Brane networks and the cobordism conjecture

Markus Dierigl[*] and Ignacio Ruiz[†]
CERN, Theoretical Physics Department, 1211 Meyrin, Switzerland

(Received 2 June 2026; accepted 17 August 2026; published 18 September 2026)

The absence of global symmetries in a quantum gravity theory often requires the introduction of (new) symmetry-breaking defects, which appear as singular objects in the low-energy description. This has been formalized in the cobordism conjecture, which further relates the asymptotics of these defects to nontrivial deformation classes of the effective theory. In this work we investigate the symmetry-breaking defects for theories with a discrete symmetry G encoded in the bordism groups $\Omega_2^\xi(BG)$ and, in particular, its subclass described in terms of the homology groups $H_2(BG;\mathbb{Z})$. Contrary to expectations, we find that the defects are naturally described in terms of networks of codimension-two objects rather than isolated objects in codimension three. While in special situations linking configurations of defects are sufficient, our strategy generically predicts the existence of junctions, thus suggesting an extended applicability of the cobordism conjecture. We demonstrate the viability of this approach in four-dimensional supergravity theories originating from string and M theory with a discrete Heisenberg group acting on its axionic degrees of freedom.

DOI: 10.1103/dk2b-rgkf

I. INTRODUCTION

Black holes and holographic theories provide strong evidence that global symmetries are absent in a theory of quantum gravity, see, e.g., [1–8]. Thus, symmetries should be either gauged or broken below the quantum gravity scale. The interplay between symmetries and conserved charges offers another perspective; in the presence of a global symmetry, the theory splits into various disconnected sectors labeled by their global charge. In this interpretation, gauging imposes consistency constraint (such as Gauss's law) that forbids certain sectors, while the breaking of a symmetry introduces connections between the formerly disconnected sectors. This logic has been made precise in the cobordism conjecture [9], which demands that quantum gravity theories (in D dimensions) should not allow for any nontrivial deformation classes, mathematically described by bordism groups

$$\Omega_k^{\mathrm{QG}} = 0, \qquad 0 < k < D. \tag{1.1}$$

In particular, this requires that one can always deform the theory to nothing, and there can be no conserved symmetry charges.

While the gauging of global symmetries is an interesting subject in its own right, it typically leads to the introduction of other potentially dangerous conserved charges, see, e.g., [10,11], and we will focus on symmetry breaking. Since we do not have the knowledge of the full structure of quantum gravity, we instead determine the nontrivial deformation classes of a low-energy effective theory described by the bordism groups $\Omega_k^\xi(BG)$, where ξ refers to the required structures on the tangent bundle of spacetime and BG allows for gauge fields of the symmetry group G.[1] If these groups are nontrivial, we need to introduce new effects, beyond the low-energy effective theory, that trivialize them. Typically, these are described by symmetry-breaking defects of codimension $(k+1)$, which have the nontrivial deformation class as asymptotic geometry. Indeed, this strategy has been successfully applied in various theories and produced various surprises, see, e.g., [16,20–28].

In this work, we investigate oriented and spin manifolds with a discrete symmetry group G, which often appear in gravitational theories. These discrete symmetries can include dualities, large diffeomorphisms of internal spaces after compactification, as well as continuous symmetries which are broken to a discrete subgroup at low energies. The presence of such symmetries immediately produces viable candidates for extended objects in codimension two,

[*]Contact author: markus.dierigl@cern.ch
[†]Contact author: ignacio.ruiz.garcia@cern.ch

Published by the American Physical Society under the terms of the Creative Commons Attribution 4.0 International license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Funded by SCOAP³.

[1]These two pieces can also mix, leading to so-called twisted tangential structures, see [12–19] for some examples.

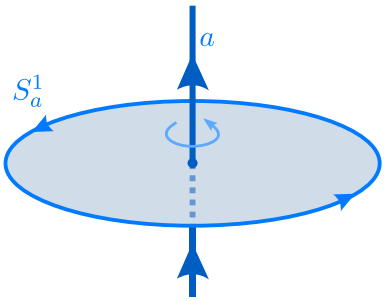


FIG. 1. Sketch of how a circle S^1_a supporting a transition function given by $a \in G$ as one winds around it requires a codimension-two defect implementing such monodromy in order to be described as a boundary.

defined by the symmetry transformation $a \in G$ experienced when looping around the object, see Fig. 1. However, for the absence of conserved charges only a subset of them is required, as captured by the associated bordism group in one dimension,

$$\Omega_1^\xi(BG) = \Omega_1^\xi(\mathrm{pt}) \oplus H_1(BG) = \Omega_1^\xi(\mathrm{pt}) \oplus \mathrm{Ab}(G). \quad (1.2)$$

The first part describes purely gravitational conserved charges encoded in $\Omega_1^\xi(\mathrm{pt})$ and is of less interest to us in this work. The charges induced by the nontrivial discrete symmetries are encoded in the first group homology $H_1(BG;\mathbb{Z})$ given by the Abelianization of G,

$$\mathrm{Ab}(G) = \frac{G}{[G,G]}. \quad (1.3)$$

As discussed, for example, in [29], the reason for this reduction of necessary defects is that all elements that can be written as a product of commutators can be bounded by smooth, but topologically nontrivial, spacetime configurations, i.e., gravitational solitons and do not require the introduction of (new) singular objects.

Continuing this strategy for higher-dimensional deformation classes $\Omega_k^\xi(BG)$ one expects to find further symmetry-breaking defects in codimension $(k+1)$. Here, we will focus on the next step, $k=2$, for which one has

$$\begin{aligned} 0 \to\ & \Omega_2^\xi(\mathrm{pt}) \oplus H_1\Big(BG; \Omega_1^\xi(\mathrm{pt})\Big) \\ & \to \Omega_2^\xi(BG) \to H_2(BG) \to 0, \end{aligned} \quad (1.4)$$

which can be deduced from the associated Atiyah-Hirzebruch spectral sequence, see, e.g., [30,31]. We see that the second group homology $H_2(BG;\mathbb{Z})$ naturally appears. These nontrivial deformation classes can be described by G gauge field backgrounds on genus-g Riemann surfaces Σ_g, with $g \geq 1$. This poses a challenge for our understanding of the dimensionality of the symmetry-breaking defect, which would appear to be codimension two rather than codimension three, by filling the

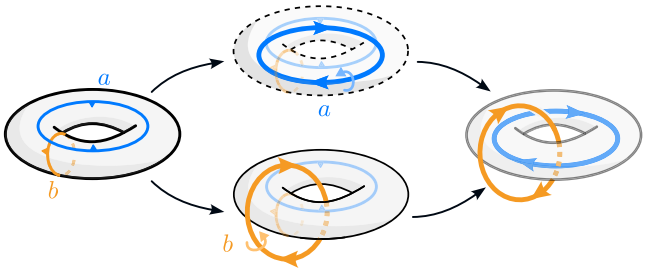


FIG. 2. Filling of the inside and outside of a two torus with transition functions a and b, such that $[a,b]=1$. Embedded in $S^3 \simeq \mathbb{R}^3 \cup \{\infty\}$, the inclusion of two codimension-two defects implementing said monodromies is needed. Note that the two defects are linked, with total linking number 1. The arrows on a given $g \in G$ transition function denote the direction of implementation (i.e., going in the arrow direction results in a g transformation, while opposite to it implements g^{-1}). This translates to the orientation of defects and their monodromy through the right-hand rule.

Riemann surface without metric singularity, see Fig. 2.[2] Indeed this behavior already appeared in [32] where the symmetry group $\mathrm{SL}(3;\mathbb{Z})$ is perfect, i.e., its Abelianization is trivial, but codimension-two defects are revived through the nontriviality of $\Omega_2^{\mathrm{Spin}}(BG)$.

In the present work, we will resolve this mismatch by identifying the obstruction to a deformation to nothing, not as the individual monodromies, but their collective interplay. In particular, we see that the codimension-two defect does not appear isolated but has to pass through other G transition functions, producing the codimension-three features expected. This also manifests on the level of the symmetry-breaking defects, which appear in nontrivial brane networks. For a very simple example consider an element of $H_2(BG;\mathbb{Z})$ which can be detected on a two torus, i.e., going around the two basis one cycles one encounters the monodromies a and b, which necessarily have to commute. Filling the interior of the torus, one needs to include a codimension-two a defect winding around the nontrivial one cycle of the solid torus. However, equally well, we could "fill" the outside by introducing a codimension-two b defect passing "through the hole" of the torus. Doing both fillings at the same time, we find a configuration of an a defect linking a b defect, which poses an obstruction to finding a pure gravitational spacetime without any defects. See Fig. 2 for an illustration.

In general, not all elements in $H_2(BG;\mathbb{Z})$ can be detected by a two torus and some require higher-genus Riemann surfaces. In these cases, while one still has the equivalence of filling the inside or outside by defects that link each other, one uncovers brane networks even as the

[2]Note that the more singular configuration of a cone over Σ_g with associated metric singularity would also be possible. We thank Jake McNamara for commenting on this.

symmetry-breaking defects themselves. Moreover, while in special situations a linking configuration of codimension-two objects can be enough to trivialize $\Omega_2^\xi(BG)$, generically one has to include brane junctions.[3] Thus, we can use the properties of the symmetry group G together with the cobordism conjecture to not only predict the existence of potentially new objects, but also suggest their configurations and networks.

The manuscript is organized as follows: In Sec. II, we describe how (part of) the second bordism group $\Omega_2^\xi(BG)$ is described in terms of the second group homology $H_2(BG;\mathbb{Z})$ and how these are described via nontrivial G bundles on genus-g Riemann manifolds, via Hopf's theorem. In Sec. III, we use this description to explore the associated symmetry-breaking defects predicted by the cobordism conjecture. In particular, we discuss the importance of brane networks, including the linking and intersection (junctions) of codimension-two objects. We apply this general strategy to type IIA compactifications on Calabi-Yau threefolds in Sec. IV, involving Heisenberg groups acting on the axionic degrees of freedom. After concluding and pointing out future research directions in Sec. V, we generalize the string theory examples in the Appendix.

II. HOPF'S THEOREM AND THE COBORDISM CONJECTURE

In this work, we want to study the defects breaking global symmetries associated with a nontrivial bordism group $\Omega_2^\xi(BG)$, for discrete groups G and tangential structure ξ. Since we do not consider twisted tangential structures, these groups split into a pure gravitational part and a part that receives gauge contributions described by the reduced bordism group $\tilde{\Omega}_2^\xi(BG)$,

$$\Omega_2^\xi(BG) = \Omega_2^\xi(\mathrm{pt}) \oplus \tilde{\Omega}_2^\xi(BG). \tag{2.1}$$

In the following, we will mainly focus on ξ denoting orientation (SO) or spin structure. Considering the second page of the Atiyah-Hirzebruch spectral sequence $E^2_{p,q}$ depicted in Fig. 3, and noting that for degree reasons the differentials are all trivial, we find

$$0 \to H_1\left(BG;\Omega_1^\xi(\mathrm{pt})\right) \to \tilde{\Omega}_2^\xi(BG) \to H_2(BG;\mathbb{Z}) \to 0. \tag{2.2}$$

Since $\Omega_1^{\mathrm{SO}}(\mathrm{pt})$ is trivial, we find that in two dimensions the oriented bordism class is fully determined by group homology, while for spin bordism one has a potential extension by $H_1(BG;\mathbb{Z}_2)$. In both cases, $H_2(BG;\mathbb{Z})$ plays

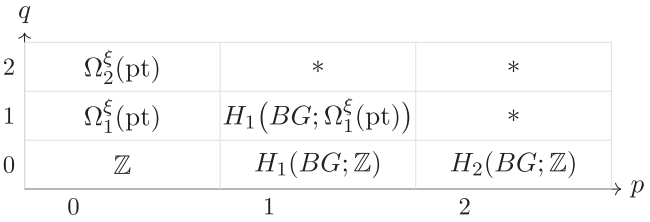


FIG. 3. $E^2_{p,q}$ for the Atiyah-Hirzebruch spectral sequence for $\Omega^\xi(BG)$.

a central role.[4] To understand the nontrivial classes in $\Omega_2^\xi(BG)$, it is therefore crucial to have a good understanding of the elements in $H_2(BG;\mathbb{Z})$ as well as their physical interpretation.

The obstruction of nontrivial classes originating from $H_2(BG;\mathbb{Z})$ is entirely encoded in the gauge sector. For discrete G and spacetime manifold X this is captured by the configurations of the associated gauge field, whose different topological sectors are given by homotopy classes of (see, e.g., [33–35])

$$[X, BG] \simeq [X, K(G,1)], \tag{2.3}$$

where we used that BG is (homotopy) equivalent to an Eilenberg-MacLane space $K(G,1)$. For Abelian groups G, this can be identified with $H^1(X;G)$, while for non-Abelian groups it can be understood as

$$[X, BG] = \frac{\mathrm{Hom}\,(\pi_1(X);G)}{G}. \tag{2.4}$$

This means that to each (homotopically) nontrivial closed path in X we associate a transition function given in terms of a group element $a \in G$, up to overall gauge transformations. Since the map is a group homomorphism, these transition functions have to respect consistency conditions encoded in the group properties of $\pi_1(X)$, the fundamental group of X.

Let us focus on closed two-dimensional spacetimes X, which for oriented and spin manifolds are given by Riemann surfaces Σ_g. Since for the two-sphere S^2 we have that $\pi_1(S^2) = 0$, there are no nontrivial gauge bundles underlining that the defects are not given in terms of codimension-three defects with S^2 as the asymptotic space. Next, we can describe gauge fields on the two torus T^2, whose fundamental group is $\mathbb{Z} \oplus \mathbb{Z}$. Gauge configurations are hence classified by two commuting elements a and b in

[3]We refer the reader to the coordinates 46°12'04.4"N 6° 07'17.8"E for a visualization of such junction.

[4]Even in the case of spin bordism it is enough to trivialize classes originating from $H_2(BG;\mathbb{Z})$, since if the extension is nontrivial they will also trivialize the classes coming from H_1. If the extension is trivial, on the other hand, these elements already appear for $\Omega_1^{\mathrm{Spin}}(BG)$ and are trivialized by the introduction of codimension-two defects there.

G, which account for the transition functions around the basis of $\pi_1(T^2)$, up to overall gauge transformations

$$[a, b] = 1. \tag{2.5}$$

Finally, for genus $g \geq 2$ Riemann surfaces the fundamental group is generated by $2g$ elements with associated transition functions determined by the group elements $\{a_1, b_1, \ldots a_g, b_g\}$, which satisfy the condition

$$[a_1, b_1]\ldots[a_g, b_g] = 1. \tag{2.6}$$

With this description of G bundles on Σ_g we can move to a description of $H_2(BG; \mathbb{Z})$ in terms of a presentation of the group G.

A. Bordism interpretation of Hopf's theorem

In the following, we will focus on discrete, but not necessarily finite, groups G, that often appear in (super) gravity theories. Such discrete groups can be described in terms of a presentation (e.g., [36])

$$G = \frac{F}{R}, \tag{2.7}$$

where F is a free group with generators $\{f_i\}$ and R is the normal closure of the set of relations, forming a normal subgroup of F. We denote the group as follows[5]:

$$G = \langle f_1, \ldots, f_n | \{r_i\} \rangle, \tag{2.8}$$

given in terms of the generators f_j and relations r_i.

Let us describe some simple examples. The cyclic group $\mathbb{Z}_n$ is given in terms of a single generator x subject to one relation

$$\mathbb{Z}_n = \langle x | x^n \rangle. \tag{2.9}$$

The dihedral group of order $2n$ can be described as

$$D_{2n} = \langle x, r | x^n, r^2, rxrx \rangle, \tag{2.10}$$

where, interpreting D_{2n} as the symmetry group of an n-gon, x is rotation by $2\pi/n$ and r is reflection along a symmetry axis of the associated n-gon. We can also describe the surface groups $\pi_1(\Sigma_g)$ in this way. As mentioned above, they are given in terms of $2g$ generators $\{a_i, b_i\}$ obeying a single relation

$$\pi_1(\Sigma_g) = \langle a_1, b_1, \ldots, a_g, b_g | [a_1, b_1]\ldots[a_g, b_g] \rangle, \tag{2.11}$$

which asserts that this combination of paths can be contracted to a point on Σ_g and hence is null homotopic.

[5]We only consider groups with a finite number of generators f_j.

The information of the presentation of the discrete group also allows one to determine the second homology group using Hopf's theorem in group homology, see, e.g., [36],

$$H_2(BG; \mathbb{Z}) \simeq \frac{R \cap [F, F]}{[R, F]}. \tag{2.12}$$

The numerator states that elements in $H_2(BG; \mathbb{Z})$ can be expressed as a product of commutators of the free generators in F which is part of the relations R. This means that an element of the second homology can be associated with

$$[f_i, f_j]\ldots[f_k, f_l] = 1 \in G. \tag{2.13}$$

Comparing to the discussion of G bundles on Σ_g above, this implies that each nontrivial element of $H_2(BG; \mathbb{Z})$ can be realized as a gauge configuration on Σ_g with g greater or equal to the number of commutators in (2.13). The denominator states that (products of) commutators with relations as one of its entries are trivial.

Let us use the formula (2.12) in the examples discussed above. For the cyclic group, we have only a single generator and thus $[F, F]$ is trivial, which correctly implies that

$$H_2(B\mathbb{Z}_n; \mathbb{Z}) = 0. \tag{2.14}$$

For the dihedral group D_{2n} and n even, one can find a relation given by

$$rx^{n/2}rx^{n/2} = rx^{n/2}r^{-1}x^{-n/2} = [r, x^{n/2}], \tag{2.15}$$

which is given by a commutator without any relation, which suggests that $H_2(BD_{2n}; \mathbb{Z})$ is nontrivial. However, we can rewrite the square of this commutator as follows:

$$[r, x^{n/2}][r, x^{n/2}] = [r, x^n], \tag{2.16}$$

showing that it is part of $[R, F]$ and hence trivial in $H_2(BG; \mathbb{Z})$. For n odd, no such element can be found and we conclude

$$H_2(BD_{2n}; \mathbb{Z}) = \begin{cases} \mathbb{Z}_2, & n \text{ even}, \\ 0, & n \text{ odd}, \end{cases} \tag{2.17}$$

reproducing the correct result, see, e.g., [37]. Finally, for the surface group $\pi_1(\Sigma_g)$, $g \geq 1$, the relation is already given in terms of a product of commutators and no power of it vanishes, leading to

$$H_2\Big(B\pi_1(\Sigma_g); \mathbb{Z}\Big) = \mathbb{Z}, \quad g \geq 1, \tag{2.18}$$

which can also be deduced from the realization that $B\pi_1(\Sigma_g) \simeq \Sigma_g$.

We therefore find that elements $H_2(BG; \mathbb{Z})$ that necessarily survive as elements in $\Omega^{\xi}_2(BG)$ are given by genus-g

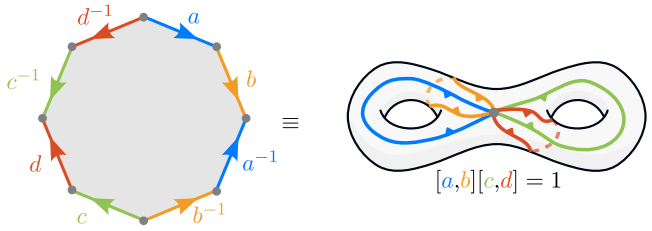


FIG. 4. Sketch of the construction of a genus $g=2$ surface Σ_2 from the relation $[a,b][c,d]=aba^{-1}b^{-1}cdc^{-1}d^{-1}=1$, where the nontrivial one cycles are interpreted as nonrelation elements from the group G.

Riemann surfaces with a nontrivial G bundle classified by the transition functions around the nontrivial closed paths. These further need to satisfy the relation of the surface group $\pi_1(\Sigma_g)$, see Fig. 4. Before we explore which defects are needed to describe such manifolds with a G bundle as the boundary of a three-dimensional manifold, we discuss what happens in case, despite the bundle on Σ_g being nontrivial, the configuration describes a trivial element in $H_2(BG;\mathbb{Z})$.

B. Bordism to nothing

To explore deformations to nothing, imagine a Riemann surface Σ_g, with $g\geq 1$, and nontrivial transition functions on each basis element of $\pi_1(\Sigma_g)$ satisfying the consistency condition (2.6). This configuration might still be trivial in $\Omega_2^{\mathrm{SO}}(BG)$, where for simplicity we focus on oriented bordism so we do not have to keep track of the spin structure (which requires pinching cycles to support anti-periodic boundary conditions for the fermions). If this is the case, it means that it describes the trivial element in $H_2(BG;\mathbb{Z})$ and by (2.12) is contained in $[R,F]$. In this section, we want to explore how one can construct a bounding three manifold, see also [38].

First we note that if one of the transition functions is described by a relation, i.e., trivial in G, one can pinch Σ_g and transition smoothly to Σ_{g-1} or split the Riemann surface into two disconnected components $\Sigma_{\tilde{g}}\sqcup\Sigma_{g-\tilde{g}}$, see Fig. 5. If, the original configuration is already in the form $[R,F]$, one can perform this operation several times to end up with the disconnected sum of spheres, which necessarily describe the trivial element in $H_2(BG;\mathbb{Z})$. In general, one first needs to execute deformations to reach this form.

For that, it will be very useful to study how one can "split open" transition functions by increasing the topological complexity of the spacetime manifold. In particular, assume that one transition function is described by $a\in G$, which can be written as a commutator[6]

$$a=[b,c]. \tag{2.19}$$

By the nucleation of a pair of gravitational solitons of genus one, see Fig. 6, one can then split the transition function. Note that the same gravitational solitons appear in the discussion of $\Omega_1^{\xi}(BG)$ as a smooth manifold bounding a circle twisted with transition function a. Implementing this operation for one of the transition functions on Σ_g, one obtains a new Riemann surface Σ_{g+2} where one nontrivial closed path, that crossed the now split transition function, carries trivial monodromy and therefore can be pinched. The result is Σ_{g+1}, where the handle with transition function given by a is replaced by two handles with transition function given by b and c satisfying (2.19). With these and similar transitions described in [38] one can always reach a configuration given by a Riemann surface with transition functions in $[R,F]$, which then can be deformed to nothing.

C. Examples

Let us illustrate this strategy in some examples.

For the dihedral group with n even we had $H_2(BD_{2n};\mathbb{Z})$ given by $\mathbb{Z}_2$, with the associated element given in terms of a single commutator

$$[r,x^{n/2}]=1. \tag{2.20}$$

From this we learn that $\Omega_2^{\mathrm{SO}}(BD_{2n})$ is generated by a two torus T^2 with transition functions r and $x^{n/2}$, both elements of order two, around the basis of one cycles. In this case, these can be directly related to the nontrivial elements of $\Omega_1^{\mathrm{SO}}(BD_{2n})$ given by the Abelianization of D_{2n},

$$\Omega_1^{\mathrm{SO}}(BD_{2n})=\mathrm{Ab}(D_{2n})=\mathbb{Z}_2\oplus\mathbb{Z}_2,\qquad n\ \text{even}, \tag{2.21}$$

which is precisely generated by the two elements r and $x^{n/2}$ above. Even for spin structure, there is no nontrivial extension and the generator above corresponds to a generator of a $\mathbb{Z}_2$ subgroup, see, e.g., [39,40].

The story changes for the group $\mathrm{SL}(n;\mathbb{Z})$ with $n\geq 5$, which has a presentation in terms of Steinberg relations [41]. Importantly, these groups are perfect, which means

$$\begin{aligned}\Omega_1^{\xi}\Big(B\mathrm{SL}(n;\mathbb{Z})\Big)&\supset H_1\Big(B\mathrm{SL}(n;\mathbb{Z});\mathbb{Z}\Big)\\&\simeq \mathrm{Ab}\Big(\mathrm{SL}(n;\mathbb{Z})\Big)=0,\qquad n\geq 3.\end{aligned} \tag{2.22}$$

This means that the cobordism conjecture would not predict any defects in codimension two. However, one finds that

$$H_2\Big(B\mathrm{SL}(n;\mathbb{Z});\mathbb{Z}\Big)=\mathbb{Z}_2,\qquad n\geq 5. \tag{2.23}$$

[6]Similar constructions also work for products of commutators in which case one would generate higher-genus configurations.

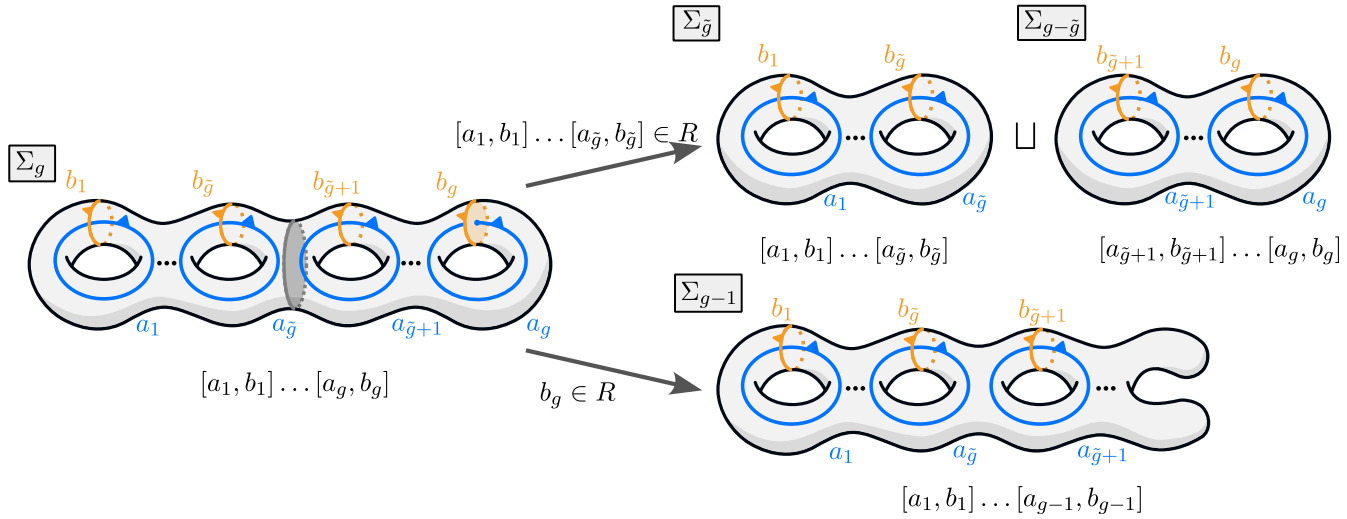


FIG. 5. Sketch of how a topology change on a $[\Sigma_g] \in H_2(BG; \mathbb{Z})$ representative can occur when certain transition function along a one cycle is trivial. We depict the two possible cases, i.e., when a set of commutators is in R, which allows the splitting of Σ_g into different connected components (with same total genus), or when a given monodromy inside a commutator is in R, which allows the pinching of a one cycle and the lowering of the genus. For the sake of simplicity, we only draw the transition functions associated with the generators of $\pi_1(\Sigma_g)$, without depicting the additional $[a_i, b_i]$ lines.

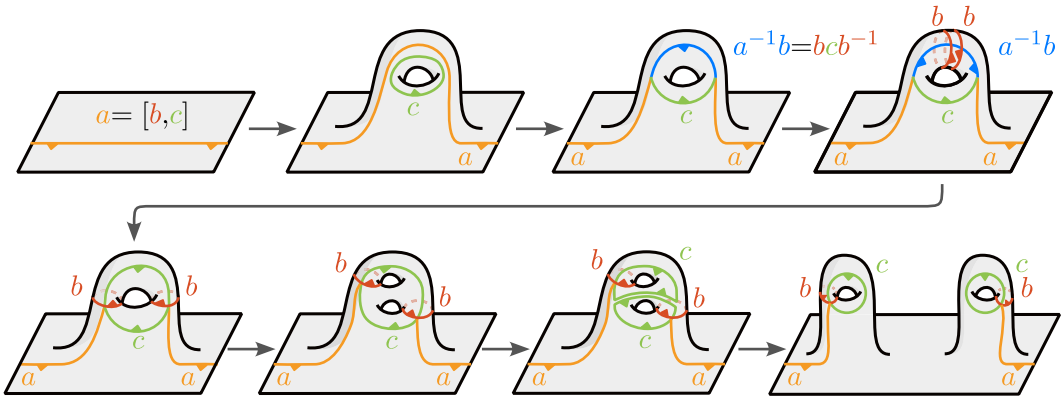


FIG. 6. Sketch of how a transition function written as a commutator $a = [b, c]$ can be broken via the nucleation of two gravitational solitons. Note that, after the above process, the one cycle dual to that previously supporting the transition function a carries trivial monodromy and thus can be pinched.

The associated element can be written as a single commutator. In particular, we can define the $\mathrm{SL}(n; \mathbb{Z})$ elements [see, e.g., the discussion in [41], Secs. 9–11],

$$h_{12} = \mathrm{diag}(-1, -1, 1, 1, \dots, 1),$$
$$h_{13} = \mathrm{diag}(-1, 1, -1, 1, \dots, 1). \tag{2.24}$$

These matrices commute and their commutator corresponds to the nontrivial element of $H_2(B\mathrm{SL}(n; \mathbb{Z}); \mathbb{Z})$ with generator given by T^2 with the associated transition functions. In the case $\mathrm{SL}(n; \mathbb{Z})$ with $n \in \{3, 4\}$, one finds $H_2(B\mathrm{SL}(n; \mathbb{Z}); \mathbb{Z}) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and a particular basis for the monodromies for $n = 3$ can be found in [32,42], see also [43]. Since these groups are perfect, the reduced spin bordism groups $\tilde{\Omega}_2^{\mathrm{Spin}}(B\mathrm{SL}(n; \mathbb{Z}))$ are identical to $H_2(B\mathrm{SL}(n; \mathbb{Z}); \mathbb{Z})$ and generated by the same manifolds as above.

With this we can also sharpen the argument of [32] stating that gravitational solitons, despite being able to bound a circle with any $\mathrm{SL}(n; \mathbb{Z})$ transition function, cannot be used to trivialize the two torus above. First, let us write one of the transition functions, say h_{12}, as a product of commutators, which always works since the group is perfect,[7]

[7]The minimal number of commutators an element of the commutator group $x \in [G, G]$ can be expressed as is known as "commutator length" of x, $\mathrm{cl}(x)$. While we will not use this concept in the paper, we refer to [44] for its role in the context of the cobordism conjecture and the problems it might entail in groups G for which $\mathrm{cl}(x)$ is not bounded for $x \in G$.

$$h_{12} = [a_1, b_1]\ldots[a_k, b_k]. \tag{2.25}$$

We can then plug this into the expression defining the nontrivial element in $H_2(B\mathrm{SL}(n;\mathbb{Z});\mathbb{Z})$,

$$1 = [h_{12}, h_{13}] = \ldots[a_k, b_k], h_{13}]. \tag{2.26}$$

Using the Leibniz rule, $[ab, c] = a[b, c]a^{-1}[a, c]$, several times, we can reduce the expression to a product of (conjugates) of the form

$$\Big[[a_j, b_j], h_{13}\Big] = \Big[b_j, [a_j, h_{13}]\Big]\Big[a_j, [b_j, h_{13}]\Big], \tag{2.27}$$

where we used the Jacobi identity. Since we started with a nontrivial element in homology, by Hopf's formula we find that not all $[a_j, h_{13}]$ and $[b_j, h_{13}]$ can be trivial, since otherwise the resulting element would be in $[F, R]$. However, this would be required for the gravitational soliton to glue appropriately across the h_{13} transition function, see Fig. 16 in [32]. Thus, these classes indeed require the introduction of (singular) defects.

Finally, one might ask the question whether all elements in $H_2(BG;\mathbb{Z})$ can be associated with a single commutator. The easiest way to see that this is not the case is the surface group $\pi_1(\Sigma_g)$ itself. It has a single relation given in terms of the product of g commutators and cannot be reduced any further, thus requiring the description in terms of a genus-g surface. We will also discuss the Heisenberg groups $\mathrm{H}_{2k+3}(\mathbb{Z})$ which require $g = 2$ in a string theory example in Sec. IV.

III. LINKING AND JUNCTIONS

Once we have identified a subset of nontrivial classes in $\Omega_2^{\xi}(BG)$ described by Riemann surfaces with a nontrivial G bundle, we switch our attention to the defects needed to trivialize them. While naively one would expect a genuine codimension-three defect playing such a role (Sec. 3.1 in [9]), since BG is aspherical, such defect cannot support a nontrivial G configuration around it[8] without metric singularities, and instead codimension-two defects will naturally appear. These are the same type one typically would have associated with classes in $\Omega_1^{\xi}(BG)$. In this section, we will resolve this conflict by noting that these codimension-two objects appear wrapped on closed curves and are generically involved in nontrivial brane configurations that capture the codimension-three nature. The importance of wrapping the symmetry defects on nontrivial cycles of the bordism generator has also been stressed in Sec. 3.3 of [45].

A. Genus one

For a nontrivial element of $H_2(BG;\mathbb{Z})$ to be described by a genus-one surface, i.e., a two torus T^2, Hopf's formula (2.12) demands that the associated element $R \cap [F, F]$ can be expressed as a single commutator. This means that the configuration is determined by two commuting transition functions $a, b \in G$, $[a, b] = 1$ (where of course both a and b need to be nontrivial elements).

As discussed in [32], one choice of defect that allows us to fill the torus and trivialize the $\Omega_2^{\xi}(BG)$ class is given by a codimension-two defect implementing the monodromy a, located in the "interior" of T^2. In this way, it implements to correct transition function on the T^2. Without a second defect, the b transition function must be continued to the interior of the solid torus and crosses the defect; however, since the two transitions commute, $[a, b] = 1$, the defect is not altered in the process, can close, and the full configuration is well defined. The solid torus given by filling the S^1 with a transition function by adding the interior defect then trivializes the associated bordism class, see Fig. 2.[9]

Alternatively, through identical arguments we could have chosen to fill the "outside" of the torus by considering a codimension-two defect implementing a monodromy b. Indeed, the (smooth)[10] embedding of T^2 in $S^3 \simeq \mathbb{R}^3 \cup \{\infty\}$ splits S^3 in two different connected components with topology $S^1 \times D^2$ and thus deformation retracts to S^1. Thus, after filling both the inside and the outside of the T^2 embedded in S^3 one ends up with two (positively) linking codimension-two defects with monodromy given by a and b, respectively, see Fig. 2. We therefore see that the defect is not only a compactified codimension-two defect, but crucially contains a nontrivial transition function that implements a linking. This explains the mismatch in the dimensionality of the bordism groups for genus-one surfaces.

B. Higher-genus and linking conditions

After describing the simple genus-one case [i.e., where the nontrivial class in $H_2(BG;\mathbb{Z})$ can be represented by a single commutator], we can consider the case with $g \geq 2$, associated with

$$[a_1, b_1]\ldots[a_g, b_g] \in \frac{R \cap [F, F]}{[R, F]} \tag{3.1}$$

[8]Note that in the case of unoriented manifolds there is one free quotient of the two-sphere given by $\mathbb{RP}^2 \simeq S^2/\mathbb{Z}_2$ which would also be a viable asymptotic for a genuine codimension-three defect. Since this manifold has $\pi_1(\mathbb{RP}^2) = \mathbb{Z}_2$ it can also support nontrivial discrete gauge fields, but only of a very special type described by $\mathrm{Hom}(\mathbb{Z}_2; G)$.

[9]From the Morse-Bott theoretical interpretation in [45], the bordism manifold corresponds to $\mathcal{B}_3 = S^1 \times D^2$, with the critical locus of the Morse-Bott function (i.e., the location of the bordism defect) being S^1.

[10]Allowing for nonsmooth embedding might result in pathological situations like Alexander's horned sphere [46], where the interior of $S^3 \backslash S^2$ is simply connected, but the exterior is not.

and represented by a genus-g surface Σ_g with the appropriate G bundle. We can then take a (smooth) embedding $\Sigma_g \subset S^3$, corresponding to the genus-g Heegaard splitting of the three-sphere [47]. The two components $V_1 \sqcup_{\Sigma_g} (-V_2) = S^3$ (where $-V_2$ denotes V_2 with the orientation flipped), with $\partial V_1 = \partial V_2 = \Sigma_g$, correspond to solid handlebodies of genus g and have homology

$$H_\bullet(V_1;\mathbb{Z}) \simeq H_\bullet(V_2;\mathbb{Z}) \simeq (\mathbb{Z}, \mathbb{Z}^g, 0, 0, \ldots). \quad (3.2)$$

This follows from the observation that they deformation retract to the wedge sum $\bigvee_{i=1}^{g} S^1$, the bouquet of g circles joining at a point. One could then try to find the defect configuration allowing the filling of the interior of V_1 and V_2 in the following way.

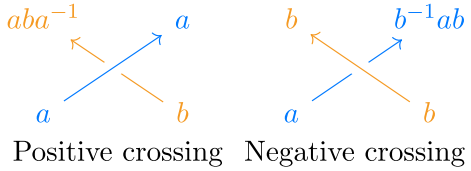
Without loss of generality, we can fill the one cycles supporting the a_i transition functions by the inclusion of a bouquet of codimension-two defects implementing the appropriate monodromy. Since $[a_i, b_i] \neq 1$. passing through the b_i transition function implements a change of the monodromy of the defect [48] (see also Fig. 7)

$$a_i \to b_i^{-1} a_i b_i, \quad (3.3)$$

in such a way that going around the handle supporting $\{a_i, b_i\}$ implements the monodromy $a_i (b_i^{-1} a_i b_i)^{-1} = [a_i, b_i]$. The joining of the g defects at the intersection point of the bouquet (notably a codimension-three point) implements the consistency condition $[a_1, b_1] \ldots [a_g, b_g] = 1$. This is illustrated in Fig. 8. Analogously, one could have "filled" the exterior component with b_i defects intersecting the a_i transition functions.

Unlike in the genus-one case, the arrangement of defects depicted in Fig. 8 is highly singular, since it involves the junction between different defects, a nontrivial property that might not be guaranteed without knowing the microscopic details of and UV realization of the defects, see [49–53] for some well-known examples in the literature. However, it might very well be the case that no defect junctions are needed at all and, inspired by the linking of defects from Fig. 2, the bouquet of defects can be resolved to a nonintersecting configuration given by a linking arrangement (albeit now in the same component of $S^3 \backslash \Sigma_g$), see, e.g., Table I.

To explore this possibility, we first take the crossing of two defects, with monodromies a and b. Depending on the sign of the crossing, i.e., whether a crossed above or below b, the monodromy of the defect changes as



(3.4)

If, instead, a defect with monodromy a passes through a codimension-one transition function b, which can be understood as the topological operator implementing the symmetry action, and orientation $\pm$, we have that the monodromy changes by $b \longrightarrow b^{\mp 1} a b^{\pm 1}$, see Fig. 7. In a nutshell, every time our defect crosses another (or passes through a transition function), the monodromy changes to a new element in the same conjugacy class.

Consider some defect with monodromy a, crossing an ordered series of defects implementing the monodromies $(d_{a,1}, d_{a,2}, \ldots, d_{a,n_a})$. Without loss of generality, we assume that the crossings are positive, since otherwise we can just take the inverse monodromy, i.e., flip the orientation of the $d_{a,i}$ defect. We also allow for transition functions, simply noting that the signs work in the opposite way. Overall, we find that the a defect changes to a different representative of the conjugacy class given by

$$a \to d_{a,1} a d_{a,1}^{-1} \to \ldots \to (d_{a,n_a} \ldots d_{a,1}) a (d_{a,n_a} \ldots d_{a,1})^{-1}. \quad (3.5)$$

For the defect to close, it needs to come back to the original element and we therefore require

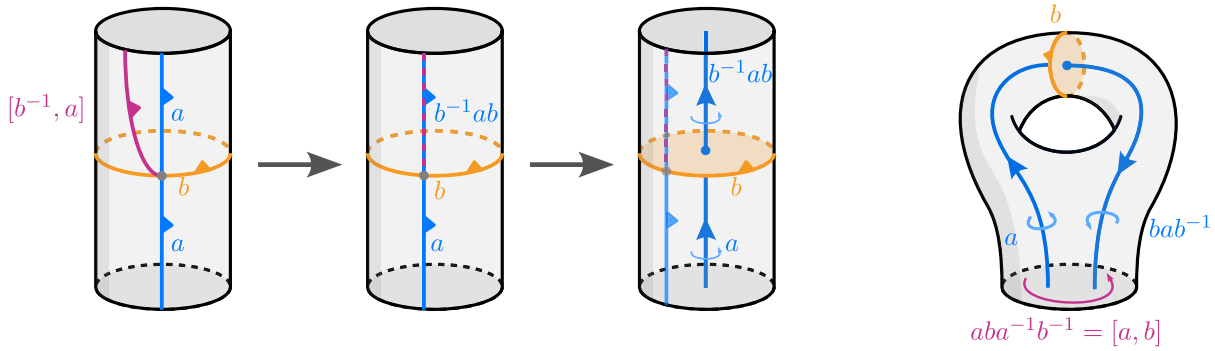


FIG. 7. Illustration of the transformation $a \to b^{-1} a b$ after a defect with monodromy a crosses a transition function b. On the right, we see how, upon "closing" the associated handle where the defect a crosses the transition function b with opposite orientation, the total monodromy is given by $a(bab^{-1})^{-1} = [a, b]$.

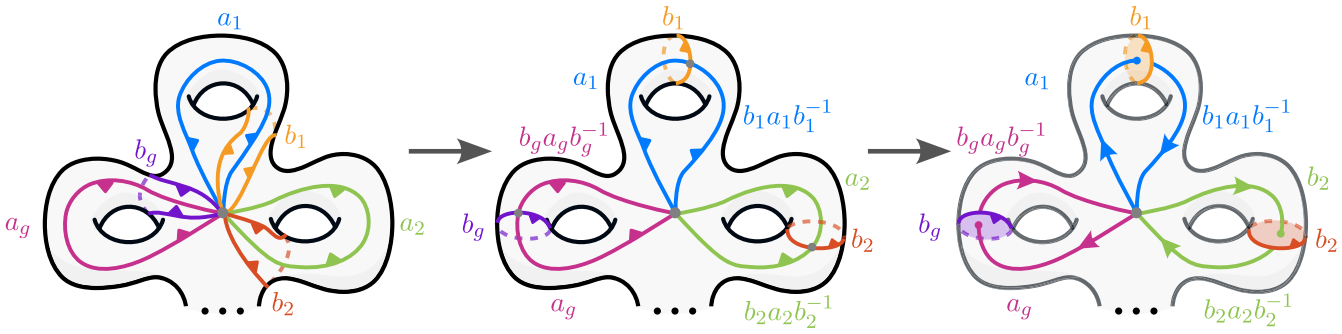
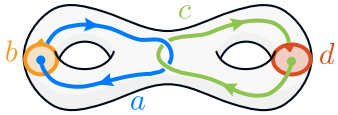
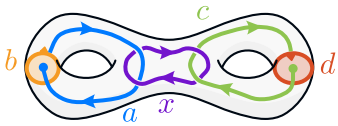
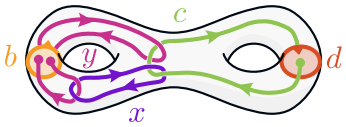
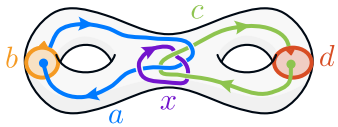


FIG. 8. Sketch of the arrangement of transition functions for a genus-g Riemann surface associated with a nontrivial element in $H_2(BG;\mathbb{Z})$ represented by $[a_1,b_1]\ldots[a_g,b_g]$. In the center, we show how the b_i transition function can be deformed to have an orientation opposite to a_i. Finally, on the right we depict a bouquet of $\{a_i\}_{i=1}^{g}$ defects allowing the filling of one component of $S^3\backslash\Sigma_g$, each of which goes through the b_i transition function (with opposite orientation).

TABLE I. Examples of possible linking defect configurations for a given G bundle on the genus-two Riemann surface Σ_2 associated with the homology class represented by $[a,b][c,d]$. Note that, for example, case (I) does not involve additional defects other than those associated with classes a and c, while case (IV) is a Borromean link, where no pair of the three defects is linked. We additionally depict the (nontrivial) linking conditions (3.7) and (3.8).

Case	Links	Link conditions
(I)		$[(d^{-1}cd)b^{-1},a]=1$ $[(b^{-1}ab)d^{-1},c]=1$
(II)		$[xb^{-1},a]=1$ $[(b^{-1}abxb^{-1}a^{-1}b)d^{-1},c]=1$ $[(d^{-1}cd)(b^{-1}ab),x]=1$
(III)		$[x,byb^{-1}]=a$ $[(d^{-1}cd)(byb^{-1}),x]=1$ $[(b^{-1}xbyb^{-1}x^{-1}b)(byb^{-1}xby^{-1}b^{-1})d^{-1},c]=1$ $[(byb^{-1}xby^{-1}b^{-1}d^{-1}cdby^{-1}bx^{-1}byb^{-1})b^{-1}xb,y]=1$
(IV)		$[x(d^{-1}c^{-1}dxd^{-1}cd)^{-1}b^{-1},a]=1$ $[(bab^{-1})(x^{-1}ax)^{-1}d,c]=1$ $[(d^{-1}cd)(b^{-1}a^{-1}bcb^{-1}ab)^{-1},x]=1$

$$[d_{a,n_a}\ldots d_{a,2}d_{a,1},a]=1. \tag{3.6}$$

$$\{[d_{x_i,n_i}\ldots d_{x_i,2}d_{x_i,1},x_i]=1\}_{i=1}^{N}. \tag{3.7}$$

The same argument holds for every other defect in the brane configuration described by the link (note that for each crossing the precise element of the conjugacy class enters). We illustrate this in Fig. 9.

Given some linking configuration involving N defects with monodromies in the conjugacy classes $\{[x_i]\}_{i=1}^{N}$ in G, the following system of equations needs to have a solution in order to describe a well-defined configuration

Furthermore, if the brane configuration in question is the defect needed to trivialize some class in $H_2(BG;\mathbb{Z})$ represented by $[a_1,b_1]\ldots[a_g,b_g]$, by filling, e.g., the one cycles associated with $\{a_i\}_{i=1}^{g}$, additional restrictions need to be included. Let $\{y_{a_i,j}\}_j$ be the monodromies of the defects (including sign accounting for their orientations) filling the one cycle associated with the monodromy a_i, then consistency requires

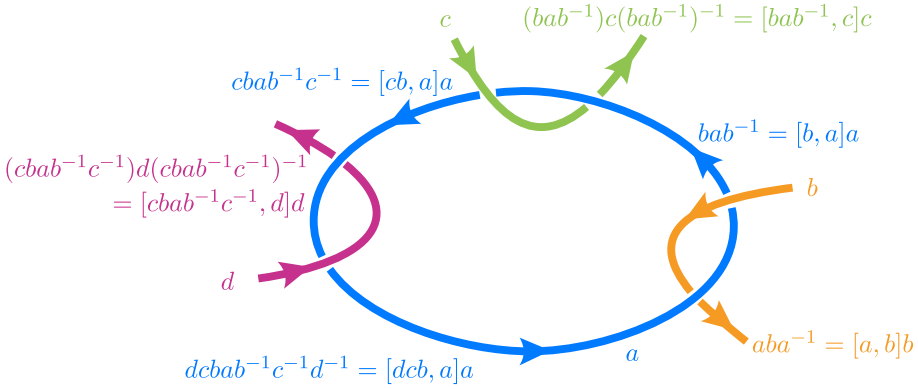


FIG. 9. Illustration of how the linking of defects changes the monodromy around them, while staying in the same conjugacy class. For the defect associated with a monodromy a (in blue) the closing condition (3.6) needs to be satisfied in order for the configuration to be consistent.

$$\prod_j y_{a_i,j} = a_i \quad \text{for } i = 1, \ldots, g. \tag{3.8}$$

Equations (3.7) and (3.8), functions of a set of N unknown variables $\{x_i\}_{i=1}^N$, define a $N+g$ relation in R. Unless R is "dense" enough, in general, the system of equations will be overdetermined, and no linking arrangement will be possible.

For intuition, in Table I we depict some examples of candidate links acting as potential symmetry-breaking defects for a nontrivial element in $H_2(BG;\mathbb{Z})$ associated with $[a,b][c,d]$ and described by a G bundle on a genus-two Riemann surface. In these cases, we spell out the consistency conditions (3.7) explicitly.

Regardless of the existence of the internal linking configurations resolving the bouquet of defects, the defect networks associated with filling the two different components of $S^3\backslash\Sigma_g$ will be linked with each other. In particular, one finds the linking of a_i and b_i defects, in generalization of Fig. 2.

One can further ask the question whether one can resolve the bouquet of defects with a single g-valent junction to a simpler brane configuration. For instance, minimal building blocks might be all possible trivalent junctions between the codimension-two defects, which can resolve the g-valent junction in a multitude of ways.[11] While we do not have an argument why such a resolution might be required, we will see that it occurs naturally in some string theory examples in Sec. IV.

C. Examples

The linking conditions (3.7) and (3.8) might seem too generic to explicitly show whether a solution exists, since the elements $d_{I,j}$ will be specific representatives of the conjugacy classes of known $\{a_i, b_i\}_{i=1}^g$ and unknown monodromies $\{x_j\}_j$, and one can take arbitrarily complicated arrangements. In this subsection, we will go through some explicit examples with groups G where the existence or absence of a solution can be settled.

1. Surface groups

We first consider the genus $g \geq 2$ surface group $\pi_1(\Sigma_g)$ (2.11), defined by a single relation

$$\pi_1(\Sigma_g) = \langle a_1, b_1, \ldots, a_g, b_g | [a_1, b_1] \ldots [a_g, b_g] \rangle, \tag{3.9}$$

which, as explained around (2.18), has $H_2(\pi_1(\Sigma_g);\mathbb{Z}) \simeq \mathbb{Z}$, generated by $[a_1, b_1]\ldots[a_g, b_g]$. Now, in trying to solve (3.7) and (3.8), an important property of surface groups with genus $g \geq 2$ comes in handy, namely, that the centralizer of y is given by (Sec. 1.1.3 [54])

$$Z_G(y) = \{x \in G : [y, x] = 1\} \simeq \mathbb{Z} \simeq \langle r_G(y) \rangle, \tag{3.10}$$

where $r_G(y)$ is the "unique primitive root" of y in G, i.e., the element such that there is an $n \geq 1$ with $r_G(y)^n = y$, and no other $w \in G$ such that $w^m = r_G(y)$ for $m > 1$. For a_i or b_i, generators of $G \simeq \pi_1(\Sigma_g)$, this implies that they are their own primitive roots. This has the nice implication that the system (3.7) can be rewritten as

$$\{d_{x_j,n_j} \ldots d_{x_j,1} = r_G(x_j)^{\alpha_j}\}_{j=1}^N, \quad \text{with} \quad \alpha_j \in \mathbb{Z}. \tag{3.11}$$

Take now the group homomorphism mapping the group to its Abelianization,

$$\begin{array}{ccccc} \mathrm{Ab}: & G \simeq \pi_1(\Sigma_g) & \longrightarrow & \mathrm{Ab}(G) \simeq \mathbb{Z}^{2g} \\ & x & \mapsto & \hat{x} \end{array}, \tag{3.12}$$

where the generators of $\mathrm{Ab}(G)$ given by $\{\hat{a}_i, \hat{b}_i\}_{i=1}^g$ can be understood as an orthonormal basis of $\mathbb{R}^{2g} \supset \mathbb{Z}^{2g}$. The system of consistency conditions (3.11) and (3.8) then maps to the linear system of equations

[11] Note that the growth of possible resolutions grows super-exponentially (in analogy to the number of Feynman diagrams for fixed asymptotic states).

$$\left\{\sum_{l=1}^{n_j}\hat{d}_{x_j,l}=\eta_j\hat{x}_j\right\}_{j=1}^{N}\cup\left\{\sum_j\hat{y}_{a_i,j}=\hat{a}_i\right\}_{i=1}^{g}, \quad (3.13)$$

where $\eta_j\in\mathbb{Q}$ are some rational numbers such that $x_j^{\eta_j}=r_G(x_j)^{\alpha_j}$ (note that upon Abelianization we switch to an additive notation). We then have that the above is a system of $N+g$ equations and N variables, thus being overdetermined and without solution. We thus conclude that it is not possible to resolve the bouquet of defects needed to trivialize the classes $\Omega_2^{\xi}(B\pi_1(\Sigma_g))$ through links, and consequently, junctions between defects must be present for $g\geq 2$.

As an illustration, if the links in Table I were associated with $G\simeq\pi_1(\Sigma_2)$, then the linking conditions would map to the following linear systems upon Abelianization:

$$\text{(I)}\begin{cases}\hat{c}-\hat{b}=\alpha\hat{a}\\ \hat{a}-\hat{d}=\gamma\hat{c}\end{cases},\qquad \text{(II)}\begin{cases}\hat{x}-\hat{b}=\alpha\hat{a}\\ \hat{x}-\hat{d}=\delta\hat{c}\\ \hat{c}+\hat{a}=\eta\hat{x}\end{cases},$$

$$\text{(III)}\begin{cases}0=\hat{a}\\ \hat{c}+\hat{y}=\eta\hat{x}\\ -\hat{d}+\hat{x}+\hat{y}=\gamma\hat{c}\\ \hat{c}+\hat{x}=\iota\hat{y}\end{cases},\qquad \text{(IV)}\begin{cases}\hat{b}=\alpha\hat{a}\\ \hat{d}=\gamma\hat{c}\\ 1=1\end{cases}, \quad (3.14)$$

none of which have solutions, since $\{\hat{a},\hat{b},\hat{c},\hat{d}\}$ is an orthogonal base in $\mathbb{R}^4$. Note that for (III) one of the equations imposes that $\hat{a}=0$, i.e., a is a product of commutators. Note that this is not possible in surface groups, since this would imply an additional relation different than the closing one.

2. Groups with torsional commutators

In general, one can argue that, given some nontrivial class in $H_2(BG;\mathbb{Z})$ generated by $[a_1,b_1]\ldots[a_g,b_g]$ (where, assuming minimality, $[a_i,b_i]\neq 1$ for all i), there can be no symmetry-breaking defect described only in terms of a link, if for some $i\in\{1,\ldots g\}$ their commutator $[a_i,b_i]$ is not a torsional element, i.e., $[a_i,b_i]^n\neq 1$ for all $n\neq 0$. In order to see this, consider the handle of the Riemann surface associated with $[a_i,b_i]\neq 1$, in such a way that defects going around it implement a_i, see (3.8), and go through the transition function b_i. Now, since

$$1\neq[a_i,b_i]=\left[\prod_j y_{a_i,j},b_i\right]=\left(\prod_j y_{a_i,j}\right)\left(\prod_j[b_i,y_{a_i,j}]y_{a_i,j}\right)^{-1}, \quad (3.15)$$

this means that at least some defect y_{a_i,j_0} must not commute with b_i (and only crosses b_i once, since said set of defects is associated with a_i). However, this implies that the centralizer $Z_G(y_{a_i,j_0})$ does not have b_i as one of its generators, and thus there must be some additional $b_i^{\pm 1}$ factor in the closing equation (3.7). Since y_{a_i,j_0} does not cross the transition function a second time, said term must come from an additional linking with a defect in the conjugacy class of b_i. In order for such linking not to be "undone" by simply moving the defects around, the b_i defect must itself be linked with the y_{a_i,j_0}. However, again because $y_{a_i,j_0}\notin Z_G(b_i)$, an additional linking with y_{a_i,j_0} in the closing relation (3.7) for b_i is needed. Every time there is a crossing, the associated monodromies change in the form

$$b_i\to[y_{a_i,j_0},b_i]b_i,\qquad y_{a_i,j_0}\to[b_i,y_{a_i,j_0}]y_{a_i,j_0}, \quad (3.16)$$

where we have omitted additional conjugating elements. If $[b_i,y_{a_i,j_0}]$ is free, no matter how many times we cross b_i and y_{a_i,j_0} (while keeping them linked), the closing relations will not be satisfied. However, if $[b_i,y_{a_i,j_0}]^n=1$ for some n, then after n mutual crossings we might be able to recover the initial y_{a_i,j_0} and b_i, which could allow for the defects to close.

We can illustrate this with a simple example given by

$$G=\langle a,b|[a,b]^g,[[a,b],a],[[a,b],b]\rangle, \quad (3.17)$$

i.e., the individual generators are not torsion, but $[a,b]$ is, and further commutes with all elements. Its first two homology groups are given by

$$H_1(BG;\mathbb{Z})\simeq\mathbb{Z}\oplus\mathbb{Z},\quad H_2(BG;\mathbb{Z})\simeq\mathbb{Z}\oplus\mathbb{Z}_g\oplus\mathbb{Z}_g, \quad (3.18)$$

with the individual factors of the second homology generated by $[a,b]^g$, $,a]$, and $,b]$. While the torsional generators can be represented by a genus-one surface, $[a,b]^g$ is associated with a genus-g Riemann manifold where all handles have identical transition functions. Using

$$(yxy^{-1})[y,x^k]y^{-1}(yxy^{-1})^{-1}=[y,x^{k+1}]y^{-1}, \quad (3.19)$$

and[12]

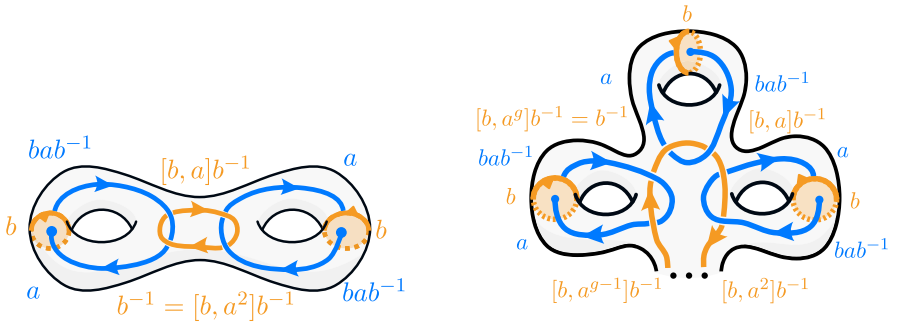
$$[x^k,y]=[x,y]^k\mod[F,F'],\quad\text{with}\quad F'=[F,F] \quad (3.22)$$

[12]To show this, we proceed by induction. Notice that

$$[x^2,y]=[x,[x,y]][x,y]^2, \quad (3.20)$$

so that $[x^2,y]=[x,y]^2 \mod [F',F]$. Now, assuming $[x^k,y]=[x,y]^k \mod [F,F']$, we have

$$[x^{k+1},y]=[x^kx,y]=[x^k,[x,y]][x,y][x,y]^k=[x,y]^{k+1}\mod[F,F']. \quad (3.21)$$



(a) Linking arrangement for $g=2$. (b) Linking arrangement for generic $g>2$.

FIG. 10. Linking configurations of defects trivializing the class in $\Omega_2^{\xi}(BG)$ represented by the generator of the second homology group of (3.17) both for genus $g=2$ in (a) and for general genus g in (b). Note that, in showing that such arrangement can be realized without junctions, we have used that for our group (3.17) $[b,a^g]=[b,a]^g=1$ and $([b,a^k]b^{-1})bab^{-1}([b,a^k]b^{-1})^{-1}=a$.

[where for our explicit group (3.17) is an exact equality, since F' is central], one can construct an explicit arrangement of linking defects that act as a symmetry-breaking defect in the interior of Σ_g without the need of junctions. We illustrate said arrangement in Fig. 10. While this group might seem a little *ad hoc*, in the Appendix we will describe an explicit M theory construction that realizes (3.17) for $g=2$.

We conclude this section by noting that, even though the generator of $H_2(BG;\mathbb{Z})$ can be represented by $[a,b]^g$, it can be shown, (see Example 2.6 in [55]), that for any group G and $u,v\in G$, $[u,v]^k$ can be expressed as the product of $\lfloor\frac{k}{2}\rfloor+1$ commutators (generally different ones). There should then be another representative of $[a,b]^g$ described by a G bundle in a Riemann manifold of genus $\lfloor\frac{g}{2}\rfloor+1$, though it is not obvious that the transition functions of the $\Sigma_{\lfloor g/2\rfloor+1}$ G bundle would admit an arrangement of defects that does not involve junctions.

IV. JUNCTIONS OF AXION STRINGS IN TYPE IIA

In this section, we will discuss a more involved top-down example resulting in a monodromy group G where some generators of $H_2(G;\mathbb{Z})$ require the product of two commutators and junctions between codimension-two defects are needed to trivialize the associated bordism classes. For this we consider type IIA string theory compactified on a Calabi-Yau threefold X, resulting in a $\mathcal{N}=2$ theory in four dimensions. We will focus on the hypermultiplet sector, which enjoys a monodromy group generically given by the form [56–58]

$$\Gamma_{\text{hyper}}=\mathrm{SL}(2;\mathbb{Z})\ltimes\left(\Gamma_{\text{mirror}}\ltimes \mathrm{H}_{2h^{2,1}+3}(\mathbb{Z})\right),\tag{4.1}$$

where $\Gamma_{\text{mirror}}\subset\mathrm{Sp}(2h^{1,1}+2;\mathbb{Z})$ is the associated monodromy group of the mirror Calabi-Yau $X^\vee$ (since the hypermultiplet moduli space $\mathcal{M}_{\text{h}}$ is fibered over the complex structure moduli space $\mathcal{M}_{\text{cs,mirror}}$ of $X^\vee$) and $\mathrm{H}_{2h^{2,1}+3}(\mathbb{Z})$ is the Heisenberg group with $2h^{2,1}+2$ generators (over the integers). We will focus on the Heisenberg group, since it will have interesting properties for our purposes. While the full structure in (4.1) might influence the bordism group due to the action of the semidirect products, here, we focus on its Heisenberg subgroup. It would be interesting to extend our analysis to also include transition functions in $\mathrm{SL}(2;\mathbb{Z})$ and Γ_{mirror}.

Let for our purposes $H_3(X;\mathbb{Z})$, the middle homology of the internal space, be free of torsion, and consider the associated symplectic base $\{\alpha^I,\beta_I\}_{I=0}^{h^{2,1}}$, in such a way that, reducing the 3-form C_3 with respect to this basis, one obtains a number of $\{\chi_I,\tilde{\chi}^I\}_{I=0}^{h^{2,1}}$ axions in the lower-dimensional effective theory. Additionally, one has the universal axion σ originating from compactifying the 6-form $\tilde{B}_6$ dual to the Neveu–Schwarz Neveu–Schwarz (NSNS) 2-form B_2 on the entire X (or equivalently dualizing the unreduced B_2 field in four dimensions). In principle, the $(2h^{2,1}+3)$ scalars would have independent axionic shift symmetries, however, the topological term of the 10D (massless) type IIA action,

$$2\kappa_{10}^2 S_{\text{IIA}}\supset-\frac{1}{2}\int B_2\wedge \mathrm{d}C_3\wedge \mathrm{d}C_3,\tag{4.2}$$

reduces to

$$\begin{aligned}2\kappa_{10}^2 S_{4\text{d}}&\supset-\frac{1}{2}\int B_2\wedge \mathrm{d}\chi_I\wedge \mathrm{d}\tilde{\chi}^I\\&=\frac{1}{4}\int \mathrm{d}B_2\wedge(\mathrm{d}\chi_I\wedge\tilde{\chi}^I-\chi_I\wedge\mathrm{d}\tilde{\chi}^I).\end{aligned}\tag{4.3}$$

This topological term requires a nontrivial mixing between the different axionic symmetries,

$$\begin{aligned}&\chi_I\to\chi_I+\zeta_I,\\&\tilde{\chi}^I\to\tilde{\chi}^I+\tilde{\zeta}^I,\\&\sigma\to\sigma+\alpha-\tfrac{1}{2}(\zeta_I\tilde{\chi}^I-\tilde{\zeta}^I\chi_I),\end{aligned}\qquad\text{for } I=0,\ldots,h^{2,1},\tag{4.4}$$

which precisely corresponds to the Heisenberg algebra of $\mathrm{H}_{2h^{2,1}+3}$, with presentation

$$\mathrm{H}_{2h^{2,1}+3}(\mathbb{Z}) = \langle a_0, b_0, \ldots, a_{h^{2,1}}, b_{h^{2,1}}, z | [z, a_I], [z, b_I], [a_I, a_J], [b_I, b_J], [a_I, b_J] z^{-\delta_{IJ}} \rangle, \tag{4.5}$$

where a_I and b_I are the shifts associated with χ_I and $\tilde{\chi}^I$ and z corresponds to the shift of σ. Noticing that, since, for example, $z = [a_0, b_0]$, we can simplify the presentation by removing z from the set of relations and it is straightforward to compute that (see also [59])

$$H_1(B\mathrm{H}_{2h^{2,1}+3}(\mathbb{Z}); \mathbb{Z}) = \mathbb{Z}^{2h^{2,1}+2}, \tag{4.6a}$$

$$H_2(B\mathrm{H}_{2h^{2,1}+3}(\mathbb{Z}); \mathbb{Z}) = \begin{cases} \mathbb{Z}^2 & \text{for } h^{2,1} = 0, \\ \mathbb{Z}^{2h^{2,1}(h^{2,1}+1)} \oplus \mathbb{Z}^{h^{2,1}} & \text{for } h^{2,1} \geq 1. \end{cases} \tag{4.6b}$$

The second homology group strongly depends on $h^{2,1}$. In the special case of vanishing $h^{2,1}$ we can write

$$\mathrm{H}_3(\mathbb{Z}) \simeq \langle a, b | [a, [a, b]], [b, [a, b]] \rangle, \tag{4.7}$$

and the nontrivial elements of the second homology group are precisely associated with] and] via Hopf's theorem. Both are therefore described in terms of a single commutator. On the other hand, for nonvanishing $h^{2,1}$ the elements in $H_2(B\mathrm{H}_{2h^{2,1}}(\mathbb{Z}); \mathbb{Z})$ fall into two classes:

(i) $[a_I, a_J], [b_I, b_J], [a_I, b_J]$ with $I \neq J$: They form

$$\binom{h^{2,1}+1}{2} + \binom{h^{2,1}+1}{2} + h^{2,1}(h^{2,1}+1) = 2h^{2,1}(h^{2,1}+1), \tag{4.8}$$

independent elements in $R \cap [F, F]$, which are not in $[R, F]$ and generate $\mathbb{Z}^{2h^{2,1}(h^{2,1}+1)}$ in (4.6b). All of them can be written in terms of a single generator.

(ii) $[a_I, b_I][a_0, b_0]^{-1}$ with $I \neq 0$: They are obtained by using $z = [a_0, b_0]$ to eliminate z, and again are elements of $R \cap [F, F]$ which are not in $[R, F]$, generating the remaining $\mathbb{Z}^{h^{2,1}}$ in (4.6b). However, these need to be written in terms of two commutators.

Note that since

$$[z, a_I] = [[a_0, b_0], a_I] = [a_0, [b_0, a_I]][[a_0, a_I], b_0] \times (\text{analogously for } [z, b_I]), \tag{4.9}$$

we have that $[z, a_I], [z, b_I] \in [R, F]$ and thus are not associated with nontrivial elements in $H_2(\mathrm{H}_{2h^{2,1}+3}(\mathbb{Z}); \mathbb{Z})$.

Summarizing, for $h^{2,1} \geq 1$ we find that $H_2(B\mathrm{H}_{2h^{2,1}+3}(\mathbb{Z}); \mathbb{Z})$ has $h^{2,1}$ generators that need to be expressed as the product of (at least) two commutators. Using the arguments in Sec. III B, the trivialization of the associated bordism class requires a $S^1 \vee S^1$ arrangement of defects that realize the appropriate monodromies of the genus-two surface representing $[a_I, b_I][a_0, b_0]^{-1}$. Moreover, as argued around (3.15) and (3.16), since the elements $\{a_I, b_I\}_{I=0}^{h^{2,1}}$ are free, and the central element $z = [a_I, b_I]$ is not torsion (i.e., $z^k \neq 1$ for any $k \neq 0$), there is no possible resolution of the $S^1 \vee S^1$ arrangement of defects in the form of linking defects and junctions are required.

These can be realized in different ways, as depicted in Fig. 11. On one hand, we can have a single four-valent junction where the defects with monodromies a_I^{-1}, $z^{-1}a_I$, zb_0, and b_0^{-1} meet (since the product of said monodromies, in any order, is the trivial element, the junction is consistent). On the other hand, we could resolve said four-valent junction into two trivalent ones, joined by a defect, with monodromy z for the resolution shown in Fig. 11. One could wonder whether this resolution is possible, and whether the defect with monodromy z exists, since it neither corresponds to a nontrivial element in $H_1(B\mathrm{H}_{2h^{2,1}+3}(\mathbb{Z}); \mathbb{Z})$ nor $H_2(B\mathrm{H}_{2h^{2,1}+3}(\mathbb{Z}); \mathbb{Z})$. However, for the type IIA example, it is obvious that indeed this is the case.

Remember that the a_I monodromy is associated with the shift of the χ_I axion obtained from reducing C_3 on the three cycle α^I. Since a D4-brane is magnetically charged under C_3, the defect implementing a_I in four dimensions will be a D4-brane wrapped on the Poincaré-dual three cycle β_I, where $\alpha^I \cdot \beta_I = 1$. Analogous arguments imply that b_I is implemented by a D4-brane wrapped on α^I. The z monodromy is instead implemented directly by the universal axion σ originating from the NSNS sector and its defect is given by an unwrapped fundamental type IIA string. Thus, the resolution of the defect network into two trivalent

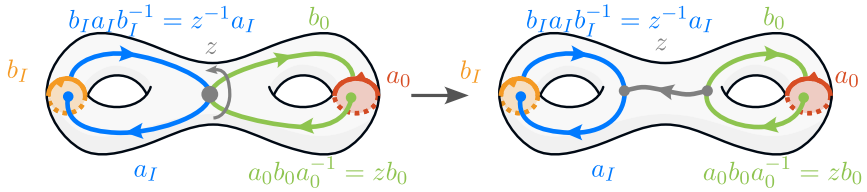


FIG. 11. Sketch of the defect network needed to kill the bordism class generated by $[a_I, b_I][a_0, b_0]^{-1}$. The defects correspond to D4-branes wrapping an appropriate three-cycle class in $H_3(X; \mathbb{Z})$. We also depict the resolution of the four-valent junction into two trivalent ones involving the defect with monodromy z, which in our setup is a fundamental string.

junctions is realized by a fundamental string stretched between two D4-branes wrapping different internal three cycles.

This demonstrates how for this case the cobordism conjecture implies the existence of junctions between wrapped D4-branes, which can be resolved through a fundamental string stretching between them. The existence of the topological term (4.2) was crucial for our derivation, since it induces a nontrivial mixing in the shift symmetries of Ramond-Ramond (RR) and NSNS axions. An analogous argument would follow in the mirror-symmetric type IIB setup, where the associated topological term, rewritten in the democratic formulation [60], is

$$2\kappa_{10}^2 S_{\mathrm{IIB}} \supset -\frac{1}{2}\int C_4 \wedge \mathrm{d}B_2 \wedge \mathrm{d}C_2 = \frac{1}{2}\int B_2 \wedge (\mathrm{d}C_2 \wedge \mathrm{d}C_4 - \mathrm{d}C_0 \wedge \mathrm{d}C_6). \tag{4.10}$$

By reducing C_2 and $-C_4 + \frac{1}{2}B_2 \wedge C_2$ on the Poincaré-dual basis of two and four cycles $\{\alpha_I, \Sigma^I\}_{I=1}^{h^{1,1}}$, as well as $-C_6 + B_2 \wedge C_4 - \frac{1}{3}B_2 \wedge B_2 \wedge C_2$ on the entire X, we obtain a set of axions $\{\chi_I, \tilde{\chi}^I\}_{I=0}^{h^{1,1}}$, see [61] for more details. Now the defects implementing the shifts would be D5- and D3-branes wrapping the dual four and two cycles, together with a D7-brane wrapping the whole Calabi-Yau (for the C_0 shift) and the unwrapped D1 string (for the C_6 axion). The central element z of the Heisenberg group is again implemented by the fundamental string, which can stretch between the axion string originating from D1-, D3-, D5-, and D7-branes resolving the four-valent junctions.

We expect this type of nonlinear topological mixing between different fields reducing to axions upon compactification to always result in Heisenberg-like groups or generalizations thereof. In the Appendix, we show how the reduction of the topological term $2\kappa_{11}^2 S_{11\mathrm{d}} \supset -\frac{1}{6}\int C_3 \wedge \mathrm{d}C_3 \wedge \mathrm{d}C_3$ in M theory results in generalized Heisenberg groups with several central elements. This opens the door to the construction of finite symmetry groups with generators of $H_2(BG;\mathbb{Z})$ associated with the product of a large number of commutators and potential resolutions by linking brane networks without the need for junctions. See also [62–64] for how said generalized Heisenberg groups appear in string theory, e.g., type IIB orientifold compactifications to $\mathcal{N}=1$ supergravity theories in four dimensions as well as their F theory lift (including torsion elements in the group relations through compactification on torsion cycles).

Finally, we would expect that similar relations could arise in heterotic string theory, where the Bianchi identity is modified to $\mathrm{d}H_3 = \frac{\alpha'}{4}(\mathrm{tr}\mathcal{R}^2 - \mathrm{tr}F^2)$, and there is the Green-Schwarz term in the action $S_{\mathrm{het}} \supset \int B_2 \wedge \mathrm{tr}F^4$. One would then expect to find generalized Heisenberg-like relations involving the NSNS axion and the gauge field degrees of freedom after compactification on a suitable six manifold.

V. CONCLUSION AND OUTLOOK

In this paper, we systematically study the symmetry-breaking defects associated with $\Omega_2^{\xi}(BG)$ for discrete symmetry groups G. The discrete nature of the symmetry group naturally suggests the involvement of codimension-two defects, which are classified by the monodromy around them given by an element in G. By using Hopf's theorem for group homology, we are able to relate the nontrivial bordism classes related to $H_2(BG;\mathbb{Z})$ to G bundles on genus-g Riemann surfaces Σ_g. This also resolves the apparent mismatch with the expected dimensionality of the cobordism defect, since it is not the individual branes that are the obstructions to the deformation to nothing, but rather their nontrivial configurations.

If the nontrivial element in $H_2(BG;\mathbb{Z})$ can be associated with a single commutator, the nontrivial deformation class is given by a two torus T^2 with commuting G transition function around the basis of closed paths. In this case, the bordism defect is the circle compactification of a codimension-two defect that passes through another G transition function. This can be interpreted as the nontrivial linking of the two codimension-two defects. For more than one generator one requires higher-genus Riemann surfaces to detect the nontrivial deformation classes, in which case the bordism defects themselves contain junctions (or linking, in special cases).

Thus, rather then predicting the appearance of genuinely codimension-three objects, the classes in $\Omega_2^{\xi}(BG)$ can be trivialized by nontrivial properties of codimension-two objects including the existence of brane networks.

We verify these predictions in Calabi-Yau compactifications of type II theories resulting in $\mathcal{N}=2$ supergravity in four dimensions, for which G is given by a Heisenberg group with nontrivial $H_2(BG;\mathbb{Z})$. We find examples involving genus-two surfaces that predict nontrivial string junctions. In the full string theory these are described nicely by branes ending on branes induced by the nontrivial Bianchi identities of the RR fields.

It might be tempting to expect that similar networks are also relevant for defects associated with $\Omega_k^{\xi}(BG)$, with $k>2$. However, there are two obstacles for such a generalization. First, it is not clear how the junctions or linking can be associated with objects of even higher codimension, since these configurations are associated with the intersection of a codimension-two defect with a codimension-one transition function, i.e., a configuration in codimension three. Second, once one goes to higher bordism groups, the differentials are generically nontrivial, and while $H_k(BG;\mathbb{Z})$ appears on the second page of the Atiyah-Hirzebruch spectral sequence, it might not survive to its ∞ page. Instead, one could imagine a generalization of the brane systems in this work that involves lower-dimensional objects, which naturally carry features in higher codimensions. Evidence of this was seen, for example, in [16,32], where some bordism generators were

fibrations, which might allow for an interpretation in terms of brane networks (see also [65]).

The most interesting configurations in our investigation have all been associated with Heisenberg groups and generalizations thereof. These originate from the compactification of theories (here, high-dimensional supergravity theories) whose action contains a topological term among the p-form fields of the parent theory. The topology of the internal space is then imprinted via an intersection matrix in the topological couplings of the low-energy effective theories. It is a natural question to ask whether the groups we encounter here are the most general or different types of compactification manifolds might lead to different classes, potentially with different origin than the topological terms in the action.

The connection of the brane configurations describing the symmetry-breaking defects and the topological terms and nontrivial Bianchi identities of the UV theory also suggests the connection to higher groups (see, e.g., [66,67] for an introduction for physicists). Indeed, since higher groups naturally mix various p-form symmetries, they are a promising arena to explore the interplay between their dual (magnetic) defects, their intersections, and brane networks.

ACKNOWLEDGMENTS

We are grateful to Damian van de Heisteeg, Miguel Montero, Ethan Torres, and Irene Valenzuela for useful discussions and comments. We especially thank Ethan Torres for comments on the manuscript. The work of the authors is supported by the European Union through ERC Starting Grant No. SymQuaG-101163591 StG-2024.

DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX: HEISENBERG GROUPS FROM M THEORY

In this Appendix, we will generalize the compactifications of type IIA string theory on a Calabi-Yau threefold described in Sec. IV to the more general setting of M theory on various seven manifolds. As we will see, the extra dimension on which we compactify will result in more than one 2-forms in the four-dimensional theory, which results in generalized Heisenberg groups describing the axion shift symmetries. We start by considering the topological term in the 11-dimensional $\mathcal{N}=1$ supergravity action, which lifts (4.2) to

$$2\kappa_{11}^2 S_{11\mathrm{d}} \supset -\frac{1}{6}\int C_3 \wedge \mathrm{d}C_3 \wedge \mathrm{d}C_3. \tag{A1}$$

Compactifying our theory on a generic manifold Y_7 allows us to expand the 3-form C_3 as

$$C_3 = B_2^i \wedge \eta_i + \chi^I \omega_I, \quad \text{with} \quad \begin{cases} H^1(Y_7;\mathbb{Z}) \simeq \langle \{\eta_i\}_{i=1}^{b_1} \rangle, \\ H^3(Y_7;\mathbb{Z}) \simeq \langle \{\omega_I\}_{I=1}^{b_3} \rangle, \end{cases} \tag{A2}$$

where for the sake of simplicity we will not consider torsion for the time being and only focus on the expansion in 1- and 3-forms giving rise to axions $\{\chi^I\}_{I=1}^{b_3}$ and 2-forms $\{B_2^i\}_{i=1}^{b_1}$ in four dimensions. From (A1) we obtain

$$\begin{aligned} 2\kappa_{11}^2 S_{4\mathrm{d}} &\supset \frac{\mathsf{c}_{iJK}}{2}\int B^i \wedge \mathrm{d}\chi^I \wedge \mathrm{d}\chi^J \\ &= -\frac{\mathsf{c}_{iJK}}{4}\int \mathrm{d}B^i \wedge (\mathrm{d}\chi^J \wedge \chi^K - \chi^J \wedge \mathrm{d}\chi^K), \end{aligned} \tag{A3}$$

where we define the intersection matrix

$$\mathsf{c}_{iJK} = \int_{Y_7} \eta_i \wedge \omega_I \wedge \omega_J, \quad \text{with} \quad \mathsf{c}_{iJK} = -\mathsf{c}_{iKJ}. \tag{A4}$$

Additionally, considering the set of axions $\{\sigma^i\}_{i=1}^{b_1}$ dual to the 2-forms B_2^i [equivalently resulting from compactifying the dual $\tilde{C}_6$ on the six-cycle classes from $H_6(Y;\mathbb{Z}) \simeq H^1(Y;\mathbb{Z})$], this translates to a transformation law for the various axions, given by

$$\begin{aligned} &\chi^I \to \chi^I + \zeta^I, \\ &\sigma^i \to \sigma^i + \alpha^i + \frac{1}{2}\mathsf{c}_{iJK}\chi^J\zeta^K, \end{aligned} \tag{A5}$$

which generalizes the type IIA axion shift symmetries in (4.4). Taking $\{x_I\}_{I=1}^{b_3}$ and $\{z_i\}_{i=1}^{b_1}$ to be the generators of the shift symmetries acting on the χ^I and σ^i axions, respectively, this results in the algebraic relations

$$[x_J, x_K] = \prod_{i=1}^{b_1} z_i^{\mathsf{c}_{iJK}}, \tag{A6}$$

which generalizes the Heisenberg group (4.5) to

$$\begin{aligned} G = \Bigg\langle x_1, \ldots, x_{b_3}, z_1, \ldots, z_{b_1} \Bigg| &[z_i, x_J], [z_i, z_j], \\ &[x_I, x_J]\left(\prod_i z_i^{\mathsf{c}_{iJK}}\right)^{-1} \Bigg\rangle. \end{aligned} \tag{A7}$$

Note that, while the intersection matrix c_{iJK} is nondegenerate (i.e., has full rank), its inverse w^{bJK}, with $\mathsf{c}_{aJK}\mathsf{w}^{bJK} = \delta_a^b$, might not be an integer matrix. We can define the following linear map[13]

[13]Remember that the elements $\hat{x}_I$ and $\hat{z}_i$ denote the image of x_I and z_i in $\mathrm{Ab}(G)$.

$$\begin{aligned} \mathsf{c}\colon\ \Lambda^2(\mathbb{Z}^{b_3}) &\to \mathbb{Z}^{b_1} \\ \hat{x}_I \wedge \hat{x}_J &\mapsto \textstyle\sum_{i=1}^{b_1} \mathsf{c}_{iJK}\tilde{z}_i, \end{aligned} \tag{A8}$$

which, even if it might have full rank, might not be surjective. Upon Abelianization, we have that (A6) translates to $\sum_i \mathsf{c}_{iJK}\hat{z}_i = 0$. This way, one obtains

$$H_1(G;\mathbb{Z}) = \mathrm{Ab}(G) = \mathbb{Z}^{b_3} \oplus \frac{\mathbb{Z}^{b_1}}{\mathrm{Im}(\mathsf{c})}. \tag{A9}$$

We now turn our attention to the second homology group of G. Similar to the traditional Heisenberg group, the commutators $[z_i, x_J]$ and $[z_i, z_j]$ are nontrivial elements in $(R \cap [F,F])/[R,F]$, thus generating some free parts $\mathbb{Z}^{b_1 b_3}$ and $\mathbb{Z}^{\frac{1}{2}b_1(b_1-1)}$ in $H_2(G;\mathbb{Z})$, with generators given in terms of a single commutator. Writing now

$$\begin{aligned} w_{\mathbf{a}} &= \prod_{1\le J<K\le b_3} [x_J, x_K]^{a_{JK}} \\ &= \prod_{1\le J<K\le b_3} \left(\prod_i z_i^{\mathsf{c}_{iJK}}\right)^{a_{JK}} = \prod_i z_i^{\sum_{J<K} \mathsf{c}_{iJK} a_{JK}}, \end{aligned} \tag{A10}$$

in order for $w_{\mathbf{a}} \in R \cap [F,F]$ we need

$$\sum_{1\le J<K\le b_3} \mathsf{c}_{iJK} a_{JK} = 0 \quad \forall\ i = 1,\ldots,b_1, \tag{A11}$$

which is precisely given by the kernel of (A8), $\langle\{w_{\mathbf{a}}\}_{\mathbf{a}}\rangle \simeq \ker(\mathsf{c}) \simeq \mathbb{Z}^{\frac{1}{2}b_3(b_3-1)-\mathrm{rank}(\mathsf{c})}$. Being clear that the elements in (A10) are not in $[R,F]$, we thus conclude that

$$H_2(G;\mathbb{Z}) \simeq \mathbb{Z}^{b_1 b_3} \oplus \mathbb{Z}^{\frac{1}{2}b_1(b_1-1)} \oplus \mathbb{Z}^{\frac{1}{2}b_3(b_3-1)-\mathrm{rank}(\mathsf{c})}. \tag{A12}$$

The generators of the first two factors can be written as single commutators, $[z_i, x_J]$ and $[z_i, z_j]$. As for $w_{\mathbf{a}} = \prod_{1\le J<K\le b_3}[x_J,x_K]^{a_{JK}}$ generators of the last factor, we see that they are represented by skew-symmetric matrices in $\Lambda^2(\mathbb{Z}^{b_3})$, whose rank is bounded by $2\lfloor\frac{b_3}{2}\rfloor$. We thus have that the minimal number of commutators the generators of $H_2(G;\mathbb{Z})$ will be expressed as will be at most $1+\lfloor\frac{b_3}{2}\rfloor$. In principle, one could expect that, choosing Y_7 with b_3 large enough, one could engineer generators of $H_2(G;\mathbb{Z})$ that needed a large number of commutators. However, we will see that, in general, things are not as straightforward.

(i) We first consider $Y_7 = T^7$, compactifying down to $\mathcal{N}=8$ supergravity in four dimensions, with $b_1 = \binom{7}{1} = 7$ and $b_3 = \binom{7}{3} = 35$. We identify each one cycle by $i = 1,\ldots,7$ and three cycle by $I \in \{[ijk], i,j,k = 1,\ldots,7\}$, so that

$$\begin{aligned} \mathsf{c}_{iJK} = \mathsf{c}_{i[jkl][mnp]} &= \int_{T^7} \mathrm{d}x^i \wedge (\mathrm{d}x^j \wedge \mathrm{d}x^k \wedge \mathrm{d}x^l) \\ &\wedge (\mathrm{d}x^m \wedge \mathrm{d}x^n \wedge \mathrm{d}x^p) = \epsilon_{ijklmnp}, \end{aligned} \tag{A13}$$

with $\epsilon_{1234567} = 1$. Since $\mathsf{c}\colon\ \Lambda^2(\mathbb{Z}^{35}) \to \mathbb{Z}^7$ is clearly surjective, we obtain

$$H_1(G;\mathbb{Z}) \simeq \mathbb{Z}^{35}, \qquad H_2(G;\mathbb{Z}) \simeq \mathbb{Z}^{245} \oplus \mathbb{Z}^{21} \oplus \mathbb{Z}^{588}. \tag{A14}$$

In order to compute the number of commutators of the generators of the $\mathbb{Z}^{588}$ factor, we see from (A13) that $\mathsf{c}_{i[jkl][mnp]} = 0$ if any index is repeated. This implies that only commutators $[x_{[jkl]}, x_{[mnp]}]$ with nonoverlapping three cycles are nontrivial, amounting to a total of $\binom{7}{6} \times \frac{1}{2}\binom{6}{3} = 70$ possibilities, and it is clear that a given commutator in (A6) equals a single $z_i^{\pm 1}$. Since we can write

$$z_1 = [x_{[234]}, x_{[567]}], \quad z_2 = [x_{[134]}, x_{[567]}], \ \text{etc.}, \tag{A15}$$

we conclude that every said 70 possibilities can be written as the product of two commutators. This was already the case in the Heisenberg groups that appeared in type IIA Calabi-Yau compactifications, so there is no surprise here.

(ii) We next consider $Y_7 = T^3 \times \mathrm{K3}$, down to a $\mathcal{N}=4$ supergravity in four dimensions, with $b_1 = 3$ and $b_3 = 1 + 3 \times 22 = 67$. Note that one cycles come from the torus, and three cycles consist of either the whole three torus or a torus one cycle times a K3 two cycle $\{[\omega_\alpha]\}_{\alpha=1}^{22}$. With this in mind, it is clear that the intersection matrix is 0 if the three cycle is the T^3, while for the other case,

$$\begin{aligned} \mathsf{c}_{iJK} = \mathsf{c}_{i(j\alpha)(k\beta)} &= \int_{T^3\times \mathrm{K3}} \mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \omega_\alpha \\ &\wedge \mathrm{d}x^k \wedge \omega_\beta = \epsilon_{ijk}\eta_{\alpha\beta}, \end{aligned} \tag{A16}$$

where $\eta_{\alpha\beta}$ is the intersection matrix of K3, symmetric and with signature (3, 19), since $H^2(\mathrm{K3};\mathbb{Z})$ is isomorphic to $2(-E_8) \oplus 3U$). Since the determinant of $\eta_{\alpha\beta} = \pm 1$, the image of $\mathsf{c}\colon\ \Lambda^2(\mathbb{Z}^{67}) \to \mathbb{Z}^3$ is $\mathbb{Z}^3$, and one recovers

$$H_1(G;\mathbb{Z}) \simeq \mathbb{Z}^{67}, \qquad H_2(G;\mathbb{Z}) \simeq \mathbb{Z}^{201} \oplus \mathbb{Z}^3 \oplus \mathbb{Z}^{2208}. \tag{A17}$$

As before, we will be interested in the generators of the $\mathbb{Z}^{2208}$ factor and the minimal number of commutators they need to be expressed as. We know that

when the whole three cycle is T^3 the intersection matrix vanishes (and thus the associated generators will have commutator length 1), so we can consider elements of the type $\mathsf{c}_{iJK} = \mathsf{c}_{i(j\alpha)(k\beta)} = \epsilon_{ijk}\eta_{\alpha\beta}$. Because of the Levi-Civita tensor appearing, we cannot have the same T^3 one cycle appearing more than once (and since we require three cycles, all of them must appear). Then from (A6)

$$[x_{(j\alpha)}, x_{(k\beta)}] = \prod_{i=1}^{3} z_i^{\epsilon_{ijk}\eta_{\alpha\beta}} = z_i^{\epsilon_{ijk}\eta_{\alpha\beta}} \tag{A18}$$

with i the element in $\{1,2,3\}$ left after removing j and k. Since $|\det\eta| = 1$, we can invert the relation and express z_1, z_2, and z_3 as single commutators. This means that, similar to the T^7 case, the generators of $H_2(G;\mathbb{Z})$ will be given by the product of at most two commutators.

(iii) For the next example, we take $Y_7 = X_6 \times S^1$, with X_6 as a Calabi-Yau threefold, down to a $\mathcal{N} = 2$ supergravity. However, we know that $b_1(X_6) = 0$, and thus $b_1(Y_7) = 1$, and we are back to the case studied in Sec. IV, consistent with type IIA string theory corresponding to M theory on S^1.

(iv) We finally consider compactifying M theory on a G_2 manifold down to a minimal $\mathcal{N} = 1$ supergravity. However, it is known that the first Betti number for this class of manifolds is exactly zero [68], which prevents us from considering this setting.

One might think that we have exhausted the possible M theory settings, since the above possibilities are the only ones that preserve supersymmetry for compactifications of 11-dimensional supergravity to four-dimensional Minkowski space, see, e.g., [69,70]. However, that would be jumping to a conclusion too fast. One important assumption in the above derivation was Y_7 not having torsion in $H_1(Y_7;\mathbb{Z})$ and $H_3(Y_7;\mathbb{Z})$. We will see that relaxing this requirement will allow us to obtain a genus-two generator of $H_2(G;\mathbb{Z})$ whose defects do not need junctions, since they admit a linking resolution. Additionally, if one allows for (not too drastic) supersymmetry breaking, we see that more freedom for the realization of the intersection matrix c_{iJK} is possible.

1. An example with links: M theory on G_2 manifold with torsion

When discussing the compactification of M theory on a G_2 manifold above, we swiftly concluded that it was not possible to get a $\int B_2 \wedge \mathrm{d}\chi \wedge \mathrm{d}\tilde{\chi}$ term for the four-dimensional action, as there was no one cycle on which to reduce $C_3 \to B_2$, since $b_1 = 0$, [68]. However, one could still find a torsion part

$$H_1(Y_7;\mathbb{Z}) = \mathrm{Ab}(\pi_1(Y_7)) \simeq \bigoplus_{i=1}^{N} \mathbb{Z}_{p_i}, \tag{A19}$$

if Y_7 is not simply connected.[14] As an illustrative example, we follow [72–74] and consider a G_2 manifold with a type IIA orientifold limit, given by

$$Y_7 = \frac{(X_6 \times S^1)/\hat{\sigma}}{\hat{\tau}}, \quad \text{with} \quad \begin{cases} \hat{\sigma}(x,y) = (\sigma(x), -y), \\ \hat{\tau}(x,y) = (\tau(y), y + \frac{1}{2}), \end{cases} \tag{A20}$$

with X_6 a Calabi-Yau threefold and where σ and τ are, respectively, commuting orientifold and freely acting involutions, with $\sigma(\tau(x)) = \tau(\sigma(x))$. Through this construction, the obtained G_2 manifold has $H_1(Y_7;\mathbb{Z}) \simeq \mathbb{Z}_2$.

We now consider two different (free) four cycles in Y_7 built by

$$\Sigma_A = S_4 \times \left\{0, \frac{1}{2}\right\}, \qquad \Sigma_B = S_3 \times S^1, \tag{A21}$$

where S_4 and S_3 are, respectively, a four cycle and three cycle in X_6. In order to have Σ_A and Σ_B invariant under the two involutions, S_4 and S_3 must be, respectively, even and odd under σ and τ. Intersecting both cycles on the Calabi-Yau results in a one-dimensional curve $C_1 = S_4 \cap S_3 \subset X_6$, which is odd under σ. Then $\tilde{\gamma} = C_1 \times \{0, \frac{1}{2}\}$ lifts to a $\mathbb{Z}_2$-torsional one cycle which precisely corresponds to the intersection $\Sigma_A \cap \Sigma_B = \gamma$, even if the four cycles were free. This translates in an intersection $\mathsf{c}_{\gamma AB} = 1 \bmod 2$.

We now compactify the topological term (A1), which results in two axions χ and $\tilde{\chi}$ coming from the reduction of $C_3 = \chi[\Sigma_A] + \tilde{\chi}[\Sigma_B] + \cdots$ [with $[\Sigma_A]$ and $[\Sigma_B]$ the $H^3(Y_7;\mathbb{Z})$ cohomology classes Poincaré dual to Σ_A and Σ_B]. Additionally, we have a $\mathbb{Z}_2$-valued phase $\sigma \in \{0, \frac{1}{2}\}$ dual to the $\mathbb{Z}_2$ 2-form in four dimensions originating from reducing C_3 on γ (equivalently from the expansion of the dual 6-form as $C_6 = \sigma[\gamma] + \cdots$). Taking a, b, and z to be the associated generators of the shifts on χ, $\tilde{\chi}$, and σ, we find the relations

$$[z,a] = [z,b] = 1, \qquad z^2 = 1, \qquad [a,b] = z, \tag{A22}$$

within the full duality group $G = F/R$. Barring fine-tuned additional relations coming from the remainder of the Y_7 geometry, the above generates a (sub)group such as the one described in (3.17) for $g = 2$, which translates to $[a,b]^2$ being one of the $H_2(G;\mathbb{Z})$ generators. As discussed there, in this case one can find a defect network without junctions, since the four-valent junction can be resolved by a linking configurations of strings in four dimensions coming from the wrapping of M5-branes on Σ_A and Σ_B, see Fig. 10(a).

[14] See [71], Sec. 12.4 for Joyce manifolds (a particular set of G_2 manifolds given by T^7 quotients) where $H_1(Y_7;\mathbb{Z}) \simeq \mathbb{Z}_2^a$.

2. More than two commutators: M theory on a twisted seven torus

In the above discussion, we have seen how compactifications of M theory on manifolds that resulted in a four-dimensional theory with $\mathcal{N} \geq 1$ supersymmetry always resulted in intersection matrices $\mathtt{c}_{iJK}$ being too simple, in such a way that generators of $H_2(G;\mathbb{Z})$ were always expressed by at most two commutators. We will see that, if we allow for Y_7 to completely break supersymmetry, there is more room to find longer commutator products. For this let us consider a twisted seven torus X_7, see, e.g., [75], only "slightly" breaking supersymmetry completely in four dimensions, spanned by 1-forms $\{e^i\}_{i=1}^7$ satisfying the Maurer-Cartan equation

$$\mathrm{d}e^a = -\frac{1}{2} f^a_{bc} e^b \wedge e^c, \tag{A23}$$

where the structure constants f^a_{bc} are understood as "metric fluxes." It is clear that then the first Betti number is given by $b_1 = 7 - \mathrm{rank}(f^a_{bc})$. We thus define a basis $\{\eta_i = A_{ia} e^a\}_{i=1}^{b_1}$. As for the 3-forms $\omega = \frac{1}{6}\Omega_{abc} e^a \wedge e^b \wedge e^c$, the Betti number b_3 will be given by the number of independent solutions to

$$\mathrm{d}\omega = 0 \Rightarrow \Omega_{a[bc} f^a_{de]} = 0, \qquad \omega \neq \mathrm{d}\beta \quad \text{with} \quad \beta = \frac{1}{2} B_{ij} e^i \wedge e^j \Rightarrow \Omega_{dec} \neq -3 B_{b[c} f^b_{de]}, \tag{A24}$$

for any B_{ij}.[15] We thus take $\{\omega_I = \frac{1}{6}\Omega_{I,abc} e^a \wedge e^b \wedge e^c\}_{I=1}^{b_3}$ as the generator set of such solutions. It is then clear that the intersection matrix would be given by

$$\mathtt{c}_{iJK} = \int_{X_7} \eta_i \wedge \omega_I \wedge \omega_J = \frac{1}{36} \epsilon^{abcdefg} A_{ia} \Omega_{I,bcd} \Omega_{J,efg}, \tag{A25}$$

from which our usual analysis follows. We illustrate this for a particular example, with Maurer-Cartan equations

$$\mathrm{d}e^1 = \mathrm{d}e^2 = \mathrm{d}e^3 = \mathrm{d}e^7 = 0, \qquad \mathrm{d}e^4 = e^1 \wedge e^2, \qquad \mathrm{d}e^5 = e^2 \wedge e^3, \qquad \mathrm{d}e^6 = e^3 \wedge e^1, \tag{A26}$$

so that the only nontrivial structure constants are $f^4_{12} = f^5_{23} = f^6_{31} = -2$. We have that clearly $b_3 = 4$, with the basis of 1-forms being $\{e^1, e^2, e^3, e^7\}$. As for the 3-forms, one iteratively computes that $b_3 = 20$, with

$$\begin{aligned}
&\omega_1 = e^{367}, \quad \omega_2 = e^{357}, \quad \omega_3 = e^{356}, \quad \omega_4 = e^{347}, \quad \omega_5 = -e^{156} + e^{346},\\
&\omega_6 = e^{256} + e^{345}, \quad \omega_7 = -e^{157} + e^{267}, \quad \omega_8 = e^{257}, \quad \omega_9 = e^{247}, \quad \omega_{10} = -e^{145} + e^{246},\\
&\omega_{11} = e^{245}, \quad \omega_{12} = e^{236}, \quad \omega_{13} = e^{235}, \quad \omega_{14} = e^{234}, \quad \omega_{15} = e^{167},\\
&\omega_{16} = e^{147}, \quad \omega_{17} = e^{146}, \quad \omega_{18} = e^{136}, \quad \omega_{19} = e^{124}, \quad \omega_{20} = e^{235},
\end{aligned} \tag{A27}$$

where $e^{abc} = e^a \wedge e^b \wedge e^c$. Then one can obtain the intersection matrix $\mathtt{c}_{iJK}$, in such a way that the only nontrivial commutators are[16]

$$[x_{20}, x_3] = [x_5, x_{14}] = [x_6, x_{19}] = [x_{10}, x_{12}] = [x_{11}, x_{18}] = [x_{17}, x_{13}] = z_1, \tag{A28a}$$

$$[x_{10}, x_4] = [x_5, x_9] = [x_6, x_{16}] = [x_{17}, x_8] = [x_{11}, x_{15}] = z_2, \tag{A28b}$$

$$[x_1, x_{10}] = [x_2, x_{17}] = [x_{16}, x_3] = [x_7, x_5] = [x_{15}, x_6] = z_3, \tag{A28c}$$

$$[x_1, x_{11}] = [x_{10}, x_2] = [x_3, x_9] = [x_4, x_6] = [x_8, x_5] = [x_6, x_7] = z_7, \tag{A28d}$$

$$[x_4, x_5] = z_3^2, \qquad [x_{10}, x_7] = z_2^2. \tag{A28e}$$

[15] In practice, in order to compute $H^k(X;\mathbb{Z})$ it is more straightforward to simply recursively take the set of closed k-forms (taking into account the Maurer-Cartan equation) and mod out by the image of all possible $(k-1)$-forms under $\mathrm{d}_{k-1}\colon \Lambda^{k-1}(X) \to \Lambda^k(X)$.

[16] We do not explicitly write the intersection matrix $\mathtt{c}_{iJK}$, since it would require four 20×20 skew-symmetric matrices. However, these are quite sparse, and the nonzero entries can be read from the commutators in (A28).

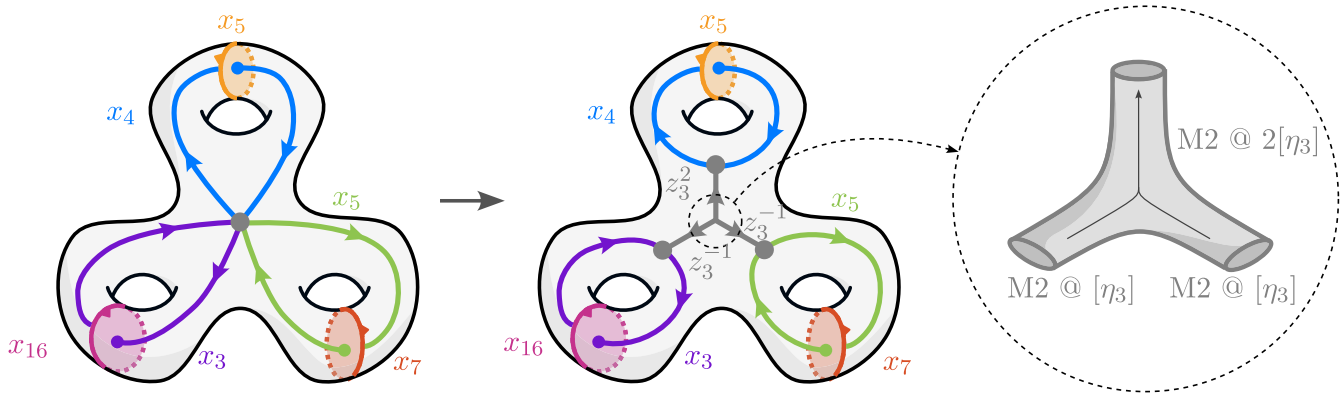


FIG. 12. Sketch of the defect arrangement for the $[x_4, x_5][x_5, x_7][x_3, x_{16}]$ representative of a nontrivial $H_2(G;\mathbb{Z})$. The defects associated with x_i correspond to M5-branes wrapped on four cycles, while the central z_i monodromies are realized by wrapped M2-branes. Note that the six-valent junction can be resolved by junctions corresponding to M2-branes ending on M5-branes. The microscopic interpretation of the "junction" where z_3^2, z_3^{-1}, and z_3^{-1} meet does not correspond to M2-branes ending on other M2-branes, but rather a single M2-brane wrapping a two chain.

Note that we can always write the central elements z_1, z_2, z_3, and z_4 as a single commutator, so we can get rid of the z_i elements and rewrite the relations $[x_J, x_K] = z_i^{\pm 1}$ in (A28a) to (A28d) as a generator of $H_2(G;\mathbb{Z})$ expressed by the product of two commutators. However, for $[x_4, x_5] = z_3^2$ and $[x_{10}, x_7] = z_2^2$ this is not possible, giving rise to two generators of $H_2(G;\mathbb{Z})$ that require a product of at least three commutators.

The resulting defects would be given by three strings in four dimensions, obtained by compactifying M5-branes on the four cycles $\{[\omega_I]\}_{I=1}^{b_3}$ Poincaré dual to the 3-forms, arranged in a $\bigvee_{i=1}^{3} S^1$ bouquet. The six-valent junction cannot be resolved through linking configuration of defects, since none of the commutators $[x_I, x_J]$ are torsion, as argued around (3.15) and (3.16). However, it is possible to resolve the six-valent junction into four trivalent ones, involving defects implementing the $\{z_i\}_{i=1}^{b_1}$ monodromies, given by strings obtained from wrapping M2-branes on the nontrivial one cycles, see Fig. 12. Three of said junctions correspond to M2-branes ending on M5-branes, as in [49], while the fourth corresponds to a single M2-brane wrapping a two chain.

[1] S. W. Hawking, Particle creation by black holes, Commun. Math. Phys. **43**, 199 (1975).
[2] Y. B. Zeldovich, A new type of radioactive decay: Gravitational annihilation of baryons, Phys. Lett. **59A**, 254 (1976).
[3] Y. B. Zeldovich, A novel type of radioactive decay: Gravitational baryon annihilation, Zh. Eksp. Teor. Fiz. **72**, 18 (1977), https://www.jetp.ras.ru/cgi-bin/dn/e_045_01_0009.pdf.
[4] T. Banks and L. J. Dixon, Constraints on string vacua with space-time supersymmetry, Nucl. Phys. **B307**, 93 (1988).
[5] R. Kallosh, A. D. Linde, D. A. Linde, and L. Susskind, Gravity and global symmetries, Phys. Rev. D **52**, 912 (1995).
[6] J. Polchinski, *String Theory. Vol. 2: Superstring Theory and Beyond*, Cambridge Monographs on Mathematical Physics (Cambridge University Press Cambridge, England, 2007).
[7] T. Banks and N. Seiberg, Symmetries and strings in field theory and gravity, Phys. Rev. D **83**, 084019 (2011).
[8] D. Harlow, TASI lectures on the emergence of bulk physics in AdS/CFT, Proc. Sci. TASI2017 (**2018**) 002.
[9] J. McNamara and C. Vafa, Cobordism classes and the Swampland, arXiv:1909.10355.
[10] R. Blumenhagen and N. Cribiori, Open-closed correspondence of K-theory and cobordism, J. High Energy Phys. 08 (2022) 037.
[11] M. Dierigl, R. Minasian, and D. Novičić, Modified Abelian gauge theories, arXiv:2602.21282.
[12] H. Sati, Twisted topological structures related to M-branes II: Twisted Wu and Wu^c structures, Int. J. Geom. Methods Mod. Phys. **09**, 1250056 (2012).
[13] T. Pantev and E. Sharpe, Duality group actions on fermions, J. High Energy Phys. 11 (2016) 171.
[14] Y. Tachikawa and K. Yonekura, Why are fractional charges of orientifolds compatible with Dirac quantization?, SciPost Phys. **7**, 058 (2019).
[15] D. S. Freed and M. J. Hopkins, Consistency of M-theory on non-orientable manifolds, Q. J. Math. Oxford Ser. **72**, 603 (2021).

[16] A. Debray, M. Dierigl, J. J. Heckman, and M. Montero, The chronicles of IIBordia: Dualities, bordisms, and the Swampland, Adv. Theor. Math. Phys. **28**, 805 (2024).
[17] I. Basile, A. Debray, M. Delgado, and M. Montero, Global anomalies & bordism of non-supersymmetric strings, J. High Energy Phys. 02 (2024) 092.
[18] A. Debray and C. Krulewski, Smith homomorphisms and spin $\hat{h}$ structures, Proc. Am. Math. Soc. **153**, 897 (2025).
[19] V. Chakrabhavi, A. Debray, M. Dierigl, and J. J. Heckman, Reflection branes, bordisms, and U dualities, Phys. Rev. D **113**, 066015 (2026).
[20] M. Dierigl, J. J. Heckman, M. Montero, and E. Torres, IIB string theory explored: Reflection 7-branes, Phys. Rev. D **107**, 086015 (2023).
[21] J. Kaidi, K. Ohmori, Y. Tachikawa, and K. Yonekura, Nonsupersymmetric heterotic branes, Phys. Rev. Lett. **131**, 121601 (2023).
[22] M. Dierigl, J. J. Heckman, M. Montero, and E. Torres, R7-branes as charge conjugation operators, Phys. Rev. D **109**, 046004 (2024).
[23] A. Debray, Bordism for the 2-group symmetries of the heterotic and CHL strings, Contemp. Math. **802**, 227 (2024).
[24] J. Kaidi, Y. Tachikawa, and K. Yonekura, On non-supersymmetric heterotic branes, J. High Energy Phys. 03 (2025) 211.
[25] M. Fukuda, S. K. Kobayashi, K. Watanabe, and K. Yonekura, Black p-branes in heterotic string theory, J. High Energy Phys. 05 (2025) 043.
[26] J. J. Heckman, J. McNamara, J. Parra-Martinez, and E. Torres, Gliozzi-Scherk-Olive defects: IIA/IIB walls and the surprisingly stable R7-brane, Phys. Rev. D **113**, 066021 (2026).
[27] E. Torres, A matrix theory construction of the IIA/IIB wall, arXiv:2603.02199.
[28] E. Anastasi, M. Montero, A. M. Uranga, and C. Wang, What IIB looks IIA string: String cobordisms via non-compact CFTs, J. High Energy Phys. 07 (2026) 182.
[29] J. McNamara, Gravitational solitons and completeness, arXiv:2108.02228.
[30] I. García-Etxebarria and M. Montero, Dai-Freed anomalies in particle physics, J. High Energy Phys. 08 (2019) 003.
[31] K. Yonekura, Heterotic global anomalies and torsion Witten index, J. High Energy Phys. 10 (2022) 114.
[32] N. Braeger, A. Debray, M. Dierigl, J. J. Heckman, and M. Montero, Cobordism utopia: U-dualities, bordisms, and the Swampland, arXiv:2505.15885.
[33] R. Dijkgraaf and E. Witten, Topological gauge theories and group cohomology, Commun. Math. Phys. **129**, 393 (1990).
[34] D. S. Freed and F. Quinn, Chern-Simons theory with finite gauge group, Commun. Math. Phys. **156**, 435 (1993).
[35] A. Hatcher, *Algebraic Topology* (Cambridge University Press, Cambridge, England, 2000).
[36] K. S. Brown, *Cohomology of Groups*, Graduate Texts in Mathematics Vol. 87 (Springer-Verlag, New York, 1982).
[37] D. Handel, On products in the cohomology of the dihedral groups, Tohoku Math. J. **45**, 13 (1993).
[38] A. Putman, Hopf' s theorem via geometry (2018), https://api.semanticscholar.org/CorpusID:202743994.
[39] R. Bruner and J. Greenlees, *Connective Real K-Theory of Finite Groups*, Mathematical Surveys and Monographs (American Mathematical Society, Providence, 2010).
[40] J. Davighi, B. Gripaios, and N. Lohitsiri, Anomalies of non-Abelian finite groups via cobordism, J. High Energy Phys. 09 (2022) 147.
[41] J. Milnor, *Introduction to Algebraic K-Theory. (AM-72)* (Princeton University Press, Princeton, NJ, 1971).
[42] C. Soulé, The cohomology of sl3(z). Topology **17**, 1 (1978).
[43] W. van der Kallen, The Schur multipliers of $\mathrm{SL}(3,\mathbb{Z})$ and $\mathrm{SL}(4,\mathbb{Z})$, Math. Ann. **212**, 47 (1974).
[44] C. las Heras and I. Ruiz, Bordisms between 9d type IIB supergravities and commutator widths of duality groups, arXiv:2605.15276.
[45] I. Ruiz, Morse-Bott inequalities, topology change and cobordisms to nothing, J. High Energy Phys. 06 (2025) 030.
[46] J. W. Alexander, An example of a simply connected surface bounding a region which is not simply connected, Proc. Natl. Acad. Sci. U.S.A. **10**, 8 (1924).
[47] F. Waldhausen, Heegaard-zerlegungen der 3-sphäre, Topology **7**, 195 (1968), https://pub.uni-bielefeld.de/download/1782178/2313707/Waldhausen_020.pdf.
[48] M. Delgado, D. van de Heisteeg, S. Raman, E. Torres, C. Vafa, and K. Xu, Finiteness and the emergence of dualities, SciPost Phys. **19**, 047 (2025).
[49] A. Strominger, Open p-branes, Phys. Lett. B **383**, 44 (1996).
[50] A. Hanany and E. Witten, Type IIB superstrings, BPS monopoles, and three-dimensional gauge dynamics, Nucl. Phys. **B492**, 152 (1997).
[51] A. Sen, String network, J. High Energy Phys. 03 (1998) 005.
[52] O. Aharony, A. Hanany, and B. Kol, Webs of (p, q) five-branes, five-dimensional field theories and grid diagrams, J. High Energy Phys. 01 (1998) 002.
[53] O. DeWolfe and B. Zwiebach, String junctions for arbitrary lie algebra representations, Nucl. Phys. **B541**, 509 (1999).
[54] B. Farb and D. Margalit, *A Primer on Mapping Class Groups*, Princeton Mathematical Series (Princeton University Press, Princeton, NJ, 2012).
[55] M. Culler, Using surfaces to solve equations in free groups, Topology **20**, 133 (1981).
[56] S. Alexandrov, D. Persson, and B. Pioline, On the topology of the hypermultiplet moduli space in type II/CY string vacua, Phys. Rev. D **83**, 026001 (2011).
[57] S. Alexandrov, Twistor approach to string compactifications: A review, Phys. Rep. **522**, 1 (2013).
[58] S. Alexandrov, J. Manschot, D. Persson, and B. Pioline, Quantum hypermultiplet moduli spaces in $N = 2$ string vacua: A review, Proc. Symp. Pure Math. **90**, 181 (2015), https://cds.cern.ch/record/1537601.
[59] S. T. Lee and J. A. Packer, The cohomology of the integer heisenberg groups, J. Algebra **184**, 230 (1996).
[60] E. Bergshoeff, R. Kallosh, T. Ortin, D. Roest, and A. Van Proeyen, New formulations of D = 10 supersymmetry and D8—O8 domain walls, Classical Quantum Gravity **18**, 3359 (2001).
[61] S. Alexandrov, B. Pioline, F. Saueressig, and S. Vandoren, D-instantons and twistors, J. High Energy Phys. 03 (2009) 044.
[62] M. Berasaluce-Gonzalez, P. G. Camara, F. Marchesano, D. Regalado, and A. M. Uranga, Non-Abelian discrete gauge

symmetries in 4d string models, J. High Energy Phys. 09 (2012) 059.
[63] T. W. Grimm, T. G. Pugh, and D. Regalado, Non-Abelian discrete gauge symmetries in F-theory, J. High Energy Phys. 02 (2016) 066.
[64] P. Corvilain, T. W. Grimm, and D. Regalado, Shift-symmetries and gauge coupling functions in orientifolds and F-theory, J. High Energy Phys. 05 (2017) 059.
[65] V. Nevoa, S. Raman, and C. Vafa, Elementary constituents conjecture, arXiv:2511.13813.
[66] C. Córdova, T. T. Dumitrescu, and K. Intriligator, Exploring 2-group global symmetries, J. High Energy Phys. 02 (2019) 184.
[67] F. Benini, C. Córdova, and P.-S. Hsin, On 2-group global symmetries and their anomalies, J. High Energy Phys. 03 (2019) 118.
[68] D. D. Joyce, Compact Riemannian 7-manifolds with holonomy G_2. I, J. Diff. Geom. **43**, 291 (1996).
[69] M. J. Duff, B. E. W. Nilsson, and C. N. Pope, Kaluza-Klein supergravity, Phys. Rep. **130**, 1 (1986).
[70] B. S. Acharya and S. Gukov, M theory and singularities of exceptional holonomy manifolds, Phys. Rep. **392**, 121 (2004).
[71] D. D. Joyce, *Compact Manifolds with Special Holonomy*, Oxford Mathematical Monographs (Oxford University Press, Oxford, UK, 2000).
[72] S. Kachru and J. McGreevy, M theory on manifolds of G(2) holonomy and type IIA orientifolds, J. High Energy Phys. 06 (2001) 027.
[73] P. G. Camara, L. E. Ibanez, and F. Marchesano, RR photons, J. High Energy Phys. 09 (2011) 110.
[74] L. Kaufmann, T. Weigand, and M. Wiesner, On quantum obstructions in type IIA orientifolds, arXiv:2604.25988.
[75] C. M. Hull and R. A. Reid-Edwards, Flux compactifications of string theory on twisted tori, Fortschr. Phys. **57**, 862 (2009).