# Reply to "Comment on 'Chiral symmetry restoration, the eigenvalue density of the Dirac operator, and the axial U(1) anomaly at finite temperature"'

Sinya Aoki

*Fundamental Quantum Science Program (FQSP), TRIP Headquarter, RIKEN Hirosawa 2-1, Wako, Saitama 351-0198, Japan*
*and Yukawa Institute for Theoretical Physics (YITP), Kyoto University, Kitashirakawa Oiwakecho, Sakyo-ku, Kyoto 606-8502, Japan*

Hidenori Fukaya

*Department of physics, The University of Osaka, Toyonaka 560-0043, Japan*

(Received 28 April 2026; accepted 22 May 2026; published 21 July 2026)

We respond to the comment by Matteo Giordano [1] on our article [2]. We find—and [1] itself acknowledges—that the proposed counterexamples intended to refute our argument violate a crucial assumption of QCD at high temperatures, namely that every gluonic observable is an analytic function of the squared quark mass, $m^2$. We further point out a technical mistake found in [1]. We conclude that the arguments presented in [1] are not valid.

DOI: 10.1103/dk71-9mzv

## I. $m^2$-ANALYTICITY OF GLUONIC OBSERVABLE

In [2], as in many other works on QCD, we assumed that every gluonic observable in the chirally symmetric phase is an analytic function of $m^2$ (which we refer to as $m^2$ analyticity for simplicity). This assumption is based on the $m^2$ analyticity of the QCD partition function above the critical temperature, together with the assumption that this $m^2$ analyticity is preserved under the insertion of gluonic operators or source terms into the action.

In Sec. f of [1] the density-of-states function

$$\langle \mathcal{O}(A)^l \rangle = \int dx x^l \rho_{\mathcal{O}}(x,m), \qquad \rho_{\mathcal{O}}(x,m) = \langle \delta(x-\mathcal{O}(A)) \rangle \tag{1}$$

was introduced and gave several examples of $\rho_{\mathcal{O}}(x,m)$ for which our argument in [2] fails. But as the author of [1] himself admits, these "counterexamples" all break the above $m^2$-analyticity.

One of the examples raised in [1] is a simple Gaussian model,

$$\langle \delta(x-O(A)) \rangle = \frac{1}{\sqrt{2\pi}\sigma(m)} \exp\left[-\frac{(x-x_0(m))^2}{2\sigma(m)^2}\right]\theta(x). \tag{2}$$

It is not difficult to confirm that for $\langle O(A)^\alpha \rangle$ with a noninteger $\alpha$ the above example gives a nonanalytic power of $m^2$ in the $m \to 0$ limit.

In general, we can assume that the partition function for the massive system without spontaneous breaking of the symmetry, which is a highly nonlocal object, is analytic in $m^2$ at $m = 0$. Any nonanalyticity in $m$ reflects an infrared divergence or criticality of the system. As far as we know, there is no example of symmetric phase showing such infrared critical behavior. If the author of [1] claim that the $m^2$ analyticity on every observable is too strong, a clear counterexample should be given, where the system respects the full symmetry of the action, still showing the non-analyticity.

## II. WELL DEFINEDNESS OF $P(m, A)$

In [2] we argued that if

$$\lim_{m\to\infty} \frac{1}{m^k} \langle \mathcal{O}(A)^{l_0} \rangle = 0, \tag{3}$$

for some non-negative integers $l_0$ and $k$, then

$$\langle \mathcal{O}(A)^{l_0} \rangle = m^{k_0} \int DA P(m,A) \mathcal{O}(A)^{l_0}, \tag{4}$$

holds, where $k_0$ satisfies $k < k_0$ and $P(0,A) \neq 0$.

*Published by the American Physical Society under the terms of the Creative Commons Attribution 4.0 International license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI.*

We believe that the above two conditions are almost equivalent, but it was argued in [1] that $P(0,A)$ is not necessarily well defined. Although the author of [1] himself admits that the equivalence is not crucial for the main discussion, we would like to show more details.

The definition of $\langle \mathcal{O}(A)^{l_0} \rangle$ at a finite quark mass $m$ is given by

$$\langle \mathcal{O}(A)^{l_0} \rangle = \frac{1}{Z(m)} \int DA \ \det(D(A)+m)^{N_f} e^{-S_G(A)} \mathcal{O}(A)^{l_0}, \tag{5}$$

$$Z(m) = \int DA \ \det(D(A)+m)^{N_f} e^{-S_G(A)}, \tag{6}$$

where we assume that $N_f = 2$, $D(A)$ is the massless Dirac operator, and $S_G(A)$ is the gluon part of the action. With a characteristic function for the support of $\mathcal{O}(A)$ denoted by $\mathcal{S}(\mathcal{O})$,

$$\chi_{\mathcal{S}(\mathcal{O})}(A) = \begin{cases} 1 & \text{for } A \in \mathcal{S}(\mathcal{O}) \\ 0 & \text{otherwise} \end{cases},$$

the above expectation value can be written as

$$\langle \mathcal{O}(A)^{l_0} \rangle = \frac{1}{Z(m)} \int DA \ \det(D(A)+m)^{N_f} \times e^{-S_G(A)} \chi_{\mathcal{S}(\mathcal{O})}(A) \mathcal{O}(A)^{l_0}, \tag{7}$$

and we may identify

$$P(m,A) = \frac{1}{m^{k_0} Z(m)} \det(D(A)+m)^{N_f} e^{-S_G(A)} \chi_{\mathcal{S}(\mathcal{O})}(A). \tag{8}$$

From the analyticity assumption with respect to $m$ above the critical temperature, the functional integral of $P(m,A)\mathcal{O}(A)^{l_0}$ over $A$ should be an analytic function, which satisfies all the required assumptions.

Note that $P(m,A)$ is given on a finite lattice with size $L$, for instance, with the overlap Dirac operator, which is well defined at the level one can write down in a simulation code. We do not see any problem in taking $L \to \infty$ limit of the above $P(m,A)$, while keeping $m > 1/L$.

In [1], an (unphysical) ill-defined example of $P(m,A)$ was suggested as

$$P(m,A) = \frac{\Theta(m^2 - |A|)}{m^{2l_0+2}}.$$

But in this case, $\langle |A|^\alpha \rangle$ is nonanalytic with respect to $m$ for noninteger $\alpha$, which can be confirmed by the nonanalytic contribution in $\frac{\partial}{\partial m^2} \langle |A|^\alpha \rangle$. Therefore, this example is against the assumption of the $m^2$ analyticity of QCD in the symmetric phase.

## III. A WRONG STATEMENT IN [1]

Section h of [1] presents a discussion that $m^2$ analyticity on nonlocal quantities is too strong, taking an example with an operator

$$\mathcal{O}_{\mathcal{L}}(A) = \frac{1}{V} \sum_x \mathcal{L}(A_x), \tag{9}$$

where $x$ is averaged over whole lattice sites, and considering

$$\rho_{\mathcal{O}_{\mathcal{L}}}(x,m) = \delta(x - l_{\mathcal{L}}(m)), \qquad l_{\mathcal{L}}(m) = \langle \mathcal{O}_{\mathcal{L}}(A) \rangle. \tag{10}$$

However, this equality is not valid, since $\langle \delta(x - O(A)) \rangle \neq \delta(x - \langle O(A) \rangle)$. This can be easily checked by setting $\mathcal{L}(A_x) = \sum_y q(y)q(x)$ where $q(x) = \tilde{F}F$ is the topological charge density and assuming a simple Gaussian model for the $\theta$ dependence of the vacuum. If (10) is assumed, we obtain a wrong result $\langle \mathcal{O}_{\mathcal{L}}(A)^2 \rangle = \langle \mathcal{O}_{\mathcal{L}}(A) \rangle^2$, while the true relation is $\langle \mathcal{O}_{\mathcal{L}}(A)^2 \rangle = 3 \langle \mathcal{O}_{\mathcal{L}}(A) \rangle^2$. It is well known in general that the second cumulant (or the topological susceptibility) and fourth cumulant are independent. The "peculiar" consequences obtained by the author of [1] do not arise from the assumption of $m^2$ analyticity, but rather from the incorrect assumption (10) about the density of states.

## IV. SUMMARY

We have found (and [1] itself admitted) that the counterexamples suggested in [1] break a crucial assumption of QCD at high temperatures that every gluonic observable is an analytic function of quark mass squared $m^2$. We have argued that the well definedness of $P(m,A)$ with the lattice regularization using the overlap Dirac operator on a finite lattice. We have also pointed out that the proposed form of $P(m,A)$ in [1] breaks the $m^2$ analyticity. Finally we found a wrong equality $\langle \delta(x - O(A)) \rangle = \delta(x - \langle O(A) \rangle)$ given in the argument of $m^2$ analyticity in [1]. We conclude that the arguments by [1] are not valid.

## ACKNOWLEDGMENTS

S. A. is supported by the RIKEN TRIP initiative. This work is supported in part by the JAPS Grant-in-Aid for Scientific Research (Grant No. JP22H00129, JP25K07283).

## DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

[1] Matteo Giordano, preceding Comment, Comment on "Chiral symmetry restoration, the eigenvalue density of the Dirac operator, and the axial U(1) anomaly at finite temperature", Phys. Rev. D **114**, 018501 (2026).

[2] S. Aoki, H. Fukaya, and Y. Taniguchi, Chiral symmetry restoration, eigenvalue density of Dirac operator and axial U(1) anomaly at finite temperature, Phys. Rev. D **86**, 114512 (2012).