

Duality of four dimensional $\mathcal{N} = 1$ gauge theory

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We show that the Seiberg-type duality of $\mathcal{N} = 1$ gauge theory coupled with tensor chiral fields and fundamental chiral fields works if the meson spectrum built from the tensor fields takes a particular form: (a) It should be truncated. (b) The R charges of tensor fields $\{R_a\}$ and the truncated mesons $\{R_j\}$ take very special values. The meson spectrum so that the duality works is encoded elegantly in the factorization of the polynomial $y^n - 1 = \Phi_+ \Phi_-$. Our consideration covers many known $\mathcal{N} = 1$ dualities and generates a large class of new examples.

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I. INTRODUCTION

One of the most important discoveries in the study of four-dimensional (4D) $\mathcal{N} = 1$ non-Abelian gauge theory was the electric-magnetic duality of supersymmetric quantum chromodynamics (SQCD) found by Seiberg [1]. The basic features of the Seiberg duality are the following: (a) The gauge group is different in the dual description. (b) The composite chiral operator of one theory is mapped to the elementary field (gauge singlet) of the dual theory. (c) A superpotential is needed so that the composite chiral operators of the dual theory are projected out. Seiberg duality plays a crucial role in understanding the strong coupling dynamics of non-Abelian gauge theory [2].

The Seiberg duality was soon generalized to many other models: theory with one adjoint chiral superfield [3,4], theory with two adjoint chiral superfields [5], and many other similar models in [6–11]. The basic picture of the duality is the same as the Seiberg duality; for example, the composite chiral operators are mapped to the elementary fields, and often a dual superpotential is needed.

A crucial ingredient in all the above models is the following: there is a truncation for the chiral spectrum of the mesons built from the tensor field (such as the adjoint chiral). The truncation was often introduced by a superpotential for the tensor fields, for example, the A_k or $(D_{k+1}, k \text{ odd})$ type superpotential for the adjoint chirals of $\mathcal{N} = 1$ theory. However, one already needs to impose the so-called quantum constraints to truncate the meson spectrum. Such a strategy has been used in [12] to find the dual description of the so-called E_7 model [13] for adjoint SQCD.

The purpose of this paper is to explore the following question: for what kind of truncation of meson spectrum can one find a sensible dual theory? Our main discovery is that the duality is essentially controlled by the truncation of the meson spectrum built from tensorial matter. Assuming our model is given by the $SU(N_c)$ gauge group coupled with N_f flavors of fundamental matter, the N_A adjoint matters. The basic picture of the duality is shown in Fig. 1. The duality works if the R charges of the adjoint matter and the truncated set of mesons satisfy the equation¹

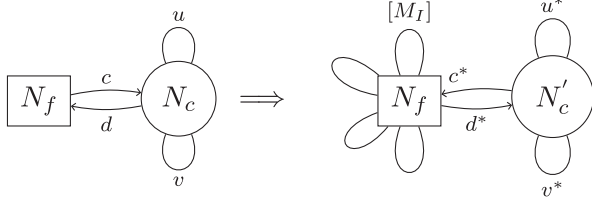
$$\sum_{j=1}^{\alpha} t^{R_j} = \frac{t^{\Delta+2} - 1}{-1 + t^2 + \sum_a (t^{R_a} - t^{2-R_a})}. \quad (1)$$

Here Δ is the pairing constant of mesons built from adjoint chirals, R_a 's are the charges for the adjoint chirals, and R_j 's are the charges for the mesons. Once the R charges satisfy the above equation, we can check that for any finite N_f, N_c , various physical quantifiers for dual theory also match: (a) the anomalies of the R symmetry such as $\text{Tr } R$ and $\text{Tr } R^3$ agree, which implies the central charges a, c agree; (b) the spectrums of mesonic and baryonic operators are mapped perfectly.

It is then quite amazing that as long as the R charges of the adjoint chirals R_a and the truncated spectrum R_j are derived from the factorization of the polynomial $y^n - 1 = \Phi^+ \Phi^-$ (where Φ^+ is a polynomial with positive coefficients), one can get an $\mathcal{N} = 1$ duality. So one immediately gets a large class of new types of $\mathcal{N} = 1$ dualities. Moreover, for each factorization, one can get many new

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¹This equation has been discussed in [14,15] and is derived from the match of large N_c, N_f limit of the superconformal index of dual theory. Our point here is that this equation will ensure that the duality works for the finite N_c, N_f limit.

FIG. 1. Basic picture of $\mathcal{N} = 1$ duality.

dualities by distributing the gauge singlets in electric and magnetic theory differently.

For other classical gauge groups with tensor fields, one often needs to introduce an involution operation on the above set of R charges so that duality could work. We also studied the duality involving exceptional gauge groups. The generalization to semisimple groups such as the quiver gauge theory is straightforward as one can perform the duality on each gauge group separately.

The present paper mainly focuses on the combinatorial nature of $\mathcal{N} = 1$ duality. The dynamical questions such as whether such a truncation is possible or not would be left for future exploration.

This paper is organized as follows: Section II discusses the basic feature of $\mathcal{N} = 1$ duality of $SU(N_c)$ gauge theory coupled with adjoint chirals and N_f chirals. Section III discusses the generalization to simple gauge groups. Section IV discusses the duality for theory with semisimple gauge groups, such as quiver gauge theory. A conclusion is given in Sec. V.

II. GENERAL DISCUSSION

A. Basic picture of $\mathcal{N} = 1$ duality

Let us consider duality for $SU(N_c)$ gauge theory coupled with two adjoint chiral fields u , v^2 and N_f pairs of fundamental and antifundamental chiral fields c , d . Notice that this gauge theory might be a part of a quiver gauge theory. There might be marginal superpotential W_E for the electric model. The duality works as follows:

- (1) Replace every chiral field of the original theory with a dual chiral field: $c \rightarrow c^*$, $d \rightarrow d^*$, $u \rightarrow u^*$, $v \rightarrow v^*$. And the dual fields are in the conjugate representation of gauge group: the orientation of the fundamental and antifundamental field is reversed; see Fig. 1.
- (2) For each oriented loop starting with the flavor node, there is an associated dressed mesonic chiral field, i.e., $M_I = cu^{a_1}v^{b_1}u^{a_2}v^{b_2}\dots d$. One adds a dual gauge singlet $[M_I]$, which is now in adjoint representation of the flavor node. Here I is a sequence of integers $[a_1, b_1, a_2, b_2, \dots]$.

²Here we use two adjoint chirals to simplify the illustration, and all the following discussions are still valid for any number of adjoint fields.

- (3) The superpotential term is changed as follows: replace the combination of letters M_I in W_E by the gauge singlet $[M_I]$, and add new terms in the superpotential as

$$\Delta W = \sum_I [M_I] d^* u^{a'_1} v^{b'_1} \dots c^* = \sum_I [M_I] M_{I'}. \quad (2)$$

Here $M_{I'}$ is the composite dressed meson in the dual theory.

- (4) Finally, one might be able to integrate out massive fields using the superpotential in the magnetic theory.

To make the above duality proposal work, the set of undressed mesons (which is formed by just adjoint chiral fields) should satisfy the following constraints: (a) The dual theory has a finite number of gauge singlets, which means the set of letters involving u , v should be finite; i.e., there are truncations on the set of mesons. The superpotential W_E is useful in truncating the set of mesons. However, it is often necessary to use quantum constraints to truncate the mesons. (b) The R charges of the allowed mesons $U_I = u^{a_1}v^{b_1}u^{a_2}v^{b_2}\dots$ are paired:

$$R(U_I) + R(U_{I'}) = \Delta. \quad (3)$$

Here Δ is a fixed constant. This condition is required so that the added superpotential (2) is marginal. This superpotential is crucial as the equation of motion for the gauge singlet $[M_I]$ would set the composite $M_{I'}$ operator of the dual theory to be zero in the chiral ring. The duality then maps the composite chiral operator to an elementary chiral operator in the dual theory, and there are no composite chiral operators in the dual theory.

The rank N'_c of the dual gauge theory can be computed easily from the above general setup of the duality. First, let us fix the R charges of adjoint chiral fields u and v as $[u]$, $[v]$ [it might be fixed by a superpotential term $f(u, v)$ in W_E]. The anomaly-free condition for the $U(1)_R$ symmetry of electric theory is

$$([u] - 1) + ([v] - 1) + \frac{N_f}{N_c}(R_c - 1) + 1 = 0,$$

$$\Rightarrow R_c = 1 - ([u] + [v] - 1) \frac{N_c}{N_f}.$$

Now for the dual theory, one has the new superpotential term (2), which should also have R charge 2:

$$2R_c + R(U_I) + R(U_{I'}) + 2R_{c^*} = 2.$$

Using the pairing condition for the mesons, the R charge of the c^* field is

$$R_{c^*} = 1 - R_c - \frac{\Delta}{2}. \quad (4)$$

This charge might be negative, but it would not cause the problem as long as the gauge invariant operator has a positive R charge. We can now compute the rank N'_c of the dual gauge theory by requiring the anomaly-free condition of the dual $U(1)_R$ symmetry:

$$\begin{aligned} ([u] + [v] - 1) + \frac{N_f}{N'_c} (R_{c^*} - 1) &= 0, \\ \Rightarrow N'_c &= \frac{N_f(R_c + \frac{\Delta}{2})}{[u] + [v] - 1} \\ &= \frac{N_f - N_c([u] + [v] - 1) + N_f \frac{\Delta}{2}}{[u] + [v] - 1} \\ &= N_f \frac{2 + \Delta}{2([u] + [v] - 1)} - N_c = aN_f - N_c. \end{aligned}$$

The crucial number a is determined by the $U(1)_R$ charge of u , v and the pairing constant Δ of the meson spectrum:

$$a = \frac{2 + \Delta}{2([u] + [v] - 1)}. \quad (5)$$

The above formula makes sense if

$$[u] + [v] > 1. \quad (6)$$

The chiral spectrum is matched as follows: First, the original meson M_I is mapped to the gauge singlet $[M_I]$, and the extra meson $M_{I'}$ in the dual theory is projected out by using the added new superpotential term. The scaling dimensions of these chiral operators are mapped straightforwardly.

The baryon spectrum is mapped as follows [12]: One first has dressed quark $c_I = U_I c$, and the baryons are formed as $B^{l_1, \dots, l_a} = c_1^{l_1} \dots c_a^{l_a}$ with $\sum_{i=1}^a l_i = N_c$. They are simply built as the determinant of the matrix formed by dressed quarks. The total number of baryons is $C_{aN_c}^{N_c}$.

The dual baryon is formed by dual dressed quark $c_I^* = U_I c^*$, and $B^{\tilde{l}_1, \dots, \tilde{l}_a} = c_1^{*\tilde{l}_1} \dots c_a^{*\tilde{l}_a}$, with $\sum_{i=1}^a \tilde{l}_i = N'_c$. The number $\tilde{l}_i = N_f - l_{I^d}$, with I^d the paired meson for I .

Let us now verify that their R charges are the same (so the scaling dimension is also the same). The R charge for electric baryon $B^{l_1, \dots, l_a}$ is

$$\sum l_j R_j + N_c R_c.$$

The R charge for dual baryon $B^{\tilde{l}_1, \dots, \tilde{l}_a}$ is

$$\begin{aligned} \sum \tilde{l}_j R_j + N'_c R_{c^*} &= \sum (N_f - l_{j^d})(\Delta - R_{j^d}) \\ &\quad + (aN_f - N_c) \left(1 - R_c - \frac{\Delta}{2}\right) \\ &= N_c R_c + \sum l_j R_j. \end{aligned}$$

First, we used $N'_c = aN_f - N_c$, $R_{c^*} = 1 - R_c - \frac{\Delta}{2}$. Next we used $\sum_j R_j = \frac{a\Delta}{2}$, $R_c = 1 - \frac{xN_c}{N_f}$, and $a = \frac{\Delta+2}{2x}$, with $x = \sum_u (R_u - 1) + 1$ to verify the equality of the R charges of the baryons.

B. Truncation by superpotential

We have seen in the last subsection that $\mathcal{N} = 1$ duality might work if the set of mesons involving adjoint chiral fields takes a particular form of truncation. We would like to find all the consistent truncations satisfying the condition (3).

The truncation can be achieved by using a superpotential $f(u, v)$. The superpotential is assumed to be a trace on two matrices u, v . We take an algebraic approach by treating u, v as two noncommutative variables. Each term in the potential $f(u, v) = \sum_\alpha \phi_\alpha(u, v)$ has a cyclic equivalence (remember that the trace function on the matrices has this property). The derivative of the superpotential is defined as

$$f_u = \sum_\alpha \partial_u \phi_\alpha(u, v) = \sum_\alpha \left[\partial_u \left(\sum_{\text{cyc}} u \dots \right) \right] = \sum_\alpha \sum_{\text{cyc}} \dots \quad (7)$$

Here the sum is over the cyclic equivalence classes of ϕ_α such that the first term is u . We then have a noncommutative algebra

$$J_f = \frac{C[u, v]}{\{f_u, f_v\}}.$$

We are looking for f such that J_f is finite dimensional, and f is required to be a quasihomogeneous polynomial (the R charge of f is two) so that there is grading for field u and v .

A necessary condition for the classical truncation is that $f(u, v)$ should define an isolated singularity if one regards u, v as commutative variables, so that J_f is finite dimensional if it is regarded as a commutative Jacobian algebra. The constraint (6) would mean that $f(u, v)$ defines an isolated Du Val singularities of type A, D, E.³

³Usually the ADE singularity is defined as surface singularity $z^2 + f(x, y) = 0$ and the ADE condition implies $[z] + [x] + [y] > d$ assuming the weight of the polynomial is d . In our case, the weight of f is two, the weight of $[z]$ is one, and so the ADE condition implies the weights of $[x]$ and $[y]$ satisfy $[x] + [y] > 2 - [z] = 1$.

TABLE I. The summary of superpotential $f(u, v)$, which would give a good truncation of mesons. Quantum constraints need to be imposed for $D_k(k \text{ even})$ and E_7 .

Duality	$f(u, v)$	$([u], [v])$	Δ	N'_c
A_1	$u^2 + v^2$	$(1, 1)$	0	$N'_c = N_f - N_c$
A_k	$u^2 + v^{k+1}$	$(1, \frac{2}{k+1})$	$\frac{2k-2}{k+1}$	$N'_c = kN_f - N_c$
D_k	$u^{k+1} + uv^2$	$(\frac{2}{k+1}, \frac{k}{k+1})$	$\frac{4k-2}{k+1}$	$N'_c = 3kN_f - N_c$
E_7	$u^3 + uv^3$	$(\frac{2}{3}, \frac{4}{9})$	$\frac{14}{3}$	$N'_c = 30N_f - N_c$

However, not every ADE singularity gives rise to a good truncation if u, v are regarded as noncommutative variables. For example, the E_6 potential $f = u^3 + v^4$ would give the relation

$$u^2 = 0, \quad v^3 = 0.$$

As a commutative algebra, J_{E_6} is finite dimensional and has only six elements, which are given by $\{1, u, v, v^2, uv, uv^2\}$. However, as a noncommutative algebra, the J_f has an infinite number of elements, i.e., words like $uvuvuvuv \dots$ are nonzero simply because u, v are noncommutative.

The set of ADE singularities that do give consistent truncation is listed in Table I. Notice that for $E_7, D_k(k \text{ even})$, one needs to impose quantum constraints to get consistent truncation. One cannot get consistent truncation for E_6, E_8 superpotential.

Example: Consider D_k superpotential $f(u, v) = u^{k+1} + uv^2$, and the ideal of J_f is given [see Eq. (7)]:

$$\begin{aligned} \frac{\partial f}{\partial v} &= uv + vu = 0, \\ \frac{\partial f}{\partial u} &= (k+1)u^k + v^2 = 0. \end{aligned} \quad (8)$$

The first equation simply means that u and v are anti-commuting, and the second equation implies that u^k is the same as v^2 in the chiral ring. The following set is certainly the part of Jacobian algebra J_f :

$$\begin{aligned} &\{1, u, \dots, u^{k-1}\}, \\ &\{v, vu, \dots, vu^{k-1}\}, \\ &\{v^2, v^2u, \dots, v^2u^{k-1}\}. \end{aligned} \quad (9)$$

Let us now consider v^3 , which one gets from two equations in (8):

$$\begin{aligned} (k+1)(u^k v + vu^k) + 2v^3 &= 0 \\ \rightarrow v^3 &= \frac{-k-1}{2}(1 + (-1)^k)u^k v. \end{aligned}$$

So for k odd, we have $v^3 = 0$, and the algebra J_f has $3k$ elements [see (9)]. For k even, one needs to truncate the set by assuming quantum constraint $v^3 = 0$. It is easy to check that the mesons have the desired pairing, and the pairing constant is listed in Table I.

C. Truncation from equality of the superconformal index

Instead of using a superpotential to find the consistent truncation of the meson spectrum, one can find a large class of new examples by using the superconformal index [14,15]. See [16,17] for more details on the superconformal index.

In the large N_c, N_f limit, the equality of the index for two dual gauge theories in Fig. 1 takes the form

$$\frac{g_E \bar{g}_E - g_M \bar{g}_M}{(1-f)} = h_M - h_E.$$

Here f is the contribution of the vector multiplet and adjoint chiral, (g_E, g_M) are the contributions of the fundamental chirals, $(\bar{g}_E, \bar{g}_M)$ are the contributions of the anti-fundamental chirals, and (h_E, h_M) are the contributions of gauge singlets. The detailed formulas for various fields are given in Appendix C. Here E denotes the contribution in the electric frame, and M denotes the contribution in the magnetic frame. The equality of the index gives the equation

$$\begin{aligned} (t^{R_Q} - t^{2-R_Q})^2 - (t^{R_q} - t^{2-R_q})^2 &= \left(1 - t^2 - \sum_a (t^{R_a} - t^{2-R_a})\right) \left(\sum_j t^{2R_Q+R_j} - t^{2-(2R_Q+R_j)}\right), \\ \Rightarrow \frac{t^{2R_Q} + t^{4-2R_Q} - t^{2R_q} - t^{4-2R_q}}{(1 - t^2 - \sum_a (t^{R_a} - t^{2-R_a}))} &= \sum_j (t^{2R_Q+R_j} - t^{2-(2R_Q+R_j)}). \end{aligned} \quad (10)$$

Here $R_Q (R_q)$ is the R charge for the fundamental fields in an electric (magnetic) frame. If there is a pairing between the mesons $R_j + R'_j = \Delta$, then we have $\sum_j t^{R_j} = t^\Delta \sum_j t^{-R_j}$ and $R_Q + R_q = 1 - \frac{\Delta}{2}$ [see Eq. (4)]. We then have the equation

$$\frac{(t^{2R_Q} - t^{2R_q}) - t^{\Delta+2}(t^{2R_Q} - t^{2R_q})}{(1 - t^2 - \sum_a (t^{R_a} - t^{2-R_a}))} = [t^{2R_Q} - t^{2R_q}] \sum_j t^{R_j}.$$

So the equality of the index is satisfied if we have

$$\sum_{j=1}^{\alpha} t^{R_j} = \frac{t^{\Delta+2} - 1}{-1 + t^2 + \sum_a (t^{R_a} - t^{2-R_a})}. \quad (11)$$

We see that whether duality works depends on the factorization of the polynomial

$$y^n - 1 = \prod_{d|n} \Phi_d(y).$$

Here $\Phi_d(y)$ is the cyclotomic polynomial. To find a consistent truncation of the meson, one just needs to factorize the cyclotomic polynomial into a positive part (all the coefficients are positive) and the negative part [15]:

$$y^n - 1 = \Phi_+ \Phi_-.$$

From the factorization, one can find the data (Δ, R_a, R_j) as follows. First, assuming the highest order of Φ_- is n_1 , which is paired with the constant term -1 , we can find the normalization constant $\delta = \frac{2}{n_1}$. Then we have

- (1) The pairing constant is $\Delta = n\delta - 2 = \frac{2}{n_1}(n - n_1)$.
- (2) The R charges R_a for the adjoint fields u_a are found from the positive exponents $(c_1, c_2, \dots)$ of Φ_- as $(c_1\delta, c_2\delta, \dots)$.
- (3) The R charges R_j for the mesons built from adjoint fields are found from the exponents $(d_0, d_1, \dots)$ of Φ_+ as $(d_0\delta, d_1\delta, \dots)$.

This is the complete set of data needed for establishing $\mathcal{N} = 1$ duality. Here are some remarks:

- (1) Although the equality of the index is computed in the large N_c, N_f limit, one can check that the central charges agree for the finite N_c, N_f value (see Sec. II E).
- (2) Once we find the R charges for the adjoint fields, it might be possible to add the marginal term for the adjoint fields u_a . One certainly recovers the known ADE superpotential as a small subset. However, one should not expect to get the mesonic spectrum from

the classical relation of the superpotential. Rather, one may simply impose the quantum constraint by hand.

- (3) The construction is symmetric under the exchange of electric and magnetic form, and so the duality is an involution. We will discuss more about this in the next subsection.
- (4) Notice that here we only get the R charges of the mesons, and knowledge about how they are formed from the adjoint chiral fields u_a is not clear: (a) there might be more than one possible combination of a given R charge R_j ; (b) even if the combination of the u_a field is given, the ordering of the fields u_a is not clear as they are noncommutative variables.

All possible factorizations of $y^8 - 1$ are shown in Table II.

Remark: For the given ADE R_a charges of the adjoint fields listed in Table I, it is possible to find an infinite sequence of R_j charges and the pairing constant Δ_n^{ADE} . These may be thought of as the rank n version of ADE theory. For example, if $R(u) = \frac{2}{3}$, then the right-hand side of Eq. (11) becomes $\frac{t^{\Delta+2}-1}{-1+t^2+t^{2/3}-t^{4/3}} = \frac{y^{\frac{3\Delta}{2}+3}-1}{-1+y^3+y-y^2} = \frac{y^{\frac{3\Delta}{2}+3}-1}{\Phi_1\Phi_4}$, so as long as $\frac{3\Delta}{2} + 3$ has a divisor 4, one can have a consistent factorization. It would be interesting to realize these spectrums from some physical models.

D. More dualities

More generally, one may assume that there are already gauge singlets in the electric frame, and the right-hand side of Eq. (10) becomes

$$\sum_j (t^{2R_Q+R_j^M} - t^{2-(2R_Q+R_j^M)}) \quad (12)$$

$$- \sum_l (t^{2R_Q+R_l^E} - t^{2-(2R_Q+R_l^E)}). \quad (13)$$

To get good duality, we first assume that the R charges of fundamental fields satisfy $R_Q + R_q = 1 - \frac{\Delta}{2}$. The above equation becomes

TABLE II. The possible products of cyclotomic polynomials with all coefficients positive $\Phi_n^+(y)$ and the corresponding antipalindromic polynomials $\Phi_n^-(y) = (y^n - 1)/\Phi_n^+(y)$ for $n = 8$.

$\Phi_8^+(y)$	$\Phi_8^-(y)$	R_a, R_j
$\Phi_2(y) = y + 1$	$\Phi_1(y)\Phi_4(y)\Phi_8(y) = y^7 - y^6 + y^5 - y^4 + y^3 - y^2 + y - 1$	$(\frac{2}{7}, \frac{6}{7}, \frac{10}{7}), (\frac{2}{7}, 0)$
$\Phi_4(y) = y^2 + 1$	$\Phi_1(y)\Phi_2(y)\Phi_8(y) = y^6 - y^4 + y^2 - 1$	$(\frac{2}{3}), (\frac{2}{3}, 0)$
$\Phi_8(y) = y^4 + 1$	$\Phi_1(y)\Phi_2(y)\Phi_4(y) = y^4 - 1$	$(1), (2, 0)$
$\Phi_2(y)\Phi_4(y) = y^3 + y^2 + y + 1$	$\Phi_1(y)\Phi_8(y) = y^5 - y^4 + y - 1$	$(\frac{2}{5}), (\frac{6}{5}, \frac{4}{5}, \frac{2}{5}, 0)$
$\Phi_2(y)\Phi_8(y) = y^5 + y^4 + y + 1$	$\Phi_1(y)\Phi_4(y) = y^3 - y^2 + y - 1$	$(\frac{2}{3}), (\frac{10}{3}, \frac{8}{3}, \frac{2}{3}, 0)$
$\Phi_4(y)\Phi_8(y) = y^6 + y^4 + y^2 + 1$	$\Phi_1(y)\Phi_2(y) = y^2 - 1$	$(1), (6, 4, 2, 0)$
$\Phi_2(y)\Phi_4(y)\Phi_8(y) = y^7 + y^6 + y^5 + y^4 + y^3 + y^2 + y + 1$	$\Phi_1(y) = y - 1$	$(1), (14, 12, 10, 8, 6, 4, 2, 0)$

$$\sum_j (t^{2R_Q+R_j^M} - t^{2-(2R_Q+R_j^M)}) \quad (14)$$

$$+ \sum_l (t^{2R_Q+\Delta-R_l^E} - t^{2-(2R_Q+\Delta-R_l^E)}). \quad (15)$$

Now define $R_l' = \Delta - R_l^E$, and the above equation would take the same form as (10). Now the electric and magnetic gauge singlets are found from the set $\{R_j\}$ in the factorization as follows:

- (1) First find a subset $\{R_j^M\}$ from the solution, and the R charge of the magnetic gauge singlet is given by $2R_Q + R_j^M$.
- (2) For the remaining elements in $\{R_j^E\} = \{R_j\} \setminus \{R_j^M\}$, the R charge of the electric gauge singlet is given by $2R_Q + \Delta - R_j^E$.

To make the duality work, one needs to add the superpotential term as follows:

$$W_E = \sum [M_l^E] M_{l^d}.$$

Here M_{l^d} is the composite meson built from the electric fundamental degree of freedoms. This superpotential would couple the gauge singlet to the fundamental fields so that the electric theory is an irreducible theory.

Example: Let us take $f = u^5$. The set of basic mesonic R charges is $R_j = \{1, u, u^2, u^3\}$. Let us take the magnetic part of gauge singlets as $R_j^M = \{u^2, u^3\}$, and the electric part as $R_j^E = \{1, u\}$. So the electric gauge singlets are $[c^*u^3d^*]$ and $[c^*u^2d^*]$ (we use these letters to indicate their R charges). An electric superpotential is

$$W_E = [c^*u^3d^*]cd + [c^*u^2d^*]cud,$$

so that the singlet is coupled with the gauge theory. Now let us do duality following the general rule, and there would be four more new magnetic mesons in the magnetic frame. The superpotential takes the form

$$W_M = [c^*u^3d^*][cd] + [c^*u^2d^*][cud] + \dots$$

The singlets $[c^*u^3d^*], [cd], [c^*u^2d^*], [cud]$ become massive. So we are left with two magnetic gauge singlets $[cu^2d], [cu^3d]$, which agree with our setup. See Fig. 2 for the illustration of this duality model.

Self-dual model: It is now possible to have a self-dual model; i.e., the dual gauge group and matter contents are the same as the original theory. The condition on N_f, N_c is $aN_f - N_c = N_c$. We also have the same R charge for the bifundamental fields: $R(c) = R(c^*) = \frac{1}{2}(1 - \frac{\Delta}{2})$. We also arrange the electric and magnetic gauge singlets symmetrically. Then we have the following possibilities: (a) A_1 : $\Delta = 0$, and so $R(c) = R(c^*) = \frac{1}{2}$, $N_f = 2N_c$; (b) A_k : $\Delta = \frac{2k-2}{k+1}$, so $R(c) = R(c^*) = \frac{1}{k+1}$, and $N_f = \frac{2N_c}{k}$. The gauge singlets are $[cd], \dots, [cu^{\lfloor \frac{k-1}{2} \rfloor}d]$. The superpotential may be used to project some mesons out.

E. General solutions

Every factorization of $y^n - 1$ would give rise to a possible duality pair: one has complete electric and magnetic descriptions. However, one often could not find a superpotential so that the truncation is derived from the equation of motion of it. One should also be aware that even for E_7 and D_k (k even) model, one needs to truncate the meson spectrum by hand [5,12]; namely a quantum constraint is needed. However, there is little if any understanding about how such quantum constraint arises.

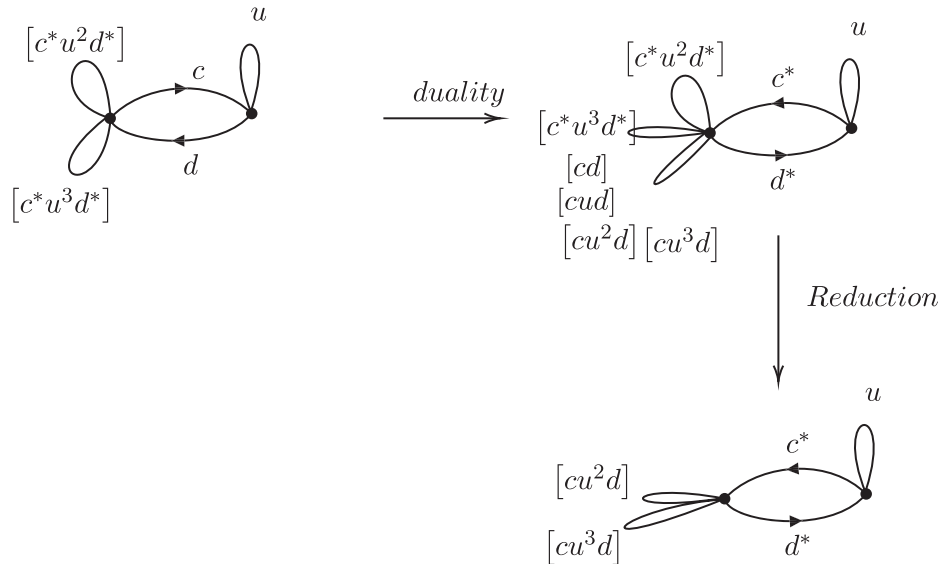


FIG. 2. Duality for superpotential $f = u^5$ with the magnetic part of gauge singlets $R_j^M = \{u^2, u^3\}$ and the electric part $R_j^E = \{1, u\}$.

The author of [15] proposed superpotential for some models from the factorization. However, the potential does just reflect the R charges of adjoint fields, and the proposed superpotential cannot give the truncation by using the equation of motion. Probably one should not take such superpotential too seriously.

Our point of view is that one should not be constrained by the existence of superpotential. Instead, the gauge theory studied above might just be part of a larger gauge theory, and the adjoint fields us could be the composite of other fundamental fields. The truncation on the mesonic spectrum could be due to some unknown quantum constraints. Of course, it would be interesting to identify specific models with these more general solutions. In any case, we just take the specified mesonic spectrum as the input of our theory.

Example: Let us discuss a model derived from one factorization, and we will show how the duality in this situation would work. Let us take the factorization $y^4 - 1 = (y^2 - 1)(y^2 + 1)$, so $\Delta = 2$ and there is no adjoint field, and $R_j = \{0, 2\}$. It is not straightforward to form the mesons from the elementary field. However, we may add an adjoint field u with R charge 1, and now the set of mesons is assumed to be cd, cu^2d . The dual theory works similarly, and a dual superpotential is formed:

$$W = [cd]c^*u^2d^* + [cu^2d]c^*d^*.$$

In the dual theory, the composite meson $c^*u^2d^*, c^*d^*$ are

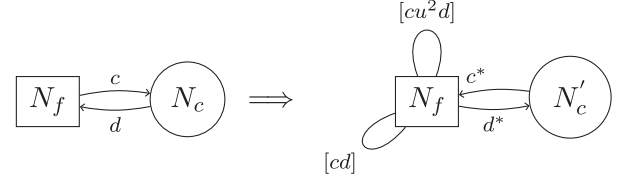


FIG. 3. A duality for a special set of the mesonic spectrum derived from factorization $y^4 - 1 = (y^2 - 1)(y^2 + 1)$.

not in the chiral ring. We then have the duality as shown in Fig. 3. The rank of the dual gauge group is $2N_f - N_c$.

We have verified that the chiral spectrum of the dual pair matches; see the discussion at the end of Sec. II A. Let us now verify that for the general duality proposal based on the factorization of the polynomial $y^n - 1$. The $\text{Tr } R$ and $\text{Tr } R^3$ anomalies match, which would ensure that the central charges match [18]. The anomaly vanishing conditions of the original and dual gauge theories are [here we take the gauge group as $U(N_c)$ without losing any generality]

$$\begin{aligned} N_c \left[\sum_u (R_u - 1) + 1 \right] + (R_c - 1)N_f &= 0, \\ N'_c \left[\sum_u (R_u - 1) + 1 \right] + (R_{c^*} - 1)N_f &= 0. \end{aligned}$$

The original and dual R anomalies are

$$\begin{aligned} \text{Tr } R &= N_c^2 \left[\sum_u (R_u - 1) + 1 \right] + 2N_c N_f (R_c - 1) = N_c N_f (R_c - 1), \\ \text{Tr } R_d &= N_c'^2 \left[\sum_u (R_u - 1) + 1 \right] + 2N'_c N_f (R_{c^*} - 1) + N_f^2 \sum_j [(2R_c + R_j) - 1] \\ &= N'_c N_f (R_{c^*} - 1) + N_f^2 \left[2aR_c + \frac{\Delta}{2}a - a \right]. \end{aligned} \quad (16)$$

Substitute $N'_c = aN_f - N_c$, $R_{c^*} = 1 - R_c - \frac{\Delta}{2}$, we have

$$\begin{aligned} \text{Tr } R_d &= (aN_f - N_c)N_f \left[-R_c - \frac{\Delta}{2} \right] + N_f^2 \left[2aR_c + \frac{\Delta}{2}a - a \right] \\ &= N_c N_f (R_c - 1) + aN_f^2 (R_c - 1) + \frac{\Delta N_c N_f}{2} + N_c N_f \\ &= N_c N_f (R_c - 1) + N_c N_f \left[1 + \frac{\Delta}{2} - a \left[\sum_u (R_u - 1) + 1 \right] \right] \\ &= N_c N_f (R_c - 1). \end{aligned}$$

Here we used the equation $R_c = 1 - [\sum_u (R_u - 1) + 1] \frac{N_c}{N_f}$, and the equation of a (5).

The original and dual R^3 anomalies are

$$\begin{aligned}\text{Tr } R^3 &= N_c^2 \left[\sum_u (R_u - 1)^3 + 1 \right] + 2N_c N_f (R_c - 1)^3 \\ &= N_c^2 \left[\sum_u (R_u - 1)^3 + 1 \right] - \frac{2N_c^4}{N_f^2} \left(\sum_u (R_u - 1) + 1 \right)^3, \\ \text{Tr } R_d^3 &= N_c'^2 \left[\sum_u (R_u - 1)^3 + 1 \right] + 2N_c' N_f (R_{c^*} - 1)^3 + N_f^2 \sum_j [(2R_c + R_j - 1)^3].\end{aligned}$$

Substitute $N_c' = aN_f - N_c$, $R_{c^*} = 1 - R_c - \frac{\Delta}{2}$. Define $x = \sum_u (R_u - 1)^3 + 1$, $y = \sum_j (R_j - 1)$, $z = \sum_j (R_j - 1)^2$, $w = \sum_j (R_j - 1)^3$. If these numbers satisfy the following equation:

$$y = \frac{1}{2}a(\Delta - 2), \quad z = \frac{a(-2ax + \Delta^3 + 8)}{3(\Delta + 2)}, \quad w = \frac{a(-4a(\Delta - 2)x + \Delta^4 - 16)}{4(\Delta + 2)}, \quad (17)$$

then we also have the equality of the cubic anomalies. We have verified that $\text{Tr } R^3 = \text{Tr } R_d^3$ match for many factorizations of polynomial $y^n - 1$,⁴ and it would be interesting to prove the above identity exactly.

Let us check that other global anomalies also match. It is easy to check that $\text{Tr } SU(N_f)^3 = 0$ and $\text{Tr } SU(N_f)^2 R^2 = 0$ agree for both sides. The last global anomaly that remains to be checked is

$$\text{Tr } SU(N_f)^2 R = \sum_i (R_i - 1) \text{ind}.$$

Here *ind* is the Dynkin index for the representation under the flavor symmetry. The original and dual $\text{Tr } SU(N_f)^2 R$ anomalies are

$$\begin{aligned}\text{Tr } RFF &= 2N_c(R_c - 1)\frac{1}{2} = N_c(R_c - 1); \\ \text{Tr } R_d F_d F_d &= 2N_c'(R_{c^*} - 1)\frac{1}{2} + N_f \sum_j (2R_c + R_j - 1) \\ &= N_c'(R_{c^*} - 1) + a(\Delta/2 + 2R_c - 1)N_f.\end{aligned}$$

One can check that they are equal from the equality of the $\text{Tr } R$ anomaly [see (16)].

There is also a baryon number $U(1)_B$ symmetry. The baryonic charge for quarks c_i and antiquarks d_i is normalized to be ± 1 . Since the duality maps baryons $B^{l_1 \cdots l_a}$ to $\tilde{B}^{\tilde{l}_1 \cdots \tilde{l}_a}$ and $\sum_i l_i = N_c$, $\sum_i \tilde{l}_i = N_c'$. Therefore the dual quarks c_i^* should have the normalized $U(1)_B$ charges $\frac{N_c}{N_c'} = \frac{N_c}{aN_f - N_c}$.

It is easy to see $\text{Tr } U(1)_B R R = \text{Tr } U(1)_B^3 = 0$ in both electric and magnetic theories. The nonzero anomalies are

$\text{Tr } RU(1)_B^2$, and the computations are

$$\begin{aligned}\text{Tr } RU(1)_B^2 &= 2N_f N_c (R_c - 1) = -2N_c^2 x, \\ \text{Tr } R_d U(1)_B^2 &= 2N_f N_c' (R_{c^*} - 1) \left(\frac{N_c}{aN_f - N_c} \right)^2 \\ &= 2N_f (aN_f - N_c) \left(-R_c - \frac{\Delta}{2} \right) \left(\frac{N_c}{aN_f - N_c} \right)^2 \\ &= -2N_c^2 x.\end{aligned}$$

Here we used the formula $a = \frac{\Delta+2}{2x}$, $R_c = 1 - x \frac{N_c}{N_f}$, with $x = \sum_u (R_u - 1) + 1$.

III. DUALITY FOR SIMPLE GAUGE GROUP

The analysis of duality for other simple gauge groups can be carried out in a similar way. The basic idea is the same as discussed in the last subsection: the composite mesons of one theory are mapped to the gauge singlets of the dual theory, and often an extra dual superpotential is needed.

To make the duality work, we also find that the R charges for tensorial fields and the meson spectrum should also satisfy Eq. (3). However, we often have more constraints, namely, sometimes a $\mathbb{Z}_2$ action is needed to further separate the mesons.

A. Classical gauge group with adjoint chiral

Let us now assume that the gauge group is of the classical type, and the matter is in fundamental or adjoint representation. The main difference is that the matter transforms in the real (or pseudoreal) representation of the gauge group, so one does not need to add an arrow for the matter in fundamental representation. The original theory and its dual are shown in Fig. 4. The dual gauge group is determined by using an anomaly-free condition for

⁴The *Mathematica* code is openly available at Zenodo [19].

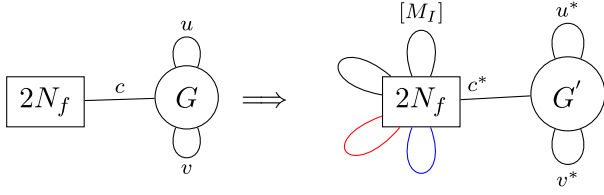


FIG. 4. Duality for classical gauge group with adjoint chiral. The gauge singlets in the dual theory could be in either symmetric or antisymmetric representation of the flavor group.

the $U(1)_R$ charge:

$$h^\vee(1 + [u] - 1 + [v] - 1) + N_f^*(R(c) - 1) = 0,$$

$$h^\vee(1 + [u] - 1 + [v] - 1) + N_f^*(R(c^*) - 1) = 0.$$

Here $h^\vee$ is the dual Coxeter number. Now assume that the adjoint chirals have a truncation, and so the R charges have a pairing $R_j + R_{j'} = \Delta$. The existence of a dual superpotential would relate the R charges of the fundamental fields as $R(c) + R(c^*) = 1 - \frac{\Delta}{2}$. Here N_f denotes the number of hypermultiplets (a pair of fundamental chiral fields).

The dual Coxeter number of the dual gauge group is then

$$h^{\vee'} = aN_f^* - h^\vee,$$

and a is determined by the pairing constant of mesons and R charges of $[u], [v]$:

$$a = \frac{2 + \Delta}{2([u] + [v] - 1)}. \quad (18)$$

And $N_f^* = N_f \times b$, where N_f denotes the number of hypermultiplets (a pair of fundamental fields), $b = 1$ for A, C gauge groups, and $b = 2$ for B, D gauge groups. The change of the gauge group is summarized in Table III.

To discuss the detailed duality, one needs to study the transformation behavior of the dressed meson under the flavor symmetry.

Example: Let us consider $SO(N_c)$ type gauge group, and the fundamental representation is denoted as V . The adjoint representation is formed from the antisymmetric tensor

TABLE III. Gauge groups of the original theory and dual theory.

Gauge group	Dual gauge group
$SU(N_c)$	$SU(aN_f - N_c)$
$SO(2N_c)$	$SO(2aN_f - 2N_c + 4)$
$SO(2N_c + 1)$	$SO(2aN_f - 2N_c + 3)$
$US_p(2N_c)$	$US_p(2aN_f - 2N_c - 4)$

product of $V: \wedge^2 V$. The basic invariant is $v^T v$ built from fundamental representation v . The dressed mesons are formed as $v^T U_I v$. When there is just one adjoint chiral u , it is easy to find the transformation behavior of the dressed meson under the flavor group: i.e., $v^T u^i v$ is a symmetric (antisymmetric) tensor of the flavor group if i is even (odd). For the $Sp(N_c)$ gauge group, the adjoint representation is the symmetric power of the fundamental representation $Sym^2 V$, and the invariant is $v^T J u^i v$, with $J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$. It is then straightforward that $v^T J u^i v$ is a symmetric (antisymmetric) tensor of the flavor group if i is even (odd).

For more than one adjoint chirals, to make duality work, one needs to define an involution on the set of mesons from the fundamental solutions: the meson is classified as positive type or negative type. There are $\frac{a+1}{2}$ ($\frac{a-1}{2}$) mesons with a positive (negative) sign for the $SO(N_c)$ gauge group. For the $Sp(N_c)$ gauge group, there are $\frac{a-1}{2}$ ($\frac{a+1}{2}$) mesons with a positive (negative) sign. Here a is the total number of mesons.

We can also distribute electric and magnetic gauge singlets to get more dualities as we did for the SU type theory in the last section.

The detailed computations for the R anomalies can be found in Appendix B 1.

B. General representations

Let us now consider a simple gauge group coupled with general representations. The representations are constrained so that there is no gauge anomaly [the local gauge anomaly exists only for the $SU(N)$ gauge group [20], and the global anomaly exists for the $Sp(2N)$ gauge group [21]].

The anomaly-free condition for the $U(1)_R$ symmetry is

$$\sum_a \text{ind}(r_a)([u_a] - 1) + \sum_i \text{ind}(r_i)(R(r_i) - 1) + h^\vee = 0.$$

Here r_a could be the general tensor (most often the two tensor) representation of gauge groups. The second sum is over the representation whose index is small, i.e., the defining representation. And $h^\vee$ is the dual Coxeter number. The index for some familiar representations of classical groups is shown in Table IV.

The duality often works as follows: First, one assumes that the dual theory involves the conjugate representation of

TABLE IV. Dynkin indices for some common representations of the classical group. The representation besides the adjoint representation is not irreducible (often one needs to subtract a trivial representation to get an irreducible representation).

Group/Index	Adjoint = $h^\vee$	Symmetric	Antisymmetric	Fundamental
$SU(N)$	N	$\frac{N}{2} + 1$	$\frac{N}{2} - 1$	$\frac{1}{2}$
$USp(2N)$	$N + 1$	Adj	$N - 1$	$\frac{1}{2}$
$SO(2N)$	$2N - 2$	$2N + 2$	Adj	1
$SO(2N + 1)$	$2N - 1$	$2N + 3$	Adj	1

the electric theory, and the gauge group is also the same type of the electric theory. The similar anomaly-free condition of the $U(1)_R$ charge of the dual theory is

$$\sum_a \text{ind}(r_a^*)([u_a] - 1) + \sum_i \text{ind}(r_i^*)(R(r_i^*) - 1) + h^\vee = 0.$$

Second, the dual theory should have gauge singlets that are identified with the composite gauge invariant chiral mesons of the original theory. Therefore, one needs to study the invariant theory involving the fundamental fields and tensorial fields, which may be solved using Weyl's invariant theory for the classical group case [22]. The case of exceptional group would be discussed in the next subsection. The meson spectrum is formed by two vector fields, and the fields involving tensor fields u_a . They would transform differently under the flavor group depending on the property of the tensor fields.

Once we find out the space of spectrum (again assuming the truncation of the meson spectrum to a finite set), a dual superpotential involving the gauge singlets is added so that the composite gauge invariant operators are projected out. The marginal condition for the added superpotential is then

$$2R(r_i) + 2R(r_i^*) + R(U_I) + R(U_{I'}) = 2.$$

Now assuming the basic meson involving the tensorial field has the pairing $R(U_I) + R(U_{I'}) = \Delta$, so

$$R(r_i^*) + R(r_i) = 1 - \frac{\Delta}{2}.$$

We have the equation for the dual Coxeter numbers

$$\begin{aligned} h^\vee + \sum_a \text{ind}(r_a)([u_a] - 1) + h^\vee \\ + \sum_a \text{ind}(r_a^*)([u_a] - 1) + \sum_i \text{ind}(r_i) \left(-1 - \frac{\Delta}{2} \right) = 0. \end{aligned} \quad (19)$$

Here we assume that the index of the vector representation does not depend on the gauge group (this is true for the classical group, as we assume the dual gauge group is of the same type).

From this formula, it is clear that the dual gauge group is determined by the R charges of tensorial fields $[u_a]$, and the pairing constant Δ . Here more data are needed, one needs to have an extra involution on the set of mesons to indicate the transformation property of the dressed meson under the flavor symmetry group.

Example: Let us consider $SO(2N_c)$ coupled with a traceless symmetric tensor X and N_f hypermultiplets in fundamental representation, and a superpotential $\text{Tr } X^{k+1}$ is added. The dual gauge group is found from the equations above:

$$\begin{aligned} 2N_c - 2 + (2N_c + 2) \left[\frac{2}{k+1} - 1 \right] \\ + 2N'_c - 2 + (2N'_c + 2) \left[\frac{2}{k+1} - 1 \right] \\ + 2N_f \left(-1 - \frac{k-1}{k+1} \right) = 0, \\ \Rightarrow 2N'_c = 2kN_f + 4k - 2N_c. \end{aligned}$$

This duality has been studied in [6]. More general classes are discussed in Appendix B 2.

General case: In the above analysis, one assumes that the dual gauge group is the same type as the original gauge group. It might be possible that the dual theory takes a different form. The duality works similarly: the composite gauge invariant operator of the electric theory is mapped to the elementary (gauge singlet) field of the dual theory. Often a superpotential is needed to project the composite chiral operator of the dual theory out. Again, a detailed knowledge of the chiral spectrum (such as the truncation to a finite set) is crucial to identify the dual theory. See [23] for some examples, and certainly more examples can be found by considering the truncation of tensorial fields discussed in the last section.

Self-dual example: One may engineer the matter representations so that the gauge groups are not changed; see [24,25] for some examples. It seems possible to find more examples using our general analysis in the last section.

C. Exceptional gauge group

For an exceptional group, one mostly needs to consider only the fundamental representation F as other representations have large indexes. The basic idea of duality is still mapping the composite chiral operators in electric theory to the elementary field in the magnetic theory. The first question is to classify the chiral spectrum, which is the study of the invariant theory for exceptional group. As in the classical group case, one can have invariant built from a fundamental field and an antifundamental field (mesons), or the invariant built from the determinant (baryons). There are more possibilities for the exceptional groups (see [26]). For example, one could have operators built from three fundamental fields for the E_6 group. This makes the study of duality more complicated; i.e., the dual gauge group is often not of the same type.

We may consider the theory where the dual gauge theory has the same type as the electric theory. The duality should work in a similar way: one adds gauge singlet $[M]$ for the composite meson in the electric theory, and adds a superpotential to project out the composite meson in the dual gauge theory. The dual gauge group might be found from the anomaly-free condition of $U(1)_R$ charge, and the self-dual condition constrains the possible matter content of the electric theory.

Example: For the E_6 group, the index of fundamental representation is 3, and the index of adjoint representation is 12. Let us assume that there are n_f fundamentals F , and it is known that there is a gauge invariant term F^3 . The duality changes the representation F to the dual representation F^* . To find the dual gauge group, one needs to modify the relation between the R charges of the fundamental fields, as now the basic invariant is a cubic invariant. According to the duality proposal, there should be a superpotential term in the magnetic description

$$W = [F^3]F^{*3}.$$

Here $[F^3]$ is the gauge singlet, and therefore the relation between R charges of F and F^* is changed as

$$R(F) + R(F^*) = \frac{2}{3}.$$

This is in contrast with the classical group case [compare (4)]. The Coxeter number of the dual gauge group takes the form

$$h^\vee = 4n_f - h^\vee.$$

When $n_f = 6$, one has the self-dual situation $h^{\vee'} = h^\vee$: one can add a marginal sextic interaction in the electric theory, and the theory is completely symmetric for electric and magnetic description. In this model, although the R charge of the fundamental fields is $\frac{1}{3}$, which is below the unitarity bound, the gauge invariant operator involves three fundamental fields and so its R charge is 1, which is above the unitarity bound. This model has been studied in [27]. More self-dual examples based on an exceptional gauge group were also studied in [28].

Let us now add one more adjoint chiral field u to the above model, so that the set of dressed mesons is $\{Fu^iF^2\}$. The dual theory should have a superpotential

$$W = \sum [Fu^iF^2]F^*u^jF^{*2}.$$

To make the duality work, the undressed mesons should be paired as $R(u^i) + R(u^j) = \Delta$, and so $R_F + R_{F^*} = \frac{2}{3} - \frac{\Delta}{3}$. The dual Coxeter number of the dual gauge group is

$$h^{\vee'} = n_f \frac{\Delta + 4}{R(u)} - h^\vee = (3k + 1)n_f - h^\vee.$$

Here we assume there is an A_k type potential for adjoint field u . It would be interesting to identify the dual theory for this more general model (notice that $k = 1, 4, 5, 7, 8, 11$).

Alternatively, we may have N_f pairs of fundamentals F and antifundamentals $\bar{F}$, and an adjoint u with an A_k type superpotential. So there could be gauge invariant $Fu^k\bar{F}$ besides the cubic invariant $F^3, \bar{F}^3$. We now add a similar dual potential as for the classical group case, and the data for the dual gauge group would be

$$h^{\vee'} = 6kN_f - h^\vee.$$

When $k = 1$ (no adjoint), one finds that $N_f = 4$ so that the dual gauge group has the same dual Coxeter number. This example is different from that in [27] as the duality maps gauge invariant $F\bar{F}$ to the gauge singlet here (the self-dual example in [27] maps a gauge invariant F^3 to a gauge singlet). When $k = 4$, $N_f = 1$, we also have a self-dual model.

IV. DUALITY FOR SEMISIMPLE GAUGE GROUPS

Let us now assume the gauge group is semisimple, i.e., the gauge group is a product of simple factors. There are two situations that one could consider: the first one regards several simple gauge groups as a single core, and there could be fields such as fundamental representations charged with individual gauge groups, and the duality acts on the gauge groups in the core simultaneously (see [6] for examples); the second one is more common: one could consider quiver gauge theory where the duality acts on each simple gauge group separately (see [29,30] for examples). The duality works in a similar way: the dual gauge groups are changed, and one adds the gauge singlet in the dual theory and a superpotential to project out the composite mesons. Again, the crucial condition is that the set of mesons is truncated, and there is a pairing between the meson spectrum. See Fig. 5 for the illustration of an

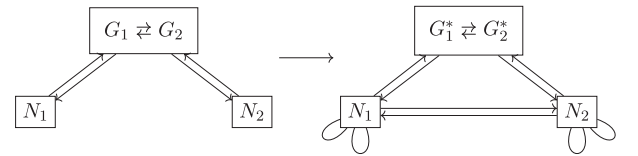


FIG. 5. The original theory has two gauge groups G_1, G_2 and two flavor groups denoted by N_1, N_2 . There are bifundamental matters between two gauge groups and fundamental charged over one of the gauge group. Dual theory has mesons as adjoint representations of two flavor groups and the bifundamental matter. The spectrum of gauge singlet is determined by the meson spectrum of original theory.

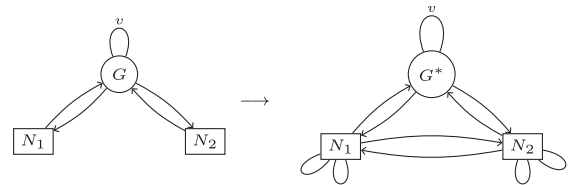


FIG. 6. The original theory has one gauge group G and two flavor groups denoted by N_1, N_2 which could be further gauged. Dual theory has mesons in adjoint representations of the $U(N_1)$ and $U(N_2)$ group, and there are also new fields in bifundamental representations of the $U(N_1)$ and $U(N_2)$ group.

example whose core has more than one gauge group, and Fig. 6 for the quiver gauge theory.

Using the result of the last sections, it is possible to have a lot more interesting dualities by considering other truncations. Some examples would be discussed in [31].

V. CONCLUSION

We discussed the general picture of duality of $\mathcal{N} = 1$ non-Abelian gauge theory. The main point is that the spectrum of undressed mesons should take a particular form to make the duality work. The spectrum is elegantly encoded in the factorization of polynomial $y^n - 1$ into a positive and a negative part $y^n - 1 = \Phi_+ \Phi_-$. It is interesting to find physical models with those spectrum.

We mainly discussed the combinatorial nature of duality, and it would be interesting to study the dynamical consequences of the dualities. Notice that the duality often works even if there are subtle dynamical questions such as the appearance of accidental symmetry, and the dual description could be useful to answer some of the difficult dynamical questions.

In this paper, we only study the duality of gauge theory coupled with free matters. It would be interesting to generalize the duality to $\mathcal{N} = 1$ gauge theory coupled with strongly coupled matters. Many interesting $\mathcal{N} = 1$ theories can be formed by relevant deformation of $\mathcal{N} = 2$ theories [32], and it is interesting to generalize the consideration of this paper to those more general cases. It is quite likely that the unusual meson spectrum studied in this paper could be realized in those models.

It would be interesting to generalize the consideration of this paper to three-dimensional (3D) $\mathcal{N} = 2$ theories.

ACKNOWLEDGMENTS

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DATA AVAILABILITY

The data that support the findings of this article are openly available in [19].

APPENDIX A: GAUGE INVARIANT OPERATORS

We summarize the results of the invariant theory of groups, from which one can get the generators for the gauge invariant operators for a single gauge group. For more details, see [33]. The basic invariants for the classical gauge groups with fundamental representations are

- (1) $SU(n)$ group: one needs to have both the fields ϕ in fundamental representation and u in antifundamental representations. The invariants are (u, ϕ) and the bracket denotes the Hermitian form;

TABLE V. The typical relations for the invariants of the classical group. Here the hat over a vector means the omission from the sequence. The Pfaffian of a matrix A is defined as $Pf(A)^2 = \det A$.

$SU(n)$	$\det((u_i, \phi_j))_{i,j=1}^n = \det(u_1, \dots, u_n) \det(\phi_1, \dots, \phi_n),$ $\Sigma_i (-1)^i \det(u_0, \dots, \hat{u}_i, \dots, u_n) (u_i, \phi) = 0,$ $\Sigma_i (-1)^i \det(\phi_0, \dots, \hat{\phi}_i, \dots, \phi_n) (u_i, \phi_i) = 0,$ $\Sigma_i (-1)^i \det(u_0, \dots, \hat{u}_i, \dots, u_n) \det(u_i, v_1, \dots, v_{n-1}) = 0,$ $\Sigma_i (-1)^i \det(\phi_0, \dots, \hat{\phi}_i, \dots, \phi_n) \det(\phi_i, \psi_1, \dots, \psi_{n-1}) = 0.$
$SO(n)$	$\det((u_i, v_j))_{i,j=0}^n = 0,$ $\det((u_i, v_j))_{i,j=1}^n = \det(u_1, \dots, u_n) \det(v_1, \dots, v_n).$
$Sp(n)$	$Pf((u_i, u_j))_{i,j=1}^{n+2} = 0.$

TABLE VI. Invariant tensors for fundamental representation of the exceptional group. Here d is the symmetric tensor, and f is the antisymmetric tensor.

Lie algebra	Invariant tensor
G_2	δ^{ab}, f^{abc}
F_4	δ^{ab}, d^{abc}
E_6	d^{abc}
E_7	f^{abc}, d^{abc}
E_8	δ^{AB}, f^{ABC}

and we also have the determinantal invariant $\det(u_1, u_2, \dots, u_n), \det(\phi_1, \phi_2, \dots, \phi_n).$

- (2) $SO(n)$ group: $v_i^T v_j$ and $\det(v_1, v_2, \dots, v_n).$

(3) $Sp(n)$ group: $v_i^T J v_j$ with J the matrix $J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$. There are also relations between these invariants, and they are summarized in Table V.

When there are tensor fields, one can build the invariants by first contracting the index and then forming the basic invariant. One uses metric g and g^{-1} in the SO type case to raise or lower the index, and antisymmetric form w (w^{-1}) in the Sp group case to raise or lower the index.

The invariant tensors for exceptional groups are listed in Table VI (see also [26]). The relations for these invariants are more complicated.

APPENDIX B: ANOMALY COMPUTATIONS FOR CLASSICAL GROUPS

In this appendix, we will compute the $\text{Tr } R$ and $\text{Tr } R^3$ anomalies for gauge theory with classical gauge group.

1. Adjoint matter

Let us do anomaly computation for the duality of classical gauge groups. We assume that the tensorial fields are all adjoint fields.

First, we have the anomaly-free condition for $U(1)_R$ symmetry:

$$h^\vee \left[\sum_u (R_u - 1) + 1 \right] + (R_c - 1)N_f^* = 0,$$

$$h'^\vee \left[\sum_u (R_u - 1) + 1 \right] + (R_{c^*} - 1)N_f^* = 0.$$

B, D gauge group. For the $SO(N_c)$ gauge group, define $x = \sum_u (R_u - 1) + 1$. We have

$$\text{Tr } R = x \dim G + 2N_f N_c (R_c - 1),$$

$$\text{Tr } R_d = x \dim G' + 2N_f N'_c (R_{c^*} - 1) + (2N_f^2 + N_f) \sum_{j \text{ even}} [(2R_c + R_j) - 1] + (2N_f^2 - N_f) \sum_{j \text{ odd}} [(2R_c + R_j) - 1].$$

Now assume that one can define a parity on the set of adjoint fields, and so the set of mesons built from an adjoint also has a parity: there are $\frac{a-1}{2}$ odd mesons (antisymmetric representation of a Sp flavor group), and $\frac{a+1}{2}$ even mesons (symmetric representation of Sp flavor group). We have

$$\begin{aligned} \text{Tr } R_d = x \dim G' + 2N_f N'_c (R_{c^*} - 1) + (2N_f^2 + N_f) & \left[\left(2R_c \frac{a+1}{2} + \Delta \frac{a+1}{4} \right) - \frac{a+1}{2} \right] \\ & + (2N_f^2 - N_f) \left[\left(2R_c \frac{a-1}{2} + \Delta \frac{a-1}{4} \right) - \frac{a-1}{2} \right]. \end{aligned}$$

Here $\dim G = \frac{1}{2}(N_c^2 - N_c)$, $N'_c = 2aN_f - N_c + 4$, and $R_{c^*} - 1 = -R_c - \frac{\Delta}{2}$. Using $R_c = 1 - \frac{(N_c-2)x}{2N_f}$, $a = \frac{\Delta+2}{2x}$, one can finally check that $\text{Tr } R_d = \text{Tr } R$.

Next we check that $\text{Tr } R^3 = \text{Tr } R_d^3$. The original and dual R^3 anomalies are

$$\begin{aligned} \text{Tr } R^3 &= \frac{1}{2} (N_c^2 - N_c) \left[\sum_u (R_u - 1)^3 + 1 \right] + 2N_c N_f (R_c - 1)^3, \\ \text{Tr } R_d^3 &= \frac{1}{2} (N_c'^2 - N'_c) \left[\sum_u (R_u - 1)^3 + 1 \right] + 2N'_c N_f (R_{c^*} - 1)^3 + (2N_f^2 + N_f) \sum_{j \text{ even}} [(2R_c + R_j) - 1]^3 \\ &+ (2N_f^2 - N_f) \sum_{j \text{ odd}} [(2R_c + R_j) - 1]^3. \end{aligned}$$

Substitute $N'_c = 2aN_f - N_c + 4$, $R_{c^*} = 1 - R_c - \frac{\Delta}{2}$. Define $x = \sum_u (R_u - 1)^3$; $y_1 = \sum_{j \text{ odd}} (R_j - 1)$, $y_2 = \sum_{j \text{ even}} (R_j - 1)$; $z_1 = \sum_{j \text{ odd}} (R_j - 1)^2$, $z_2 = \sum_{j \text{ even}} (R_j - 1)^2$; $w_1 = \sum_{j \text{ odd}} (R_j - 1)^3$; $w_2 = \sum_{j \text{ even}} (R_j - 1)^3$. Then we have $\text{Tr } R^3 = \text{Tr } R_d^3$ if these numbers satisfy the following equations:

$$\begin{aligned} y_1 &= \frac{1}{4}(a-1)(\Delta-2), & y_2 &= \frac{1}{4}(a+1)(\Delta-2), \\ z_1 &= -\frac{4a^2x + 4a^2 - 2a\Delta^3 - 12ax - 28a + 3\Delta^3 + 6\Delta^2 + 12\Delta + 24}{12(\Delta+2)}, \\ z_2 &= -\frac{4a^2x + 4a^2 - 2a\Delta^3 + 12ax - 4a - 3\Delta^3 - 6\Delta^2 - 12\Delta - 24}{12(\Delta+2)}, \\ w_1 &= -\frac{4a^2\Delta x + 4a^2\Delta - 8a^2x - 8a^2 - a\Delta^4 - 12a\Delta x - 12a\Delta + 24ax + 40a + 2\Delta^4 + 4\Delta^3 - 16\Delta - 32}{8(\Delta+2)}, \\ w_2 &= -\frac{4a^2\Delta x + 4a^2\Delta - 8a^2x - 8a^2 - a\Delta^4 + 12a\Delta x + 12a\Delta - 24ax - 8a - 2\Delta^4 - 4\Delta^3 + 16\Delta + 32}{8(\Delta+2)}. \end{aligned} \tag{B1}$$

One can check that other global anomalies also match.

C gauge group. For $Sp(N_c)$ gauge group, define $x = \sum_u (R_u - 1) + 1$. We have

$$\begin{aligned} \text{Tr } R &= x \dim G + 2N_f N_c (R_c - 1), \\ \text{Tr } R_d &= x \dim G' + 2N_f N'_c (R_{c^*} - 1) \\ &\quad + (2N_f^2 - N_f) \sum_{j \text{ even}} [(2R_c + R_j) - 1] \\ &\quad + (2N_f^2 + N_f) \sum_{j \text{ odd}} [(2R_c + R_j) - 1]. \end{aligned}$$

Now assume that one can define a parity on the set of adjoint fields, and so the set of mesons built from the adjoint also has a parity: there are $\frac{a-1}{2}$ odd mesons (symmetric representation of the flavor group), and $\frac{a+1}{2}$ even mesons (antisymmetric representation of the flavor group). We have

$$\begin{aligned} \text{Tr } R_d &= x \dim G' + 2N_f N'_c (R_{c^*}) \\ &\quad + (2N_f^2 - N_f) \left[\left(2R_c \frac{a+1}{2} + \Delta \frac{a+1}{4} \right) - \frac{a+1}{2} \right] \\ &\quad + (2N_f^2 + N_f) \left[\left(2R_c \frac{a-1}{2} + \Delta \frac{a-1}{4} \right) - \frac{a-1}{2} \right]. \end{aligned}$$

We have $\dim G = \frac{1}{2}(N_c^2 + N_c)$, $N'_c = 2aN_f - N_c - 4$, and $R_{c^*} - 1 = -R_c - \frac{\Delta}{2}$. Using $R_c = 1 - \frac{(N_c+2)x}{2N_f}$, $a = \frac{\Delta+2}{2x}$, one can finally check that $\text{Tr } R_d = \text{Tr } R$.

We can also check that $\text{Tr } R^3 = \text{Tr } R_d^3$ if these numbers $x, y_1, y_2, z_1, z_2, w_1, w_2$ satisfy the same equation (B1).

2. Other matter

a. B, D gauge group with symmetric matter

First, we have the anomaly-free condition for $U(1)_R$ symmetry:

$$\begin{aligned} h^\vee + (h^\vee + 4) \left[\sum_u (R_u - 1) \right] + (R_c - 1)N_f^* &= 0, \\ h^{\vee'} + (h^{\vee'} + 4) \left[\sum_u (R_u - 1) \right] + (R_{c^*} - 1)N_f^* &= 0. \end{aligned}$$

So we have

$$h^{\vee'} = aN_f^* - h^\vee - \frac{8x - 8}{x}.$$

Here $a = \frac{\Delta+2}{2x}$, $x = \sum_u (R_u - 1) + 1$. The anomalies are

$$\begin{aligned} \text{Tr } R &= \frac{1}{2}(N_c^2 - N_c) + \frac{1}{2}(N_c^2 + N_c) \sum_u (R_u - 1) + 2N_f N_c (R_c - 1), \\ \text{Tr } R_d &= \frac{1}{2}(N_c'^2 - N_c') + \frac{1}{2}(N_c'^2 + N_c') \sum_u (R_u - 1) + 2N_f N'_c (R_{c^*} - 1) \\ &\quad + (2N_f^2 + N_f) \sum_{j \in I} [(2R_c + R_j) - 1] + (2N_f^2 - N_f) \sum_{j \notin I} [(2R_c + R_j) - 1]. \end{aligned}$$

For A type potential, I is the whole set of mesons. For D type potential, I denotes the combination $u^a v^b$ such that ab is even.

More generally, assume that there are m symmetric mesons, and n antisymmetric mesons under the flavor symmetry. We may fix m, n by matching the anomalies. First, we need to have the match of the $\text{Tr } R$ anomaly:

$$\begin{aligned} \text{Tr } R_d &= \frac{1}{2}(N_c'^2 - N_c') + \frac{1}{2}(N_c'^2 + N_c') \sum_u (R_u - 1) + 2N_f N'_c (R_{c^*} - 1) \\ &\quad + (2N_f^2 + N_f)(m) \left[2R_c + \frac{\Delta}{2} - 1 \right] + (2N_f^2 - N_f)(n) \left[2R_c + \frac{\Delta}{2} - 1 \right]. \end{aligned}$$

The $\text{Tr } RUSp(2N_f)^2$ anomalies in electric and magnetic frames are

$$\begin{aligned} \text{Tr } RFF &= \frac{1}{2} N_c (R_c - 1), \\ \text{Tr } R_d F_d F_d &= \left(\frac{\Delta}{2} + 2R_c - 1 \right) (m(N_f + 1) + n(N_f - 1)) \\ &\quad + \frac{1}{2} N'_c (R_{c^*} - 1). \end{aligned}$$

By solving $\text{Tr } R = \text{Tr } R_d$ and $\text{Tr } RFF = \text{Tr } R_d FF$ at the same time, we find m, n :

$$\begin{aligned} m &= -\frac{-d + 2x - 6}{4x} = \frac{a}{2} + \frac{4 - 2x}{4x}, \\ n &= -\frac{-d - 2x + 2}{4x} = \frac{a}{2} - \frac{4 - 2x}{4x}. \end{aligned}$$

For A_k type potential, $x = \frac{2}{k+1}$, $a = k$, we have $m = k = a$, $n = 0$.

For D_k type potential, $x = \frac{1}{k+1}$, $a = 3k$, we have $m = \frac{5k+1}{2}$, $n = \frac{k-1}{2}$. When k is odd, it gives a possible duality. It is possible to find other solutions from the factorization of the polynomials $y^n - 1$.

b. C gauge group with antisymmetric matter

First, we have the anomaly-free condition for $U(1)_R$ symmetry:

$$h^\vee + (h^\vee - 2) \left[\sum_u (R_u - 1) \right] + (R_c - 1)N_f^* = 0,$$

$$h'^\vee + (h'^\vee - 2) \left[\sum_u (R_u - 1) \right] + (R_{c^*} - 1)N_f^* = 0.$$

So we have

$$h'^\vee = aN_f^* - h^\vee + \frac{4x - 4}{x},$$

$$R_c = 1 - \frac{(N_c + 2)x - 4(x - 1)}{2N_f}.$$

Here $a = \frac{\Delta+2}{2x}$, $x = \sum_u (R_u - 1) + 1$. So $N'_c = 2aN_f - N_c + \frac{8x-8}{x} - 4$. The anomalies are

$$\text{Tr } R = \frac{1}{2}(N_c^2 + N_c) + \frac{1}{2}(N_c^2 - N_c) \sum_u (R_u - 1) + 2N_f N_c (R_c - 1),$$

$$\text{Tr } R_d = \frac{1}{2}(N_c'^2 + N_c') + \frac{1}{2}(N_c'^2 - N_c') \sum_u (R_u - 1) + 2N_f N_c' (R_{c^*} - 1)$$

$$+ (2N_f^2 - N_f) \sum_{j \in I} [(2R_c + R_j) - 1] + (2N_f^2 + N_f) \sum_{j \notin I} [(2R_c + R_j) - 1].$$

For the A type potential, I is the whole set of mesons. For the D type potential, I denotes the combination $u^a v^b$ such that ab is even.

c. A gauge group with a pair of symmetric (and its conjugate) $(X, \tilde{X})$

First, we have the anomaly-free condition for $U(1)_R$ symmetry:

$$N_c + \left(\frac{N_c}{2} + 1 \right) \sum_u (R_u - 1) + (R_c - 1)N_f = 0,$$

$$N'_c + \left(\frac{N'_c}{2} + 1 \right) \sum_u (R_u - 1) + (R_{c^*} - 1)N_f = 0.$$

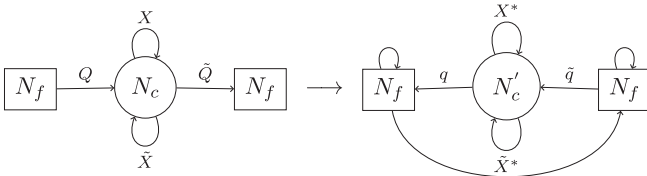


FIG. 7. A gauge group with a pair of symmetric and its conjugate matters.

We have

$$N'_c = aN_f - N_c - \frac{4x}{x+2},$$

$$R_c = 1 - \frac{N_c(1 + x/2) + x}{N_f}.$$

Here $x = \sum_u (R_u - 1)$ and $a = \frac{\Delta+2}{x+2}$. R_c and R_{c^*} are still paired as $R_c + R_{c^*} = 1 - \frac{\Delta}{2}$, and Δ is the pairing constant of the (undressed) mesons.

If there is a single pair of symmetric matter, with the superpotential $\text{Tr}(X\tilde{X})^{k+1}$, then the dressed mesons are $(M_j)^{f\dot{g}} \equiv Q^f (\tilde{X}X)^j \tilde{Q}^{\dot{g}}$, $j = 0, \dots, k$, $(P_r)^{f\dot{g}} \equiv Q^f (\tilde{X}X)^r \tilde{Q}^{\dot{g}}$, and $(\tilde{P}_r)^{\dot{f}g} \equiv \tilde{Q}^{\dot{f}} X (\tilde{X}X)^r \tilde{Q}^g$, $r = 0, \dots, k-1$, where M_j 's are adjoint and $P_r, \tilde{P}_r$'s are symmetric tensors of the flavor group. There are $k+1$ adjoint mesons and $2k$ symmetric mesons. The transformation property is that of the flavor group.

For Fig. 7 we have $x = 2R_u - 2$, $\Delta = 2kR_u = (2+x)k$, and thus $a = 2k + 1$.⁵ So the anomalies are

⁵In this case, the number a appeared in the dual gauge group rank is not equal to the number of mesons in the spectrum.

$$\begin{aligned}
\text{Tr } R &= N_c^2 + \frac{1}{2}(N_c^2 + N_c) \sum_u (R_u - 1) + 2N_f N_c (R_c - 1), \\
\text{Tr } R_d &= N_c'^2 + \frac{1}{2}(N_c'^2 + N_c') \sum_u (R_u - 1) + 2N_f N_c' (R_{c^*} - 1) + N_f^2 \sum_{\text{adjoint mesons}} (2R_c + R_j - 1) \\
&\quad + \frac{N_f^2 + N_f}{2} \sum_{\text{symmetric mesons}} (2R_c + R_j - 1) \\
&= N_c'^2 + \frac{1}{2}(N_c'^2 + N_c') \sum_u (R_u - 1) + 2N_f N_c' (R_{c^*} - 1) + N_f^2 (k+1) \left(2R_c - 1 + \frac{\Delta}{2} \right) \\
&\quad + \frac{N_f^2 + N_f}{2} 2k \left(2R_c - 1 + \frac{\Delta}{2} \right).
\end{aligned}$$

Then one can finally check that $\text{Tr } R_d = \text{Tr } R$.

d. A gauge group with pairs of antisymmetric (and its conjugate) $(X, \tilde{X})$

First, we have the anomaly-free condition for $U(1)_R$ symmetry:

$$\begin{aligned}
N_c + \left(\frac{N_c}{2} - 1 \right) \sum_u (R_u - 1) + (R_c - 1) N_f &= 0, \\
N_c' + \left(\frac{N_c'}{2} - 1 \right) \sum_u (R_u - 1) + (R_{c^*} - 1) N_f &= 0.
\end{aligned}$$

We have

$$\begin{aligned}
N_c' &= aN_f - N_c + \frac{4x}{x+2}, \\
R_c &= 1 - \frac{N_c(1+x/2) - x}{N_f}.
\end{aligned}$$

Here $x = \sum_u (R_u - 1)$ and $a = \frac{\Delta+2}{x+2}$. R_c and R_{c^*} are still paired as $R_c + R_{c^*} = 1 - \frac{\Delta}{2}$, and Δ is the pairing constant of the (undressed) mesons.

If there is a single pair of symmetric matter, with the superpotential $\text{Tr}(X\tilde{X})^{k+1}$, then the dressed mesons are $(M_j)^{f\dot{g}} \equiv Q^f (\tilde{X}X)^j \tilde{Q}^{\dot{g}}$, $j=0, \dots, k$, $(P_r)^{fg} \equiv Q^f (\tilde{X}X)^r \tilde{X}Q^g$, and $(\tilde{P}_r)^{\dot{f}\dot{g}} \equiv \tilde{Q}^{\dot{f}} X(\tilde{X}X)^r \tilde{Q}^{\dot{g}}$, $r=0, \dots, k-1$, where M_j 's are adjoint and $P_r, \tilde{P}_r$'s are antisymmetric tensors of the flavor group.

For Fig. 7 we have $x = 2R_u - 2$, $\Delta = 2kR_u = (2+x)k$, and thus $a = 2k+1$. So the anomalies are

$$\begin{aligned}
\text{Tr } R &= N_c^2 + \frac{1}{2}(N_c^2 - N_c) \sum_u (R_u - 1) + 2N_f N_c (R_c - 1), \\
\text{Tr } R_d &= N_c'^2 + \frac{1}{2}(N_c'^2 - N_c') \sum_u (R_u - 1) + 2N_f N_c' (R_{c^*} - 1) \\
&\quad + N_f^2 \sum_{\text{adjoint mesons}} (2R_c + R_j - 1) + \frac{N_f^2 - N_f}{2} \sum_{\text{anti-symmetric mesons}} (2R_c + R_j - 1) \\
&= N_c'^2 + \frac{1}{2}(N_c'^2 - N_c') \sum_u (R_u - 1) + 2N_f N_c' (R_{c^*} - 1) \\
&\quad + N_f^2 (k+1) \left(2R_c - 1 + \frac{\Delta}{2} \right) + \frac{N_f^2 - N_f}{2} 2k \left(2R_c - 1 + \frac{\Delta}{2} \right).
\end{aligned}$$

Then one can finally check that $\text{Tr } R_d = \text{Tr } R$.

APPENDIX C: FORMULA FOR SUPERCONFORMAL INDEX

The formula relevant for the computation of the large N_c , N_f index:

$$\begin{aligned}
 g_E &= \bar{g}_E = \frac{t^{R_c} - t^{2-R_c}}{(1-tx)(1-tx^{-1})}, \\
 g_M &= \bar{g}_M = \frac{t^{R_{c^*}} - t^{2-R_{c^*}}}{(1-tx)(1-tx^{-1})}, \\
 f &= \frac{2t^2 - t(x+x^{-1}) + \sum_{u=1}^{N_A} (t^{R_u} - t^{2-R_u})}{(1-tx)(1-tx^{-1})}, \\
 h(M_I) &= \frac{t^{2R_c+R_j} - t^{2-(2R_c+R_j)}}{(1-tx)(1-tx^{-1})}.
 \end{aligned}$$

Here R_c is the R charge for the fundamental fields c , and R_{c^*} is the R charge for the dual fundamental fields c^* . R_u 's are the R charges for the adjoint fields, and their values are the same for electric and magnetic theory. Finally, M_I denotes the dressed meson whose R charge is $2R_c + R_j$, with R_j the R charge for the meson u_I formed by adjoint chiral field.

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