

Yang-Mills kinematic algebra via homotopy transfer from a worldline operator algebra

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The homotopy Lie or L_∞ algebra encoding Yang-Mills theory is the tensor product of a color Lie algebra with the kinematic C_∞ algebra. We derive this C_∞ algebra, via homotopy transfer, from a strict operator algebra of a worldline theory, realized as an associative star product algebra. This gives a homotopy transfer interpretation to worldline vertex operators introduced in previous work.

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I. INTRODUCTION

The kinematic algebra of Yang-Mills theory denotes a conjectural structure suggested by the color-kinematics duality discovered at the level of scattering amplitudes, which in turn is the basis for the double copy construction of gravity amplitudes as the square of gauge theory amplitudes [1–3]. While these properties are well understood at the level of on-shell objects like tree-level scattering amplitudes, it was only more recently that significant progress was made in finding a first-principle derivation that is off-shell, local and gauge invariant [4–14]. In this, the formulation of field theory in terms of homotopy algebras is instrumental [15–18]. In particular, an algebraic structure capturing color-kinematics duality was proposed by Reiterer [19] and, being a homotopy generalization of a Batalin-Vilkovisky (BV) algebra, was named $BV_\infty^\square$, with $\square$ referring to the d'Alembert wave operator. Except for special theories and formulations [8,14,19], this $BV_\infty^\square$ algebra is only partially understood.

Our goal in this paper is to make progress in finding a general all-order description of this elusive algebra. One would like to derive the $BV_\infty^\square$ algebra, including its infinitely many maps of ever increasing arity, from something simple, ideally an algebra that is strict in that all maps or brackets are at most bilinear. Here we will follow an approach that seems particularly promising to us, based on a worldline QFT and its operator algebra. In fact, recently

three of us showed that part of the Yang-Mills kinematic algebra can be derived from the strict algebra of vertex operators associated to this worldline QFT [20]. (In this we were greatly inspired by the work of Lian-Zuckermann and Zeitlin [21–26].) However, these vertex operators were unconventional, requiring nonlinear contributions, see also [27,28] for the role of nonlinear vertex operators in worldline theories.

In this paper we will show that the vertex operators of [20], notably their nonlinear contributions, have an interpretation in terms of homotopy transfer from a strictly associative operator algebra. (See [29–35] for introductions and applications of homotopy transfer.) In this the vertex operators will be interpreted as the inclusion map that injects the nonstrict color-stripped Yang-Mills homotopy algebra (a C_∞ algebra that is a subsector of the $BV_\infty^\square$ algebra) into the strict operator algebra. Concretely, we will define a strictly associative algebra $\mathcal{F}$ based on a Moyal-Weyl-type star product, using the methods of deformation quantization [36–40], and then show that there is a homotopy transfer to the kinematic space $\mathcal{K}$. Apart from the vertex operators being interpreted as the (nontrivial) inclusion map $V = I: \mathcal{K} \rightarrow \mathcal{F}$, we have the projector $\Pi: \mathcal{F} \rightarrow \mathcal{K}$ and the homotopy $H: \mathcal{F} \rightarrow \mathcal{F}$. The star algebra is transported to $\mathcal{K}$ via:

$$\mu_2(\psi_1, \psi_2) = \Pi(V(\psi_1) \star V(\psi_2)), \quad (1.1)$$

for $\psi_1, \psi_2 \in \mathcal{K}$, which yields (up to homotopy) the kinematic C_∞ product of Yang-Mills theory. Acting with V on this equation and using the homotopy relation $V\Pi = I\Pi = 1_{\mathcal{F}} + [Q, H]$ one obtains

$$V(\mu_2(\psi_1, \psi_2)) = V(\psi_1) \star V(\psi_2) + [Q, H](V(\psi_1) \star V(\psi_2)), \quad (1.2)$$

which gives a *derived* construction of the bilinear vertex operator $V_2(\psi_1, \psi_2) = H(V(\psi_1) \star V(\psi_2))$ previously determined by hand in [20].

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Let us emphasize from the outset that the central goal here is not so much to derive the C_∞ algebra on the kinematic space $\mathcal{K}$ of Yang-Mills theory from something strict. This could simply be achieved by making Yang-Mills theory cubic, say by introducing an auxiliary two-form field. Rather, the goal is to do so in the context of the worldline theory, as this has proven to provide the right maps suggested by string theory, including the b -ghost to be discussed below, which are instrumental for the understanding of the full $BV_\infty^\square$ algebra. So far the derived construction is restricted to the C_∞ subsector, but we will comment in the conclusions on possible strategies to eventually capture the full kinematic algebra.

The rest of this paper is organized as follows. In Sec. II we introduce the worldline theory and define its operator algebra, which we realize as a star product algebra of functions. Moreover, we define a state space on which this algebra acts. In Sec. III we consider subspaces and ideals of this operator algebra, which is needed to define the subspace onto which the homotopy transfer maps. This homotopy transfer is spelled out in Sec. IV. We close with an outlook in Sec. V.

II. WORLDLINE INSPIRED PHASE SPACE

In this section, we review the phase space of the bosonic spinning particle, which has been treated in more detail in [41]. We present the phase space of the gauge-fixed classical theory, which is enriched with fermionic coordinates corresponding to Becchi-Rouet-Stora-Tyutin (BRST) ghosts. Quantization is performed through the deformation of the pointwise product on phase space into a star product, yielding an associative and noncommutative algebra on the quantized phase space.

A. Classical phase space

The free bosonic spinning particle is described by the following symplectic structure

$$S_{\text{symp}} = \int d\tau [p_\mu \dot{x}^\mu + \bar{\alpha}_\mu \dot{\alpha}^\mu], \quad (2.1)$$

in the Hamiltonian action. The phase space of the theory contains, first of all, the bosonic canonical pairs of the position and momentum of the particle, (x^μ, p_μ) , as well as the worldline oscillators $(\alpha^\mu, \bar{\alpha}_\mu)$ which generate spin degrees of freedom. We denote spacetime indices with Greek letters, which take values $\mu = 0, \dots, D-1$. The following phase space functions

$$H := p^2, \quad L := \alpha \cdot p, \quad \bar{L} := \bar{\alpha} \cdot p, \quad (2.2)$$

form a first-class algebra of Hamiltonian constraints under Poisson brackets. In the quantized theory, the operators associated to these constraints form a contraction of the $sl(2, \mathbb{R})$ subalgebra of the Virasoro algebra, as discussed in

[41], and more extensively in [42,43]. To ensure unitarity of the quantum theory, we need to gauge these constraints, yielding the action of the bosonic spinning particle

$$S = \int d\tau [p_\mu \dot{x}^\mu + \bar{\alpha}^\mu \dot{\alpha}_\mu - eH - u\bar{L} - \bar{u}L], \quad (2.3)$$

where e , u , and $\bar{u}$ are Lagrange multipliers. For the purpose of Hamiltonian BRST quantization, we assign to each constraint a canonical pair of fermionic ghosts (c^i, b_i) , with $i \in \{1, 2, 3\}$ where $c^i := (\mathcal{C}, c, \bar{\mathcal{C}})$ and $b_i := (\bar{\mathcal{B}}, b, \mathcal{B})$. Upon gauge fixing $e = 1$, $u = 0$, and $\bar{u} = 0$, the full action in first-order form is given by

$$S = \int d\tau [p_\mu \dot{x}^\mu + \bar{\alpha}^\mu \dot{\alpha}_\mu - b_i \dot{c}^i - p^2]. \quad (2.4)$$

All in all, we identify our graded phase space to be $\mathcal{P} := \mathbb{R}^{4D|6}$, meaning that we have $4D$ bosonic coordinates, with D being the dimension of spacetime, and 6 fermionic ones. We describe this space with coordinates (X^A, P_A) , where

$$X^A := (x^\mu, \alpha^\nu, c, \mathcal{C}, \mathcal{B}), \quad P_A := (p_\mu, \bar{\alpha}_\nu, b, \bar{\mathcal{B}}, \bar{\mathcal{C}}). \quad (2.5)$$

Capital Latin indices will from now on run over the phase space coordinates. Note that $P_A = 0$ is a Lagrangian submanifold of the phase space $\mathcal{P}$, which we will use in order to define our state space. However, any such choice will lead to an equivalent quantization of the theory.

The phase space $\mathcal{P}$ naturally admits a $\mathbb{Z}_2$ grading, induced by the fermionic or bosonic nature of each coordinate. In fact, the theory possesses a $U(1)_\mathcal{G} \times U(1)_\mathcal{J}$ symmetry. The $U(1)_\mathcal{G}$ sector comes from further refining our grading by assigning *ghost numbers* $\mathcal{G}$ as $\mathcal{G}(c^i) = +1$, $\mathcal{G}(b_i) = -1$, while choosing all bosonic coordinates to have degree zero. We denote indices of the ghost or antighost multiplets with lowercase Latin letters i and j .

Furthermore, the $U(1)_\mathcal{J}$ symmetry is related to another charge $\mathcal{J}$, which is such that $\mathcal{J}(x^\mu, p_\mu, c, b) = 0$, while $\mathcal{J}(\alpha^\mu, \mathcal{C}, \mathcal{B}) = +1$ and $\mathcal{J}(\bar{\alpha}^\mu, \bar{\mathcal{C}}, \bar{\mathcal{B}}) = -1$. The ghost and antighost triplets, c^i and b_i , are thus organized with decreasing and increasing charge $\mathcal{J}$, respectively. We can also reorganize our coordinates as $\mathbf{a}^m := (\mathcal{B}, \alpha^\mu, \mathcal{C})$ with $\mathcal{J} = 1$, $\bar{\mathbf{a}}_m := (\bar{\mathcal{C}}, \bar{\alpha}_\mu, \bar{\mathcal{B}})$ with $\mathcal{J} = -1$. We also have the $U(1)_\mathcal{J}$ singlets x^μ , p_μ , c , and b . We represent indices of the charge multiplets with lowercase Latin letters starting with m . The $U(1)_\mathcal{J}$ symmetry expresses the invariance of the action (2.4) under simultaneously transforming $\mathbf{a}^m \rightarrow e^{i\theta} \mathbf{a}^m$ and $\bar{\mathbf{a}}_m \rightarrow e^{-i\theta} \bar{\mathbf{a}}_m$.

We restrict the class of functions on phase space to the subspace

$$\mathcal{O} := C^\infty(\mathbb{R}^D)[p^\mu, \mathbf{a}^m, \bar{\mathbf{a}}_m, c, b], \quad (2.6)$$

of polynomials in $\{p^\mu, \alpha^m, \bar{\alpha}_n, c, b\}$ with coefficients that are smooth functions of x^μ . This space is naturally equipped with the usual pointwise product of functions, which defines the graded commutative algebra $(\mathcal{O}, \cdot)$. Simultaneously, it can be imbued with a Poisson bracket, defined as

$$\{F, G\} := (-1)^{\mathcal{G}(X)} F \frac{\bar{\partial}}{\partial X^A} \frac{\bar{\partial}}{\partial P_A} G - F \frac{\bar{\partial}}{\partial P_A} \frac{\bar{\partial}}{\partial X^A} G, \quad (2.7)$$

where the right- and left-derivatives are related by $F \frac{\bar{\partial}}{\partial X^A} = (-1)^{X(F+1)} \frac{\partial F}{\partial X^A}$, and from now on we will use the short-hand notation $(-1)^{\mathcal{G}(F)} \equiv (-1)^F$. With these conventions, the canonical Poisson bracket is $\{P_A, X^B\} = -\delta_A^B$, which is the inverse of the symplectic structure given through the action (2.4). The algebra $(\mathcal{O}, \{-, -\})$ is a graded Poisson algebra.

The gauge fixed theory features a BRST symmetry, with an associated BRST charge of ghost number +1. This charge is given as

$$Q := c p^2 + (\bar{C} \alpha^\mu + C \bar{\alpha}^\mu) p_\mu - C \bar{C} b. \quad (2.8)$$

We can use this charge to construct a nilpotent differential as the Hamiltonian vector field $\{Q, -\}$, given that $\{Q, Q\} = 0$. This is a degree +1 differential that turns $(\mathcal{O}, \{-, -\}, \{Q, -\})$ into a differential graded Poisson algebra. In particular, $(\mathcal{O}, \{Q, -\})$ is a chain complex and its cohomology encodes the classical observables of the theory.

Note that, at this point, we can assign mass dimension to the phase space variables. Demanding that the BRST charge is dimensionless, $[Q] = 0$, and knowing that $[p_\mu] = +1$, $[x^\mu] = -1$, as well as the requirement of antighosts to have the opposite mass dimension than the corresponding ghosts, we get the following assignments¹:

$$[c] = -2, \quad [b] = +2, \quad (2.9)$$

$$[C] = [\bar{C}] = +1 \quad [B] = [\bar{B}] = -1, \quad (2.10)$$

$$[\alpha^\mu] = [\bar{\alpha}^\mu] = 0. \quad (2.11)$$

B. Deformation quantization

To quantize our theory, we deform the graded commutative algebra $(\mathcal{O}, \cdot)$ into a noncommutative graded algebra $(\mathcal{O}, \star)$. The new algebra is equipped with a star product, a bilinear map

$$\star: \mathcal{O} \times \mathcal{O} \rightarrow \mathcal{O}, \quad (F, G) \mapsto F \star G := \sum_{n \geq 0} \hbar^n L_n(F, G), \quad (2.12)$$

where we take $\hbar$ to be a general complex number. This product has the following properties:

- (i) It is associative: $(F \star G) \star H = F \star (G \star H)$ for $F, G, H \in \mathcal{O}$,
- (ii) There is a unit element, $1 \star F = F \star 1 = F$,
- (iii) The map L_0 is the pointwise product, $L_0(F, G) = F \cdot G$,
- (iv) The map L_1 is related to the Poisson bracket: $L_1(F, G) - (-1)^{FG} L_1(G, F) = i\{F, G\}$.

As the algebra $(\mathcal{O}, \star)$ is associative, one can further define a graded Lie bracket by introducing the graded star commutator

$$[F, G]_\star := F \star G - (-1)^{FG} G \star F. \quad (2.13)$$

One recovers the classical algebra by taking the classical limit, $\hbar \rightarrow 0$:

$$\lim_{\hbar \rightarrow 0} F \star G = F \cdot G, \quad \lim_{\hbar \rightarrow 0} \frac{1}{i\hbar} [F, G]_\star = \{F, G\}. \quad (2.14)$$

We can use the star commutator in order to define a (left) adjoint action of $(\mathcal{O}, \star)$ on itself. Given $W, F \in \mathcal{O}$, we define

$$\mathcal{W}: \mathcal{O} \longrightarrow \mathcal{O}, \quad \mathcal{W}(F) := [W, F]_\star, \quad (2.15)$$

so that $\mathcal{W} \in \text{End}(\mathcal{O})$ is a derivation of the star product.

The choice of star product for the algebra of functions in deformation quantization is equivalent to a choice of ordering for operators in the canonical formalism. We will elaborate on the relation of the two quantization schemes further below. We first exemplify the relevant concepts for the simplest case of a two-dimensional bosonic phase space, with coordinates (X, P) , and then give their concrete realizations for our model of interest. Two typical choices for the star product are the Weyl-Moyal product $\star_w$ and the normal ordered product $\star_n$, given by

$$\begin{aligned} F \star_w G &:= F \exp \left[-\frac{i\hbar}{2} \left(\frac{\bar{\partial}}{\partial P} \frac{\bar{\partial}}{\partial X} - \frac{\bar{\partial}}{\partial X} \frac{\bar{\partial}}{\partial P} \right) \right] G, \\ F \star_n G &:= F \exp \left[-i\hbar \frac{\bar{\partial}}{\partial P} \frac{\bar{\partial}}{\partial X} \right] G, \end{aligned} \quad (2.16)$$

where $F(X, P)$ and $G(X, P)$ are functions on phase space. Notice that, when acting on functions that are polynomials in P , the above exponentials truncate at some finite order and the star product closes on polynomials of finite, but arbitrary, order. This ensures that the space $\mathcal{O}$ is closed

¹The only constraint for the mass dimension for $\alpha^\mu, \bar{\alpha}^\mu, C, \bar{C}$ is that they should satisfy $[C] + [\bar{\alpha}^\mu] = [\bar{C}] + [\alpha^\mu] = -1$, from demanding $[Q] = 0$, as well as that $[\alpha^\mu] + [\bar{\alpha}^\mu] = 0$ from $[S] = 0$. There are other choices for the specific mass dimensions of these variables which are consistent, but they are all equivalent.

under the star product. With the above choices, the identity element is the constant function 1.

The product $\star_w$ treats X and P on equal footing and is related to the Weyl (symmetric) ordering of position and momentum operators in canonical quantization. The product $\star_n$, in which we take only X -derivatives of G and P -derivatives of F , is related to the normal ordering at the level of operators, where all momenta are moved to the right. The two star products provide isomorphic realizations of the algebra of operators. They can be mapped to one another via

$$e^G(F\star_w H) = (e^G F)\star_n(e^G H), \quad G = -\frac{i\hbar}{2}\partial_X\partial_P, \quad (2.17)$$

where e^G essentially takes Wick contractions. This is most commonly seen in quantum field theory, where the equivalent of (2.17) reexpresses time-ordered products of operators into normal-ordered ones, via Wick's theorem.

We can now define the states of the theory to be functions on a Lagrangian submanifold of the phase space. With the polarization (X, P) of phase space coordinates, we can for example choose states to be given by functions of X . The space of states $\mathcal{H}$ then carries a representation of the star algebra of functions: given a function $F(X, P)$ on phase space, we can construct an operator $\rho(F): \mathcal{H} \rightarrow \mathcal{H}$, whose action on a state $\phi(X)$ we denote as $\rho(F;\phi) \in \mathcal{H}$. The requirement that ρ furnishes a representation of the star algebra amounts to

$$\rho(F) \circ \rho(G) = \rho(F\star G). \quad (2.18)$$

Depending on the choice of star product, the representation map is different. In normal ordering one simply has

$$\rho_n(F;\phi) := (F\star_n \phi)|_{P=0}, \quad (2.19)$$

which obeys (2.18) because $\star_n$ does not take P -derivatives of the right argument. To use the Weyl-Moyal product, instead, one has to employ the isomorphism (2.17) to define the representation

$$\rho_w(F;\phi) := [e^G(F\star_w \phi)]|_{P=0}. \quad (2.20)$$

Going back to our actual model, for the purposes of this paper we will use a star product that implements a mix of Weyl and normal ordering. Specifically, we use $\star_w$ for the $\mathcal{J} = 0$ variables, while we employ $\star_n$ for the $\mathcal{J} = \pm 1$ variables. To that end, we introduce the product

$$F\star G := F \exp \left[-\frac{i\hbar}{2} (\vec{\partial}_P \cdot \vec{\partial}_X - \vec{\partial}_X \cdot \vec{\partial}_P + \vec{\partial}_b \cdot \vec{\partial}_c + \vec{\partial}_c \cdot \vec{\partial}_b) - i\hbar \vec{\partial}_a \cdot \vec{\partial}_a \right] G, \quad (2.21)$$

where we recall that F and G are functions of the graded coordinates $X^A = (x^\mu, c, \mathbf{a}^m)$ and momenta $P_A = (p_\mu, b, \bar{\mathbf{a}}_m)$. The star product above is the one we refer to when using the symbol $\star$ from now on. We define the graded associative algebra $\mathcal{O}_q := (\mathcal{O}, \star)$, that is the space of functions (2.6) endowed with the star product (2.21).

For the corresponding state space $\mathcal{H}$, we choose functions $\phi(X)$ which are smooth functions of x^μ and polynomials in $\mathbf{a}^m$ and c , i.e.,

$$\mathcal{H} := C^\infty(\mathbb{R}^D)[\mathbf{a}^m, c], \quad (2.22)$$

which is indeed the subspace of $\mathcal{O}$ that does not depend on the momenta P_A . Given the mixed form of the star product (2.21), the action of the star algebra $\mathcal{O}_q$ on $\mathcal{H}$ is given by

$$\rho(F;\phi) := [e^G(F\star \phi)]|_{P_A=0}, \quad (2.23)$$

where we now extend G to

$$G := -\frac{i\hbar}{2}(\partial_x \cdot \partial_p + \partial_c \partial_b), \quad (2.24)$$

in agreement with the Weyl representation (2.20) for the $\mathcal{J} = 0$ variables. Because of the algebra isomorphism $\rho: \mathcal{O}_q \rightarrow \text{End}(\mathcal{H})$, we often refer to $\mathcal{O}_q$ as the *operator algebra*, although elements in $\mathcal{O}_q$ are functions on the phase space. Similarly, since $\rho(F)$ is an operator acting on $\mathcal{H}$, we will often use the notation

$$\rho(F) \equiv \hat{F} \in \text{End}(\mathcal{H}), \quad F \in \mathcal{O}_q. \quad (2.25)$$

The graded star commutators computed through (2.21) are the usual ones imposed through canonical quantization, meaning that

$$[P_A, X^B]_\star = -i\hbar \delta_A^B. \quad (2.26)$$

In the following we will set $-i\hbar = 1$, so that $[P_A, -]_\star = \partial_{X^A}$ without phase factors. With this normalization, both gradings under $U(1)_{\mathcal{G}}$ and $U(1)_{\mathcal{J}}$ are counted by the $\star$ -vector fields $\mathcal{G}$ and $\mathcal{J}$, respectively, generated by the functions

$$\begin{aligned} G &:= cb + c\bar{B} + \bar{C}B \equiv c^i b_i, \\ J &:= \alpha^\mu \bar{\alpha}_\mu + C\bar{B} + B\bar{C} \equiv \mathbf{a}^m \bar{\mathbf{a}}_m, \end{aligned} \quad (2.27)$$

such that

$$\begin{aligned} \mathcal{G} &:= [G, -]_\star = c^i \frac{\partial}{\partial c^i} - b_i \frac{\partial}{\partial b_i}, \\ \mathcal{J} &:= [J, -]_\star = \mathbf{a}^m \frac{\partial}{\partial \mathbf{a}^m} - \bar{\mathbf{a}}_m \frac{\partial}{\partial \bar{\mathbf{a}}_m}, \end{aligned} \quad (2.28)$$

yield $\mathcal{G}(c^i) = +c^i$, $\mathcal{G}(b_i) = -b_i$, $\mathcal{J}(\mathbf{a}^m) = +\mathbf{a}^m$, and $\mathcal{J}(\bar{\mathbf{a}}_m) = -\bar{\mathbf{a}}_m$, as desired.

Moreover, since the BRST charge (2.8) remains nilpotent in the quantized theory, i.e., it obeys $Q \star Q \equiv \frac{1}{2}[Q, Q]_\star = 0$, we can endow both the operator algebra $\mathcal{O}_q$ and the state space $\mathcal{H}$ with differentials given by

$$Q := [Q, -]_\star \in \text{End}(\mathcal{O}_q), \quad \hat{Q} := \rho(Q) \in \text{End}(\mathcal{H}), \quad (2.29)$$

thus making $\mathcal{O}_q$ into a differential graded algebra and $\mathcal{H}$ into a chain complex.

To summarize, we have identified the state space $\mathcal{H}$ of the bosonic spinning particle as the space $\mathcal{H} = C^\infty(\mathbb{R}^D)[\mathbf{a}^m, c]$. We introduced the graded associative algebra $\mathcal{O}_q = (\mathcal{O}, \star)$ on the quantized phase space through a deformation of the pointwise product into a noncommutative star product. The algebra $\mathcal{O}_q$ is additionally a Lie algebra under star commutator, which one can use to define an adjoint action on itself. Finally, using the appropriate representation (2.23), we defined the action of $\mathcal{O}_q$ on states in $\mathcal{H}$.

C. Differential operators

We now relate the above construction in terms of the star algebra of functions to differential operators, which are the more usual representation of abstract operator algebras in quantum mechanics. The representations given in (2.19) and (2.20) can be viewed as maps

$$\begin{aligned} \mathcal{O}_q &\longrightarrow \text{End}(\mathcal{H}), \\ F &\mapsto \hat{F}, \end{aligned} \quad (2.30)$$

where on the left hand side $\mathcal{O}_q$ is equipped with either $\star_n$ or $\star_w$. The formula for the star product shows that for a function $F \in \mathcal{O}_q$, the operator $\hat{F} \in \text{End}(\mathcal{H})$ is actually a differential operator on the algebra $\mathcal{H}$. This means that $\hat{F}$ is of the form

$$\hat{F} = \sum_{i=0}^n F^{A_1 \dots A_i} \frac{\partial}{\partial X^{A_1}} \cdots \frac{\partial}{\partial X^{A_i}}. \quad (2.31)$$

Here, the first term in the sum (i.e., $i = 0$) acts via left multiplication. The number n is called the order of the operator $\hat{F}$, which is finite but can be arbitrarily large. The algebra of all differential operators on $\mathcal{H}$ is called $\text{Diff}(\mathcal{H})$, and it is a subalgebra of $\text{End}(\mathcal{H})$.

Since any $F \in \mathcal{O}_q$ is polynomial in momenta, the map $F \mapsto \hat{F}$ is an isomorphism of associative algebras, given by the ρ in (2.25). This isomorphism is such that when expanding $\hat{F}$ in powers of $\hbar$

$$\hat{F} = \sum_{k \geq 0} \hbar^k \hat{F}_k, \quad (2.32)$$

the operator $\hat{F}_k$ is an order k differential operator.

We can think of a quantization procedure on $\mathcal{O}$ as a choice of isomorphism of vector spaces $\mathcal{O} \rightarrow \text{Diff}(\mathcal{H})$, with the condition that a function of the form $F^{A_1 \dots A_k}(X) P_{A_1} \cdots P_{A_k}$ gets mapped to

$$F^{A_1 \dots A_k}(X) \frac{\partial}{\partial X^{A_1}} \cdots \frac{\partial}{\partial X^{A_k}} + \text{lower order terms}. \quad (2.33)$$

The isomorphism of vector spaces then induces a product $\star$ on $\mathcal{O}$ induced by the composition product on $\text{Diff}(\mathcal{H})$. The products $\star_n$ and $\star_w$ could be derived in this way. For example, the isomorphism reproducing $\star_w$ maps $x^\mu p_\nu$ to $x^\mu \partial_\nu + \frac{1}{2} \delta_\nu^\mu$, while $\star_n$ is reproduced by mapping $x^\mu p_\nu$ to $x^\mu \partial_\nu$.

We can also talk about the order of a differential operator with respect to $\star$. Although higher order operators with respect to noncommutative products can be defined [44], for us it is important that the commutator $\mathcal{W} := [W, -]_\star$, for $W \in \mathcal{O}_q$, is a derivation of $\star$, but not necessarily of the pointwise product.

III. STATE SPACE DECOMPOSITION AND OPERATOR ALGEBRA QUOTIENTS

In this section, we decompose the state space $\mathcal{H}$ of the bosonic spinning particle into sectors of different spin, focusing on the spin-1 sector $\mathcal{H}_1$. Within the corresponding operator algebra, we identify an ideal $\mathfrak{F}$ under star product, which corresponds to the annihilator of the space $\mathcal{H}_1$. In the coming sections, our goal will be to relate a special subalgebra of $\mathcal{O}_q/\mathfrak{F}$ to the C_∞ structure appearing in Yang-Mills theory upon color stripping.

A. The spin-1 sector and its operator algebra

We now introduce gradings for the state space $\mathcal{H}$ associated with the $U(1)_G$ and $U(1)_J$ symmetries of the theory. The grading by ghost number is implemented on $\mathcal{H}$ by the operator

$$\hat{G} := \rho(G) = c \frac{\partial}{\partial c} + \mathcal{C} \frac{\partial}{\partial \mathcal{C}} - \mathcal{B} \frac{\partial}{\partial \mathcal{B}}. \quad (3.1)$$

We can decompose the space of states as the bounded direct sum $\mathcal{H} = \bigoplus_{k=-1}^2 \mathcal{H}_k$, where elements ψ_k of $\mathcal{H}_k$ have homogeneous ghost number $\hat{G} = k$. We also define the operator $\hat{J}$ associated with the $U(1)_J$ charge as

$$\hat{J} := \rho(J) = \mathbf{a}^m \frac{\partial}{\partial \mathbf{a}^m}, \quad (3.2)$$

which counts the number of $\mathbf{a}^m$ modes in a state. This grading induces a decomposition for $\mathcal{H}$ that is bounded from below, since the coordinates $X^A = (x^\mu, c, \mathbf{a}^m)$ have positive or zero charge. We thus obtain

$$\mathcal{H} = \bigoplus_{j=0}^{\infty} \bigoplus_{k=-1}^2 \mathcal{H}_{j,k}. \quad (3.3)$$

The differential on $\mathcal{H}$ is $\hat{Q}$, which is nilpotent and has ghost number $+1$, as inherited from the charge Q . This means that $[\hat{G}, \hat{Q}] = \hat{Q}$. The differential is also neutral under $U(1)_{\mathcal{J}}$, since $[\hat{J}, \hat{Q}] = 0$. Thus, $\hat{Q}$ maps between sectors of the state space with ghost number differing by one, but with the same $U(1)_{\mathcal{J}}$ charge, as

$$\hat{Q}: \mathcal{H}_{j,k} \rightarrow \mathcal{H}_{j,k+1}. \quad (3.4)$$

As such, $(\mathcal{H}_j, \hat{Q})$ is a chain complex for every j , with $\mathcal{H}_j := \bigoplus_{k=-1}^2 \mathcal{H}_{j,k}$. The physical content of its cohomology is a tower of free massless particles of spins ranging from j to one or zero, decreasing in steps of 2, see e.g., [41] for details.

We now set $j = 1$, limiting ourselves to the space $\mathcal{H}_1$. This space is related to free Yang-Mills theory, since the cohomology of $(\mathcal{H}_1, \hat{Q})$ contains a single massless spin-1 particle. A general element of $\mathcal{H}_1$ is a function $\psi(X)$ of the form

$$\psi(X) = \psi_m(x, c) \alpha^m. \quad (3.5)$$

The operator algebra $\mathcal{O}_q$ has a subalgebra of functions F with $U(1)_{\mathcal{J}}$ charge equal to zero, given by the subspace $\ker \mathcal{J} \subset \mathcal{O}_q$. Under the map ρ , this subalgebra gives rise to diagonal operators in $\mathcal{H}$, in that

$$\rho|_{\ker \mathcal{J}}: \mathcal{H}_j \rightarrow \mathcal{H}_j \quad \forall j, \quad (3.6)$$

which, in particular, are endomorphisms of $\mathcal{H}_1$. We recall from (2.28) that

$$\mathcal{J} = \alpha^m \partial_{\alpha^m} - \bar{\alpha}_m \partial_{\bar{\alpha}_m} = N_{\alpha} - N_{\bar{\alpha}}, \quad (3.7)$$

where we defined the separate number operators $N_{\alpha} := \alpha^m \partial_{\alpha^m}$ and $N_{\bar{\alpha}} := \bar{\alpha}_m \partial_{\bar{\alpha}_m}$. Using this, we can write a general function F in $\ker \mathcal{J}$ as

$$F = \sum_{k=0}^{\infty} F_{(k)}, \quad N_{\alpha}(F_{(k)}) = N_{\bar{\alpha}}(F_{(k)}) = k F_{(k)},$$

$$F_{(k)} = F_{m_1 \dots m_k}^{n_1 \dots n_k}(x, p, c, b) \alpha^{m_1} \dots \alpha^{m_k} \bar{\alpha}_{n_1} \dots \bar{\alpha}_{n_k}. \quad (3.8)$$

A crucial remark is that any $\hat{F}_{(k)} \equiv \rho(F_{(k)})$ with $k > 1$ annihilates any element of $\mathcal{H}_1$, since any $\psi(X) \in \mathcal{H}_1$ is linear in α^m , while

$$\hat{F}_{(k)} = \hat{F}_{m_1 \dots m_k}^{n_1 \dots n_k} \alpha^{m_1} \dots \alpha^{m_k} \frac{\partial}{\partial \alpha^{n_1}} \dots \frac{\partial}{\partial \alpha^{n_k}}, \quad (3.9)$$

which can be derived from (2.21) and (2.23). We thus see that the annihilator of $\mathcal{H}_1$ is isomorphic (through the map ρ)

to the space of functions in $\ker \mathcal{J}$ with $k > 1$, which we denote by $\mathfrak{F}$. Moreover, the subspace $\mathfrak{F}$ is a two-sided ideal of $\ker \mathcal{J}$, meaning that for every $F \in \ker \mathcal{J}$ one has

$$\mathfrak{F} \star F \subset \mathfrak{F}, \quad F \star \mathfrak{F} \subset \mathfrak{F}. \quad (3.10)$$

This can be proven as follows: given two functions $F_{(k_1)}$ and $F_{(k_2)}$ in $\ker \mathcal{J}$, with $N_{\alpha} = k_1$ and $N_{\bar{\alpha}} = k_2$, respectively, their star product decomposes as

$$F_{(k_1)} \star F_{(k_2)} = F_{(k_1+k_2)} + F_{(k_1+k_2-1)} + \dots + F_{(\max(k_1, k_2))}, \quad (3.11)$$

so that, if either k_1 or k_2 is larger than 1, all terms in the right-hand side of (3.11) have $N_{\bar{\alpha}} > 1$ and thus belong to the ideal.

We are thus led to define the quotient space $\mathcal{R}$ obtained by modding out the ideal from the space $\ker \mathcal{J} \subset \mathcal{O}_q$

$$\mathcal{R} := \frac{\ker \mathcal{J}}{\mathfrak{F}}. \quad (3.12)$$

The space $\mathcal{R}$ is thus effectively the space of operators acting nontrivially on $\mathcal{H}_1$. Given that the elements of $\mathfrak{F}$ are the functions in $\ker \mathcal{J}$ with $N_{\bar{\alpha}} > 1$, one can define a projector that picks the obvious representatives in the equivalence classes in $\mathcal{R}$:

$$P_{\mathcal{R}} \left(\sum_{k=0}^{\infty} F_{(k)} \right) := F_{(0)} + F_{(1)}. \quad (3.13)$$

Notice that the ideal is the kernel of this projector: $\mathfrak{F} = \ker P_{\mathcal{R}}$. Although $\mathcal{R}$ is defined as the quotient space (3.12), we will practically work with the representatives (3.13) and project again the results after taking the star product, which is equivalent to using a projected product $\star_{\mathcal{R}}$ defined by

$$F \star_{\mathcal{R}} G := P_{\mathcal{R}}(F \star G), \quad \forall F, G \in \text{Im } P_{\mathcal{R}}, \quad (3.14)$$

so that $(\text{Im } P_{\mathcal{R}}, \star_{\mathcal{R}})$ is also an associative algebra, but we will never write $P_{\mathcal{R}}$ or $\star_{\mathcal{R}}$ explicitly.

B. Decomposition of the differential

We now present in more detail the structure of the operator $\mathcal{Q} = [\mathcal{Q}, -]_{\star}$, which is the differential on the operator algebras $\mathcal{O}_q$ and $\mathcal{R}$. We first introduce the following decomposition of $\mathcal{Q}$

$$\mathcal{Q} = \gamma + \delta, \quad \gamma = d + \sigma + \Delta, \quad (3.15)$$

where we present its components explicitly as differential operators in $\text{Diff}(\mathcal{O}_q)$, with respect to the pointwise product, as

$$d := C^\mu \partial_\mu, \quad (3.16)$$

$$\sigma := \mathcal{C}(p \cdot \partial_\alpha + b \partial_{\bar{B}}) - \bar{\mathcal{C}}(p \cdot \partial_{\bar{\alpha}} + b \partial_{\bar{B}}) - \mathcal{M} \partial_c, \quad (3.17)$$

$$\Delta := \frac{1}{2}(\mathcal{C} \partial_\alpha^\mu + \bar{\mathcal{C}} \partial_{\bar{\alpha}}^\mu) \partial_\mu, \quad (3.18)$$

$$\begin{aligned} \delta := & p^2 \partial_b + \alpha \cdot p \partial_B + \bar{\alpha} \cdot p \partial_{\bar{B}} \\ & + \frac{1}{2}[\alpha^\mu \partial_\mu \partial_B - \bar{\alpha}^\mu \partial_\mu \partial_{\bar{B}} - \mathcal{C} \partial_B \partial_c - \bar{\mathcal{C}} \partial_{\bar{B}} \partial_c] + \frac{1}{4} \square \partial_b, \end{aligned} \quad (3.19)$$

where we defined

$$C^\mu := \mathcal{C} \bar{\alpha}^\mu + \bar{\mathcal{C}} \alpha^\mu + 2c p^\mu, \quad \mathcal{M} := \mathcal{C} \bar{\mathcal{C}}. \quad (3.20)$$

The split of $\mathcal{Q}$ into the operators γ and δ comes from refining the ghost degree into pure ghost and pure antighost number, such that $\mathcal{G} = N_{\text{gh}} - N_{\text{agh}}$. The ghost and antighost counting operators can be expressed as differential operators in the following way

$$N_{\text{gh}} := c^i \frac{\partial}{\partial c^i}, \quad N_{\text{agh}} := b_i \frac{\partial}{\partial b_i}, \quad (3.21)$$

in agreement with (2.28), where we imply summation over the ghost and antighost multiplets respectively. This refinement of the ghost degree assigns

$$\begin{aligned} N_{\text{gh}}(\gamma) &= +1, & N_{\text{agh}}(\gamma) &= 0, \\ N_{\text{gh}}(\delta) &= 0, & N_{\text{agh}}(\delta) &= -1. \end{aligned} \quad (3.22)$$

Therefore, the split is induced by either pure ghost or pure antighost number. Expanding the $\mathcal{Q}^2 = 0$ condition into

$$0 = \gamma^2 + \delta^2 + [\gamma, \delta], \quad (3.23)$$

we see that the three terms above have, for instance, pure ghost number +2, 0 and +1, respectively, and must thus vanish separately. This implies that both γ and δ are nilpotent differentials that commute with each other: $\gamma^2 = \delta^2 = [\gamma, \delta] = 0$.

The introduction of pure ghost and antighost number grading leads us to a decomposition of $\mathcal{O}_q$, depending on whether operators contain antighost variables

$$\begin{aligned} \mathcal{O}_q &\equiv \mathcal{O}_{q,b} \oplus \mathcal{O}_{q,c}, \\ \mathcal{O}_{q,b} &:= \{F \in \mathcal{O}_q | N_{\text{agh}}(F) > 0\}, \\ \mathcal{O}_{q,c} &:= \{F \in \mathcal{O}_q | N_{\text{agh}}(F) = 0\}. \end{aligned} \quad (3.24)$$

While the subspace $\mathcal{O}_{q,b}$ is *not* closed under star product, the space $\mathcal{O}_{q,c}$ is, and forms a subalgebra. The differential on $\mathcal{O}_{q,c}$ is γ , in that $\mathcal{Q}(\mathcal{O}_{q,c}) \equiv \gamma(\mathcal{O}_{q,c})$. The pure ghost

number grading is inherited by the subspaces $\ker \mathcal{J}$ as well as $\mathcal{R}$, leading to the spaces $\ker \mathcal{J}_c$ and $\mathcal{R}_c$. As for the star product, when performing computations in $\mathcal{R}$ or $\mathcal{R}_c$ we always pick the representatives (3.13) but omit the projector $P_{\mathcal{R}}$, implying that all results are presented up to elements in the ideal $\mathfrak{I}$.

The differential γ can be further split into $d + \sigma$, which are first-order differential operators, and Δ , which is second order. We have separated d and σ based on their derivative structure: d contains spacetime derivatives $\frac{\partial}{\partial x^\mu}$, while σ does not. One can further verify that these operators are themselves nilpotent differentials: $d^2 = \sigma^2 = \Delta^2 = 0$ and that they commute with one another.

IV. HOMOTOPY TRANSFER TO THE C_∞ ALGEBRA OF YANG-MILLS

In this section, we briefly review the C_∞ algebra of Yang-Mills theory and its relation to the spin-one sector of the state space $\mathcal{H}_1$. Elements of the algebra are generated from the vacuum of $\mathcal{H}_1$ through the action of vertex operators as presented in [20]. The functions corresponding to these operators define the algebra $(\mathcal{F}, \mathcal{Q}, \star)$, a strict differential, associative and noncommutative algebra which contains all Yang-Mills vertex operators and is homotopy equivalent to the C_∞ algebra of the theory.

A. A subalgebra for Yang-Mills vertex operators

Perturbative Yang-Mills theory, upon stripping off color degrees of freedom, exhibits a C_∞ algebra structure [7,18,24]. A C_∞ algebra consists of a graded vector space $\mathcal{K}$ endowed with a collection of multilinear maps $m_n: \mathcal{K}^{\otimes n} \rightarrow \mathcal{K}$. It is a generalization of a differential graded commutative algebra, where the map m_1 serves as a differential and the bilinear product m_2 is graded commutative, but associative only up to homotopy.

In the case of Yang-Mills theory, the underlying vector space is isomorphic to a suspension (degree shift) of the state space $\mathcal{H}_1$

$$\mathcal{K} = \bigoplus_{i=0}^3 \mathcal{K}_i, \quad \mathcal{K}_i := \mathcal{H}_{1,i-1}. \quad (4.1)$$

As explained in more detail in [20], elements of $\mathcal{K}$ are given as states in $\mathcal{H}_1$ by

$$\lambda := \lambda(x) \mathcal{B} \in \mathcal{K}_0, \quad (4.2)$$

$$\mathcal{A} := A_\mu(x) \alpha^\mu + \varphi(x) c \mathcal{B} \in \mathcal{K}_1, \quad (4.3)$$

$$\mathcal{E} := E_\mu(x) c \alpha^\mu + E(x) \mathcal{C} \in \mathcal{K}_2, \quad (4.4)$$

$$\mathcal{N} := N(x) c \mathcal{C} \in \mathcal{K}_3, \quad (4.5)$$

which corresponds to the Yang-Mills theory for a gauge boson A_μ equipped with an auxiliary scalar field φ as presented in [7,41]. In the following, we will denote generic elements of $\mathcal{K}$ as ψ , intended as functions $\psi(X) \in \mathcal{H}_1$.

The differential map m_1 for this algebra is given through the action of the BRST operator on states in $\mathcal{H}_1$ as

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{K}_0 & \xrightarrow{m_1} & \mathcal{K}_1 & \xrightarrow{m_1} & \mathcal{K}_2 & \xrightarrow{m_1} & \mathcal{K}_3 & \longrightarrow & 0 \\ & & \lambda & & \mathcal{A} & & \mathcal{E} & & \mathcal{N} & & \end{array}$$

The physical meaning of this chain complex is the following: the fields of our theory, A_μ and φ , are contained in the element $\mathcal{A} \in \mathcal{K}_1$, at ghost number 0. At ghost number -1 , so in the space $\mathcal{K}_0$, we find gauge parameters, which induce gauge transformations given as $m_1(\lambda)$. The action of the differential on fields results in (linearized) equations of motion, which live in the space $\mathcal{K}_2$ and have the form (4.4). Finally, the gauge symmetry leads to relations between the equations of motion themselves, the Noether identities, which are given by $m_1(\mathcal{E})$ and will be elements of $\mathcal{K}_3$ of the form (4.5). We further assign target space mass dimension to $\{\lambda, A_\mu, \varphi, E, E_\mu, N\}$ as follows: $[\lambda] = 0$, $[A_\mu] = 1$, $[\varphi] = 2$, $[E_\mu] = 3$, $[E] = 2$, $[N] = 4$. The absence of mass parameters in the theory helps in controlling locality of all the maps to be defined in the following.

A different point of view on this algebra was introduced in [20]. The space $\mathcal{H}_1$, hence $\mathcal{K}$, contains a “vacuum state” represented by the function $\psi_{\text{vac}}(X) = \mathcal{B}$. This state is interpreted as a constant gauge parameter in $\mathcal{K}_0$ and obeys $m_1(\psi_{\text{vac}}) = 0$. As a closed element of degree 0, it belongs to the cohomology of m_1 . Arbitrary elements of $\mathcal{K}$ can then be generated from the vacuum by applying suitable vertex operators, in the sense

$$\psi = \rho(V(\psi); \mathcal{B}) = \hat{V}(\psi)\psi_{\text{vac}} \in \mathcal{H}_1, \quad (4.7)$$

where $V(\psi)$ is a function in $\mathcal{R}_c \subset \mathcal{O}_q$. Since the vacuum ψ_{vac} belongs to $\mathcal{H}_1$, the vertex operators must be elements of $\ker \mathcal{J}$ and have $\mathcal{G}(V(\psi)) = |\psi|$, where $|\cdot|$ is the degree in $\mathcal{K}$. We further demand V to be a chain map between $(\text{End}(\mathcal{K}), m_1)$ and $(\mathcal{R}_c, \mathcal{Q})$, meaning that, for $\psi \in \mathcal{K}$

$$\mathcal{Q}(V(\psi)) = V(m_1(\psi)). \quad (4.8)$$

These requirements lead to the following functions $V(\psi)$ for elements of $\mathcal{K}$ of different degrees:

$$m_1 \equiv \hat{Q} = c\Box + \left(\mathcal{C} \frac{\partial}{\partial \alpha_\mu} + \alpha^\mu \frac{\partial}{\partial \mathcal{B}} \right) \partial_\mu - \mathcal{C} \frac{\partial}{\partial \mathcal{B}} \frac{\partial}{\partial c}, \quad (4.6)$$

which is derived from the map ρ (2.23). Therefore, color-stripped Yang-Mills theory is encoded in the chain complex

$$\begin{aligned} V(\lambda) &= \lambda(x), \\ V(\mathcal{A}) &= C^\mu A_\mu(x) + C^{\mu\nu} f_{\mu\nu}(x) + c(\varphi(x) - \partial^\mu A_\mu(x)), \\ V(\mathcal{E}) &= \mathcal{M}E(x) + \mathcal{M}^\mu \partial_\mu E(x) + \tilde{\mathcal{M}}^\mu (E_\mu - \partial_\mu E), \\ V(\mathcal{N}) &= eN(x), \end{aligned} \quad (4.9)$$

where $f_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the Abelian field strength, and we introduced the following notation for specific elements of $\mathcal{R}_c$,

$$C^{\mu\nu} := 2c\alpha^{[\mu}\tilde{\alpha}^{\nu]}, \quad (4.10)$$

$$\mathcal{M}^\mu := c(\tilde{\mathcal{C}}\alpha^\mu + \mathcal{C}\tilde{\alpha}^\mu), \quad (4.11)$$

$$\tilde{\mathcal{M}}^\mu := c(\tilde{\mathcal{C}}\alpha^\mu - \mathcal{C}\tilde{\alpha}^\mu), \quad (4.12)$$

$$e := c\mathcal{C}\tilde{\mathcal{C}}. \quad (4.13)$$

As remarked, all of these vertex operators belong to $\mathcal{R}_c \subset \mathcal{R}$, meaning that they contain no dependence on the b_i , equivalently $N_{\text{agh}}V(\psi) = 0$ for all $\psi \in \mathcal{K}$. Moreover, all of their x -dependence is given through polynomials in $\{c^i, \alpha^\mu, \tilde{\alpha}^\mu, p_\mu\}$ with coefficients in $C^\infty(\mathbb{R}^D)$. Given the target space mass dimensions, all $V(\psi)$ have total mass dimension zero.

These vertex operators do not form a closed subalgebra of $(\mathcal{R}_c, \mathcal{Q}, \star)$. We now seek their minimal extension to a set of operators that create elements of $\mathcal{K}$ from the vacuum ψ_{vac} , while simultaneously forming a closed algebra under $\star$ and $\mathcal{Q}$. We identify the following such elements within $\mathcal{R}_c$:

$$F_0 = \lambda(x), \quad (4.14)$$

$$F_1 = C^\mu A_\mu(x) + C^{\mu\nu} B_{\mu\nu}(x) + c\pi(x), \quad (4.15)$$

$$F_2 = \mathcal{M}w(x) + \mathcal{M}^\mu w_\mu(x) + \tilde{\mathcal{M}}^\mu \tilde{w}_\mu(x) + \mathcal{M}^{\mu\nu} w_{\mu\nu}(x), \quad (4.16)$$

$$F_3 = eN(x) + e^\mu N_\mu(x), \quad (4.17)$$

where we further defined

$$\mathcal{M}^{\mu\nu} := \frac{1}{2} C^\mu C^\nu = 2c p^{[\mu} (\bar{\mathcal{C}} \alpha^{\nu]} + \mathcal{C} \bar{\alpha}^{\nu]}), \quad (4.18)$$

$$e^\mu := e p^\mu. \quad (4.19)$$

These elements span a space $\mathcal{F} := C^\infty(\mathbb{R}^D)[B_{\mathcal{F}}]$, where $B_{\mathcal{F}}$ is the following basis:

$$B_{\mathcal{F}} := \{1; c, C^\mu, C^{\mu\nu}; \mathcal{M}, \mathcal{M}^\mu, \tilde{\mathcal{M}}^\mu, \mathcal{M}^{\mu\nu}; e, e^\mu\}. \quad (4.20)$$

The space $\mathcal{F}$ inherits the grading of $\mathcal{R}_c \subset \mathcal{O}_q$ and we have $\mathcal{F} = \bigoplus_{i=0}^3 \mathcal{F}_i$, with the F_i of (4.14)–(4.17) belonging to the space $\mathcal{F}_i$.

We can now check that $\mathcal{F}$ is closed under $\star$ by computing the products at each degree to be

$$F_{0,1} \star F_{0,2} = \lambda_1 \lambda_2, \quad (4.21)$$

$$F_0 \star F_1 = C^\mu (\lambda A_\mu) + C^{\mu\nu} (\lambda B_{\mu\nu}) + c(\pi - A \cdot \partial) \lambda, \quad (4.22)$$

$$\begin{aligned} F_0 \star F_2 &= \mathcal{M}(\lambda w) + \mathcal{M}^\mu (w_\mu + w_{\mu\nu} \partial^\nu) \lambda \\ &\quad + \tilde{\mathcal{M}}^\mu (\lambda \tilde{w}_\mu) + \mathcal{M}^{\mu\nu} (\lambda w_{\mu\nu}), \end{aligned} \quad (4.23)$$

$$F_0 \star F_3 = e \left(N \lambda - \frac{1}{2} N^\mu \partial_\mu \lambda \right) + e^\mu (\lambda N_\mu), \quad (4.24)$$

$$\begin{aligned} F_{1,1} \star F_{1,2} &= \mathcal{M} A_1 \cdot A_2 \\ &\quad + 2\mathcal{M}^\mu [A_{(1} \cdot \partial A_{2)\mu} - A_{(1}^\nu B_{2)\nu\mu} + \pi_{[1} A_{2]\mu}] \\ &\quad + 2\mathcal{M}^{\mu\nu} A_{1\mu} A_{2\nu} + 2\tilde{\mathcal{M}}^\mu A_{1\mu}^\nu B_{2\nu\mu}, \end{aligned} \quad (4.25)$$

$$\begin{aligned} F_1 \star F_2 &= e[(\pi + A \cdot \partial)w - A_\mu (w^\mu + \tilde{w}^\mu) + \partial^\mu A^\nu w_{\mu\nu}] \\ &\quad + 2e^\mu (A_\mu w - w_{\mu\nu} A^\nu). \end{aligned} \quad (4.26)$$

The star product in $\mathcal{F}$ is still not commutative, with graded commutators given by

$$[F_0, F_1]_\star = -2c A \cdot \partial \lambda, \quad (4.27)$$

$$[F_0, F_2]_\star = 2\mathcal{M}^\mu w_{\mu\nu} \partial^\nu \lambda, \quad (4.28)$$

$$[F_0, F_3]_\star = -e N^\mu \partial_\mu \lambda, \quad (4.29)$$

$$\begin{aligned} [F_{1,1}, F_{1,2}]_\star &= 2\mathcal{M} A_1 \cdot A_2 \\ &\quad + 4\mathcal{M}^\mu (A_{(1} \partial \cdot A_{2)\mu} - A_{(1}^\nu B_{2)\nu\mu}), \end{aligned} \quad (4.30)$$

$$[F_1, F_2]_\star = 2e A^\mu (\partial_\mu w - w_\mu) - 2e^\mu w_{\mu\nu} A^\nu. \quad (4.31)$$

We further verify that $\mathcal{F}$ is closed under the action of the differential $\mathcal{Q}$

$$\mathcal{Q}(F_0) = C^\mu \partial_\mu \lambda, \quad (4.32)$$

$$\begin{aligned} \mathcal{Q}(F_1) &= -\mathcal{M}\pi - \mathcal{M}^\mu \partial_\mu \pi - \tilde{\mathcal{M}}^\mu \partial^\nu B_{\mu\nu} \\ &\quad + \mathcal{M}^{\mu\nu} (f_{\mu\nu} - B_{\mu\nu}), \end{aligned} \quad (4.33)$$

$$\mathcal{Q}(F_2) = 2e^\mu (\partial_\mu w - w_\mu) - e \partial^\mu \tilde{w}_\mu. \quad (4.34)$$

In conclusion, $(\mathcal{F}, \mathcal{Q}, \star)$ defines a differential graded associative algebra. Since all basis elements of $\mathcal{F}$ are in $\mathcal{R}_c$, the effective differential on the space is γ . We remark that, if one would interpret the x -dependent functions at degree $+1$ as the field content of a theory, it would be a theory with a vector field A_μ , as well as a scalar field π and two-form $B_{\mu\nu}$ as auxiliary fields. This suggests that the minimal extension of the algebra of worldline vertex operators (4.9) that is closed under $\mathcal{Q}$ and $\star$ is related to a generalization of first order Yang-Mills theory.

We will now show how the basis elements of $\mathcal{F}$ are connected to the cohomology of the differential σ , and that the latter encodes the “spectrum” of the physical sector of the theory. Since σ is insensitive to the x – dependence, it is sufficient to compute it on the span of the basis $\mathbb{C}[B_{\mathcal{F}}]$ with x -independent coefficients, yielding

$$\begin{aligned} \sigma(1) &= 0, \\ \sigma(C^\mu) &= 0, \quad \sigma(C^{\mu\nu}) = -\mathcal{M}^{\mu\nu}, \quad \sigma(c) = -\mathcal{M}, \\ \sigma(\tilde{\mathcal{M}}^\mu) &= 0, \quad \sigma(\mathcal{M}^\mu) = -2e^\mu, \\ \sigma(e) &= 0, \end{aligned} \quad (4.35)$$

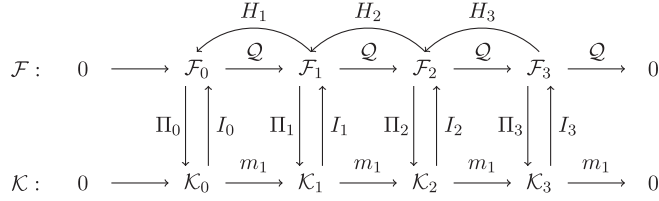
where we omitted σ on exact elements, such as $\mathcal{M}$ and $\mathcal{M}^{\mu\nu}$, since it is obviously zero. It turns out that the elements of $B_{\mathcal{F}}$ split into a subspace in the σ -cohomology, generated by $\{1, C^\mu, \tilde{\mathcal{M}}^\mu, e\}$, plus the trivial pairs

$$\begin{aligned} \sigma(c) + \mathcal{M} &= 0, \quad \sigma(\mathcal{M}^\mu) + 2e^\mu = 0, \\ \sigma(C^{\mu\nu}) + \mathcal{M}^{\mu\nu} &= 0. \end{aligned} \quad (4.36)$$

We can now make the observation that physical objects, meaning A_μ with its gauge parameter, together with the corresponding equation of motion and Noether identity, appear with a basis element that is in cohomology. On the contrary, the auxiliary fields π and $B_{\mu\nu}$ and their associated elements in other degrees appear with basis elements that form trivial pairs. The underlying reason is that the auxiliary scalar field and two-form can be integrated out from the theory in a local way, without altering its physical content. In fact, in the next subsection we will relate the C_∞ algebra of color stripped Yang-Mills theory to the star algebra of $\mathcal{F}$ via a local homotopy transfer.

B. Homotopy transfer from $\mathcal{F}$ to $\mathcal{K}$

We will now perform homotopy transfer between the chain complexes $(\mathcal{F}, \mathcal{Q})$ and $(\mathcal{K}, m_1)$ by constructing a

FIG. 1. Homotopy transfer from $\mathcal{F}$ to $\mathcal{K}$.

quasi-isomorphism between the two algebras. Such a map is the natural morphism between homotopy algebras, the existence of which guarantees that the algebras have isomorphic cohomologies. In physics terms, these algebras describe theories with an equivalent physical content at the perturbative level. For more details on homotopy transfer, we refer the reader to [32–34].

To perform homotopy transfer between the chain complexes, we need to construct the diagram in Fig. 1.

In essence, we need to first introduce a projection $\Pi: \mathcal{F} \rightarrow \mathcal{K}$, as well as an inclusion $I: \mathcal{K} \rightarrow \mathcal{F}$ which are chain maps between their differentials

$$\Pi \circ \mathcal{Q} = m_1 \circ \Pi, \quad \mathcal{Q} \circ I = I \circ m_1. \quad (4.37)$$

The projection and inclusion maps are then required to be inverting each other up to corrections by a map $H: \mathcal{F} \rightarrow \mathcal{F}$, meaning that

$$\Pi \circ I = 1_{\mathcal{K}}, \quad I \circ \Pi = 1_{\mathcal{F}} + \mathcal{Q}H + H\mathcal{Q}. \quad (4.38)$$

If this holds, the homotopy transfer theorem ensures that if $(\mathcal{F}, \star)$ has a homotopy associative (or A_∞) structure, then so should $(\mathcal{K}, m_1)$, with higher maps μ_n that can be constructed using the homotopy data (Π, I, H) . Note that in this case, $(\mathcal{F}, \star)$ is a strict associative algebra, meaning it is a special case of an A_∞ algebra where all products of arity greater than two vanish. Even so, the resulting algebra after homotopy transfer is expected to be only associative up to homotopy.

For the projector $\Pi: \mathcal{F} \rightarrow \mathcal{K} \simeq \mathcal{H}_1$, we take the action of the operator $\rho(F)$ on the vacuum

$$\Pi: \mathcal{F} \rightarrow \mathcal{K}, \quad \Pi(F) := \rho(F; \psi_{\text{vac}}), \quad (4.39)$$

where ρ is the representation of $(\mathcal{O}_q, \star)$ on states in $\mathcal{H}$ as defined in (2.23) and we recall that $\psi_{\text{vac}}(X) = \mathcal{B}$. To verify that the chain map condition holds we first compute

$$\begin{aligned} m_1(\Pi(F)) &= \rho(\mathcal{Q}; \Pi(F)) = \rho(\mathcal{Q}; \rho(F; \psi_{\text{vac}})) \\ &= \rho(\mathcal{Q} \star F; \psi_{\text{vac}}), \end{aligned} \quad (4.40)$$

where we used the algebra morphism property (2.18). We now write $\mathcal{Q} \star F = [\mathcal{Q}, F]_\star + (-1)^F F \star \mathcal{Q}$ and make use of (2.18) again to obtain

$$\begin{aligned} \rho(\mathcal{Q} \star F; \psi_{\text{vac}}) &= \rho([\mathcal{Q}, F]_\star; \psi_{\text{vac}}) + (-1)^F \rho(F; \rho(\mathcal{Q}; \psi_{\text{vac}})) \\ &= \rho(\mathcal{Q}(F); \psi_{\text{vac}}) = \Pi(\mathcal{Q}(F)), \end{aligned} \quad (4.41)$$

which concludes the proof, upon recognizing that $\rho(\mathcal{Q}; \psi_{\text{vac}}) \equiv m_1(\psi_{\text{vac}}) = 0$. The explicit action of the projection on the general elements F_i of $\mathcal{F}$ is

$$\Pi(F_0) = \lambda \mathcal{B}, \quad (4.42)$$

$$\Pi(F_1) = A_\mu \alpha^\mu + c \mathcal{B}(\pi + \partial \cdot A), \quad (4.43)$$

$$\Pi(F_2) = c \alpha^\mu (w_\mu + \tilde{w}_\mu - \partial^\nu w_{\mu\nu}) + \mathcal{C}w, \quad (4.44)$$

$$\Pi(F_3) = c \mathcal{C} \left(N + \frac{1}{2} \partial^\mu N_\mu \right). \quad (4.45)$$

On the other hand, the natural way to include elements of $\mathcal{K}$ as given in (4.2)–(4.5) in $\mathcal{F}$ is through the corresponding vertex operators, so we have $I \equiv V$, with $V(\psi)$ given in (4.9), and the chain map condition for I is the property (4.8).

Coming to the homotopy map, the defining relation (4.38) implies that it needs to have ghost number -1 and $U(1)_{\mathcal{F}}$ charge zero. We also require mass dimension $[H] = 0$, since this is the case for all functions $V(\psi)$ as well as for the differential $\mathcal{Q}$. Since this map is in $\text{End}(\mathcal{F})$, it is not expected to depend on any of the b_i . We identify the map

$$H := \bar{\alpha} \cdot \partial_p \partial_{\bar{c}}, \quad (4.46)$$

which fulfills all previously listed requirements and satisfies (4.38), as can be shown by direct computation

$$\begin{aligned} (I\Pi - 1)F_0 &= 0 = H\mathcal{Q}F_0, \\ (I\Pi - 1)F_1 &= C^{\mu\nu}(f_{\mu\nu} - B_{\mu\nu}) = (H\mathcal{Q} + \mathcal{Q}H)F_1, \\ (I\Pi - 1)F_2 &= \mathcal{M}^\mu(\partial_\mu w - w_\mu) - \tilde{\mathcal{M}}^\mu(\partial_\mu w - w_\mu + \partial^\nu w_{\mu\nu}) \\ &\quad - \mathcal{M}^{\mu\nu}w_{\mu\nu} = (H\mathcal{Q} + \mathcal{Q}H)F_2, \\ (I\Pi - 1)F_3 &= \frac{1}{2}e\partial^\mu N_\mu - e^\mu N_\mu = \mathcal{Q}H F_3. \end{aligned} \quad (4.47)$$

The homotopy H acts trivially on the x -dependent part of elements of $\mathcal{F}$, but relates different basis elements as

$$H(\mathcal{M}^{\mu\nu}) = C^{\mu\nu}, \quad H(e^\mu) = \frac{1}{2}(\mathcal{M}^\mu - \tilde{\mathcal{M}}^\mu). \quad (4.48)$$

Essentially, comparing with the trivial pairs (4.36), we see that H is the map that inverts the action of σ on the specific basis elements that appear along with the auxiliary two-form and its related objects in other degrees of the chain complex. As a result, H implements the process of integrating out of $B_{\mu\nu}$.

We have thus verified that the homotopy data (Π, I, H) define a quasi-isomorphism between the chain complexes $(\mathcal{F}, \mathcal{Q})$ and $(\mathcal{K}, m_1)$. Note that the choice of homotopy is in general not unique, however the map we present seems to be the minimal choice achieving the desired homotopy transfer. We now use the homotopy data to construct the 2-product of $(\mathcal{K}, m_1)$ out of the original product $\star$. Note that, since the star product is associative, but not commutative, we do not expect the transferred product to be the one of the C_∞ algebra of Yang-Mills (as presented for example in [20]), but an A_∞ product μ_2 , given by

$$\mu_2 := \Pi \circ \star \circ (I \otimes I). \quad (4.49)$$

The explicit form of this product on different sectors of $\mathcal{K}$ is the following

$$\mu_2(\lambda_1, \lambda_2) = \lambda_1 \lambda_2, \quad (4.50)$$

$$\mu_2(\lambda, \mathcal{A}) = (\lambda A_\mu) \alpha^\mu + (\lambda \varphi) c \mathcal{B} = \mu_2(\mathcal{A}, \lambda) - 2A \cdot \partial \lambda c \mathcal{B}, \quad (4.51)$$

$$\mu_2(\lambda, \mathcal{E}) = (\lambda E_\mu) c \alpha^\mu + \lambda E C = \mu_2(\mathcal{E}, \lambda), \quad (4.52)$$

$$\mu_2(\lambda, \mathcal{N}) = (\lambda N) c C = \mu_2(\mathcal{N}, \lambda). \quad (4.53)$$

$$\begin{aligned} \mu_2(\mathcal{A}_1, \mathcal{A}_2) &= (A_1 \cdot A_2) C \\ &+ 2(\varphi_{[1} A_{2]\mu} + 2A_{[1} \cdot \partial A_{2]\mu} + \partial_\mu A_1 \cdot A_2) c \alpha^\mu, \end{aligned} \quad (4.54)$$

$$\mu_2(\mathcal{A}, \mathcal{E}) = (\varphi E + 2A \cdot \partial E - A^\mu E_\mu) c C = \mu_2(\mathcal{E}, \mathcal{A}), \quad (4.55)$$

and it is identical to the μ_2 computed in [20] from the composition of two vertex operators acting on the vacuum state. Applying the inclusion map $I \equiv V$ as given in (4.9) to the products μ_2 of two elements $\psi_1, \psi_2 \in \mathcal{K}$, we find that

$$V(\mu_2(\psi_1, \psi_2)) = V(\psi_1) \star V(\psi_2) + \partial V_2(\psi_1, \psi_2), \quad (4.56)$$

where we defined

$$\begin{aligned} V_2(\psi_1, \psi_2) &:= H(V(\psi_1) \star V(\psi_2)), \\ \partial V_2(\psi_1, \psi_2) &:= \mathcal{Q} V_2(\psi_1, \psi_2) + V_2(m_1(\psi_1), \psi_2) \\ &+ (-1)^{\psi_1} V_2(\psi_1, m_1(\psi_2)). \end{aligned} \quad (4.57)$$

This bilinear map V_2 coincides with the bilinear vertex operator which was defined in [20]. We now showed that this object is derived through the homotopy transfer of $(\mathcal{F}, \mathcal{Q}, \star)$ to $(\mathcal{K}, m_1, \mu_2)$.

In [20], it was also shown that μ_2 can be mapped to a commutative product m_2 on $\mathcal{K}$ via

$$\begin{aligned} m_2(\psi_1, \psi_2) &:= \mu_2(\psi_1, \psi_2) + m_1(f_2(\psi_1, \psi_2)) \\ &+ f_2(m_1(\psi_1), \psi_2) \\ &+ (-1)^{\psi_1} f_2(\psi_1, m_1(\psi_2)). \end{aligned} \quad (4.58)$$

Since the shift is exact, the two products are homotopy equivalent, with homotopy f_2 given in [20]. In fact, $\text{id} + f_2$ is an isomorphism of A_∞ algebras, so that $(\mathcal{K}, m_2, \dots)$ is a C_∞ algebra, the one which describes color stripped Yang-Mills theory. We can show that the relation between μ_2 and the commutative m_2 in $\mathcal{K}$ also descends from an analogous relation in $\mathcal{F}$: to this end, consider the symmetrization of the star product

$$M_2(F_1, F_2) := \frac{1}{2}(F_1 \star F_2 + (-1)^{F_1 F_2} F_2 \star F_1). \quad (4.59)$$

One can construct a degree -1 graded antisymmetric map $N_2: \mathcal{F}^{\otimes 2} \rightarrow \mathcal{F}$ such that

$$\begin{aligned} M_2(F_1, F_2) &= F_1 \star F_2 + \mathcal{Q} N_2(F_1, F_2) \\ &+ N_2(\mathcal{Q} F_1, F_2) \\ &+ (-1)^{F_1} N_2(F_1, \mathcal{Q} F_2), \end{aligned} \quad (4.60)$$

meaning that the star product on $\mathcal{F}$ is commutative up to the homotopy N_2 . On the other hand, the product M_2 is only associative up to homotopy and $\text{id} + N_2$ is an isomorphism of A_∞ algebras. The general form of this map is

$$N_2(F_1, F_2) = \frac{1}{4} \partial_c \partial_p F_1 \cdot \partial_p F_2, \quad (4.61)$$

with the following explicit form at each degree

$$\begin{aligned} N_2(F_{1,1}, F_{1,2}) &= c A_1 \cdot A_2, \\ N_2(F_1, F_2) &= \mathcal{M}^\mu A^\rho w_{\rho\mu}, \\ N_2(F_1, F_3) &= \frac{1}{2} e A^\mu N_\mu. \end{aligned} \quad (4.62)$$

The homotopy for commutativity N_2 can then be projected to $\mathcal{K}$, yielding f_2 as

$$f_2 = \Pi \circ N_2 \circ (I \otimes I). \quad (4.63)$$

In conclusion, we have established the following relations between our different algebras:

$$\begin{array}{ccc}
& N_2 & \\
(\mathcal{F}, \mathcal{Q}, \star) & \xrightarrow{\quad} & (\mathcal{F}, \mathcal{Q}, M_2) \\
\downarrow & & \downarrow \\
(\Pi, I, H) & & \\
\downarrow & & \downarrow \\
(\mathcal{K}, m_1, \mu_2) & \xrightarrow{f_2} & (\mathcal{K}, m_1, m_2)
\end{array}$$

In this diagram, horizontal arrows are homotopy equivalences between products, while vertical maps represent quasi-isomorphisms under homotopy transfer. The right-most arrow indeed represents a homotopy transfer akin to the one we presented in this section, mapping between C_∞ algebras in that case.

V. OUTLOOK

In this paper, we have established that the C_∞ algebra of color-stripped Yang-Mills theory is homotopy equivalent to a strict algebra of quantum worldline operators. In doing so, we have clarified how the nonlinear vertex operators introduced in [20] arise from the standard formulas for homotopy transfer. We view these results as partial progress toward clarifying the origin of the kinematic algebra of gauge theories. The main challenge, however, lies still ahead: the C_∞ algebra of Yang-Mills theory constitutes only the first layer of the much richer, and much more complicated, off-shell kinematic algebra, the $BV_\infty^\square$ algebra identified by Reiterer in [19].

In this respect, the first ingredient to go beyond the C_∞ sector is a second nilpotent differential of degree -1 . This operator plays the role of a BV Laplacian and defines the bracket of the kinematic algebra as its associated BV antibracket. In our worldline setting, the homotopy BV Laplacian is given by the antighost b , which acts as a simple degree shift in the space of states $\mathcal{K}$ carrying the C_∞ algebra structure. Despite its simplicity, determining its *order* with respect to m_2 is highly nontrivial, given that the latter is not a pointwise product. This has been a major hurdle in the recent attempts to explicitly construct the kinematic $BV_\infty^\square$ algebra.

To simplify this state of affairs, a possible route would be to find an analogous operator at the level of the quantized phase space, since it is generally simpler to determine the order of an operator with respect to $\star$. The antighost b itself is a coordinate of the phase space, so a natural ansatz for such an operator would be the adjoint action $[b, -]_\star = \frac{\partial}{\partial c}$, which is a derivation of $\star$, i.e., it is first order with respect to $\star$. However, $\frac{\partial}{\partial c}$ is *not* an endomorphism of the subalgebra $\mathcal{F}$, meaning that $[b, \mathcal{F}]_\star \not\subseteq \mathcal{F}$. Nevertheless, given the established homotopy equivalence with $\mathcal{K}$, b on $\mathcal{K}$ can be transported to an endomorphism in $\mathcal{F}$. To this end one first views b as an operator $\hat{b}$ acting on $\mathcal{K}$ and then defines

$$b_{\mathcal{F}} := I \circ \hat{b} \circ \Pi \in \text{End}(\mathcal{F}). \quad (5.1)$$

It remains to be seen whether this operator, or perhaps a suitable modification of it, will help simplify the $BV_\infty^\square$ algebra.

More generally, part of the main challenge is to guess what the proper algebraic structure is that in its ‘strict’ form captures all of color-kinematics duality. For instance, one may speculate that the framework of factorization algebras may be useful, as there are very simple such algebras that are quasi-isomorphic to the kind of $\star$ product algebras that are at the heart of the present results [45]. We plan to explore these and other directions in the future.

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DATA AVAILABILITY

No data were created or analyzed in this study.

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