\left[\frac{g^2N}{16\pi^2}\beta_1 + \left(\frac{g^2N}{16\pi^2}\right)^2\beta_2 + O(g^6)\right]. \tag{34}$$

In fact, in computing Eq. (34), we neglect the higher powers of the logarithm in Eq. (33), because they are automatically taken into account in β_{g_R}, if one replaces $g \to g_R$ in the right-hand side of Eq. (34). Furthermore, as far as the coefficients $O(g^3)$ and $O(g^5)$ of β_{g_R} in Eq. (34) are concerned, we can neglect the factor $Z_A^{-\frac{3}{2}}$ as it is proportional to powers of the logarithm.

In computing the one-loop contribution from the PT flow we cannot use Eq. (4), since in this case the full action consists of a bosonic sector, S_A, and a fermionic sector, S_G, so that the Hessian contains nondiagonal terms. In fact, by recalling that S_A does not contain ghost fields, while S_G is quadratic in the ghost field and linear in the gluon fluctuations Q, we have,

$$\begin{aligned}
K^{ab}_{\mu\nu}(A) &= \frac{\delta^2 S_A(A,Q)}{\delta Q^a_\mu(x)\delta Q^b_\nu(y)}\bigg|_{Q=c=0};\\
(\mathcal{D}_{\bar c,Q})^{ab}_\nu(A,\chi) &= \frac{\delta^2 S_G(A,Q,\chi,c)}{\delta \bar c^a(x)\delta Q^b_\nu(y)}\bigg|_{Q=c=0}\\
(\mathcal{D}_{Q,c})^{ab}_\mu(A,\chi) &= \frac{\delta^2 S_G(A,Q,\chi,c)}{\delta Q^a_\mu(x)\delta c^b(y)}\bigg|_{Q=c=0};\\
H^{ab}(A) &= \frac{\delta^2 S_G(A,Q,\chi,c)}{\delta \bar c^a(x) c^b(y)}\bigg|_{Q=c=0}
\end{aligned} \tag{35}$$

with diagonal parts

$$\begin{aligned}
K^{ab}_{\mu\nu}(A) = \{&[-\delta^{ab}\partial_x^2 + 2gf^{abc}A^c\cdot\partial_x + gf^{abc}(\partial_x\cdot A^c)\\
&- g^2 f^{amc}f^{clb}A^m\cdot A^l]\delta_{\mu\nu}\\
&+ 2gf^{abc}F^c_{\mu\nu}\}\delta^4(x-y)
\end{aligned} \tag{36}$$

and

$$\begin{aligned}
H^{ab}(A) = [&-\delta^{ab}\partial_x^2 + 2gf^{abc}A^c\cdot\partial_x + gf^{abc}(\partial_x\cdot A^c)\\
&- g^2 f^{amc}f^{clb}A^m\cdot A^l]\delta^4(x-y).
\end{aligned} \tag{37}$$

Then, the PT representation requires the computation of the supertrace of the nondiagonal Hessian, which yields the one loop contribution to the Yang-Mills effective action

$$S_{\mathrm{YM1}} = -\frac{1}{2}\int_{1/\Lambda^2}^{1/k^2}\frac{ds}{s}\mathrm{TR}[e^{-sK^{ab}_{\mu\nu}}] + \int_{1/\Lambda^2}^{1/k^2}\frac{ds}{s}\mathrm{TR}[e^{-s[H^{ab}+M^{ab}]}], \tag{38}$$

where, again, TR stands for the trace on all internal indices and integrals over spacetime and momentum coordinates, and the coefficients $-(1/2)$ and 1 in front of the traces respectively provide the weight of the bosonic (gluon) and fermionic (ghost) degrees of freedom. Finally, the operator M^{ab}, generated by the nondiagonal part, is

$$M^{ab} = -(\mathcal{D}_{\bar c,Q})^{ac}_{\mu_1}(A,\chi)\cdot[K^{-1}]^{cd}_{\mu_1\mu_2}(A)\cdot(\mathcal{D}_{Q,c})^{db}_{\mu_2}(A,\chi). \tag{39}$$

However, at least at one-loop order, we are interested in the effective action S_{YM1} with zero background ghost field, $\bar\chi = \chi = 0$, and therefore we get no contribution from the nondiagonal part as, in this case, it is $M^{ab} = 0$. The following step is to recover Z_A from the expansion of the exponential in Eq. (38) with $M^{ab} = 0$. We notice that, for the sole purpose of determining β_1, it is sufficient to reconstruct just the $O(A^4)$ part of the product $F(A)F(A)$, and this can be achieved by considering constant fields A, i.e., by neglecting all space-derivatives of A. Therefore, the expansion of the exponential is straightforward and Z_A at order $O(g^2)$ is easily determined. Then, Eqs. (33) and (34) give the correct one-loop coefficient of the Yang-Mills β function, as it was already obtained in [18],

$$\beta_1 = \frac{11}{3}. \tag{40}$$

Additionally, we verified that this procedure on the one hand does not generate any term quadratic in the fields $O(A^2)$ and, on the other hand, it does not produce any renormalization of the gauge-fixing action S_{gf}. This is in agreement with the gauge invariance property of S_{YM1} proved in [37], as the generation of either term would break gauge symmetry.

So far the computation is quite simple because Eq. (40) only needs the projection of Eq. (38) on constant fields, while the determination of the full one loop effective action S_{YM1}, essential to compute the two-loop effects, requires a more careful analysis. In fact, S_{YM1} not only contains the field derivative contribution to the term proportional to F^2 but it also includes novel gauge invariant terms, e.g., $O(F^3)$ terms.

Adapting Eq. (6) to contain both the bosonic (gluon) and fermionic (ghost) degrees of freedom, it is possible to extract the two-loop effective action of the Yang-Mills theory within PT regularization as

$$S_{\mathrm{YM2}} = \int_{1/\Lambda^2}^{1/k^2} \frac{ds}{s} \left\{ -\frac{1}{2} \mathrm{TR}[-sS''_{\mathrm{YM1}} e^{-sK}] + \mathrm{TR}[-s\ddot{S}_{\mathrm{YM1}} e^{-sH}] \right\}, \tag{41}$$

where color and Lorentz indices are omitted for simplicity, the double prime indicates the double functional derivative with respect to the field A while the double dot refers to the ghost and antighost functional derivatives. We recall that in Eq. (41), the full S_{YM1} must be retained in the form given in Eq. (38), without setting the ghost background to zero the subsequent computations. Therefore, S''_{YM1} and $\ddot{S}_{\mathrm{YM1}}$ in Eq. (41) must be written according to the expansion in Eq. (8),

$$\begin{aligned} S''_{\mathrm{YM1}} = & -\frac{1}{2} \int_{1/\Lambda^2}^{s} \frac{dt}{t} \left\{ \mathrm{TR}\left[\int_0^1 du t^2 K' e^{-t(1-u)K} K' e^{-tuK} \right] + \mathrm{TR}[-tK'' e^{-tK}] \right\} \\ & + \int_{1/\Lambda^2}^{s} \frac{dt}{t} \left\{ \mathrm{TR}\left[\int_0^1 du t^2 R' e^{-t(1-u)R} R' e^{-tuR} \right] + \mathrm{TR}[-tR'' e^{-tR}] \right\}, \end{aligned} \tag{42}$$

$$\ddot{S}_{\mathrm{YM1}} = \int_{1/\Lambda^2}^{s} \frac{dt}{t} \left\{ \mathrm{TR}\left[\int_0^1 du t^2 \dot{R} e^{-t(1-u)R} \dot{R} e^{-tuR} \right] + \mathrm{TR}[-t\ddot{R} e^{-tR}] \right\}, \tag{43}$$

where we used the compact notation $R = (H + M)$.

Equation (41) is very difficult to treat and the reconstruction of the $O(A^4)$ sector of the term F^2 is very intricate. Therefore, in order to minimize the effort in determining β_2, we exploit the relation between the renormalization factor of the field A and of the coupling g, valid in the background gauge [37]. Then, instead of looking at the full S_{YM2}, we can extract the contribution of the two-loop field renormalization from the coefficient of the square external momentum q^2 in the two-point Green function at two-loop order. The latter is obtained by taking two functional derivatives of S_{YM2} in Eq. (41) and then setting all fields to zero, analogously to the computation of Z in Eq. (22). With vanishing fields, the computation becomes much simpler.

Moreover, of all the possible ways of performing two functional derivatives in Eq. (41), only the application of both derivatives to S''_{YM1} contributes to β_2, while the derivatives of other field dependent terms, such as $\ddot{S}_{\mathrm{YM1}}$ or the exponential factors e^{-sK} and e^{-sH}, either give zero contribution to Z_A in the limit $\Lambda \to \infty$, or produce terms proportional to $\log^2(k^2/\Lambda^2)$, which are irrelevant to β_2. On the other hand, the application of the two functional derivatives to S''_{YM1} in Eq. (41) produces three relevant contributions.

The first contribution, indicated below as Σ_1, comes from the application of the two derivatives to the nonexponential part of S''_{YM1} in Eq. (42),

$$\begin{aligned} \Sigma_1 = & -\int_{1/\Lambda^2}^{s} dt t \int \frac{d^4 p_i}{(2\pi)^4} \left\{ \mathrm{Tr}\left[\int_0^1 du \delta[K'] e^{-t(1-u)(p_i+q)^2} \delta[K'] e^{-tup_i^2} \right] \right. \\ & \left. - 2\mathrm{Tr}\left[\int_0^1 du \delta[H'] e^{-t(1-u)(p_i+q)^2} \delta[H'] e^{-tup_i^2} \right] \right\}. \end{aligned} \tag{44}$$

In Eq. (44) we indicate the functional derivatives that define the two-point function with a δ, and it is understood that eventually all fields are set to zero which, in turn, implies the replacement of R' with H', because of the dependence of M on the ghost fields, and also implies the suppression of the fields in the exponential terms displayed above. The momentum q is the external momentum related to the functional derivatives and the trace in the first and second line are on different sets of internal indices, as in the first line it covers both color and Lorentz indices, while in the second line it refers to color indices only. Finally, in Eq. (44), an overall factor 2 is included to count the two equivalent ways of performing the functional derivatives.

The second contribution, Σ_2, is obtained by applying both derivatives on the exponential part of S''_{YM1} in Eq. (42),

$$\begin{aligned} \Sigma_2 = & -\int_{1/\Lambda^2}^{s} \frac{dt}{2t} \int \frac{d^4 p_i}{(2\pi)^4} \left\{ \mathrm{Tr}\left[(t^4 K'K' - t^3 K'') \int_0^1 du \delta[K] e^{-t(1-u)(p_i+q)^2} \delta[K] e^{-tup_i^2} \right] \right. \\ & \left. - 2\mathrm{Tr}\left[(t^4 H'H' - t^3 H'') \int_0^1 du \delta[H] e^{-t(1-u)(p_i+q)^2} \delta[H] e^{-tup_i^2} \right] \right\}. \end{aligned} \tag{45}$$

In this case, the exponentials in Eq. (42) were first rearranged into a single factor as they all depend on the same momentum [this produces the brackets in front of the integral in the variable u, both in the first and second line of Eq. (45)], and then

$\delta[K]$ and $\delta[H]$ are obtained from the double derivation of the exponential factors. Then, as before, all fields are set to zero. Note that, in Σ_2, the multiplicity factor 2, due to the two functional derivatives, is canceled by the factor $(1/2)$ coming from the further expansion of the exponential, according to Eq. (8).

The third contribution, Σ_3 is obtained by applying one derivative on the exponential factor, and the other derivative on one of the two factors K' (or R') in Eq. (42). In this case, all possible ways of arranging the two derivatives produces a multiplicative factor 4, so that,

$$\Sigma_3 = 2\int_{1/\Lambda^2}^{s}\frac{dt}{t}\int\frac{d^4p_i}{(2\pi)^4}\left\{\mathrm{Tr}\left[t^3K'\int_0^1 du\delta[K]e^{-t(1-u)(p_i+q)^2}\delta[K']e^{-tup_i^2}\right]\right.$$
$$\left.-2\mathrm{Tr}\left[t^3H'\int_0^1 du\delta[H]e^{-t(1-u)(p_i+q)^2}\delta[H']e^{-tup_i^2}\right]\right\}. \tag{46}$$

Similarly as for the first two contributions, Eq. (46) must be evaluated for vanishing fields.

When Σ_1, Σ_2, Σ_3 are rearranged within Eq. (41), the corresponding coefficients of the external momentum q^2 produce a contribution to β_2 in Eq. (33), which we indicate as $\beta_2^{(i)}$, with $i = 1, 2, 3$. We find $\beta_2^{(1)} = -5/2$, $\beta_2^{(2)} = 53/6$ and $\beta_2^{(3)} = 5$, so that we finally obtain the well-known two-loop coefficient of the Yang-Mills β function,

$$\beta_2 = \beta_2^{(1)} + \beta_2^{(2)} + \beta_2^{(3)} = \frac{34}{3}. \tag{47}$$

V. CONCLUSIONS

The reduction of the PT flow to evaluate the perturbative series that defines the β-function proved successful in determining the first two coefficients of the expansion in powers of the coupling by means of a compact and rather simple technique, both for the scalar and the Yang-Mills field theory. This somehow explains the reliability of the PT flow predictions in several applications. In fact, since the one- and two-loop coefficients of the β function are universal, in the sense that they do not depend on the details of the regularization procedure at least for mass-independent regularization schemes, the present computation indicates that the PT flow preserves the main physical features associated with the renormalization of these theories, with the warning for the case of the scalar theory, where mass terms are unavoidably generated by the renormalization procedure, that the running scale must be kept very large with respect to the mass, so that every residual regulator dependence is negligible.

On the other hand, in the case of the Yang-Mills theory, we checked that no gauge symmetry-breaking "mass" term (i.e., proportional to A^2), is generated in the analyzed quantum corrections, and this warrants that no spurious regulator and mass dependence affects the determination of the β function.

Certainly, the successful computation of one- and two-loop β function is not sufficient to warrant the reconstruction of the full perturbative structure of the effective action as it was stated in [7,8]. In fact, we verified that such a reconstruction requires a careful expansion and rearrangement of the exponential structure of the PT flow, and it is not evident at all if this is realizable in each single diagram. In particular, for the Yang-Mills case we are not able to verify that a manifestly gauge-invariant effective action is generated at each order of the perturbative expansion. Fortunately, the latter drawback does not affect the two-loop β function, which is strictly related to the computable gauge invariant part of the two-point function.

ACKNOWLEDGMENTS

We are grateful to A. Bonanno and G. Oglialoro for several clarifying discussions. The work of D. R. is supported by the National Science Centre (Poland) under the research Grant No. 2017/26/E/ST2/00470.

DATA AVAILABILITY

No data were created or analyzed in this study.

[1] K. G. Wilson and J. B. Kogut, The renormalization group and the epsilon expansion, Phys. Rep. **12**, 75 (1974).

[2] J. Berges, N. Tetradis, and C. Wetterich, Nonperturbative renormalization flow in quantum field theory and statistical physics, Phys. Rep. **363**, 223 (2002).

[3] N. Dupuis, L. Canet, A. Eichhorn, W. Metzner, J. M. Pawlowski, M. Tissier, and N. Wschebor, The nonperturbative functional renormalization group and its applications, Phys. Rep. **910**, 1 (2021).

[4] C. Wetterich, Exact evolution equation for the effective potential, Phys. Lett. B **301**, 90 (1993).

[5] T. R. Morris, The exact renormalization group and approximate solutions, Int. J. Mod. Phys. A **09**, 2411 (1994).

[6] M. Reuter and C. Wetterich, Effective average action for gauge theories and exact evolution equations, Nucl. Phys. **B417**, 181 (1994).

[7] D. F. Litim and J. M. Pawlowski, Perturbation theory and renormalization group equations, Phys. Rev. D **65**, 081701 (2002).

[8] D. F. Litim and J. M. Pawlowski, Completeness and consistency of renormalisation group flows, Phys. Rev. D **66**, 025030 (2002).

[9] T. Papenbrock and C. Wetterich, Two loop results from one loop computations and nonperturbative solutions of exact evolution equations, Z. Phys. C **65**, 519 (1995).

[10] F. J. Wegner and A. Houghton, Renormalization group equation for critical phenomena, Phys. Rev. A **8**, 401 (1973).

[11] S.-B. Liao and J. Polonyi, Blocking transformation in field theory, Ann. Phys. (N.Y.) **222**, 122 (1993).

[12] S.-B. Liao, J. Polonyi, and D.-p. Xu, Quantum and thermal fluctuations in field theory, Phys. Rev. D **51**, 748 (1995).

[13] S.-B. Liao, On connection between momentum cutoff and the proper time regularizations, Phys. Rev. D **53**, 2020 (1996).

[14] A. Bonanno and D. Zappala, Nonperturbative renormalization group approach for a scalar theory in higher derivative gravity, Phys. Rev. D **55**, 6135 (1997).

[15] A. Bonanno and D. Zappala, Two loop results from the derivative expansion of the blocked action, Phys. Rev. D **57**, 7383 (1998).

[16] A. Bonanno, V. Branchina, H. Mohrbach, and D. Zappala, Wegner-Houghton equation and derivative expansion, Phys. Rev. D **60**, 065009 (1999).

[17] M. Oleszczuk, A symmetry preserving cutoff regularization, Z. Phys. C **64**, 533 (1994).

[18] S.-B. Liao, Operator cutoff regularization and renormalization group in Yang-Mills theory, Phys. Rev. D **56**, 5008 (1997).

[19] A. Bonanno and D. Zappala, Towards an accurate determination of the critical exponents with the renormalization group flow equations, Phys. Lett. B **504**, 181 (2001).

[20] D. F. Litim and J. M. Pawlowski, Predictive power of renormalization group flows: A comparison, Phys. Lett. B **516**, 197 (2001).

[21] M. Mazza and D. Zappala, Proper time regulator and renormalization group flow, Phys. Rev. D **64**, 105013 (2001).

[22] D. F. Litim and D. Zappala, Ising exponents from the functional renormalisation group, Phys. Rev. D **83**, 085009 (2011).

[23] A. Bonanno, A. Codello, and D. Zappalà, Structural aspects of FRG in quantum tunneling computations, Ann. Phys. (Amsterdam) **445**, 169090 (2022).

[24] A. Bonanno and M. Reuter, Proper time flow equation for gravity, J. High Energy Phys. 02 (2005) 035.

[25] A. Bonanno and F. Guarnieri, Universality and symmetry breaking in conformally reduced quantum gravity, Phys. Rev. D **86**, 105027 (2012).

[26] A. Bonanno, M. Conti, and D. Zappalà, The conformal sector of quantum Einstein gravity beyond the local potential approximation, Phys. Lett. B **847**, 138311 (2023).

[27] A. Bonanno, G. Oglialoro, and D. Zappalà, Gauge and parametrization dependence of quantum Einstein gravity within the proper time flow, Phys. Rev. D **112**, 026002 (2025).

[28] A. Bonanno, E. Glaviano, and G. P. Vacca, Proper-time functional renormalization in $O(N)$ scalar models coupled to gravity, arXiv:2508.00807.

[29] A. M. Bonanno, R. A. Konoplya, G. Oglialoro, and A. Spina, Regular black holes from proper-time flow in quantum gravity and their quasinormal modes, shadow and Hawking radiation, J. Cosmol. Astropart. Phys. 12 (2025) 042.

[30] K. Falls and R. Ferrero, Asymptotic safety within on-shell perturbation theory, J. High Energy Phys. 08 (2025) 173.

[31] D. F. Litim and J. M. Pawlowski, Wilsonian flows and background fields, Phys. Lett. B **546**, 279 (2002).

[32] C. Wetterich, Simplified functional flow equation, Phys. Lett. B **864**, 139435 (2025).

[33] S. P. de Alwis, Exact RG flow equations and quantum gravity, J. High Energy Phys. 03 (2017) 118.

[34] A. Bonanno, S. Lippoldt, R. Percacci, and G. P. Vacca, On exact proper time Wilsonian RG flows, Eur. Phys. J. C **80**, 249 (2020).

[35] J. S. Schwinger, On gauge invariance and vacuum polarization, Phys. Rev. **82**, 664 (1951).

[36] D. Zappala, Perturbative and nonperturbative aspects of the proper time renormalization group, Phys. Rev. D **66**, 105020 (2002).

[37] L. F. Abbott, The background field method beyond one loop, Nucl. Phys. **B185**, 189 (1981).