

Anomaly flow and anomaly cancellation

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(Received 3 June 2026; accepted 30 July 2026; published 19 August 2026)

In gauge theory on 5D orbifolds the magnitude of chiral anomalies of 4D gauge fields changes with the value of the Aharonov-Bohm (AB) phase θ_H in the fifth dimension. Anomaly flows with the AB phase. In particular in the Randall-Sundrum (RS) warped space gauge couplings of 4D fermions depend on bulk mass parameters of 5D fermion multiplets. We show that in the GUT-inspired $SO(5) \times U(1) \times SU(3)$ gauge-Higgs unification model in the RS warped space the total anomalies including contributions of all Kaluza-Klein excited modes of fermions become universal, being independent of the values of the bulk mass parameters of fermions and expressed in terms of the values of W and Z wave functions at the ultraviolet and infrared branes in the RS space. It is shown that cancellation of gauge anomalies is achieved in each generation even for $\theta_H \neq 0$.

DOI: [10.1103/ds9g-5m21](https://doi.org/10.1103/ds9g-5m21)

I. INTRODUCTION

In 4D gauge theory cancellation of gauge anomalies is necessary for the consistency of theory. In the standard model (SM) of $SU(3)_C \times SU(2)_L \times U(1)_Y$ gauge theory the cancellation is achieved in each generation [1–4]. Although the SM has been successful in describing phenomena at low energies, it has a gauge hierarchy problem. Gauge-Higgs unification (GHU) models have been proposed to solve the gauge hierarchy problem by unifying the Higgs boson and gauge fields in five dimensions. The gauge symmetry $SU(2)_L \times U(1)_Y$ is dynamically broken by an Aharonov-Bohm (AB) phase, θ_H , in the fifth dimension. The Higgs boson appears as a 4D fluctuation mode of θ_H , and its mass is generated by quantum effects of the AB phase θ_H [5–22]. In this paper we reexamine the issue of anomaly cancellation in GHU models.

Among various GHU models the $SO(5) \times U(1) \times SU(3)$ GHU in the Randall-Sundrum (RS) warped space [22], which is inspired by $SO(11)$ gauge-Higgs grand unification models [23], is a promising realistic model. The grand unified theory (GUT)-inspired GHU gives nearly the same phenomenology at low energies as in the SM. It has been shown recently that the Cabibbo-Kobayashi-Maskawa (CKM) matrix in the quark sector is reproduced [24] and that neutrino oscillations are explained and the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix in

the normal ordering with $\delta_{CP} = \pi$ naturally arises [25]. Further it predicts distinct deviation from the SM in forward-backward asymmetry in fermion pair production in electron-positron (e^-e^+) collisions and smaller Higgs cubic and quartic self-couplings than those in the SM [26].

One of the characteristics of GHU models in the RS space is that gauge couplings of quarks and leptons to W and Z bosons at $\theta_H \neq 0$ are not exactly the same as those in the SM, showing tiny deviation from those in the SM. Furthermore those couplings depend not only on θ_H , but also on bulk mass parameters of fermion multiplets. It implies that if only contributions of quarks and leptons to gauge anomalies, namely those of the lowest modes of their Kaluza-Klein (KK) towers, were taken into account, then gauge anomalies would not be canceled in a rigorous sense. From the viewpoint of effective 4D gauge theory this would cause a serious danger for theoretical consistency.

GHU models on an orbifold such as the RS space exhibit the phenomenon of anomaly flow. The magnitude of gauge anomalies varies as the AB phase θ_H changes. Previously this phenomenon was investigated in an $SU(2)$ toy model [27–29]. It was shown there that one has to incorporate, for gauge anomalies, contributions of all KK excited modes of fermions running along internal triangular loops. With all those contributions taken into account, total anomalies become universal. We note that the phenomenon of anomaly flow is different from that of anomaly inflow [30]. Further it was shown in Ref. [28] that the magnitude of anomalies is expressed in terms of the values of the wave functions of gauge fields on the ultraviolet (UV) and infrared (IR) branes. In this paper we show that the magnitude of gauge anomalies in the GUT-inspired GHU model is expressed in terms of the values of the W and Z wave functions on the UV- and IR-branes, holographic formulas for anomalies

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being established. With these formulas cancellation of gauge anomalies in the GUT-inspired GHU is achieved.

In Sec. II the GUT-inspired $SO(5) \times U(1)_X \times SU(3)_C$ GHU model is described. With a general θ_H mass spectra and wave functions of gauge field and fermion multiplets are given. In Sec. III gauge couplings of all KK modes of fermions are given. In Sec. IV chiral anomalies associated with gauge bosons in the SM are expressed in terms of gauge couplings derived in Sec. III. In Sec. V holographic formulas for chiral anomalies are derived. The validity of the formulas is checked by numerical evaluations of gauge couplings of all KK fermions as well. The θ_H -dependence of anomalies is also investigated. With the use of holographic formulas for anomalies we show the cancellation of gauge anomalies in the GUT-inspired GHU in Sec. VI. A brief summary is given in Sec. VII. In Appendix A basis functions used to express wave functions of gauge and fermion fields in the RS space are given. In Appendix B wave functions of fermions with a vanishing bulk mass parameter $c = 0$ are explained. In Appendix C numerical results for establishing universality of anomaly flow in the GUT-inspired GHU model are summarized.

II. GUT-INSPIRED GHU

We analyze gauge anomaly in the GUT inspired $SO(5) \times U(1)_X \times SU(3)_C (\equiv \mathcal{G})$ GHU [22,26]. The model is defined in the RS warped space with the metric [31]

$$ds^2 = G_{MN} dx^M dx^N = e^{-2\sigma(y)} \eta_{\mu\nu} dx^\mu dx^\nu + dy^2, \quad (2.1)$$

where $M, N = 0, 1, 2, 3, 5$, $\mu, \nu = 0, 1, 2, 3$, $y = x^5$, $\eta_{\mu\nu} = \text{diag}(-1, +1, +1, +1)$, $\sigma(y) = \sigma(y + 2L) = \sigma(-y)$, and $\sigma(y) = ky$ for $0 \leq y \leq L$. In terms of the conformal coordinate $z = e^{ky}$ ($0 \leq y \leq L$, $1 \leq z \leq z_L = e^{kL}$) the metric becomes

$$ds^2 = \frac{1}{z^2} \left(\eta_{\mu\nu} dx^\mu dx^\nu + \frac{dz^2}{k^2} \right). \quad (2.2)$$

The bulk region $0 < y < L$ is anti-de Sitter spacetime with a cosmological constant $\Lambda = -6k^2$, which is sandwiched by the UV brane at $y = 0$ and the IR brane at $y = L$. The warp factor z_L is large. The KK mass scale is given by $m_{\text{KK}} = \pi k / (z_L - 1) \simeq \pi k z_L^{-1}$. Typical values are $z_L \sim 10^{11}$ and $m_{\text{KK}} \sim 13$ TeV.

Gauge fields $A_M^{SO(5)}$, $A_M^{U(1)_X}$ and $A_M^{SU(3)_C}$ of $SO(5) \times U(1)_X \times SU(3)_C$ with gauge couplings g_A , g_B and g_C satisfy the orbifold boundary conditions (BCs)

$$\begin{aligned} \begin{pmatrix} A_\mu \\ A_y \end{pmatrix} (x, y_j - y) &= P_j \begin{pmatrix} A_\mu \\ -A_y \end{pmatrix} (x, y_j + y) P_j^{-1} \quad (j = 0, 1), \\ (y_0, y_1) &= (0, L). \end{aligned} \quad (2.3)$$

Here $P_0 = P_1 = P_5^{SO(5)} = \text{diag}(I_4, -I_1)$ for $A_M^{SO(5)}$ in the vector representation and $P_0 = P_1 = P_4^{SO(5)} = \text{diag}(I_2, -I_2)$ for $A_M^{SO(5)}$ in the spinorial representation. $P_0 = P_1 = 1$ for $A_M^{U(1)_X}$ and $A_M^{SU(3)_C}$. The orbifold BCs break $SO(5)$ to $SO(4) \simeq SU(2)_L \times SU(2)_R$. In the following we write $A_M^{SO(5)} = A_M$ and $A_M^{U(1)_X} = B_M$ unless confusion arises.

The 4D Higgs boson doublet $\phi_H(x)$ is contained in $A_z = (kz)^{-1} A_y$;

$$\begin{aligned} A_z^{(j5)}(x, z) &= \frac{1}{\sqrt{k}} \phi_j(x) u_H(z) + \dots, \\ u_H(z) &= \sqrt{\frac{2}{z_L^2 - 1}} z, \\ \phi_H(x) &= \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_2 + i\phi_1 \\ \phi_4 - i\phi_3 \end{pmatrix}. \end{aligned} \quad (2.4)$$

Without loss of generality one can assume $\langle \phi_1 \rangle, \langle \phi_2 \rangle, \langle \phi_3 \rangle = 0$ and $\langle \phi_4 \rangle \neq 0$. The AB phase θ_H in the fifth dimension is given by $\langle \phi_4 \rangle = \theta_H f_H$ where

$$\begin{aligned} P \exp \left\{ ig \oint dy A_y \right\} &\sim \exp \left\{ i \left(\theta_H + \frac{H(x)}{f_H} \right) 2T^{(45)} \right\}, \\ f_H &= \frac{2}{g_w} \sqrt{\frac{k}{L(z_L^2 - 1)}}, \quad g_w = \frac{g_A}{\sqrt{L}}. \end{aligned} \quad (2.5)$$

Physics is periodic in θ_H with a period 2π . (Note that eigenvalues of $2T_{\text{sp}}^{(45)}$ in the spinorial representation are ± 1 .) The neutral Higgs field $H(x)$ is a four-dimensional fluctuation mode of θ_H .

The matter field content of the model is summarized in Table I. The orbifold BCs for fermion fields in the bulk are

$$\begin{aligned} \Psi_{(3,4)}^{q,\alpha}(x, y_j - y) &= -P_4^{SO(5)} \gamma^5 \Psi_{(3,4)}^{q,\alpha}(x, y_j + y), \\ \Psi_{(1,4)}^{\ell,\alpha}(x, y_j - y) &= -P_4^{SO(5)} \gamma^5 \Psi_{(1,4)}^{\ell,\alpha}(x, y_j + y), \\ \Psi_{(3,1)^\pm}^{q,\alpha}(x, y_j - y) &= \mp \gamma^5 \Psi_{(3,1)^\pm}^{q,\alpha}(x, y_j + y), \\ \alpha &= 1, 2, 3, \end{aligned} \quad (2.6)$$

TABLE I. Matter field content in the GUT-inspired GHU model. The $(SU(3)_C, SO(5))_{U(1)_X}$ content of each field is shown in the last column.

In the bulk ($1 \leq z \leq z_L$)	Quark Ψ^q	$(\mathbf{3}, \mathbf{4})_{\frac{1}{6}} (\mathbf{3}, \mathbf{1})_{-\frac{1}{3}}^+ (\mathbf{3}, \mathbf{1})_{-\frac{1}{3}}^-$
	Lepton Ψ^ℓ	$(\mathbf{1}, \mathbf{4})_{-\frac{1}{2}}$
	Dark fermion $\Psi^{F_q}, \Psi^{F_\ell}$	$(\mathbf{3}, \mathbf{4})_{\frac{1}{6}}, (\mathbf{1}, \mathbf{4})_{-\frac{1}{2}}$
	Dark fermion Ψ^V	$(\mathbf{1}, \mathbf{5})_0^+ (\mathbf{1}, \mathbf{5})_0^-$
On the UV brane (at $z = 1$)	Majorana fermion $\hat{\chi}$	$(\mathbf{1}, \mathbf{1})_0$
	Brane scalar $\hat{\Phi}$	$(\mathbf{1}, \mathbf{4})_{\frac{1}{2}}$

TABLE II. Parity assignment (P_0, P_1) of quark-lepton multiplets in the bulk is shown. In the second column $(SU(3)_C, SO(5))_{U(1)_X}$ content is shown. In the third column $G_{22} = SU(2)_L \times SU(2)_R$ content is shown.

Field	$\mathcal{G}$	G_{22}	Left handed	Right handed	Name		
$\Psi_{(3,4)}^{q,\alpha}$	$(\mathbf{3}, \mathbf{4})_{\frac{1}{6}}$	$[\mathbf{2}, \mathbf{1}]$	$(+, +)$	$(-, -)$	u	c	t
					d	s	b
					u'	c'	t'
$\Psi_{(3,1)}^{q,\alpha\pm}$	$(\mathbf{3}, \mathbf{1})_{\frac{1}{3}}$	$[\mathbf{1}, \mathbf{2}]$	$(-, -)$	$(+, +)$	d'	s'	b'
					$D_d^\pm$	$D_s^\pm$	$D_b^\pm$
$\Psi_{(1,4)}^{\ell,\alpha}$	$(\mathbf{1}, \mathbf{4})_{-\frac{1}{2}}$	$[\mathbf{2}, \mathbf{1}]$	$(+, +)$	$(-, -)$	ν_e	ν_μ	ν_τ
					e	μ	τ
					ν'_e	ν'_μ	ν'_τ
$\hat{\chi}^\alpha$	$(\mathbf{1}, \mathbf{1})_0$	$[\mathbf{1}, \mathbf{1}]$	$(-, -)$	$(+, +)$	e'	μ'	τ'
					η_e	η_μ	η_τ

for quark and lepton multiplets and

$$\begin{aligned}
 \Psi_{(3,4)}^{F_q,\alpha}(x, y_j - y) &= (-1)^j P_4^{SO(5)} \gamma^5 \Psi_{(3,4)}^{F_q,\alpha}(x, y_j + y), \\
 \Psi_{(1,4)}^{F_\ell,\alpha}(x, y_j - y) &= (-1)^j P_4^{SO(5)} \gamma^5 \Psi_{(1,4)}^{F_\ell,\alpha}(x, y_j + y), \\
 \Psi_{(1,5)^\pm}^{V,\beta}(x, y_j - y) &= \mp P_5^{SO(5)} \gamma^5 \Psi_{(1,5)^\pm}^{V,\beta}(x, y_j + y), \\
 \alpha &= 1, \dots, n_F, \quad \beta = 1, \dots, n_V, \quad (2.7)
 \end{aligned}$$

for dark fermion multiplets. The kinetic part of the action in the bulk of each fermion multiplet Ψ takes the form of

$$\begin{aligned}
 \int d^5x \sqrt{-G} \bar{\Psi} \mathcal{D}(c) \Psi, \\
 \mathcal{D}(c) = \Gamma^a e_a^M \left(D_M + \frac{1}{8} \omega_{bcM} [\Gamma^b, \Gamma^c] \right) - c \sigma'(y), \\
 D_M = \partial_M - i g_A A_M - i g_B Q_X B_M - i g_C A_M^{SU(3)_C}. \quad (2.8)
 \end{aligned}$$

Here $\omega_{bc} = \omega_{bcM} dx^M$ is the spin-connection 1-form. Q_X and c are $U(1)_X$ charge and bulk mass parameter of the Ψ field.

Parity assignment and $G_{22} = SU(2)_L \times SU(2)_R$ content of quark-lepton multiplets with their names are tabulated in Table II. A consistent model is obtained for small θ_H . As typical values let us take $\theta_H = 0.1$ and KK mass scale $m_{KK} = \pi k / (z_L - 1) = 13$ TeV. The parameters of the model are fixed such that m_Z , quark-lepton masses, fine structure constant α_{EM} , QCD coupling α_S , Fermi constant G_F , weak mixing angle $\sin^2 \theta_W$, and Higgs boson mass m_H are reproduced. With the mixing in the mass matrix of $\Psi_{(3,4)}^{q,\alpha}$ fields in the quark sector and the mixing in Majorana mass terms for $\hat{\chi}^\alpha$ fields, both of the CKM matrix and PMNS matrix are obtained [24,25]. The bulk mass parameters of $\Psi_{(3,4)}^{q,\alpha}$ and $\Psi_{(1,4)}^{\ell,\alpha}$ are $(c_u, c_c, c_t) = (-0.85912, -0.71913, -0.27455)$ and $(c_e, c_\mu, c_\tau) = (-1.00684, -0.79302, -0.67539)$ for $\theta_H = 0.1$ and

$m_{KK} = 13$ TeV. The mass spectra of W and top quark KK towers are depicted in Fig. 1.

Gauge couplings of quarks and leptons to W and Z bosons are determined by integrating over the overlaps of respective wave functions in the fifth dimension. Manipulations are simplified in the twisted gauge defined by

$$\begin{aligned}
 \tilde{A}_M(x, z) &= \Omega A_M \Omega^{-1} - \frac{i}{g_A} \Omega \partial_M \Omega^{-1}, \\
 \Omega(z) &= \exp\{i\theta(z) T^{45}\}, \quad \theta(z) = \theta_H \frac{z_L^2 - z^2}{z_L^2 - 1}, \quad (2.9)
 \end{aligned}$$

where T^{jk} 's are $SO(5)$ generators and $A_M = 2^{-1/2} \sum_{1 \leq j < k \leq 5} A_M^{(jk)} T^{jk}$. In the twisted gauge the

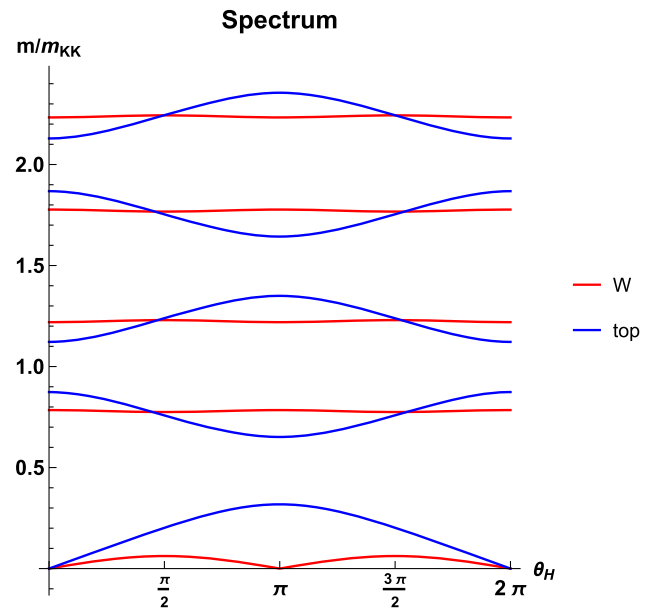


FIG. 1. Mass spectra of the W and top quark KK towers in the RS space in units of $m_{KK} = 13$ TeV. The observed m_W and m_t are reproduced at $\theta_H = 0.1$.

background field vanishes ($\tilde{\theta}_H = 0$), whereas boundary conditions at the UV-brane are modified. Boundary conditions at the IR brane remain the same as in the original gauge. Quantities in the twisted gauge are denoted by the tilde sign $\sim$.

It is convenient to adopt the $SO(5)$ basis of $\{SU(2)_L, SU(2)_R, SO(5)/SO(4)\}$;

$$\begin{aligned} \{T^\alpha, \alpha = 1 \sim 10\} &= \{T^{a_L}, T^{a_R} (a = 1, 2, 3), T^{\hat{p}} (p = 1 \sim 4)\}, \\ \begin{pmatrix} T^{a_L} \\ T^{a_R} \end{pmatrix} &= \frac{1}{4} \epsilon^{abc} T^{bc} \pm \frac{1}{2} T^{a4}, \quad T^{\hat{p}} = \frac{1}{\sqrt{2}} T^{p5}. \end{aligned} \quad (2.10)$$

$SO(5)$ gauge fields are expressed as

$$A_M = \sum_{\alpha=1}^{10} A_M^\alpha T^\alpha, \quad \begin{pmatrix} A_M^{a_L} \\ A_M^{a_R} \end{pmatrix} = \frac{1}{\sqrt{2}} \left(\frac{1}{2} \epsilon^{abc} A_M^{bc} \pm A_M^{a4} \right), \quad A_M^{\hat{p}} = A_M^{p5}. \quad (2.11)$$

The W boson field, $W_\mu(x)$, is contained in $\sum_{a=1}^2 (A_\mu^{a_L} T^{a_L} + A_\mu^{a_R} T^{a_R} + A_\mu^{\hat{a}} T^{\hat{a}})$. In the twisted gauge its KK expansion in the region $1 \leq z \leq z_L$ is given by

$$\begin{aligned} \frac{1}{\sqrt{2k}} \begin{pmatrix} \tilde{A}_\mu^{1_L} + i\tilde{A}_\mu^{2_L} \\ \tilde{A}_\mu^{1_R} + i\tilde{A}_\mu^{2_R} \\ \tilde{A}_\mu^{\hat{1}} + i\tilde{A}_\mu^{\hat{2}} \end{pmatrix} &= \sum_{n=0}^{\infty} W_\mu^{(n)}(x) \begin{pmatrix} \tilde{h}_{W^{(n)}}^L(z) \\ \tilde{h}_{W^{(n)}}^R(z) \\ \tilde{\tilde{h}}_{W^{(n)}}(z) \end{pmatrix} + \cdots, \quad \begin{pmatrix} \tilde{h}_{W^{(n)}}^L(z) \\ \tilde{h}_{W^{(n)}}^R(z) \\ \tilde{\tilde{h}}_{W^{(n)}}(z) \end{pmatrix} = \frac{1}{\sqrt{2r_{W^{(n)}}}} \begin{pmatrix} (1 + c_H)C(z; \lambda_{W^{(n)}}) \\ (1 - c_H)C(z; \lambda_{W^{(n)}}) \\ \sqrt{2}s_H \check{S}(z; \lambda_{W^{(n)}}) \end{pmatrix}, \\ c_H &= \cos \theta_H, \quad s_H = \sin \theta_H, \end{aligned} \quad (2.12)$$

where the spectrum $m_{W^{(n)}} = k\lambda_{W^{(n)}}$ is determined by

$$2SC'(1; \lambda_{W^{(n)}}) + \lambda_{W^{(n)}} \sin^2 \theta_H = 0. \quad (2.13)$$

$C(z; \lambda)$, $S(z; \lambda)$, and $\check{S}(z; \lambda)$ are defined in Eq. (A1). The normalization factor $r_{W^{(n)}}$ is determined by

$$\int_1^{z_L} \frac{dz}{z} \{ (|\tilde{h}_{W^{(n)}}^L|^2 + |\tilde{h}_{W^{(n)}}^R|^2 + |\tilde{\tilde{h}}_{W^{(n)}}|^2) \} = 1. \quad (2.14)$$

The W boson field is $W_\mu(x) = W_\mu^{(0)}(x)$. Wave functions in the original gauge are given by

$$\begin{aligned} \begin{pmatrix} h_{W^{(n)}}^L(z) \\ h_{W^{(n)}}^R(z) \end{pmatrix} &= \frac{1 \pm \cos \theta(z)}{2} \tilde{h}_{W^{(n)}}^L(z) + \frac{1 \mp \cos \theta(z)}{2} \tilde{h}_{W^{(n)}}^R(z) \pm \frac{\sin \theta(z)}{\sqrt{2}} \tilde{\tilde{h}}_{W^{(n)}}(z), \\ \hat{h}_{W^{(n)}}(z) &= -\frac{\sin \theta(z)}{\sqrt{2}} \tilde{h}_{W^{(n)}}^L(z) + \frac{\sin \theta(z)}{\sqrt{2}} \tilde{h}_{W^{(n)}}^R(z) + \cos \theta(z) \tilde{\tilde{h}}_{W^{(n)}}(z). \end{aligned} \quad (2.15)$$

Here $\theta(z)$ is given in Eq. (2.9).

Similarly the spectrum of the Z boson tower, $\{m_{Z^{(n)}} = k\lambda_{Z^{(n)}}\}$, is determined by

$$2SC'(1; \lambda_{Z^{(n)}}) + \lambda_{Z^{(n)}} \frac{\sin^2 \theta_H}{\cos^2 \theta_W^0} = 0, \quad \sin \theta_W^0 = \frac{g_B}{\sqrt{g_A^2 + 2g_B^2}}. \quad (2.16)$$

The Z boson field is contained in $A_\mu^{3_L}, A_\mu^{3_R}, A_\mu^{\hat{3}}$, and B_μ . Wave functions of the Z -boson tower in the twisted gauge are given by [26]

$$\begin{aligned}
\frac{1}{\sqrt{k}} \begin{pmatrix} \tilde{A}_\mu^{3L} \\ \tilde{A}_\mu^{3R} \\ \tilde{A}_\mu^{\hat{3}} \\ B_\mu \end{pmatrix} &= \sum_{n=0}^{\infty} Z_\mu^{(n)}(x) \begin{pmatrix} \tilde{h}_{Z^{(n)}}^L(z) \\ \tilde{h}_{Z^{(n)}}^R(z) \\ \tilde{h}_{Z^{(n)}}(z) \\ h_{Z^{(n)}}^B(z) \end{pmatrix} + \cdots, \quad \begin{pmatrix} \tilde{h}_{Z^{(n)}}^L(z) \\ \tilde{h}_{Z^{(n)}}^R(z) \\ \tilde{h}_{Z^{(n)}}(z) \\ h_{Z^{(n)}}^B(z) \end{pmatrix} = \begin{pmatrix} \tilde{h}_{Z^{(n)}}^{L,su2}(z) \\ \tilde{h}_{Z^{(n)}}^{R,su2}(z) \\ \tilde{h}_{Z^{(n)}}^{su2}(z) \\ 0 \end{pmatrix} - \sin \theta_W^0 \begin{pmatrix} \sin \theta_W^0 \\ \sin \theta_W^0 \\ 0 \\ \sqrt{1-2\sin^2 \theta_W^0} \end{pmatrix} h_{Z^{(n)}}^{em}(z), \\
\begin{pmatrix} \tilde{h}_{Z^{(n)}}^{L,su2}(z) \\ \tilde{h}_{Z^{(n)}}^{R,su2}(z) \\ \tilde{h}_{Z^{(n)}}^{su2}(z) \end{pmatrix} &= \frac{1}{\sqrt{2}r_{Z^{(n)}}} \begin{pmatrix} (1+c_H)C(z, \lambda_{Z^{(n)}}) \\ (1-c_H)C(z, \lambda_{Z^{(n)}}) \\ \sqrt{2}s_H \check{S}(z, \lambda_{Z^{(n)}}) \end{pmatrix}, \quad h_{Z^{(n)}}^{em}(z) = \sqrt{\frac{2}{r_{Z^{(n)}}}} C(z, \lambda_{Z^{(n)}}). \tag{2.17}
\end{aligned}$$

Wave functions are normalized by

$$\int_1^{z_L} \frac{dz}{z} \{(|\tilde{h}_{Z^{(n)}}^L|^2 + |\tilde{h}_{Z^{(n)}}^R|^2 + |\tilde{h}_{Z^{(n)}}|^2 + |h_{Z^{(n)}}^B|^2)\} = 1. \tag{2.18}$$

The Z boson field is $Z_\mu(x) = Z_\mu^{(0)}(x)$. Wave functions in the original gauge are given by

$$\begin{pmatrix} h_{Z^{(n)}}^L(z) \\ h_{Z^{(n)}}^R(z) \\ \hat{h}_{Z^{(n)}}(z) \\ h_{Z^{(n)}}^B(z) \end{pmatrix} = \begin{pmatrix} h_{Z^{(n)}}^{L,su2}(z) \\ h_{Z^{(n)}}^{R,su2}(z) \\ \hat{h}_{Z^{(n)}}^{su2}(z) \\ 0 \end{pmatrix} - \sin \theta_W^0 \begin{pmatrix} \sin \theta_W^0 \\ \sin \theta_W^0 \\ 0 \\ \sqrt{1-2\sin^2 \theta_W^0} \end{pmatrix} h_{Z^{(n)}}^{em}(z). \tag{2.19}$$

Here $(h_{Z^{(n)}}^{L,su2}, h_{Z^{(n)}}^{R,su2}, \hat{h}_{Z^{(n)}}^{su2})$ is related to $(\tilde{h}_{Z^{(n)}}^{L,su2}, \tilde{h}_{Z^{(n)}}^{R,su2}, \tilde{h}_{Z^{(n)}}^{su2})$, as in the formula (2.15), by

$$\begin{aligned}
\begin{pmatrix} h_{Z^{(n)}}^{L,su2}(z) \\ h_{Z^{(n)}}^{R,su2}(z) \end{pmatrix} &= \frac{1 \pm \cos \theta(z)}{2} \tilde{h}_{Z^{(n)}}^{L,su2}(z) + \frac{1 \mp \cos \theta(z)}{2} \tilde{h}_{Z^{(n)}}^{R,su2}(z) \pm \frac{\sin \theta(z)}{\sqrt{2}} \tilde{h}_{Z^{(n)}}^{su2}(z), \\
\hat{h}_{Z^{(n)}}^{su2}(z) &= -\frac{\sin \theta(z)}{\sqrt{2}} \tilde{h}_{Z^{(n)}}^{L,su2}(z) + \frac{\sin \theta(z)}{\sqrt{2}} \tilde{h}_{Z^{(n)}}^{R,su2}(z) + \cos \theta(z) \tilde{h}_{Z^{(n)}}^{su2}(z). \tag{2.20}
\end{aligned}$$

The $U(1)_{\text{EM}}$ part $h_{Z^{(n)}}^{em}(z)$ is unchanged. Note that $h_{Z^{(n)}}^{L,su2}(z) + h_{Z^{(n)}}^{R,su2}(z) = h_{Z^{(n)}}^{em}(z)$.

In this paper we restrict ourselves to the case in which there is no flavor mixing in quarks and leptons, to make clear the argument of anomaly flow and anomaly cancellation. The extension to a general case with flavor mixing is straightforward. Up, charm, and top quarks are zero modes contained in $\Psi_{(3,4)}^\alpha$. The mass spectrum $m_{q^{(n)}} = k\lambda_{q^{(n)}} (q = u, c, t)$ is determined by

$$S_L(1; \lambda_{q^{(n)}}, c_q) S_R(1; \lambda_{q^{(n)}}, c_q) + \sin^2 \frac{1}{2} \theta_H = 0, \tag{2.21}$$

where $S_{L/R}(z, \lambda, c)$ and $C_{L/R}(z, \lambda, c)$ are given by (A3). The lowest modes $u^{(0)}$, $c^{(0)}$ and $t^{(0)}$ are u , c , and t quarks. With m_u , m_c , and m_t given, the corresponding bulk mass parameters c_u , c_c , and c_t are fixed. Let us define $\check{\Psi} = z^{-2}\Psi$. The KK expansion of 5D $u(x, z)$ and $u'(x, z)$ fields is given, in the twisted gauge, by

$$\begin{aligned}
\begin{pmatrix} \check{u} \\ \check{u}' \end{pmatrix} &= \sqrt{k} \sum_{n=0}^{\infty} \left\{ u_L^{(n)}(x) \begin{pmatrix} \check{f}_L^{u(n)}(z) \\ \check{g}_L^{u(n)}(z) \end{pmatrix} + u_R^{(n)}(x) \begin{pmatrix} \check{f}_R^{u(n)}(z) \\ \check{g}_R^{u(n)}(z) \end{pmatrix} \right\}, \\
\begin{pmatrix} \check{f}_L^{u(n)}(z) \\ \check{g}_L^{u(n)}(z) \end{pmatrix} &= \frac{1}{\sqrt{r_{u(n)L}}} \begin{pmatrix} \bar{c}_H C_L(z, \lambda_{u(n)}, c_u) \\ -i \bar{s}_H \check{S}_L(z, \lambda_{u(n)}, c_u) \end{pmatrix}, \\
\begin{pmatrix} \check{f}_R^{u(n)}(z) \\ \check{g}_R^{u(n)}(z) \end{pmatrix} &= \frac{1}{\sqrt{r_{u(n)R}}} \begin{pmatrix} \bar{c}_H S_R(z, \lambda_{u(n)}, c_u) \\ -i \bar{s}_H \check{C}_R(z, \lambda_{u(n)}, c_u) \end{pmatrix},
\end{aligned} \tag{2.22}$$

where $\check{S}_L$ and $\check{C}_R$ are given in (A3) and $(\bar{c}_H, \bar{s}_H) = (\cos \frac{1}{2} \theta_H, \sin \frac{1}{2} \theta_H)$. The normalization factor is determined by the condition

$$\int_1^{z_L} dz \{ |f(z)|^2 + |g(z)|^2 \} = 1 \quad \text{for} \quad \begin{pmatrix} f(z) \\ g(z) \end{pmatrix}. \tag{2.23}$$

One can show that $r_{u(n)L} = r_{u(n)R}$ for $\lambda_{u(n)} \neq 0$. Wave functions in the original gauge are given by

$$\begin{pmatrix} f_{L/R}^{u(n)}(z) \\ g_{L/R}^{u(n)}(z) \end{pmatrix} = \begin{pmatrix} \cos \frac{1}{2} \theta(z) & i \sin \frac{1}{2} \theta(z) \\ i \sin \frac{1}{2} \theta(z) & \cos \frac{1}{2} \theta(z) \end{pmatrix} \begin{pmatrix} \check{f}_{L/R}^{u(n)}(z) \\ \check{g}_{L/R}^{u(n)}(z) \end{pmatrix}. \tag{2.24}$$

In the down-type quark sector, d, d', D_d^+ , and D_d^- intertwine with each other in the first generation. The KK expansion is given by

$$\begin{aligned}
\begin{pmatrix} \check{d} \\ \check{d}' \\ \check{D}_d^+ \\ \check{D}_d^- \end{pmatrix} &= \sqrt{k} \sum_{n=0}^{\infty} \left\{ d_L^{(n)}(x) \begin{pmatrix} f_L^{d(n)}(z) \\ g_L^{d(n)}(z) \\ h_L^{d(n)}(z) \\ k_L^{d(n)}(z) \end{pmatrix} + d_R^{(n)}(x) \begin{pmatrix} f_R^{d(n)}(z) \\ g_R^{d(n)}(z) \\ h_R^{d(n)}(z) \\ k_R^{d(n)}(z) \end{pmatrix} \right\} \\
&+ \sqrt{k} \sum_{n=1}^{\infty} \left\{ D_{dL}^{(n)}(x) \begin{pmatrix} f_L^{D_d(n)}(z) \\ g_L^{D_d(n)}(z) \\ h_L^{D_d(n)}(z) \\ k_L^{D_d(n)}(z) \end{pmatrix} + D_{dR}^{(n)}(x) \begin{pmatrix} f_R^{D_d(n)}(z) \\ g_R^{D_d(n)}(z) \\ h_R^{D_d(n)}(z) \\ k_R^{D_d(n)}(z) \end{pmatrix} \right\}
\end{aligned} \tag{2.25}$$

where the wave functions are normalized by

$$\int_1^{z_L} dz \{ |f|^2 + |g|^2 + |h|^2 + |k|^2 \} = 1 \tag{2.26}$$

in each mode. The lowest mode $d^{(0)}$ is the d quark. In the wave functions of the d and D towers, (f, g) and (h, k) components are dominant, respectively. The mass spectrum $\{m_n = k\lambda_n\}$ is determined by

$$\begin{aligned}
&\left(S_L S_R + \sin^2 \frac{\theta_H}{2} \right) (S_{L1} S_{R1} - S_{L2} S_{R2}) + |\mu_d|^2 C_R S_R (S_{L1} C_{L1} - S_{L2} C_{L2}) = 0, \\
&(C, S)_{L/R} = (C, S)_{L/R}(z; \lambda_n, c_u)|_{z=1}, \quad (C, S)_{L/Rj} = (C, S)_{L/Rj}(z; \lambda_n, c_{D_d}, \tilde{m}_{D_d})|_{z=1},
\end{aligned} \tag{2.27}$$

where $(C, S)_{L/Rj}(z; \lambda, c, \tilde{m})$ is given in Eq. (A5). μ_d arises from a brane interaction. c_{D_d} is the bulk mass parameter of $D_d^\pm$ fields. $k\tilde{m}_{D_d}$ is a vectorlike mass connecting D_d^+ and D_d^- fields. Details of the wave functions are found in Ref. [26].

The wave functions $(f_{L/R}^{d(n)}, g_{L/R}^{d(n)})$ and $(f_{L/R}^{D_d(n)}, g_{L/R}^{D_d(n)})$ in the original gauge are related to $(\tilde{f}_{L/R}^{d(n)}, \tilde{g}_{L/R}^{d(n)})$ and $(\tilde{f}_{L/R}^{D_d(n)}, \tilde{g}_{L/R}^{D_d(n)})$ in the twisted gauge by the same formula as in Eq. (2.24).

The spectrum $\{m_{e^{(n)}} = k\lambda_{e^{(n)}}\}$ of the electron KK tower is determined by

$$S_L(1; \lambda_{e^{(n)}}, c_e) S_R(1; \lambda_{e^{(n)}}, c_e) + \sin^2 \frac{1}{2} \theta_H = 0. \quad (2.28)$$

The electron mass $m_e = m_{e^{(0)}}$ fixes c_e . The KK expansion is given by

$$\begin{pmatrix} \tilde{e} \\ \tilde{e}' \end{pmatrix} = \sqrt{k} \sum_{n=0}^{\infty} \left\{ e_L^{(n)}(x) \begin{pmatrix} f_L^{e^{(n)}}(z) \\ g_L^{e^{(n)}}(z) \end{pmatrix} + e_R^{(n)}(x) \begin{pmatrix} f_R^{e^{(n)}}(z) \\ g_R^{e^{(n)}}(z) \end{pmatrix} \right\}. \quad (2.29)$$

Wave functions take the same form as those in Eq. (2.22) where $u^{(n)}$ and c_u are replaced by $e^{(n)}$ and c_e .

In the neutrino sector ν_e, ν_e' and $\hat{\chi}^1 = (\eta_e^c, \eta_e)$ fields intertwine with each other. The spectrum of the ν_e KK tower, $\{\nu_e^{+(n)}, \nu_e^{-\ell}\}; n \geq 0, \ell \geq 1\}$, is determined by

$$\det K_{\nu_e^\pm} = (k\lambda_{\nu_e^\pm} \mp M_e) \left\{ S_L^e S_R^e + \sin^2 \frac{\theta_H}{2} \right\} + \frac{|m_{B_e}|^2}{2k} S_R^e C_R^e = 0. \quad (2.30)$$

Here $S_{L/R}^e = S_{L/R}(1; \lambda_{\nu_e^\pm}^{(n)}, c_e)$ etc. M_e is a Majorana mass of $\hat{\chi}^1$ and m_{B_e} arises from a brane interaction. $\nu_e^{+(0)}$ is the light neutrino field. Its tiny mass generated by a gauge-Higgs seesaw mechanism is $m_{\nu_e} \sim m_e^2 M_e / (|c_e| - \frac{1}{2}) m_{B_e}^2$. The KK expansion is given by

$$\begin{aligned} \begin{pmatrix} \check{\nu}_e \\ \check{\nu}_e' \end{pmatrix} &= \sqrt{k} \sum_{n=0}^{\infty} \left\{ \nu_{eL}^{+(n)}(x) \begin{pmatrix} f_L^{\nu_e^{+(n)}}(z) \\ g_L^{\nu_e^{+(n)}}(z) \end{pmatrix} + \nu_{eR}^{+(n)}(x) \begin{pmatrix} f_R^{\nu_e^{+(n)}}(z) \\ g_R^{\nu_e^{+(n)}}(z) \end{pmatrix} \right\} \\ &\quad + \sqrt{k} \sum_{n=1}^{\infty} \left\{ \nu_{eL}^{-(n)}(x) \begin{pmatrix} f_L^{\nu_e^{-(n)}}(z) \\ g_L^{\nu_e^{-(n)}}(z) \end{pmatrix} + \nu_{eR}^{-(n)}(x) \begin{pmatrix} f_R^{\nu_e^{-(n)}}(z) \\ g_R^{\nu_e^{-(n)}}(z) \end{pmatrix} \right\}, \\ \eta_e &= \sum_{n=0}^{\infty} \nu_{eL}^{+(n)}(x) h^{\nu_e^{+(n)}} + \sum_{n=1}^{\infty} \nu_{eL}^{-(n)}(x) h^{\nu_e^{-(n)}}, \quad \nu_{eR}^{\pm(n)} = \pm (\nu_{eL}^{\pm(n)})^c = \pm e^{i\delta_c} \sigma^2 (\nu_{eL}^{\pm(n)})^*. \end{aligned} \quad (2.31)$$

In each mode $\int_1^{z_L} dz \{ |f_L|^2 + |g_L|^2 + |f_R|^2 + |g_R|^2 \} + |h|^2 = 1$. Details of wave functions have been given in Ref. [26].

III. GAUGE COUPLINGS

Gauge couplings of fermion fields are contained in the part of the action

$$\int d^4x \int_1^{z_L} \frac{dz}{k} \sum_J \tilde{\Psi}^J \gamma^\mu (-i)(g_A A_\mu + g_B Q_X B_\mu + g_C A_\mu^{SU(3)_C}) \tilde{\Psi}^J \quad (3.1)$$

where $\tilde{\Psi}^J = z^{-2} \Psi^J$. Inserting the KK expansions for $A_\mu, B_\mu, A_\mu^{SU(3)_C}, \tilde{\Psi}^J$, and $\tilde{\Psi}^J$, obtained in the twisted gauge, into (3.1) and integrating over z , one finds gauge couplings of all fermion modes.

The W interaction is evaluated from

$$\mathcal{L}_{\text{int}}^W = -ig_A \sum_J \int_1^{z_L} \frac{dz}{\sqrt{k}} \tilde{\tilde{\Psi}}^J \gamma^\mu \sum_{a=1}^2 (\tilde{A}_\mu^{a_L} T^{a_L} + \tilde{A}_\mu^{a_R} T^{a_R} + \tilde{A}_\mu^{\hat{a}} T^{\hat{a}}) \tilde{\tilde{\Psi}}^J. \quad (3.2)$$

The Wud couplings are given by

$$\begin{aligned}
\mathcal{L}^{Wud} &= -i \frac{g_w}{\sqrt{2}} \sum_{\ell=0}^{\infty} W_{\mu}^{(\ell)\dagger} \sum_{n,m=0}^{\infty} \{ \hat{g}_{L\ell nm}^{Wud} \bar{u}_L^{(n)} \gamma^{\mu} d_L^{(m)} + \hat{g}_{R\ell nm}^{Wud} \bar{u}_R^{(n)} \gamma^{\mu} d_R^{(m)} \} + \text{H.c.}, \\
\hat{g}_{L/R\ell nm}^{Wud} &= G_W[(\tilde{h}^L, \tilde{h}^R, \tilde{h})_{W^{(\ell)}}; (\tilde{f}, \tilde{g})_{L/R}^{u(n)}, (\tilde{f}, \tilde{g})_{L/R}^{d(m)}] \\
&= G_W[(h^L, h^R, \hat{h})_{W^{(\ell)}}; (f, g)_{L/R}^{u(n)}, (f, g)_{L/R}^{d(m)}],
\end{aligned} \tag{3.3}$$

where

$$G_W[(h^L, h^R, \hat{h})_{\alpha}; (f, g)_1, (f, g)_2] = \sqrt{kL} \int_1^{z_L} dz \left\{ h_{\alpha}^* f_1^* f_2 + h_{\alpha}^{R*} g_1^* g_2 + \frac{i}{\sqrt{2}} \hat{h}_{\alpha}^* (f_1^* g_2 - g_1^* f_2) \right\}. \tag{3.4}$$

The last equality in Eq. (3.3) is confirmed with the aid of Eqs. (2.15) and (2.24). The couplings to the $D^{(m)}$ ($m \geq 1$) modes are given by replacing $d^{(m)}$ by $D^{(m)}$ in the above formula. The $W\nu_e e$ couplings are given by

$$\begin{aligned}
\mathcal{L}^{W\nu_e e} &= -i \frac{g_w}{\sqrt{2}} \sum_{\ell=0}^{\infty} W_{\mu}^{(\ell)\dagger} \left[\sum_{n,m=0}^{\infty} \{ \hat{g}_{L\ell nm}^{W\nu_e^+ e} \bar{\nu}_{eL}^{+(n)} \gamma^{\mu} e_L^{(m)} + \hat{g}_{R\ell nm}^{W\nu_e^+ e} \bar{\nu}_{eR}^{+(n)} \gamma^{\mu} e_R^{(m)} \} \right. \\
&\quad \left. + \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \{ \hat{g}_{L\ell nm}^{W\nu_e^- e} \bar{\nu}_{eL}^{-(n)} \gamma^{\mu} e_L^{(m)} + \hat{g}_{R\ell nm}^{W\nu_e^- e} \bar{\nu}_{eR}^{-(n)} \gamma^{\mu} e_R^{(m)} \} \right] + \text{H.c.}, \\
\hat{g}_{L/R\ell nm}^{W\nu_e^{\pm} e} &= G_W[(\tilde{h}^L, \tilde{h}^R, \tilde{h})_{W^{(\ell)}}; (\tilde{f}, \tilde{g})_{L/R}^{\nu_e^{\pm(n)}}, (\tilde{f}, \tilde{g})_{L/R}^{e(m)}] \\
&= G_W[(h^L, h^R, \hat{h})_{W^{(\ell)}}; (f, g)_{L/R}^{\nu_e^{\pm(n)}}, (f, g)_{L/R}^{e(m)}].
\end{aligned} \tag{3.5}$$

Similarly the Z interaction is evaluated from

$$\mathcal{L}_{\text{int}}^Z = -ig_A \sum_J \int_1^{z_L} \frac{dz}{\sqrt{k}} \tilde{\Psi}^J \gamma^{\mu} \left(\tilde{A}_{\mu}^{3L} T^{3L} + \tilde{A}_{\mu}^{3R} T^{3R} + \tilde{A}_{\mu}^{\hat{3}} T^{\hat{3}} + \frac{g_B}{g_A} B_{\mu} Q_X \right) \tilde{\Psi}^J. \tag{3.6}$$

The Zuu and Zdd couplings are given by

$$\mathcal{L}^{Z,u,d} = -i \frac{g_w}{\cos \theta_W^0} \sum_{\ell=0}^{\infty} Z_{\mu}^{(\ell)} \sum_{n,m=0}^{\infty} \{ \hat{g}_{L\ell nm}^{Zuu} \bar{u}_L^{(n)} \gamma^{\mu} u_L^{(m)} + \hat{g}_{R\ell nm}^{Zuu} \bar{u}_R^{(n)} \gamma^{\mu} u_R^{(m)} + \hat{g}_{L\ell nm}^{Zdd} \bar{d}_L^{(n)} \gamma^{\mu} d_L^{(m)} + \hat{g}_{R\ell nm}^{Zdd} \bar{d}_R^{(n)} \gamma^{\mu} d_R^{(m)} \}, \tag{3.7}$$

where

$$\begin{aligned}
\begin{pmatrix} \hat{g}_{L/R\ell nm}^{Zuu} \\ \hat{g}_{L/R\ell nm}^{Zdd} \end{pmatrix} &= \begin{pmatrix} \hat{g}_{L/R\ell nm}^{Zuu, su2} \\ \hat{g}_{L/R\ell nm}^{Zdd, su2} \end{pmatrix} - \sin^2 \theta_W^0 \begin{pmatrix} \hat{g}_{L/R\ell nm}^{Zuu, em} \\ \hat{g}_{L/R\ell nm}^{Zdd, em} \end{pmatrix}, \\
\begin{pmatrix} \hat{g}_{L/R\ell nm}^{Zuu, su2} \\ \hat{g}_{L/R\ell nm}^{Zdd, su2} \end{pmatrix} &= \cos \theta_W^0 \begin{pmatrix} T_u^3 G_W[(\tilde{h}^L, \tilde{h}^R, \tilde{h})_{Z^{(\ell)}}^{su2}; (\tilde{f}, \tilde{g})_{L/R}^{u(n)}, (\tilde{f}, \tilde{g})_{L/R}^{u(m)}] \\ T_d^3 G_W[(\tilde{h}^L, \tilde{h}^R, \tilde{h})_{Z^{(\ell)}}^{su2}; (\tilde{f}, \tilde{g})_{L/R}^{d(n)}, (\tilde{f}, \tilde{g})_{L/R}^{d(m)}] \end{pmatrix}, \\
&= \cos \theta_W^0 \begin{pmatrix} T_u^3 G_W[(h^L, h^R, \hat{h})_{Z^{(\ell)}}^{su2}; (f, g)_{L/R}^{u(n)}, (f, g)_{L/R}^{u(m)}] \\ T_d^3 G_W[(h^L, h^R, \hat{h})_{Z^{(\ell)}}^{su2}; (f, g)_{L/R}^{d(n)}, (f, g)_{L/R}^{d(m)}] \end{pmatrix}, \\
\begin{pmatrix} \hat{g}_{L/R\ell nm}^{Zuu, em} \\ \hat{g}_{L/R\ell nm}^{Zdd, em} \end{pmatrix} &= \cos \theta_W^0 \begin{pmatrix} Q_u G_{\gamma}^u[h_{Z^{(\ell)}}^{em}; (\tilde{f}, \tilde{g})_{L/R}^{u(n)}, (\tilde{f}, \tilde{g})_{L/R}^{u(m)}] \\ Q_d G_{\gamma}^d[h_{Z^{(\ell)}}^{em}; (\tilde{f}, \tilde{g}, h, k)_{L/R}^{d(n)}, (\tilde{f}, \tilde{g}, h, k)_{L/R}^{d(m)}] \end{pmatrix}, \\
&= \cos \theta_W^0 \begin{pmatrix} Q_u G_{\gamma}^u[h_{Z^{(\ell)}}^{em}; (f, g)_{L/R}^{u(n)}, (f, g)_{L/R}^{u(m)}] \\ Q_d G_{\gamma}^d[h_{Z^{(\ell)}}^{em}; (f, g, h, k)_{L/R}^{d(n)}, (f, g, h, k)_{L/R}^{d(m)}] \end{pmatrix},
\end{aligned} \tag{3.8}$$

$(T_u^3, T_d^3) = (\frac{1}{2}, -\frac{1}{2})$ and $(Q_u, Q_d) = (\frac{2}{3}, -\frac{1}{3})$. Here

$$\begin{aligned} G_\gamma^u[h_\gamma; (f, g)_1, (f, g)_2] &= \sqrt{kL} \int_1^{z_L} dz h_\gamma (f_1^* f_2 + g_1^* g_2), \\ G_\gamma^d[h_\gamma; (f, g, h, k)_1, (f, g, h, k)_2] &= \sqrt{kL} \int_1^{z_L} dz h_\gamma (f_1^* f_2 + g_1^* g_2 + h_1^* h_2 + k_1^* k_2). \end{aligned} \quad (3.9)$$

The $Z^{(0)}$ couplings to $d^{(n)} D_d^{(m)}$ and $D_d^{(n)} D_d^{(m)}$ are very small ($|\hat{g}_{L/R0nm}^{ZdD_d}|, |\hat{g}_{L/R0nm}^{ZD_dD_d}| < O(10^{-6})$) except for diagonal elements $\hat{g}_{L/R0nn}^{ZD_dD_d}$ ($n \geq 1$) for which there are contributions from the $U(1)$ part. Numerically $\hat{g}_{L/R0nn}^{ZD_dD_d,em} \sim -0.333385$ and $|\hat{g}_{L/R0nn}^{ZD_dD_d,su2}| < O(10^{-12})$ for $\theta_H = 0.1$ and $m_{KK} = 13$ TeV.

In a similar manner the Zee and $Z\nu_e\nu_e$ couplings are found to be

$$\begin{aligned} \hat{g}_{L/R\ell nm}^{Zee} &= \hat{g}_{L/R\ell nm}^{Zee,su2} - \sin^2 \theta_W^0 \hat{g}_{L/R\ell nm}^{Zee,em}, & \hat{g}_{L/R\ell nm}^{Zee,su2} &= \cos \theta_W^0 T_e^3 G_W[(h^L, h^R, \hat{h})_{Z^{(\ell)}}^{su2}; (f, g)_{L/R}^{e(n)}, (f, g)_{L/R}^{e(m)}], \\ \hat{g}_{L/R\ell nm}^{Zee,em} &= \cos \theta_W^0 Q_e G_\gamma^u[h_{Z^{(\ell)}}^{em}; (f, g)_{L/R}^{e(n)}, (f, g)_{L/R}^{e(m)}], \\ \hat{g}_{L/R\ell nm}^{Z\nu_e^a\nu_e^b} &= \hat{g}_{L/R\ell nm}^{Z\nu_e^a\nu_e^b,su2} = \cos \theta_W^0 T_\nu^3 G_W[(h^L, h^R, \hat{h})_{Z^{(\ell)}}^{su2}; (f, g)_{L/R}^{\nu_e^{a(n)}}, (f, g)_{L/R}^{\nu_e^{b(m)}}], \end{aligned} \quad (3.10)$$

where $a, b = +$ or $-$, $Q_e = -1$ and $(T_\nu^3, T_e^3) = (\frac{1}{2}, -\frac{1}{2})$. Here $\hat{g}_{L/R\ell 0m}^{Z\nu_e^+\nu_e^-} = \hat{g}_{L/R\ell n0}^{Z\nu_e^+\nu_e^-} = 0$ has been understood.

We note that all gauge couplings depend on θ_H . The $Z = Z^{(0)}$ coupling of up quarks $u^{(0)}$, $\hat{g}_{L/R000}^{Zuu}$, for instance, takes the SM value $\hat{g}_{L000}^{Zuu} = T_u^3 - \sin^2 \theta_W^0 Q_u$ and $\hat{g}_{R000}^{Zuu} = -\sin^2 \theta_W^0 Q_u$ at $\theta_H = 0$. However it varies with θ_H . The θ_H -dependence of $\hat{g}_{L/R000}^{Zuu}$ is depicted in Fig. 2. The couplings become vectorlike, $\hat{g}_{L000}^{Zuu} = \hat{g}_{R000}^{Zuu}$, at $\theta_H = \pi$, where the Z boson is massless and the u quark is massive. The θ_H dependence of m_Z and m_u is qualitatively similar to that of m_W and m_t in Fig. 1.

IV. CHIRAL ANOMALIES

Chiral anomalies in four dimensions are expressed in terms of gauge couplings obtained in the previous section. Gauge anomalies for $\gamma\gamma Z^{(\ell)}$ depicted in Fig. 3 are proportional to

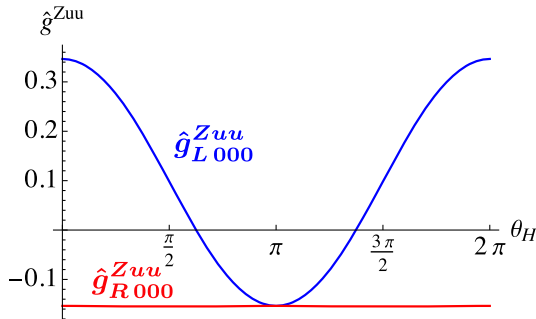


FIG. 2. θ_H dependence of Z -boson couplings of u quarks, $\hat{g}_{L000}^{Zuu}$ and $\hat{g}_{R000}^{Zuu}$, is displayed. $\hat{g}_{R000}^{Zuu}(\theta_H)$ is nearly constant; $-0.15495 < \hat{g}_{R000}^{Zuu} < -0.15357$.

$$\begin{aligned} J_{\gamma\gamma Z^{(\ell)}} &= 3\{Q_u^2(\text{Tr}\hat{g}_{L\ell}^{Zuu} - \text{Tr}\hat{g}_{R\ell}^{Zuu}) + Q_d^2(\text{Tr}\hat{g}_{L\ell}^{Zdd} - \text{Tr}\hat{g}_{R\ell}^{Zdd}) \\ &\quad + Q_d^2(\text{Tr}\hat{g}_{L\ell}^{ZD_dD_d} - \text{Tr}\hat{g}_{R\ell}^{ZD_dD_d})\} \\ &\quad + Q_e^2(\text{Tr}\hat{g}_{L\ell}^{Zee} - \text{Tr}\hat{g}_{R\ell}^{Zee}), \\ (\hat{g}_{L/R\ell}^{Zuu})_{nm} &= \hat{g}_{L/R\ell nm}^{Zuu} \text{ etc.} \end{aligned} \quad (4.1)$$

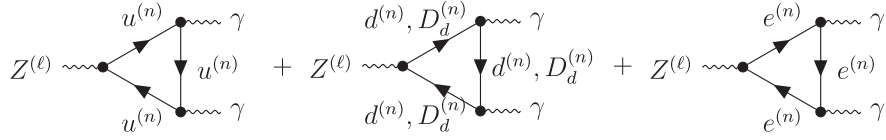
Note that KK excited modes of quarks and leptons contribute to anomalies. In the down quark sector both $\{d^{(n)}\}$ and $\{D_d^{(n)}\}$ towers contribute. Similarly for $ggZ^{(\ell)}$ (where g is a gluon) anomalies depicted in Fig. 4 are proportional to

$$\begin{aligned} J_{ggZ^{(\ell)}} &= \text{Tr}\hat{g}_{L\ell}^{Zuu} - \text{Tr}\hat{g}_{R\ell}^{Zuu} + \text{Tr}\hat{g}_{L\ell}^{Zdd} - \text{Tr}\hat{g}_{R\ell}^{Zdd} \\ &\quad + \text{Tr}\hat{g}_{L\ell}^{ZD_dD_d} - \text{Tr}\hat{g}_{R\ell}^{ZD_dD_d}. \end{aligned} \quad (4.2)$$

Cancellation of these two anomalies is achieved when

$$\begin{aligned} \text{Tr}\hat{g}_{L\ell}^{Zuu} - \text{Tr}\hat{g}_{R\ell}^{Zuu} &= -(\text{Tr}\hat{g}_{L\ell}^{Zdd} - \text{Tr}\hat{g}_{R\ell}^{Zdd} + \text{Tr}\hat{g}_{L\ell}^{ZD_dD_d} - \text{Tr}\hat{g}_{R\ell}^{ZD_dD_d}) \\ &= -(\text{Tr}\hat{g}_{L\ell}^{Zee} - \text{Tr}\hat{g}_{R\ell}^{Zee}). \end{aligned} \quad (4.3)$$

Anomalies for the vertex γWW depicted in Fig. 5 are proportional to

FIG. 3. Chiral anomalies for $\gamma\gamma Z^{(\ell)}$ in GHU.

$$\begin{aligned}
 J_{\gamma WW} &= 3Q_u \sum_{\tilde{d}=d, D_d} (\text{Tr} \hat{g}_{L0}^{Wu\tilde{d}} \hat{g}_{L0}^{W\tilde{d}u} - \text{Tr} \hat{g}_{R0}^{Wu\tilde{d}} \hat{g}_{R0}^{W\tilde{d}u}) \\
 &\quad + 3Q_d \sum_{\tilde{d}=d, D_d} (\text{Tr} \hat{g}_{L0}^{W\tilde{d}u} \hat{g}_{L0}^{Wu\tilde{d}} - \text{Tr} \hat{g}_{R0}^{W\tilde{d}u} \hat{g}_{R0}^{Wu\tilde{d}}) \\
 &\quad + Q_e \sum_{a=+,-} (\text{Tr} \hat{g}_{L0}^{W\tilde{e}a} \hat{g}_{L0}^{W\nu_e^a} - \text{Tr} \hat{g}_{R0}^{W\tilde{e}a} \hat{g}_{R0}^{W\nu_e^a}), \\
 J_{\gamma ZZ} &= 3Q_u (\text{Tr} \hat{g}_{L0}^{Zuu} \hat{g}_{L0}^{Zuu} - \text{Tr} \hat{g}_{R0}^{Zuu} \hat{g}_{R0}^{Zuu}) \\
 &\quad + 3Q_d \sum_{\tilde{d}_1, \tilde{d}_2=d, D_d} (\text{Tr} \hat{g}_{L0}^{Z\tilde{d}_1\tilde{d}_2} \hat{g}_{L0}^{Z\tilde{d}_2\tilde{d}_1} - \text{Tr} \hat{g}_{R0}^{Z\tilde{d}_1\tilde{d}_2} \hat{g}_{R0}^{Z\tilde{d}_2\tilde{d}_1}) \\
 &\quad + Q_e (\text{Tr} \hat{g}_{L0}^{Zee} \hat{g}_{L0}^{Zee} - \text{Tr} \hat{g}_{R0}^{Zee} \hat{g}_{R0}^{Zee}). \quad (4.5)
 \end{aligned}$$

$$\hat{g}_{L/R\ell}^{W\tilde{d}u} = (\hat{g}_{L/R\ell}^{Wud})^\dagger \text{ etc.} \quad (4.4)$$

Anomalies for the vertex γZZ depicted in Fig. 6 are proportional to

Anomalies for the vertex ZWW depicted in Fig. 7 are proportional to

$$\begin{aligned}
 J_{ZWW} &= 3 \sum_{\tilde{d}=d, D_d} (\text{Tr} \hat{g}_{L0}^{Wu\tilde{d}} \hat{g}_{L0}^{W\tilde{d}u} \hat{g}_{L0}^{Zuu} - \text{Tr} \hat{g}_{R0}^{Wu\tilde{d}} \hat{g}_{R0}^{W\tilde{d}u} \hat{g}_{R0}^{Zuu}) + 3 \sum_{\tilde{d}_1, \tilde{d}_2=d, D_d} (\text{Tr} \hat{g}_{L0}^{W\tilde{d}_1u} \hat{g}_{L0}^{Wu\tilde{d}_2} \hat{g}_{L0}^{Z\tilde{d}_2\tilde{d}_1} - \text{Tr} \hat{g}_{R0}^{W\tilde{d}_1u} \hat{g}_{R0}^{Wu\tilde{d}_2} \hat{g}_{R0}^{Z\tilde{d}_2\tilde{d}_1}) \\
 &\quad + \sum_{a,b=+,-} (\text{Tr} \hat{g}_{L0}^{W\nu_e^a} \hat{g}_{L0}^{W\tilde{e}b} \hat{g}_{L0}^{Z\nu_e^b\nu_e^a} - \text{Tr} \hat{g}_{R0}^{W\nu_e^a} \hat{g}_{R0}^{W\tilde{e}b} \hat{g}_{R0}^{Z\nu_e^b\nu_e^a}) + \sum_{a=+,-} (\text{Tr} \hat{g}_{L0}^{W\tilde{e}a} \hat{g}_{L0}^{W\nu_e^a} \hat{g}_{L0}^{Zee} - \text{Tr} \hat{g}_{R0}^{W\tilde{e}a} \hat{g}_{R0}^{W\nu_e^a} \hat{g}_{R0}^{Zee}). \quad (4.6)
 \end{aligned}$$

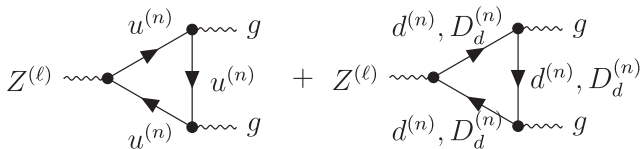
Finally anomalies for the vertex ZZZ depicted in Fig. 8 are proportional to

$$\begin{aligned}
 J_{ZZZ} &= 3(\text{Tr} \hat{g}_{L0}^{Zuu} \hat{g}_{L0}^{Zuu} \hat{g}_{L0}^{Zuu} - \text{Tr} \hat{g}_{R0}^{Zuu} \hat{g}_{R0}^{Zuu} \hat{g}_{R0}^{Zuu}) + 3 \sum_{\tilde{d}_1, \tilde{d}_2, \tilde{d}_3=d, D_d} (\text{Tr} \hat{g}_{L0}^{Z\tilde{d}_1\tilde{d}_2} \hat{g}_{L0}^{Z\tilde{d}_2\tilde{d}_3} \hat{g}_{L0}^{Z\tilde{d}_3\tilde{d}_1} - \text{Tr} \hat{g}_{R0}^{Z\tilde{d}_1\tilde{d}_2} \hat{g}_{R0}^{Z\tilde{d}_2\tilde{d}_3} \hat{g}_{R0}^{Z\tilde{d}_3\tilde{d}_1}) \\
 &\quad + \sum_{a,b,c=+,-} (\text{Tr} \hat{g}_{L0}^{Z\nu_e^a\nu_e^b} \hat{g}_{L0}^{Z\nu_e^b\nu_e^c} \hat{g}_{L0}^{Z\nu_e^c\nu_e^a} - \text{Tr} \hat{g}_{R0}^{Z\nu_e^a\nu_e^b} \hat{g}_{R0}^{Z\nu_e^b\nu_e^c} \hat{g}_{R0}^{Z\nu_e^c\nu_e^a}) + (\text{Tr} \hat{g}_{L0}^{Zee} \hat{g}_{L0}^{Zee} \hat{g}_{L0}^{Zee} - \text{Tr} \hat{g}_{R0}^{Zee} \hat{g}_{R0}^{Zee} \hat{g}_{R0}^{Zee}). \quad (4.7)
 \end{aligned}$$

We show below that all of $J_{\gamma\gamma Z^{(\ell)}}$, $J_{gg Z^{(\ell)}}$, $J_{\gamma WW}$, $J_{\gamma ZZ}$, J_{ZWW} , and J_{ZZZ} vanish. It is necessary to incorporate contributions of KK fermions running inside loops in triangular diagrams.

V. HOLOGRAPHY IN ANOMALY

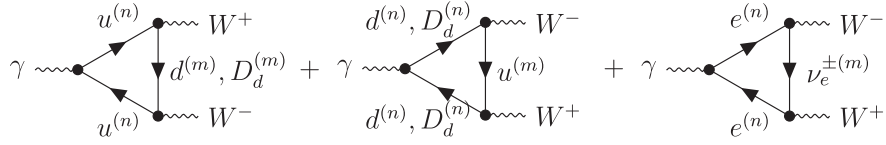
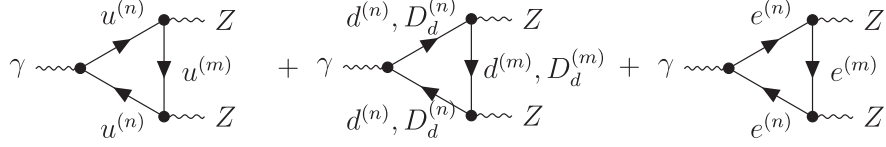
In GHU in the RS warped space gauge couplings of quarks and leptons in four dimensions slightly deviate from those in the SM. They depend on particle species. Although deviations are very tiny, cancellation of gauge anomalies would be spoiled if only contributions of quarks and leptons are taken into account. In GHU there are KK

FIG. 4. Chiral anomalies for $gg Z^{(\ell)}$ in GHU.

excitation modes of quarks and leptons, which also give nonvanishing contributions to anomalies as summarized in the previous section. In the RS space, or more generally in curved space, gauge couplings are not diagonal in the KK levels, and evaluation of total anomalies becomes highly nontrivial.

It was shown in Ref. [28] that in an $SU(2)$ GHU model in the RS space total anomaly is expressed in terms of the values of wave functions of gauge fields at the UV brane (at $y=0$) and IR brane (at $y=L$) and orbifold boundary conditions satisfied by fermion multiples. It does not depend on the fermion bulk mass parameter. θ_H dependence is also described by wave functions of gauge fields. In this paper we generalize the argument of Ref. [28] to the GUT-inspired GHU model, and derive holographic formulas for anomalies. Anomalies are expressed in terms of the values of W and Z wave functions at the UV and IR branes.

First let us take a look at $J_{\gamma\gamma Z^{(\ell)}}$ in (4.1) and $J_{gg Z^{(\ell)}}$ in (4.2). We decompose as

FIG. 5. Chiral anomalies for γWW in GHU.FIG. 6. Chiral anomalies for γZZ in GHU.

$$\begin{pmatrix} \text{Tr} \hat{g}_{L\ell}^{Zuu} - \text{Tr} \hat{g}_{R\ell}^{Zuu} \\ \sum_{\tilde{d}=d,D_d} (\text{Tr} \hat{g}_{L\ell}^{Z\tilde{d}\tilde{d}} - \text{Tr} \hat{g}_{R\ell}^{Z\tilde{d}\tilde{d}}) \\ \text{Tr} \hat{g}_{L\ell}^{Zee} - \text{Tr} \hat{g}_{R\ell}^{Zee} \end{pmatrix} = \begin{pmatrix} T_u^3 F_{Z^{(\ell)}}^{u1} \\ T_d^3 F_{Z^{(\ell)}}^{d1} \\ T_e^3 F_{Z^{(\ell)}}^{e1} \end{pmatrix} - \sin^2 \theta_W^0 \begin{pmatrix} Q_u F_{Z^{(\ell)}}^{u2} \\ Q_d F_{Z^{(\ell)}}^{d2} \\ Q_e F_{Z^{(\ell)}}^{e2} \end{pmatrix}. \quad (5.1)$$

Let us introduce

$$\begin{pmatrix} G_W[(h^L, h^R, \hat{h})_\alpha; (f, g)_1, (f, g)_2] \\ G_\gamma^u[h_\gamma; (f, g)_1, (f, g)_2] \\ G_\gamma^d[h_\gamma; (f, g, h, k)_1, (f, g, h, k)_2] \end{pmatrix} = \frac{k}{2} \sqrt{kL} \int_{-a}^{2L-a} dy e^{\sigma(y)} \begin{pmatrix} \mathcal{K}_W[(h^L, h^R, \hat{h})_\alpha; (f, g)_1, (f, g)_2](y) \\ \mathcal{K}_\gamma^u[h_\gamma; (f, g)_1, (f, g)_2](y) \\ \mathcal{K}_\gamma^d[h_\gamma; (f, g, h, k)_1, (f, g, h, k)_2](y) \end{pmatrix},$$

$$\mathcal{K}_W[(h^L, h^R, \hat{h})_\alpha; (f, g)_1, (f, g)_2] = h_\alpha^{L*} f_1^* f_2 + h_\alpha^{R*} g_1^* g_2 + \frac{i}{\sqrt{2}} \hat{h}_\alpha^* (f_1^* g_2 - g_1^* f_2),$$

$$\mathcal{K}_\gamma^u[h_\gamma; (f, g)_1, (f, g)_2] = h_\gamma (f_1^* f_2 + g_1^* g_2),$$

$$\mathcal{K}_\gamma^d[h_\gamma; (f, g, h, k)_1, (f, g, h, k)_2] = h_\gamma (f_1^* f_2 + g_1^* g_2 + h_1^* h_2 + k_1^* k_2), \quad (5.2)$$

in the y coordinate in the original gauge. Note that the functions $\mathcal{K}_W$, $\mathcal{K}_\gamma^u$, and $\mathcal{K}_\gamma^d$ in (5.2) are even under parity $y \rightarrow -y$ and periodic with a period $2L$. a is arbitrary. With Eq. (3.7) one finds that

$$\begin{pmatrix} F_{Z^{(\ell)}}^{u1} \\ F_{Z^{(\ell)}}^{u2} \end{pmatrix} = \frac{k}{2} \sqrt{kL} \cos \theta_W^0 \int_{-a}^{2L-a} dy e^{\sigma(y)} \sum_n \left\{ \begin{pmatrix} \mathcal{K}_W[(h^L, h^R, \hat{h})_{Z^{(\ell)}}^{su2}; (f, g)_L^{u(n)}, (f, g)_L^{u(n)}] \\ \mathcal{K}_\gamma^u[h_{Z^{(\ell)}}^{em}; (f, g)_L^{u(n)}, (f, g)_L^{u(n)}] \end{pmatrix} - [(f, g)_L \rightarrow (f, g)_R] \right\}. \quad (5.3)$$

For $J_{\gamma ZZ}$ in Eq. (4.5) we decompose the coupling factors as

$$\begin{pmatrix} \text{Tr} \hat{g}_{L0}^{Zuu} \hat{g}_{L0}^{Zuu} - \text{Tr} \hat{g}_{R0}^{Zuu} \hat{g}_{R0}^{Zuu} \\ \sum_{\tilde{d}_1, \tilde{d}_2=d,D_d} (\text{Tr} \hat{g}_{L0}^{Z\tilde{d}_1\tilde{d}_2} \hat{g}_{L0}^{Z\tilde{d}_2\tilde{d}_1} - \text{Tr} \hat{g}_{R0}^{Z\tilde{d}_1\tilde{d}_2} \hat{g}_{R0}^{Z\tilde{d}_2\tilde{d}_1}) \\ \text{Tr} \hat{g}_{L0}^{Zee} \hat{g}_{L0}^{Zee} - \text{Tr} \hat{g}_{R0}^{Zee} \hat{g}_{R0}^{Zee} \end{pmatrix} = \begin{pmatrix} (T_u^3)^2 F_{\gamma ZZ}^{u1} \\ (T_d^3)^2 F_{\gamma ZZ}^{d1} \\ (T_e^3)^2 F_{\gamma ZZ}^{e1} \end{pmatrix} - 2 \sin^2 \theta_W^0 \begin{pmatrix} T_u^3 Q_u F_{\gamma ZZ}^{u2} \\ T_d^3 Q_d F_{\gamma ZZ}^{d2} \\ T_e^3 Q_e F_{\gamma ZZ}^{e2} \end{pmatrix} + \sin^4 \theta_W^0 \begin{pmatrix} (Q_u)^2 F_{\gamma ZZ}^{u3} \\ (Q_d)^2 F_{\gamma ZZ}^{d3} \\ (Q_e)^2 F_{\gamma ZZ}^{e3} \end{pmatrix}. \quad (5.4)$$

With Eq. (3.7) one finds that

$$\begin{aligned}
\begin{pmatrix} F_{\gamma ZZ}^{u1} \\ F_{\gamma ZZ}^{u2} \\ F_{\gamma ZZ}^{u3} \end{pmatrix} &= \left(\frac{k}{2} \sqrt{kL} \cos \theta_W^0 \right)^2 \int_{-a}^{2L-a} dy_1 \int_{-a}^{2L-a} dy_2 e^{\sigma(y_1) + \sigma(y_2)} \sum_{n,m=0}^{\infty} \\
&\times \begin{pmatrix} \mathcal{K}_{W,Z(0)}^{u_L^{(n)}, u_L^{(m)}}(y_1) \mathcal{K}_{W,Z(0)}^{u_L^{(m)}, u_L^{(n)}}(y_2) - \mathcal{K}_{W,Z(0)}^{u_R^{(n)}, u_R^{(m)}}(y_1) \mathcal{K}_{W,Z(0)}^{u_R^{(m)}, u_R^{(n)}}(y_2) \\ \mathcal{K}_{W,Z(0)}^{u_L^{(n)}, u_L^{(m)}}(y_1) \mathcal{K}_{\gamma,Z(0)}^{u_L^{(m)}, u_L^{(n)}}(y_2) - \mathcal{K}_{W,Z(0)}^{u_R^{(n)}, u_R^{(m)}}(y_1) \mathcal{K}_{\gamma,Z(0)}^{u_R^{(m)}, u_R^{(n)}}(y_2) \\ \mathcal{K}_{\gamma,Z(0)}^{u_L^{(n)}, u_L^{(m)}}(y_1) \mathcal{K}_{\gamma,Z(0)}^{u_L^{(m)}, u_L^{(n)}}(y_2) - \mathcal{K}_{\gamma,Z(0)}^{u_R^{(n)}, u_R^{(m)}}(y_1) \mathcal{K}_{\gamma,Z(0)}^{u_R^{(m)}, u_R^{(n)}}(y_2) \end{pmatrix}, \\
\mathcal{K}_{W,Z(0)}^{u_L^{(n)}, u_L^{(m)}}(y) &= \mathcal{K}_W[(h^L, h^R, \hat{h})_{Z(0)}^{su2}; (f, g)_{L/R}^{u(n)}, (f, g)_{L/R}^{u(m)}](y), \\
\mathcal{K}_{\gamma,Z(0)}^{u_L^{(n)}, u_L^{(m)}}(y) &= \mathcal{K}_\gamma[h_{Z(0)}^{em}; (f, g)_{L/R}^{u(n)}, (f, g)_{L/R}^{u(m)}](y).
\end{aligned} \tag{5.5}$$

At this stage we recognize that there are two ways to evaluate $(F_{Z(c)}^{u1}, F_{Z(c)}^{u2})$ and $(F_{\gamma ZZ}^{u1}, F_{\gamma ZZ}^{u2}, F_{\gamma ZZ}^{u3})$.

Method 1 First integrate over y or (y_1, y_2) . Then do the KK sum $\sum_{n=0}^{\infty}$ or $\sum_{n,m=0}^{\infty}$.

Method 2 First do the KK sum $\sum_{n=0}^{\infty}$ or $\sum_{n,m=0}^{\infty}$. Then integrate over y or (y_1, y_2) .

Method 1 corresponds to evaluating each gauge coupling first and do the infinite KK sum. It was shown in Ref. [28] that Method 2 leads to astonishing formulas in an $SU(2)$ model.

In performing the KK summation $\sum_{n=0}^{\infty}$ we encounter

$$\begin{pmatrix} A_{L/R}^{uu} \\ B_{L/R}^{uu} \\ C_{L/R}^{uu} \\ D_{L/R}^{uu} \end{pmatrix}(y_j, y_k) = \sum_{n=0}^{\infty} \begin{pmatrix} f_{L/R}^{u(n)}(y_j) f_{L/R}^{u(n)}(y_k)^* \\ g_{L/R}^{u(n)}(y_j) g_{L/R}^{u(n)}(y_k)^* \\ f_{L/R}^{u(n)}(y_j) g_{L/R}^{u(n)}(y_k)^* \\ g_{L/R}^{u(n)}(y_j) f_{L/R}^{u(n)}(y_k)^* \end{pmatrix}. \tag{5.6}$$

Wave functions $f_L^{u(n)}$ and $g_R^{u(n)}$ are parity even, whereas $f_R^{u(n)}$ and $g_L^{u(n)}$ are parity odd. The KK sum in (5.6) gives completeness relations on orbifolds. We shall confirm below by numerical evaluation that

$$\begin{aligned}
F_{Z(c)}^{\alpha j} &= F_{Z(c)}^j, & F_{\gamma ZZ}^{\alpha k} &= F_{\gamma ZZ}^k, \\
(\alpha &= u, d, c, s, t, b \text{ and } e, \mu, \tau).
\end{aligned} \tag{5.7}$$

In other words the quantities $F_{Z(c)}^{\alpha j}, F_{\gamma ZZ}^{\alpha k}$ do not depend on the bulk mass parameter of fermions. With this fact in mind one can evaluate the $(A_{L/R}, B_{L/R}, C_{L/R}, D_{L/R})$ functions in Eq. (5.6) in the case of a vanishing bulk mass parameter $c = 0$ for which all wave functions are expressed in terms of trigonometric functions. The derivation was given in Ref. [28] and also is explained in Appendix B. One finds

for $c = 0$

$$\begin{aligned}
A_L(y, y') &= B_R(y, y') = \frac{e^{-\sigma(y)}}{k} \{ \delta_{2L}(y - y') + \delta_{2L}(y + y') \}, \\
A_R(y, y') &= B_L(y, y') = \frac{e^{-\sigma(y)}}{k} \{ \delta_{2L}(y - y') - \delta_{2L}(y + y') \}, \\
C_{L/R}(y, y') &= D_{L/R}(y, y') = 0, \\
\delta_L(x) &= \sum_{n=-\infty}^{\infty} \delta(x - nL).
\end{aligned} \tag{5.8}$$

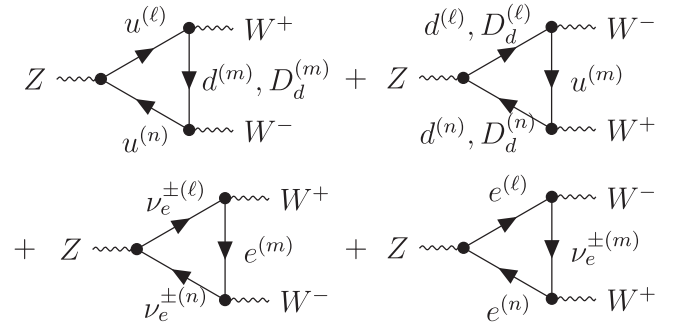


FIG. 7. Chiral anomalies for ZWW in GHU.

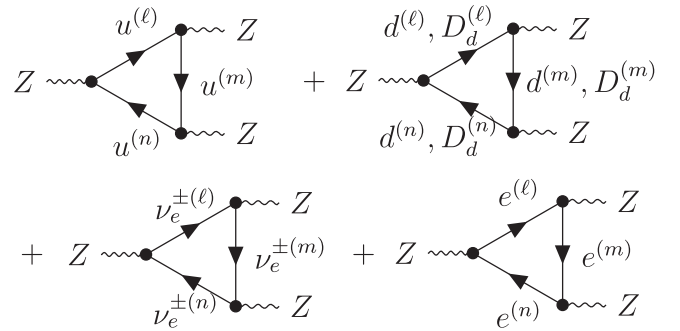


FIG. 8. Chiral anomalies for ZZZ in GHU.

With the aid of the relations (5.7) and (5.8) one finds for $F_{Z^{(\ell)}}^j$ that

$$\begin{aligned}
 F_{Z^{(\ell)}}^1 &= \frac{k}{2} \sqrt{kL} \cos \theta_W^0 \int_{-a}^{2L-a} dy e^{\sigma(y)} \{h_{Z^{(\ell)}}^{L,su2}(y)(A_L - A_R)(y, y) + h_{Z^{(\ell)}}^{R,su2}(y)(B_L - B_R)(y, y)\} \\
 &= \sqrt{kL} \cos \theta_W^0 \int_{-a}^{2L-a} dy \{h_{Z^{(\ell)}}^{L,su2}(y) - h_{Z^{(\ell)}}^{R,su2}(y)\} \delta_{2L}(2y) \\
 &= \frac{1}{2} \sqrt{kL} \cos \theta_W^0 [h_{Z^{(\ell)}}^{L,su2}(0) - h_{Z^{(\ell)}}^{R,su2}(0) + h_{Z^{(\ell)}}^{L,su2}(L) - h_{Z^{(\ell)}}^{R,su2}(L)], \\
 F_{Z^{(\ell)}}^2 &= \frac{k}{2} \sqrt{kL} \cos \theta_W^0 \int_{-a}^{2L-a} dy e^{\sigma(y)} h_{Z^{(\ell)}}^{em}(y)(A_L + B_L - A_R - B_R)(y, y) \\
 &= 0.
 \end{aligned} \tag{5.9}$$

Similarly for $F_{\gamma ZZ}^k$ we find that

$$\begin{aligned}
 F_{\gamma ZZ}^1 &= \left(\frac{k}{2} \sqrt{kL} \cos \theta_W^0 \right)^2 \int_{-a}^{2L-a} dy_1 \int_{-a}^{2L-a} dy_2 e^{\sigma(y_1)+\sigma(y_2)} \\
 &\quad \times \left(h_1^{L,su2} h_2^{L,su2} \{A_L^{2,1} A_L^{1,2} - A_R^{2,1} A_R^{1,2}\} + h_1^{R,su2} h_2^{R,su2} \{B_L^{2,1} B_L^{1,2} - B_R^{2,1} B_R^{1,2}\} \right. \\
 &\quad \left. + \frac{1}{2} \hat{h}_1^{su2} \hat{h}_2^{su2} \{A_L^{2,1} B_L^{1,2} + B_L^{2,1} A_L^{1,2} - A_R^{2,1} B_R^{1,2} - B_R^{2,1} A_R^{1,2}\} \right) \\
 &= \left(\frac{k}{2} \sqrt{kL} \cos \theta_W^0 \right)^2 \int_{-a}^{2L-a} dy_1 \int_{-a}^{2L-a} dy_2 e^{\sigma(y_1)+\sigma(y_2)} (h_1^{L,su2} h_2^{L,su2} - h_1^{R,su2} h_2^{R,su2}) \{A_L^{2,1} A_L^{1,2} - A_R^{2,1} A_R^{1,2}\} \\
 &= kL \cos^2 \theta_W^0 \int_{-a}^{2L-a} dy_1 dy_2 (h_1^{L,su2} h_2^{L,su2} - h_1^{R,su2} h_2^{R,su2}) \delta_{2L}(y_1 - y_2) \delta_{2L}(y_1 + y_2) \\
 &= \frac{kL}{2} \cos^2 \theta_W^0 [h_{Z^{(0)}}^{L,su2}(0)^2 - h_{Z^{(0)}}^{R,su2}(0)^2 + h_{Z^{(0)}}^{L,su2}(L)^2 - h_{Z^{(0)}}^{R,su2}(L)^2], \\
 F_{\gamma ZZ}^2 &= \frac{kL}{2} \cos^2 \theta_W^0 [\{h_{Z^{(0)}}^{L,su2} - h_{Z^{(0)}}^{R,su2}\} h_{Z^{(0)}}^{em}|_{y=0} + \{h_{Z^{(0)}}^{L,su2} - h_{Z^{(0)}}^{R,su2}\} h_{Z^{(0)}}^{em}|_{y=L}] \\
 &= F_{\gamma ZZ}^1, \\
 F_{\gamma ZZ}^3 &= 0,
 \end{aligned} \tag{5.10}$$

where $h_j^{L,su2} = h_{Z^{(0)}}^{L,su2}(y_j)$, $A_{L/R}^{j,k} = A_{L/R}(y_j, y_k)$ etc. The equality $F_{\gamma ZZ}^2 = F_{\gamma ZZ}^1$ follows as $h_{Z^{(0)}}^{em} = h_{Z^{(0)}}^{L,su2} + h_{Z^{(0)}}^{R,su2}$. We stress that F 's in Eqs. (5.9) and (5.10) are expressed in terms of the values of the Z wave functions at $y = 0$ and $y = L$.

As mentioned above, the independence of F 's on the fermion species, namely the relations (5.7), is numerically checked.

Let us define $F_{Z^{(\ell)}}^{\alpha j}|_{n_{\max}}$ and $F_{\gamma ZZ}^{\alpha k}|_{n_{\max}}$ by

$$\begin{aligned}
 &\sum_{n=0}^{n_{\max}} \{(\hat{g}_{L\ell}^{Zuu})_{nn} - (\hat{g}_{R\ell}^{Zuu})_{nn}\} = T_u^3 F_{Z^{(\ell)}}^{u1}|_{n_{\max}} - \sin^2 \theta_W^0 Q_u F_{Z^{(\ell)}}^{u2}|_{n_{\max}}, \\
 &\sum_{n,m=0}^{n_{\max}} \{(\hat{g}_{L0}^{Zuu})_{nm} (\hat{g}_{L0}^{Zuu})_{mn} - (\hat{g}_{R0}^{Zuu})_{nm} (\hat{g}_{R0}^{Zuu})_{mn}\} \\
 &= (T_u^3)^2 F_{\gamma ZZ}^{u1}|_{n_{\max}} - 2 \sin^2 \theta_W^0 T_u^3 Q_u F_{\gamma ZZ}^{u2}|_{n_{\max}} + \sin^4 \theta_W^0 (Q_u)^2 F_{\gamma ZZ}^{u3}|_{n_{\max}}.
 \end{aligned} \tag{5.11}$$

In other words $F|_{n_{\max}}$ is defined by doing the KK sum up to the KK level $n_{\max}$. The relations (5.7) are confirmed by showing

$$\lim_{n_{\max} \rightarrow \infty} F_{Z^{(\ell)}}^{\alpha j}|_{n_{\max}} = F_{Z^{(\ell)}}^j, \quad \lim_{n_{\max} \rightarrow \infty} F_{\gamma ZZ}^{\alpha k}|_{n_{\max}} = F_{\gamma ZZ}^k. \tag{5.12}$$

The result for $F_{Z^{(0)}}^{\alpha 1}$ ($\alpha = u, d, e, t, b, \tau$) is tabulated in Table III. It is seen that the convergence of $F_{Z^{(0)}}^{\alpha 1}|_{n_{\max}}$ to $F_{Z^{(0)}}^1$ is fast. The result for $F_{Z^{(1)}}^{\alpha 1}$ is given in Table V in Appendix C. There is large parity violation in the $Z^{(1)}$ couplings of the low-lying KK modes of fermions. The convergence of $F_{Z^{(1)}}^{\alpha 1}|_{n_{\max}}$ to $F_{Z^{(1)}}^1$ is slower than that of $F_{Z^{(0)}}^{\alpha 1}|_{n_{\max}}$ to $F_{Z^{(0)}}^1$.

TABLE III. $F_{Z^{(0)}}^{\alpha 1}|_{n_{\max}}$ ($\alpha = u, d, e, t, b, \tau$) is tabulated for $\theta_H = 0.1$ and $m_{\text{KK}} = 13$ TeV. The bottom row $n_{\max} = \infty$ represents the value of the formula in Eq. (5.9).

$n_{\max}$	$F_{Z^{(0)}}^{u1}$	$F_{Z^{(0)}}^{d1}$	$F_{Z^{(0)}}^{e1}$	$F_{Z^{(0)}}^{t1}$	$F_{Z^{(0)}}^{b1}$	$F_{Z^{(0)}}^{\tau 1}$
0	0.997688	0.997688	0.997691	0.997671	0.997671	0.997684
6	0.997659	0.997659	0.997661	0.997653	0.997653	0.997657
12	0.997655	0.997655	0.997656	0.997651	0.997651	0.997653
18	0.997653	0.997653	0.997654	0.997651	0.997651	0.997652
24	0.997652	0.997652	0.997653	0.997650	0.997650	0.997651
∞			0.997649			

TABLE IV. $F_{\gamma ZZ}^{\alpha 1}|_{n_{\max}}$ ($\alpha = u, d, e, t, b, \tau$) is tabulated for $\theta_H = 0.1$ and $m_{\text{KK}} = 13$ TeV. The bottom row $n_{\max} = \infty$ represents the value of the formula in Eq. (5.10).

$n_{\max}$	$F_{\gamma ZZ}^{u1}$	$F_{\gamma ZZ}^{d1}$	$F_{\gamma ZZ}^{e1}$	$F_{\gamma ZZ}^{t1}$	$F_{\gamma ZZ}^{b1}$	$F_{\gamma ZZ}^{\tau 1}$
0	0.995381	0.995381	0.995386	0.996691	0.995348	0.995374
6	0.997479	0.997479	0.997433	0.997692	0.997679	0.997539
12	0.997628	0.997628	0.997601	0.997742	0.997734	0.997661
18	0.997682	0.997682	0.997663	0.997759	0.997753	0.997704
24	0.997709	0.997709	0.997695	0.997768	0.997762	0.997726
∞			0.997796			

The result for $F_{\gamma ZZ}^{\alpha 1}$ ($\alpha = u, d, e, t, b, \tau$) is tabulated in Table IV. It is seen that $F_{\gamma ZZ}^{\alpha 1}|_{n_{\max}}$ converges to $F_{\gamma ZZ}^{\alpha 1}$ as $n_{\max} \rightarrow \infty$. Similarly one can check that $F_{\gamma ZZ}^{\alpha 2}|_{n_{\max} \rightarrow \infty} = F_{\gamma ZZ}^{\alpha 2} = F_{\gamma ZZ}^{\alpha 1}$. See Table VI in Appendix C. We observe that

$$F_{\gamma ZZ}^{\alpha 1}|_{n_{\max}} < F_{\gamma ZZ}^{\alpha 1} = F_{\gamma ZZ}^{\alpha 2} < F_{\gamma ZZ}^{\alpha 2}|_{n_{\max}} \quad \text{for } n_{\max} < \infty. \quad (5.13)$$

For $J_{\gamma WW}$ in (4.4) relevant quantities are

$$\begin{aligned} F_{\gamma WW}^{ud} &= \sum_{\tilde{d}=d, D_d} (\text{Tr} \hat{g}_{L0}^{Wu\tilde{d}} \hat{g}_{L0}^{W\tilde{d}u} - \text{Tr} \hat{g}_{R0}^{Wu\tilde{d}} \hat{g}_{R0}^{W\tilde{d}u}), \\ F_{\gamma WW}^{\nu_e e} &= \sum_{a=+,-} (\text{Tr} \hat{g}_{L0}^{W\nu_e^a} \hat{g}_{L0}^{W\nu_e^a} - \text{Tr} \hat{g}_{R0}^{W\nu_e^a} \hat{g}_{R0}^{W\nu_e^a}). \end{aligned} \quad (5.14)$$

Again one can check numerically that

$$\begin{aligned} F_{\gamma WW}^{\alpha} &= F_{\gamma WW}, \\ \alpha &= ud, \quad cs, \quad tb, \quad \nu_e e, \quad \nu_\mu \mu, \quad \nu_\tau \tau. \end{aligned} \quad (5.15)$$

The analytic form of $F_{\gamma WW}$ is found for the limiting case when the bulk mass parameter $c = 0$ and brane interactions are absent. In this case u and d quarks are degenerate and $g_{L/R0}^{Wu D_d} = 0$. Using $\hat{g}_{L/R0nm}^{Wud} = G_W[(h^L, h^R, \hat{h})_{W^{(0)}}; (f, g)_{L/R}^{u(n)}, (f, g)_{L/R}^{d(m)}]$ and doing Tr (the KK sum $\sum_{n,m}$) first, one finds, as in Eq. (5.10), that

$$\begin{aligned} F_{\gamma WW} &= \left(\frac{k}{2}\sqrt{kL}\right)^2 \int_{-a}^{2L-a} dy_1 \int_{-a}^{2L-a} dy_2 e^{\sigma(y_1) + \sigma(y_2)} \left(h_1^{L*} h_2^L \{A_L^{2,1} A_L^{1,2} - A_R^{2,1} A_R^{1,2}\} + h_1^{R*} h_2^R \{B_L^{2,1} B_L^{1,2} - B_R^{2,1} B_R^{1,2}\} \right. \\ &\quad \left. + \frac{1}{2} \hat{h}_1^* \hat{h}_2 \{A_L^{2,1} B_L^{1,2} + B_L^{2,1} A_L^{1,2} - A_R^{2,1} B_R^{1,2} - B_R^{2,1} A_R^{1,2}\} \right) \\ &= \frac{kL}{2} [|h_{W^{(0)}}^L(0)|^2 - |h_{W^{(0)}}^R(0)|^2 + |h_{W^{(0)}}^L(L)|^2 - |h_{W^{(0)}}^R(L)|^2], \end{aligned} \quad (5.16)$$

where $h_j^L = h_{W^{(0)}}^L(y_j)$ etc. As in Eq. (5.11) we define $F_{\gamma WW}^{\alpha}|_{n_{\max}}$ ($\alpha = ud, \nu_e e$, etc.) by doing the KK sum up to the KK level $n_{\max}$. The relation (5.15) is confirmed by showing

$$\lim_{n_{\max} \rightarrow \infty} F_{\gamma WW}^{\alpha}|_{n_{\max}} = F_{\gamma WW}. \quad (5.17)$$

The result for $F_{\gamma WW}^\alpha$ ($\alpha = ud, \nu_e e, tb, \nu_\tau \tau$) is tabulated in Table VII in Appendix C.

Arguments are generalized to J_{ZZZ} in (4.7). For the u loop we decompose as

$$\begin{aligned} \text{Tr} \hat{g}_{L0}^{Zuu} \hat{g}_{L0}^{Zuu} \hat{g}_{L0}^{Zuu} - \text{Tr} \hat{g}_{R0}^{Zuu} \hat{g}_{R0}^{Zuu} \hat{g}_{R0}^{Zuu} &= (T_u^3)^3 F_{ZZZ}^{u1} - 3 \sin^2 \theta_W^0 (T_u^3)^2 Q_u F_{ZZZ}^{u2} \\ &+ 3 \sin^4 \theta_W^0 T_u^3 (Q_u)^2 F_{ZZZ}^{u3} - \sin^6 \theta_W^0 (Q_u)^3 F_{ZZZ}^{u4}. \end{aligned} \quad (5.18)$$

With similar decompositions one can check numerically that

$$F_{ZZZ}^{uj} = F_{ZZZ}^{dj} = F_{ZZZ}^{ej} \equiv F_{ZZZ}^j, \quad F_{ZZZ}^{\nu_e 1} = F_{ZZZ}^1. \quad (5.19)$$

The analytic form of F_{ZZZ}^j is conveniently found for $c = 0$. One finds for F_{ZZZ}^1 that

$$\begin{aligned} F_{ZZZ}^1 &= \left(\frac{k}{2} \sqrt{kL} \cos \theta_W^0 \right)^3 \int_{-a}^{2L-a} dy_1 \int_{-a}^{2L-a} dy_2 \int_{-a}^{2L-a} dy_3 e^{\sigma(y_1) + \sigma(y_2) + \sigma(y_3)} \\ &\times \left(h_1^{L,su2} h_2^{L,su2} h_3^{L,su2} \{ A_L^{1,2} A_L^{2,3} A_L^{3,1} - A_R^{1,2} A_R^{2,3} A_R^{3,1} \} + h_1^{R,su2} h_2^{R,su2} h_3^{R,su2} \{ B_L^{1,2} B_L^{2,3} B_L^{3,1} - B_R^{1,2} B_R^{2,3} B_R^{3,1} \} \right. \\ &\left. + \frac{3}{2} h_1^{L,su2} \hat{h}_2^{su2} \hat{h}_3^{su2} \{ A_L^{1,2} B_L^{2,3} A_L^{3,1} - A_R^{1,2} B_R^{2,3} A_R^{3,1} \} + \frac{3}{2} h_1^{R,su2} \hat{h}_2^{su2} \hat{h}_3^{su2} \{ B_L^{1,2} A_L^{2,3} B_L^{3,1} - B_R^{1,2} A_R^{2,3} B_R^{3,1} \} \right). \end{aligned} \quad (5.20)$$

Inserting the expressions in Eq. (5.8) into the above, one finds that

$$\begin{aligned} F_{ZZZ}^1 &= \frac{1}{4} (\sqrt{kL} \cos \theta_W^0)^3 \int_{-a}^{2L-a} dy_1 \int_{-a}^{2L-a} dy_2 \int_{-a}^{2L-a} dy_3 \\ &\times (h_1^{L,su2} h_2^{L,su2} h_3^{L,su2} - h_1^{R,su2} h_2^{R,su2} h_3^{R,su2}) [\delta_{2L}(y_1 - y_2) \delta_{2L}(y_2 - y_3) \delta_{2L}(y_3 + y_1) \\ &+ \delta_{2L}(y_1 - y_2) \delta_{2L}(y_2 + y_3) \delta_{2L}(y_3 - y_1) + \delta_{2L}(y_1 + y_2) \delta_{2L}(y_2 - y_3) \delta_{2L}(y_3 - y_1) \\ &+ \delta_{2L}(y_1 + y_2) \delta_{2L}(y_2 + y_3) \delta_{2L}(y_3 + y_1)] \\ &= \frac{1}{2} (\sqrt{kL} \cos \theta_W^0)^3 [h_{Z(0)}^{L,su2}(0)^3 - h_{Z(0)}^{R,su2}(0)^3 + h_{Z(0)}^{L,su2}(L)^3 - h_{Z(0)}^{R,su2}(L)^3]. \end{aligned} \quad (5.21)$$

Similarly one finds that

$$\begin{aligned} F_{ZZZ}^2 &= \frac{1}{2} (\sqrt{kL} \cos \theta_W^0)^3 [\{ h_{Z(0)}^{L,su2}(0)^2 - h_{Z(0)}^{R,su2}(0)^2 \} h_{Z(0)}^{em}(0) + \{ h_{Z(0)}^{L,su2}(L)^2 - h_{Z(0)}^{R,su2}(L)^2 \} h_{Z(0)}^{em}(L)], \\ F_{ZZZ}^3 &= \frac{1}{2} (\sqrt{kL} \cos \theta_W^0)^3 [\{ h_{Z(0)}^{L,su2}(0) - h_{Z(0)}^{R,su2}(0) \} h_{Z(0)}^{em}(0)^2 + \{ h_{Z(0)}^{L,su2}(L) - h_{Z(0)}^{R,su2}(L) \} h_{Z(0)}^{em}(L)^2] \\ &= F_{ZZZ}^2, \\ F_{ZZZ}^4 &= 0. \end{aligned} \quad (5.22)$$

For $\theta_H = 0.1$ and $m_{KK} = 13$ TeV, $F_{ZZZ}^1 = 0.996703$ and $F_{ZZZ}^2 = F_{ZZZ}^3 = 0.997943$. The relation (5.19) is checked numerically. By defining $F_{ZZZ}^{\alpha j}|_{n_{\max}}$ as before, one confirms that $\lim_{n_{\max} \rightarrow \infty} F_{ZZZ}^{\alpha j}|_{n_{\max}} = F_{ZZZ}^j$. The result for $F_{ZZZ}^{\alpha 1}|_{n_{\max}}$ ($\alpha = u, d, \nu_e, e, t, b, \nu_\tau, \tau$) is tabulated in Table VIII in Appendix C. The result for $F_{ZZZ}^{\alpha 2}|_{n_{\max}}$ ($\alpha = u, d, e, t, b, \tau$) is tabulated in Table IX in Appendix C.

Similarly $\lim_{n_{\max} \rightarrow \infty} F_{ZZZ}^{\alpha 3}|_{n_{\max}} = F_{ZZZ}^3$ is checked. It is found that

$$F_{ZZZ}^{\alpha 2}|_{n_{\max}} < F_{ZZZ}^2 = F_{ZZZ}^3 < F_{ZZZ}^{\alpha 3}|_{n_{\max}} \quad \text{for } n_{\max} < \infty. \quad (5.23)$$

The result for $F_{ZZZ}^{\alpha 4}|_{n_{\max}}$ ($\alpha = u, d, e, t, b, \tau$) is tabulated in Table X in Appendix C. We note that in evaluating $F_{ZZZ}^{\alpha 4}|_{n_{\max}}$ ($\alpha = d, s, b$) contributions coming from D_d, D_s, D_b towers are very important.

For J_{ZWW} in (4.6) we decompose as

$$\begin{aligned}
& \sum_{\tilde{d}=d,D_d} (\text{Tr} \hat{g}_{L0}^{Wu\tilde{d}} \hat{g}_{L0}^{W\tilde{d}u} \hat{g}_{L0}^{Zuu} - \text{Tr} \hat{g}_{R0}^{Wu\tilde{d}} \hat{g}_{R0}^{W\tilde{d}u} \hat{g}_{R0}^{Zuu}) = T_u^3 F_{ZWW}^{u1} - \sin^2 \theta_W^0 Q_u F_{ZWW}^{u2}, \\
& \sum_{\tilde{d}_1, \tilde{d}_2=d,D_d} (\text{Tr} \hat{g}_{L0}^{W\tilde{d}_1 u} \hat{g}_{L0}^{Wu\tilde{d}_2} \hat{g}_{L0}^{Z\tilde{d}_2 \tilde{d}_1} - \text{Tr} \hat{g}_{R0}^{W\tilde{d}_1 u} \hat{g}_{R0}^{Wu\tilde{d}_2} \hat{g}_{R0}^{Z\tilde{d}_2 \tilde{d}_1}) = T_d^3 F_{ZWW}^{d1} - \sin^2 \theta_W^0 Q_d F_{ZWW}^{d2}, \\
& \sum_{a,b=+,-} (\text{Tr} \hat{g}_{L0}^{W\nu_e^a e} \hat{g}_{L0}^{W^\dagger e \nu_e^b} \hat{g}_{L0}^{Z\nu_e^b \nu_e^a} - \text{Tr} \hat{g}_{R0}^{W\nu_e^a e} \hat{g}_{R0}^{W^\dagger e \nu_e^b} \hat{g}_{R0}^{Z\nu_e^b \nu_e^a}) = T_\nu^3 F_{ZWW}^{\nu_e 1}, \\
& \sum_{a=+,-} (\text{Tr} \hat{g}_{L0}^{W^\dagger e \nu_e^a} \hat{g}_{L0}^{W\nu_e^a e} \hat{g}_{L0}^{Zee} - \text{Tr} \hat{g}_{R0}^{W^\dagger e \nu_e^a} \hat{g}_{R0}^{W\nu_e^a e} \hat{g}_{R0}^{Zee}) = T_e^3 F_{ZWW}^{e1} - \sin^2 \theta_W^0 Q_e F_{ZWW}^{e2}. \tag{5.24}
\end{aligned}$$

We find numerically that

$$\begin{aligned}
F_{ZWW}^{\alpha 1} &= F_{ZWW}^1 \quad \text{for } \alpha = u, d, \nu_e, e, \\
F_{ZWW}^{\beta 2} &= F_{ZWW}^2 \quad \text{for } \beta = u, d, e. \tag{5.25}
\end{aligned}$$

Analytic form of F_{ZWW}^j is found, as in the case of $F_{\gamma WW}$, by considering the case with vanishing bulk mass parameter and brane interactions. Manipulations are similar to those for F_{ZZZ}^j . One finds that

$$\begin{aligned}
F_{ZWW}^1 &= \frac{1}{2} (kL)^{3/2} \cos \theta_W^0 [|h_{W(0)}^L(0)|^2 h_{Z(0)}^{L, su2}(0) - |h_{W(0)}^R(0)|^2 h_{Z(0)}^{R, su2}(0) + |h_{W(0)}^L(L)|^2 h_{Z(0)}^{L, su2}(L) - |h_{W(0)}^R(L)|^2 h_{Z(0)}^{R, su2}(L)], \\
F_{ZWW}^2 &= \frac{1}{2} (kL)^{3/2} \cos \theta_W^0 [(|h_{W(0)}^L(0)|^2 - |h_{W(0)}^R(0)|^2) h_{Z(0)}^{em}(0) + (|h_{W(0)}^L(L)|^2 - |h_{W(0)}^R(L)|^2) h_{Z(0)}^{em}(L)]. \tag{5.26}
\end{aligned}$$

The relation (5.25) is checked numerically. By defining $F_{ZWW}^{\alpha j}|_{n_{\max}}$ as before, one confirms that $\lim_{n_{\max} \rightarrow \infty} F_{ZWW}^{\alpha j}|_{n_{\max}} = F_{ZWW}^j$. The results for F_{ZWW}^1 and F_{ZWW}^2 are tabulated in Tables XI and XII in Appendix C, respectively.

We have established holographic formulas (5.9), (5.10), (5.16), (5.21), (5.22), and (5.26) for chiral anomalies. The F factors are expressed in terms of the values of the W and Z wave functions at the UV and IR branes, irrespective of fermion species running along triangular loops. It is important to include the contributions of all KK fermion modes.

Anomalies flow with θ_H . The θ_H -dependence of the anomalies can be easily seen from the holographic formulas derived above. The wave functions of the W and Z bosons depend on θ_H and z_L so that various F factors are functions of θ_H and z_L ; $F = F(\theta_H, z_L)$. The θ_H -dependence of the wave functions of W and Z at $y = 0$ and L with z_L kept fixed is displayed in Figs. 9 and 10. Each anomaly coefficient is proportional to the F factor. The θ_H -dependence of $F_{\gamma\gamma Z}^1 \equiv F_{Z(0)}^1$, F_{ZZZ}^1 , F_{ZWW}^1 , and F_{ZWW}^2 is displayed in Figs. 11 and 12. One sees that F 's smoothly change with θ_H in the RS space ($z_L > 1$). These F 's vanish at $\theta_H = \pi$ where quark and lepton multiplet fields become vectorlike in gauge couplings. In the flat spacetime limit,

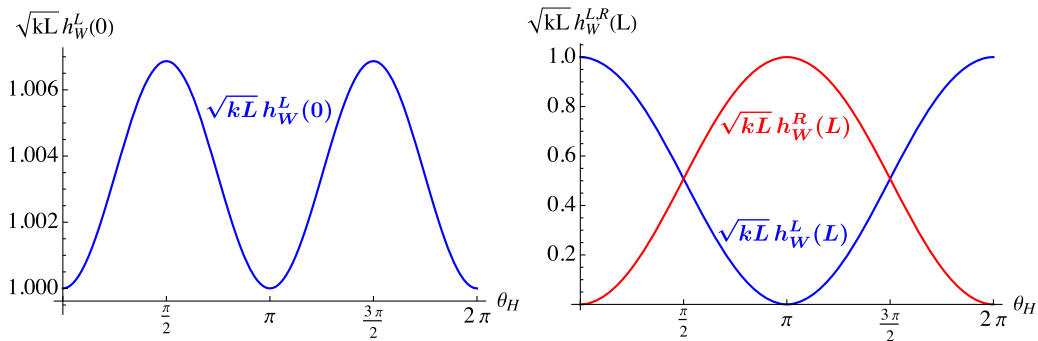


FIG. 9. θ_H dependence of $\sqrt{kL}\{h_W^L(0, \theta_H), h_W^L(L, \theta_H), h_W^R(L, \theta_H)\}$ is shown. $h_W^R(0, \theta_H) \sim 0$. The behavior of $\cos \theta_W^0 \sqrt{kL} h_Z^{L/R, su2}(y, \theta_H)$ at $y = 0, L$ is similar to that of $\sqrt{kL} h_W^{L/R}(y, \theta_H)$ at $y = 0, L$.

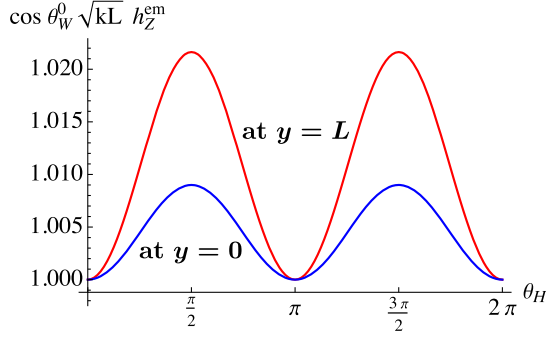


FIG. 10. θ_H dependence of $\cos\theta_W^0\sqrt{kL}\{h_Z^{em}(0,\theta_H), h_Z^{em}(L,\theta_H)\}$ is displayed.

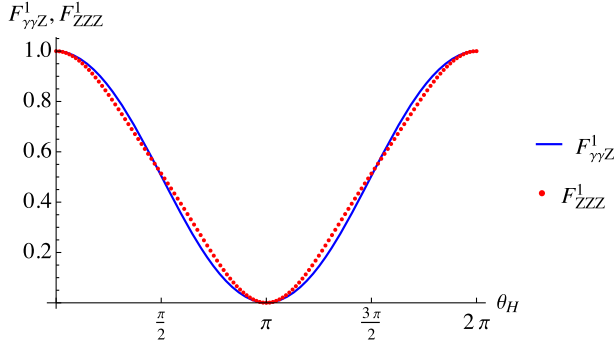


FIG. 11. θ_H dependence of $F_{\gamma Z}^1 = F_{Z^{(0)}}^1$ and F_{ZZZ}^1 is displayed.

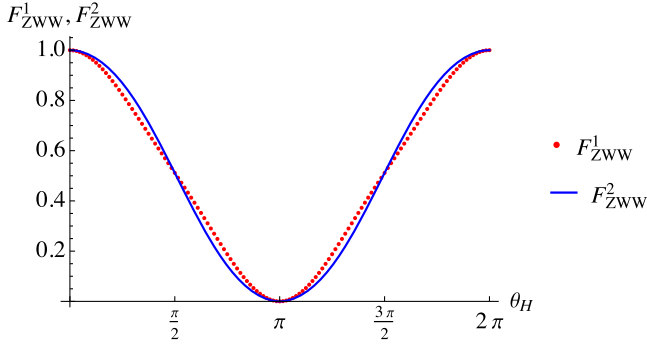


FIG. 12. θ_H dependence of F_{ZWW}^1 and F_{ZWW}^2 is displayed.

namely in the $k \rightarrow 0$ and $z_L \rightarrow 1$ limit, the F factors show the behavior of step functions at $\theta_H = 0, \pm\frac{1}{2}\pi, \pm\pi$ where level crossing in the mass spectrum of gauge bosons takes place, as was shown in the $SU(2)$ model in Ref. [28].

VI. ANOMALY CANCELLATION

We have seen in the previous section that total chiral anomalies, which take into account the contributions of all fermion KK modes, are expressed in terms of the values of wave functions of gauge bosons at the UV and IR branes. The magnitude of the total anomalies is independent of

bulk mass parameters of fermions running along internal triangular loops. With this fact the cancellation of gauge anomalies in the GUT-inspired GHU model is achieved as shown below.

Gauge anomalies for $\gamma\gamma Z^{(\ell)}$ and $ggZ^{(\ell)}$ are proportional to $J_{\gamma\gamma Z^{(\ell)}}$ and $J_{ggZ^{(\ell)}}$ in Eqs. (4.1) and (4.2), respectively. With Eqs. (5.7) and (5.9) one finds that

$$\begin{aligned} J_{\gamma\gamma Z^{(\ell)}} &= \{3[(Q_u)^2 T_u^3 + (Q_d)^2 T_d^3] + (Q_e)^2 T_e^3\} F_{Z^{(\ell)}}^1 \\ &= 0, \\ J_{ggZ^{(\ell)}} &= 3(T_u^3 + T_d^3) F_{Z^{(\ell)}}^1 \\ &= 0. \end{aligned} \quad (6.1)$$

For γZZ , with Eqs. (5.7) and (5.10), one finds that

$$\begin{aligned} J_{\gamma ZZ} &= \{3[Q_u(T_u^3)^2 + Q_d(T_d^3)^2] + Q_e(T_e^3)^2\} F_{\gamma ZZ}^1 \\ &\quad - 2\sin^2\theta_W^0 \{3[(Q_u)^2 T_u^3 + (Q_d)^2 T_d^3] + (Q_e)^2 T_e^3\} F_{\gamma ZZ}^2 \\ &= 0. \end{aligned} \quad (6.2)$$

For γWW , with Eqs. (5.15) and (5.10),

$$\begin{aligned} J_{\gamma WW} &= \{3[Q_u + Q_d] + Q_e\} F_{\gamma WW} \\ &= 0. \end{aligned} \quad (6.3)$$

For ZZZ , with Eqs. (5.19), (5.21), and (5.22),

$$\begin{aligned} J_{ZZZ} &= \{3[(T_u^3)^3 + (T_d^3)^3] + (T_\nu^3)^3 + (T_e^3)^3\} F_{ZZZ}^1 \\ &\quad - 3\sin^2\theta_W^0 \{3[Q_u(T_u^3)^2 + Q_d(T_d^3)^2] + Q_e(T_e^3)^2\} F_{ZZZ}^2 \\ &\quad + 3\sin^4\theta_W^0 \{3[(Q_u)^2 T_u^3 + (Q_d)^2 T_d^3] + (Q_e)^2 T_e^3\} F_{ZZZ}^3 \\ &= 0. \end{aligned} \quad (6.4)$$

For ZWW , with Eqs. (5.25) and (5.26),

$$\begin{aligned} J_{ZWW} &= \{3[T_u^3 + T_d^3] + T_\nu^3 + T_e^3\} F_{ZWW}^1 \\ &\quad - \sin^2\theta_W^0 \{3[Q_u + Q_d] + Q_e\} F_{ZWW}^2 \\ &= 0. \end{aligned} \quad (6.5)$$

Thus all gauge anomalies are canceled in each generation.

So far we have mainly presented the results of gauge anomalies associated with gauge bosons γ , W and Z . Generalization to anomalies associated with KK modes $\gamma^{(n)}$, $W^{(n)}$, and $Z^{(n)}$ is straightforward. In the F factors wave functions of gauge bosons are replaced by those of KK gauge bosons. One comment is in order. In the GUT-inspired GHU there are dark fermion multiplets listed in Eq. (2.7). The multiplets $\Psi_{(3,4)}^{F_q,\alpha}$ and $\Psi_{(1,4)}^{F_\ell,\alpha}$ give contributions to chiral anomalies. Their contributions are summarized by formulas similar to those presented above. For instance, the F_{ZZZ}^1 part in J_{ZZZ} , contributions from quark-lepton multiplets are

decomposed as $F_{ZZZ}^1 = F_{ZZZ}^1|_{y=0} + F_{ZZZ}^1|_{y=L}$ in Eq. (5.21). Contributions from dark fermion multiplets are cast as $F_{ZZZ}^{\text{IDF}} = -F_{ZZZ}^1|_{y=0} + F_{ZZZ}^1|_{y=L}$. Thorough discussions of contributions coming from fermion multiplets with various orbifold boundary conditions in an $SU(2)$ model are given in Ref. [28]. Gauge anomalies must be canceled in the dark fermion sector as well. In the model specified in this paper with boundary conditions (2.6) and (2.7) gauge anomalies are canceled both in the quark-lepton and dark fermion sectors.

VII. SUMMARY

In the GUT-inspired GHU in the RS space the W and Z couplings of quarks and leptons vary with θ_H . The magnitude of these couplings depends on fermion species at $\theta_H \neq 0, \pi$. This fact leads to a question of whether or not the cancellation of gauge anomalies is spoiled. First we noted that all KK modes of quark-lepton multiplets contribute to chiral anomalies. It has been shown in this paper that total chiral anomalies including contributions of all KK fermion modes are expressed in terms of the values of wave functions of W and Z bosons at the UV and IR branes. We derived holographic formulas for chiral anomalies. The total chiral anomalies, which are originally expressed as sums of products of gauge couplings, are decomposed into products of group factors (T^3 and Q) and various F factors. We showed that F factors are universal and independent of fermion species running along internal triangular loops. As the F factors are universal, cancellation of anomalies is achieved when sums of group factors of fermion multiplets vanish. This is indeed what happens in the GUT-inspired GHU model.

In this paper we have focused on gauge anomalies. The technique developed in deriving holographic formulas for anomalies is general, and can be applied to global anomalies as well. It is of great interest to derive a formula for an anomaly in baryon number current. In the GUT-inspired GHU in the RS space anomaly terms of the form $F_{\mu\nu}\tilde{F}^{\mu\nu}$ contain KK gauge bosons whose anomaly coefficients are expected to become large for the first KK modes. It may affect the process of baryon number generation in the universe. We hope to come back to this issue in the near future.

DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: BASIS FUNCTIONS IN THE RS SPACE

Wave functions of gauge and fermion fields in the RS space are expressed in terms of Bessel functions. For gauge fields we introduce

$$\begin{aligned} F_{\alpha\beta}(u, v) &\equiv J_\alpha(u)Y_\beta(v) - Y_\alpha(u)J_\beta(v), \\ C(z; \lambda) &= \frac{\pi}{2}\lambda z z_L F_{1,0}(\lambda z, \lambda z_L), \\ S(z; \lambda) &= -\frac{\pi}{2}\lambda z F_{1,1}(\lambda z, \lambda z_L), \\ C'(z; \lambda) &= \frac{\pi}{2}\lambda^2 z z_L F_{0,0}(\lambda z, \lambda z_L), \\ S'(z; \lambda) &= -\frac{\pi}{2}\lambda^2 z F_{0,1}(\lambda z, \lambda z_L), \\ \check{S}(z; \lambda) &= \frac{C(1; \lambda)}{S(1; \lambda)} S(z; \lambda), \end{aligned} \quad (\text{A1})$$

where $J_\alpha(u)$ and $Y_\alpha(u)$ are Bessel functions of the first and second kind. $C(z; \lambda)$ and $S(z; \lambda)$ satisfy

$$-z \frac{d}{dz} \frac{1}{z} \frac{d}{dz} \begin{pmatrix} C \\ S \end{pmatrix} = \lambda^2 \begin{pmatrix} C \\ S \end{pmatrix}. \quad (\text{A2})$$

Boundary conditions are given by $C = z_L$, $C' = S = 0$, and $S' = \lambda$ at $z = z_L$. A relation $CS' - SC' = \lambda z$ holds.

For fermion fields with a bulk mass parameter c , we define

$$\begin{aligned} \begin{pmatrix} C_L \\ S_L \end{pmatrix}(z; \lambda, c) &= \pm \frac{\pi}{2} \lambda \sqrt{z z_L} F_{c+\frac{1}{2}, c \mp \frac{1}{2}}(\lambda z, \lambda z_L), \\ \begin{pmatrix} C_R \\ S_R \end{pmatrix}(z; \lambda, c) &= \mp \frac{\pi}{2} \lambda \sqrt{z z_L} F_{c-\frac{1}{2}, c \pm \frac{1}{2}}(\lambda z, \lambda z_L), \\ \begin{pmatrix} \check{S}_L \\ \check{C}_R \end{pmatrix}(z; \lambda, c) &= \frac{C_L(1; \lambda, c)}{S_L(1; \lambda, c)} \begin{pmatrix} S_L \\ C_R \end{pmatrix}(z; \lambda, c). \end{aligned} \quad (\text{A3})$$

These functions satisfy

$$\begin{aligned} D_+(c) \begin{pmatrix} C_L \\ S_L \end{pmatrix} &= \lambda \begin{pmatrix} S_R \\ C_R \end{pmatrix}, \\ D_-(c) \begin{pmatrix} S_R \\ C_R \end{pmatrix} &= \lambda \begin{pmatrix} C_L \\ S_L \end{pmatrix}, \\ D_\pm(c) &= \pm \frac{d}{dz} + \frac{c}{z}, \end{aligned} \quad (\text{A4})$$

with the boundary conditions $C_{R/L} = 1$, $D_-(c)C_R = D_+(c)C_L = S_{R/L} = 0$ at $z = z_L$, and $C_L C_R - S_L S_R = 1$. For fermion fields with a vector-like mass $m = k\tilde{m}$, mode functions are expressed in terms of

$$\begin{aligned} C_{L/R1}(z; \lambda, c, \tilde{m}) &= C_{L/R}(z; \lambda, c + \tilde{m}) + C_{L/R}(z; \lambda, c - \tilde{m}), \\ S_{L/R1}(z; \lambda, c, \tilde{m}) &= S_{L/R}(z; \lambda, c + \tilde{m}) + S_{L/R}(z; \lambda, c - \tilde{m}), \\ C_{L/R2}(z; \lambda, c, \tilde{m}) &= S_{L/R}(z; \lambda, c + \tilde{m}) - S_{L/R}(z; \lambda, c - \tilde{m}), \\ S_{L/R2}(z; \lambda, c, \tilde{m}) &= C_{L/R}(z; \lambda, c + \tilde{m}) - C_{L/R}(z; \lambda, c - \tilde{m}). \end{aligned} \quad (\text{A5})$$

APPENDIX B: FERMIONS WITH $c=0$

For fermions with a vanishing bulk mass parameter $c=0$ wave functions are expressed in terms of trigonometric functions. Basis functions become

$$\begin{pmatrix} C_L \\ S_L \end{pmatrix}(z; \lambda, 0) = \begin{pmatrix} \cos \lambda(z - z_L) \\ \sin \lambda(z - z_L) \end{pmatrix}, \\ \begin{pmatrix} C_R \\ S_R \end{pmatrix}(z; \lambda, 0) = \begin{pmatrix} \cos \lambda(z - z_L) \\ -\sin \lambda(z - z_L) \end{pmatrix}. \quad (\text{B1})$$

The spectrum $\{\lambda_n\}$ determined by Eq. (2.21) is

$$\lambda_n = \frac{1}{z_L - 1} |n\pi + \frac{1}{2}\theta_H| \quad (-\infty < n < \infty). \quad (\text{B2})$$

The corresponding wave functions in $1 \leq z = e^{ky} \leq z_L$ in the original gauge are

$$\begin{pmatrix} f_{Ln}(y) \\ g_{Ln}(y) \end{pmatrix} = \frac{1}{\sqrt{z_L - 1}} \begin{pmatrix} \cos \alpha_n(z) \\ i \sin \alpha_n(z) \end{pmatrix}, \\ \begin{pmatrix} f_{Rn}(y) \\ g_{Rn}(y) \end{pmatrix} = \frac{1}{\sqrt{z_L - 1}} \begin{pmatrix} i \sin \alpha_n(z) \\ \cos \alpha_n(z) \end{pmatrix}, \\ \alpha_n(z) = \left(n\pi + \frac{1}{2}\theta_H\right) \frac{z - z_L}{z_L - 1} + \frac{1}{2}\theta(z). \quad (\text{B3})$$

Note that $\alpha_n(1) = -n\pi$ and $\alpha_n(z_L) = 0$. From the boundary conditions

$$\begin{pmatrix} f_{Ln}(y) \\ g_{Ln}(y) \end{pmatrix} = \begin{pmatrix} f_{Ln}(-y) \\ -g_{Ln}(-y) \end{pmatrix} = \begin{pmatrix} f_{Ln}(y + 2L) \\ g_{Ln}(y + 2L) \end{pmatrix}, \\ \begin{pmatrix} f_{Rn}(y) \\ g_{Rn}(y) \end{pmatrix} = \begin{pmatrix} -f_{Rn}(-y) \\ g_{Rn}(-y) \end{pmatrix} = \begin{pmatrix} f_{Rn}(y + 2L) \\ g_{Rn}(y + 2L) \end{pmatrix}. \quad (\text{B4})$$

Making use of Eqs. (B3) and (B4), one finds the relations in Eq. (5.8).

Boundary conditions of dark fermions are given by Eq. (2.7). For $\Psi_{(3,4)}^{F_q, \alpha}$ and $\Psi_{(1,4)}^{F_\ell, \alpha}$ with $c_{F_q} = c_{F_\ell} = 0$, $\{\lambda_n\}$ and $\alpha_n(z)$ in Eqs. (B2) and (B3) are replaced by $\{\lambda_n^{DF}\}$ and $\beta_n(z)$ where

$$\lambda_n^{DF} = \frac{1}{z_L - 1} \left| \left(n + \frac{1}{2}\right)\pi + \frac{1}{2}\theta_H \right| \quad (-\infty < n < \infty), \\ \beta_n(z) = \left(n\pi + \frac{1}{2}\pi + \frac{1}{2}\theta_H\right) \frac{z - z_L}{z_L - 1} + \frac{1}{2}\theta(z). \quad (\text{B5})$$

APPENDIX C: UNIVERSALITY OF F FACTORS

In Sec. V we argued that various F factors appearing in the formulas of chiral anomalies are universal. For $\gamma\gamma Z^{(\ell)}$ and γZZ anomalies, for instance, the relations $F_{Z^{(\ell)}}^{\alpha j} = F_{Z^{(\ell)}}^j$ and $F_{\gamma ZZ}^{\alpha k} = F_{\gamma ZZ}^k$ ($\alpha = u, d, c, s, t, b, e, \mu, \tau$) in Eq. (5.7)

are confirmed by showing the relations in Eq. (5.12), $F_{Z^{(\ell)}}^{\alpha j}|_{n_{\max}} \rightarrow F_{Z^{(\ell)}}^j$ and $F_{\gamma ZZ}^{\alpha k}|_{n_{\max}} \rightarrow F_{\gamma ZZ}^k$ as $n_{\max} \rightarrow \infty$, where $F|_{n_{\max}}$ is defined by doing the KK sum up to the KK level $n_{\max}$. $F|_{n_{\max}}$ is numerically evaluated. The results for $F_{Z^{(0)}}^{\alpha 1}|_{n_{\max}}$ and $F_{\gamma ZZ}^{\alpha 1}|_{n_{\max}}$ have been given in Tables III and IV in Sec. V, respectively. In this Appendix we present the results for other F factors.

(i) $F_{Z^{(\ell)}}^{\alpha 1}$

The result for $F_{Z^{(0)}}^{\alpha 1}|_{n_{\max}}$ is given in Table III in Sec. V. The result for $F_{Z^{(1)}}^{\alpha 1}|_{n_{\max}}$ is given in Table V. It converges to $F_{Z^{(1)}}^1$ as $n_{\max} \rightarrow \infty$, though the convergence is slower than in the case of $F_{Z^{(0)}}^{\alpha 1}|_{n_{\max}}$.

(ii) $F_{\gamma ZZ}^{\alpha 2}$ and $F_{\gamma WW}^{\alpha}$

The result for $F_{\gamma ZZ}^{\alpha 1}|_{n_{\max}}$ ($\alpha = u, d, e, t, b, \tau$) is given in Table IV in Sec. V. The result for $F_{\gamma ZZ}^{\alpha 2}|_{n_{\max}}$ is given in Table VI. The result for $F_{\gamma WW}^{\alpha}|_{n_{\max}}$ ($\alpha = ud, \nu_e, tb, \nu_\tau$) is tabulated in Table VII.

(iii) $F_{ZZZ}^{\alpha 1}$, $F_{ZZZ}^{\alpha 2}$, and $F_{ZZZ}^{\alpha 4}$

The result for $F_{ZZZ}^{\alpha 1}|_{n_{\max}}$ ($\alpha = u, d, \nu_e, e, t, b, \nu_\tau$) is given in Table VIII. $F_{ZZZ}^{u1}|_{n_{\max}} \sim F_{ZZZ}^{d1}|_{n_{\max}}$ and $F_{ZZZ}^{\nu_e 1}|_{n_{\max}} \sim F_{ZZZ}^{e1}|_{n_{\max}}$ in six-digit accuracy. The result for $F_{ZZZ}^{\alpha 2}|_{n_{\max}}$ ($\alpha = u, d, e, t, b, \tau$) is tabulated in Table IX. $F_{ZZZ}^{u2}|_{n_{\max}} \sim F_{ZZZ}^{d2}|_{n_{\max}}$ in six-digit accuracy.

The result for $F_{ZZZ}^{\alpha 4}|_{n_{\max}}$ ($\alpha = u, d, e, t, b, \tau$) is given in Table X. As remarked in the text, in evaluating $F_{ZZZ}^{\alpha 4}|_{n_{\max}}$ ($\alpha = d, s, b$) contributions coming from D_d, D_s, D_b towers are very important. For instance, for $n_{\max} = 24$, $F_{ZZZ}^{b4}|_{b \text{ tower loop}} = -0.000188$ whereas $F_{ZZZ}^{b4}|_{D_b \text{ tower loop}} = 0.000179$.

(iv) $F_{ZWW}^{\alpha 1}$ and $F_{ZWW}^{\alpha 2}$

The result for $F_{ZWW}^{\alpha 1}|_{n_{\max}}$ ($\alpha = u, d, \nu_e, e, t, b, \nu_\tau$) is given in Table XI. $F_{ZWW}^{u1}|_{n_{\max}} \sim F_{ZWW}^{d1}|_{n_{\max}}$ and $F_{ZWW}^{\nu_e 1}|_{n_{\max}} \sim F_{ZWW}^{e1}|_{n_{\max}}$ in six-digit accuracy. The result for $F_{ZWW}^{\alpha 2}|_{n_{\max}}$ ($\alpha = u, d, e, t, b, \tau$) is given in Table XII. $F_{ZWW}^{u2}|_{n_{\max}} \sim F_{ZWW}^{d2}|_{n_{\max}}$ and $F_{ZWW}^{\nu_e 2}|_{n_{\max}} \sim F_{ZWW}^{e2}|_{n_{\max}}$ in six-digit accuracy.

TABLE V. $F_{Z^{(1)}}^{\alpha 1}|_{n_{\max}}$ ($\alpha = u, d, e, t, b, \tau$) is tabulated for $\theta_H = 0.1$ and $m_{\text{KK}} = 13$ TeV. The bottom row $n_{\max} = \infty$ represents the value of the formula in Eq. (5.9).

$n_{\max}$	$F_{Z^{(1)}}^{u1}$	$F_{Z^{(1)}}^{d1}$	$F_{Z^{(1)}}^{e1}$	$F_{Z^{(1)}}^{t1}$	$F_{Z^{(1)}}^{b1}$	$F_{Z^{(1)}}^{\tau 1}$
0	5.56610	5.56610	5.74876	4.40691	4.41348	5.29107
10	3.84398	3.84398	3.89273	3.63898	3.63901	3.78169
20	3.70283	3.70283	3.73028	3.59060	3.59062	3.66819
30	3.65039	3.65039	3.66949	3.57312	3.57315	3.62640
40	3.62303	3.62303	3.63766	3.56411	3.56413	3.60468
∞	3.53591					

TABLE VI. $F_{\gamma ZZ}^{\alpha 2}|_{n_{\max}}$ ($\alpha = u, d, e, t, b, \tau$) is tabulated for $\theta_H = 0.1$ and $m_{KK} = 13$ TeV. The bottom row $n_{\max} = \infty$ represents the value of the formula in Eq. (5.10).

$n_{\max}$	$F_{\gamma ZZ}^{u2}$	$F_{\gamma ZZ}^{d2}$	$F_{\gamma ZZ}^{e2}$	$F_{\gamma ZZ}^{t2}$	$F_{\gamma ZZ}^{b2}$	$F_{\gamma ZZ}^{\tau 2}$
0	0.997874	0.997874	0.997879	0.997841	0.997841	0.997866
6	0.997817	0.997817	0.997820	0.997804	0.997804	0.997813
12	0.997808	0.997808	0.997810	0.997800	0.997800	0.997805
18	0.997805	0.997805	0.997806	0.997799	0.997799	0.997802
24	0.997803	0.997803	0.997804	0.997798	0.997798	0.997800
∞						0.997796

TABLE VII. $F_{\gamma WW}^{\alpha}|_{n_{\max}}$ ($\alpha = ud, \nu_e e, tb, \nu_\tau \tau$) is tabulated for $\theta_H = 0.1$ and $m_{KK} = 13$ TeV. The bottom row $n_{\max} = \infty$ represents the value of the formula in Eq. (5.16).

$n_{\max}$	$F_{\gamma WW}^{ud}$	$F_{\gamma WW}^{\nu_e e}$	$F_{\gamma WW}^{tb}$	$F_{\gamma WW}^{\nu_\tau \tau}$
0	0.995296	0.995300	0.995942	0.995290
6	0.997407	0.997360	0.997616	0.997468
12	0.997558	0.997531	0.997669	0.997592
18	0.997612	0.997593	0.997688	0.997636
24	0.997640	0.997626	0.997697	0.997658
∞				0.997728

TABLE VIII. $F_{ZZZ}^{\alpha 1}|_{n_{\max}}$ ($\alpha = u, d, \nu_e, e, t, b, \nu_\tau, \tau$) is tabulated for $\theta_H = 0.1$ and $m_{KK} = 13$ TeV. The bottom row $n_{\max} = \infty$ represents the value of the formula in Eq. (5.21).

$n_{\max}$	$F_{ZZZ}^{u1}, F_{ZZZ}^{d1}$	$F_{ZZZ}^{\nu_e 1}, F_{ZZZ}^{e1}$	F_{ZZZ}^{t1}	F_{ZZZ}^{b1}	$F_{ZZZ}^{\nu_\tau 1}$	$F_{ZZZ}^{\tau 1}$
0	0.993080	0.993088	0.995041	0.993030	0.993069	0.993069
6	0.996227	0.996158	0.996547	0.996527	0.996318	0.996317
12	0.996450	0.996410	0.996621	0.996610	0.996502	0.996500
18	0.996531	0.996503	0.996648	0.996638	0.996567	0.996565
24	0.996573	0.996551	0.996661	0.996652	0.996600	0.996598
∞						0.996703

TABLE IX. $F_{ZZZ}^{\alpha 2}|_{n_{\max}}$ ($\alpha = u, d, e, t, b, \tau$) is tabulated for $\theta_H = 0.1$ and $m_{KK} = 13$ TeV. The bottom row $n_{\max} = \infty$ represents the value of the formula in Eq. (5.22).

$n_{\max}$	$F_{ZZZ}^{u2}, F_{ZZZ}^{d2}$	F_{ZZZ}^{e2}	F_{ZZZ}^{t2}	F_{ZZZ}^{b2}	$F_{ZZZ}^{\tau2}$
0	0.995567	0.995574	0.996860	0.995517	0.995556
6	0.997637	0.997592	0.997843	0.997830	0.997695
12	0.997781	0.997755	0.997891	0.997883	0.997813
18	0.997833	0.997815	0.997908	0.997901	0.997855
24	0.997860	0.997846	0.997916	0.997910	0.997876
∞	0.997943				

TABLE X. $F_{ZZZ}^{\alpha 4}|_{n_{\max}}$ ($\alpha = u, d, e, t, b, \tau$) is tabulated for $\theta_H = 0.1$ and $m_{KK} = 13$ TeV. The bottom row $n_{\max} = \infty$ represents the value of the formula in Eq. (5.22).

$n_{\max}$	F_{ZZZ}^{u4}	F_{ZZZ}^{d4}	F_{ZZZ}^{e4}	F_{ZZZ}^{t4}	F_{ZZZ}^{b4}	$F_{ZZZ}^{\tau 4}$
0	0.000299	0.000209	0.000306	0.000127	0.000054	0.000287
6	0.000066	0.000034	0.000075	0.000023	-0.000026	0.000054
12	0.000037	0.000019	0.000043	0.000012	-0.000016	0.000030
18	0.000026	0.000014	0.000030	0.000009	-0.000011	0.000021
24	0.000020	0.000011	0.000023	0.000006	-0.000009	0.000016
∞	0					

TABLE XI. $F_{ZWW}^{\alpha 1}|_{n_{\max}}$ ($\alpha = u, d, \nu_e, e, t, b, \nu_\tau, \tau$) is tabulated for $\theta_H = 0.1$ and $m_{KK} = 13$ TeV. The bottom row $n_{\max} = \infty$ represents the value of the formula in Eq. (5.26).

$n_{\max}$	$F_{ZWW}^{u1}, F_{ZWW}^{d1}$	$F_{ZWW}^{\nu_e 1}, F_{ZWW}^{e1}$	F_{ZWW}^{t1}	F_{ZWW}^{b1}	$F_{ZWW}^{\nu_\tau 1}$	$F_{ZWW}^{\tau 2}$
0	0.992995	0.993001	0.994292	0.993622	0.992985	0.992985
6	0.996155	0.996085	0.996471	0.996464	0.996247	0.996246
12	0.996380	0.996340	0.996549	0.996545	0.996431	0.996431
18	0.996462	0.996433	0.996576	0.996573	0.996497	0.996496
24	0.996504	0.996482	0.996590	0.996587	0.996530	0.996530
∞	0.996635					

TABLE XII. $F_{ZWW}^{\alpha 2}|_{n_{\max}}$ ($\alpha = u, d, e, t, b, \tau$) is tabulated for $\theta_H = 0.1$ and $m_{KK} = 13$ TeV. The bottom row $n_{\max} = \infty$ represents the value of the formula in Eq. (5.26).

$n_{\max}$	$F_{ZWW}^{u2}, F_{ZWW}^{d2}$	F_{ZWW}^{e2}	$F_{ZWW}^{t2}, F_{ZWW}^{b2}$	$F_{ZWW}^{\tau 2}$
0	0.995482	0.995488	0.996111	0.995472
6	0.997564	0.997519	0.997767	0.997624
12	0.997711	0.997684	0.997819	0.997744
18	0.997763	0.997745	0.997836	0.997786
24	0.997791	0.997777	0.997845	0.997808
∞	0.997876			

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