

Unexpected symmetries of Kerr black hole scattering

Dogan Akpinar^{1,*} , Graham R. Brown^{1,†} , Riccardo Gonzo^{2,‡} , and Mao Zeng^{1,§} 

¹*Higgs Centre for Theoretical Physics, School of Physics and Astronomy,
University of Edinburgh, Edinburgh EH9 3FD, United Kingdom*

²*Centre for Theoretical Physics, Department of Physics and Astronomy,
Queen Mary University of London, Mile End Road, London E1 4NS, United Kingdom*



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Motivated by the recent introduction of the Dirac bracket framework to compute spinning observables for the scattering of Kerr black holes, we initiate the study of conserved quantities from an on shell amplitude perspective. We establish new results for the conservation of energy, angular momentum, the Rüdiger invariant, and the quadrupolar Carter constant using the spinning radial action extracted from the literature both in the probe limit and beyond, up to third post-Minkowskian order in the conservative sector. Furthermore, we offer a new perspective on the spin-shift symmetry of the radial action, clarifying its role in the dynamics. Finally, we define a new on shell notion of asymptotic integrability in the Liouville sense and present strong evidence that it is surprisingly satisfied by a spinning probe in Kerr up to quartic order in the probe spin, to all orders in the post-Minkowskian expansion. We further establish integrability beyond the probe limit at low post-Minkowskian orders. Our results suggest important new implications for the dynamics of Kerr black holes.

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Motivation and introduction. The groundbreaking detection of gravitational waves by the LIGO-Virgo-KAGRA Collaboration [1] has opened new avenues for exploring the Universe, enabling high-precision investigations of compact astrophysical objects such as black holes and neutron stars. As next-generation detectors extend observational capabilities, they demand substantial theoretical progress in our understanding of classical two-body dynamics [2].

A particularly promising approach has been the reformulation of the classical two-body problem within quantum field theory. In this setting, the *post-Minkowskian* (PM) expansion—where dynamics are treated perturbatively in Newton’s constant G —has proven fruitful, facilitating the use of advanced techniques from collider physics to tackle gravitational interactions. Remarkable progress has been achieved in the spinless sector, both through on shell scattering amplitude methods [3–6] and worldline methods [7]. Extending these approaches to include spin is essential for modeling astrophysical

systems. This challenge has spurred activity in both amplitude-based and worldline-based approaches; see Refs. [8–16].

The study of spinning binaries has a long and rich history within traditional perturbative approaches to general relativity, such as the post-Newtonian expansion in the slow-velocity regime and the gravitational self-force framework in the small mass-ratio limit. Analytical and numerical techniques have been developed in these complementary regimes, often aided by symmetries and the integrability of probe motion in Kerr spacetime [17,18]. For the latter, conservation of energy, azimuthal angular momentum and the Carter constant follow from Killing vector [19] and tensor [20] symmetries of the background spacetime. Such conservation laws have been extended to a spinning probe in Kerr, through the introduction of the Rüdiger constant at linear in spin order [21,22] and an extension of the Carter constant at quadrupolar order [23]; see Ref. [24] for a review. These conservation laws hold only for probes whose spin-induced multipole moments match those of a Kerr black hole, while for generic compact objects with generic Wilson coefficients—notably neutron stars—integrability is expected to be broken at quadratic order in spin.

Inspired by the simplicity of Kerr black hole dynamics, and given the role of such symmetries in both the reformulation of the self-force program in action-angle variables [25,26] and the development of the effective-one-body formalism for spinning binaries [27,28], one might ask whether this problem can be studied from the on

*Contact author: dogan.akpinar@ed.ac.uk

†Contact author: graham.brown@ed.ac.uk

‡Contact author: r.gonzo@qmul.ac.uk

§Contact author: mao.zeng@ed.ac.uk

shell scattering amplitude perspective. This question is motivated by the recent discovery of a hidden spin-shift symmetry—wherein the amplitude remains invariant under a shift of the spin vector along the momentum transfer—first identified at one-loop order [10,11], and later confirmed at two-loop order [14,29]. The origin of this symmetry remains to be understood.

A natural setup to answer these questions is the Dirac bracket framework [30] (see also Refs. [31–34]), which allows one to compute the variation of dynamical variables in the two-body spinning phase space for scattering orbits—including observables such as the impulse and spin kick—directly from the gauge-invariant radial action extracted through the amplitude-action relation [4,14,29,30,35]. In light of this, we start by defining conservation laws from the asymptotic scattering perspective, clarifying their relation with the usual conserved quantities in Kerr spacetime. After extracting from the literature the relevant spinning radial actions in the probe limit and beyond—up to 3PM at quartic in spin order—we establish the validity of such symmetries, comparing them with a reformulation of the spin-shift symmetry in phase space. Finally, we define a notion of asymptotic integrability, satisfied by a spinning probe in Kerr to all PM orders and for generic spinning binaries at low PM orders.

Conventions: In this Letter, we work in the negative metric signature $(+ - \dots -)$ with $\epsilon^{0123} = +1$ and $c = 1$.

Conserved quantities from the Dirac bracket formalism. In this section, we review the covariant Dirac bracket formalism discussed in Ref. [30] (see also Refs. [31,32]) which we use to define conserved charges for the scattering of Kerr black holes. We begin by specifying the relevant kinematic phase space for the classical gravitational scattering of two spinning bodies, given by $\mathcal{H}_{2B} = \{v_1^\mu, v_2^\mu, s_1^\mu, s_2^\mu, b^\mu\}$. Here, v_i^μ denotes the particle velocities, s_i^μ their normalized spin vectors, and $b^\mu = b_2^\mu - b_1^\mu$ is the initial impact parameter of the system. Then we consider the variation of an operator $\mathcal{Q}(v_i, s_i, b)$ within the Kosower-Maybee-O’Connell formalism [3]

$$\Delta \hat{\mathcal{Q}} = \hat{\mathcal{S}}^\dagger \hat{\mathcal{Q}} \hat{\mathcal{S}} - \hat{\mathcal{Q}} = \sum_{n=1}^{\infty} \frac{(-i)^n}{\hbar^n n!} \underbrace{[\hat{N}, [\hat{N}, \dots, [\hat{N}, \hat{\mathcal{Q}}]]]}_{n \text{ times}}, \quad (1)$$

where we have used the exponential representation of the $\mathcal{S}$ matrix: $\hat{\mathcal{S}} = \exp(i\hat{N}/\hbar)$ [5,35]. Focusing on the case of elastic $2 \rightarrow 2$ scattering of two massive particles, we parametrize the incoming and outgoing momenta as p_1^μ, p_2^μ and $p_1'^\mu, p_2'^\mu$. In the classical limit, we parametrize them as $p_1^\mu = \bar{p}_1^\mu + \frac{\hbar}{2} \bar{q}^\mu$, $p_2^\mu = \bar{p}_2^\mu - \frac{\hbar}{2} \bar{q}^\mu$, $p_1'^\mu = \bar{p}_1^\mu - \frac{\hbar}{2} \bar{q}^\mu$ and $p_2'^\mu = \bar{p}_2^\mu + \frac{\hbar}{2} \bar{q}^\mu$, where $\bar{p}_j^\mu = m_j v_j^\mu$ are classical momenta and $q = \hbar \bar{q}^\mu$ is the momentum transfer. The radial action is

then defined via the Fourier transform of the elastic four-point matrix element of the operator $\hat{N}$ in the classical limit

$$I_r(\{v_i, s_i, b\}) = \int_q e^{i(q \cdot b)/\hbar} \langle p_1' p_2' | \hat{N} | p_1 p_2 \rangle |_{\hbar \rightarrow 0}, \quad (2)$$

as proven rigorously in the spinless case [5]; we define $\int_q := \frac{d^D q}{(2\pi)^D} \delta(2\bar{p}_1 \cdot q) \delta(2\bar{p}_2 \cdot q)$. Introducing the rapidity variable $y = v_1 \cdot v_2 > 1$ and retracing the connection between the radial action and the amplitude defined from the expansion $\hat{\mathcal{S}} = 1 + i\hat{T}$, we arrive at the amplitude-action relation

$$i\mathcal{M}_4(q) = \int_b e^{-iq \cdot b/\hbar} (e^{iI_r(b)/\hbar} - 1), \quad (3)$$

with $\int_b := 4 \frac{m_1 m_2 \sqrt{y^2 - 1}}{\hbar^2} \int d^{D-2} b$. This provides a prescription to obtain the radial action $I_r(b)$ from the four-point scattering amplitude $\mathcal{M}_4(q)$ for two massive spinning point particles by dropping certain divergent integrals after soft expansion [36].

The Dirac bracket formalism [30] provides the classical limit of commutators in the phase space $\mathcal{H}_{2B}$:

$$[f(\cdot), g(\cdot)] \xrightarrow{\hbar \rightarrow 0} i\hbar \{f(\cdot), g(\cdot)\}_{\text{DB}}, \quad (4)$$

so that the total variation of the expectation value $\langle \Delta \hat{\mathcal{Q}} \rangle$ (1) in the scattering process becomes, after inserting the two-body phase space completeness relation and Fourier transforming to impact parameter space,

$$\langle \Delta \hat{\mathcal{Q}} \rangle |_{\text{cons}} = \sum_{n=1}^{\infty} \frac{1}{n!} \underbrace{\{I_r, \{I_r, \dots, \{I_r, \mathcal{Q}\}_{\text{DB}}\}_{\text{DB}}\}_{\text{DB}}}_{n \text{ times}}. \quad (5)$$

This expression admits a natural extension to the dissipative sector upon including appropriate radiative kernels; see Ref. [33] for more details. However, in this work we focus exclusively on the conservative regime. For the classical two-body dynamics of spinning particles with spin tensors $S_i^{\mu\nu} = m_i \epsilon^{\mu\nu}_{\rho\sigma} v_i^\rho s_i^\sigma$, the Dirac brackets can be written as [33]

$$\{b^\mu, v_i^\nu\}_{\text{DB}} = -\text{sgn}_i \frac{(y^2 - 1)b^\mu b^\nu + l^\mu l^\nu}{m_i b^2 (y^2 - 1)}, \quad (6)$$

$$\{b^\mu, s_i^\nu\}_{\text{DB}} = \text{sgn}_i \frac{v_i^\nu ((y^2 - 1)b^\mu (b \cdot s_i) + l^\mu (l \cdot s_i))}{m_i b^2 (y^2 - 1)}, \quad (7)$$

$$\begin{aligned} \{b^\mu, b^\nu\}_{\text{DB}} &= \frac{(b^2(y v_2^\mu - v_1^\mu) - l^\mu (s_1 \cdot v_2))b^\nu - (\mu \leftrightarrow \nu)}{m_1 b^2 (y^2 - 1)} \\ &\quad + (1 \leftrightarrow 2, b \rightarrow -b), \end{aligned} \quad (8)$$

$$\{s_i^\mu, s_i^\nu\}_{\text{DB}} = \frac{\epsilon^{\mu\nu}_{\rho\sigma} v_i^\rho s_i^\sigma}{m_i}, \quad (9)$$

with $l^\mu = \epsilon^{\mu\nu\rho\sigma} b_\nu v_{1\rho} v_{2\sigma}$, $\text{sgn}_1 := -1$ and $\text{sgn}_2 := +1$.

As such, given a quantity $\mathcal{Q}(\{v_i, s_i, b\})$ in our phase space $\mathcal{H}_{2B}$, we say that such a quantity is conserved if

$$\{\mathcal{Q}, I_r\}_{DB} = 0 \rightarrow \langle \Delta \mathcal{Q} \rangle = 0, \quad (10)$$

just as we would do in a Hamiltonian formulation.

Asymptotic vs local conservation laws: Now we review how exact local conservation laws [23,24] can be formulated in the probe limit for a spinning test particle in a Kerr background at quadrupolar order, and then discuss their connection to the vanishing of the Dirac brackets in the asymptotic on shell expansion (10).

The Kerr spacetime admits two Killing vectors, $\xi^\mu = (\partial_t)^\mu$ and $\zeta^\mu = (\partial_\phi)^\mu$, associated respectively with stationarity and axisymmetry, as well as a rank-2 Killing tensor $K_{\mu\nu} = Y_{\mu\rho} Y_\nu{}^\rho$, constructed from an antisymmetric Killing-Yano tensor $Y_{\mu\nu} = -Y_{\nu\mu}$ that encodes a hidden symmetry of the background. These geometric structures give rise to all the conserved quantities for the motion of a spinning test particle in the quadrupolar approximation, and—as we will show—also beyond [21,22,24].

We define the covariant four-momentum for the probe particle as $\pi_\mu = p_\mu - \omega_{\mu ab} S^{ab}/2$, where S^{ab} is the spin tensor in the local Lorentz frame and $\omega_{\mu ab}$ is the spin connection, ensuring local Lorentz invariance in curved spacetime. The symmetries of Kerr then yield four conserved quantities. First, the Killing vectors lead to spin-dependent energy and angular momentum [21,22]:

$$E = -\pi_\mu \xi^\mu + \frac{1}{2} S^{\alpha\beta} \nabla_\beta \xi_\alpha, \quad L = \pi_\mu \zeta^\mu - \frac{1}{2} S^{\alpha\beta} \nabla_\beta \zeta_\alpha. \quad (11)$$

Second, the Killing tensor gives rise to the spin-corrected Carter constant [17,23,24]:

$$\begin{aligned} Q = & K_{\mu\nu} p^\mu p^\nu + 4\xi^\lambda \epsilon_{\lambda\mu\sigma[\rho} Y_{\nu]}{}^\sigma S^{\mu\nu} p^\rho \\ & - \left[g_{\mu\rho} \left(\xi_\nu \xi_\sigma - \frac{1}{2} g_{\nu\sigma} \xi^2 \right) \right. \\ & \left. - \frac{1}{2} Y_\mu{}^\lambda \left(Y_\rho{}^\kappa R_{\lambda\nu\kappa\sigma} + \frac{1}{2} Y_\lambda{}^\kappa R_{\kappa\nu\rho\sigma} \right) \right] S^{\mu\nu} S^{\rho\sigma}. \end{aligned} \quad (12)$$

Finally, there is another hidden symmetry which yields a conserved projection of the spin along the principal plane of the Kerr geometry, the Rüdiger linear invariant:

$$Q_Y = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} S_{\mu\nu} Y_{\alpha\beta}. \quad (13)$$

These four quantities— E , L , Q and Q_Y —fully characterize the integrable structure of spinning particle dynamics in Kerr at quadratic order in spin [23]. To make a connection with the asymptotic radial action, we evaluate these four quantities in the limit $r \rightarrow +\infty$. In this limit, denoting for simplicity their asymptotic forms with the same symbols,

the conserved quantities reduce to expressions that depend only on the two-particle kinematics [30,38]:

$$\begin{aligned} \frac{E}{m_1} = & y, \quad \tilde{L} = l \cdot s_2 - y(s_1 \cdot s_2) + (v_2 \cdot s_1)(v_1 \cdot s_2), \\ Q_Y = & l \cdot s_1 + y(s_1 \cdot s_2) - (v_2 \cdot s_1)(v_1 \cdot s_2), \\ Q = & -l^2 + 2y(l \cdot s_1 - l \cdot s_2) - (y^2 - 1)(s_1^2 + s_2^2) - (v_2 \cdot s_1)^2 \\ & - (v_1 \cdot s_2)^2 + 2(y^2 + 1)(s_1 \cdot s_2) - 2y(v_1 \cdot s_2)(v_2 \cdot s_1), \end{aligned} \quad (14)$$

where we define $m_1 \tilde{L} = |s_2|L$ and choose particle 1 as the probe while treating particle 2 as the background.

A word of caution is necessary here regarding the distinction between local and asymptotic conservation laws in the sense of (10): while the former implies the latter, the converse need not hold. Starting at quadratic order in spin, curvature-dependent contributions generate subleading corrections in the $1/r$ expansion to the asymptotic quantities in (14). For example, the kinematic mass $m_1^2 = \eta_{\mu\nu} p_1^\mu p_1^\nu$ is not conserved off shell at $\mathcal{O}(S_1^2)$ in the worldline setup. The conserved quantity at this order is instead the dynamical mass [12]

$$\mu_1^2 = m_1^2 - \frac{1}{4} R_{\mu\nu\rho\sigma} S_1^{\mu\nu} S_1^{\rho\sigma} + \mathcal{O}(S_1^3), \quad (15)$$

whose curvature-dependent correction vanishes in the $r \rightarrow \infty$ limit, effectively giving $\mu_1 \rightarrow m_1$ in our on shell approach. Similar curvature terms appear on the last line of the quadratic-in-spin correction to the Carter constant (12), but likewise drop out asymptotically. Nevertheless, since our primary goal is to analyze the simplicity of the radial action and other on shell quantities relevant for Kerr black hole scattering, the asymptotic formulation adopted here provides precisely the gauge-invariant notion of conservation laws appropriate for our purposes.

Unexpected symmetries of Kerr black hole scattering. In this section, we test the symmetries of the radial action using data available in the literature [39]:

- (1) 1PM and all orders in spin [9],
- (2) 2PM through $\mathcal{O}(s^{11})$ [15],
- (3) 2PM and 3PM through $\mathcal{O}(s^2)$ [13],
- (4) 2PM and 3PM through $\mathcal{O}(s_1^4 s_2^0)$ [29],
- (5) ∞ PM through $\mathcal{O}(s_1^4 s_2^0)$ in the probe limit [30], and
- (6) 3PM through $\mathcal{O}(s^4)$, 4PM through $\mathcal{O}(s^3)$, and 5PM through $\mathcal{O}(s^2)$ in the probe limit [40].

with the total spin power denoted by $\mathcal{O}(s^n)$, e.g., $\mathcal{O}(s^2) = \mathcal{O}(s_1^2) + \mathcal{O}(s_1 s_2) + \mathcal{O}(s_2^2)$. For Refs. [13,40], we extracted the radial action uniquely via an ansatz matched to their impulse. We have checked that the radial actions listed above are in agreement where they overlap

(up to metric signature and Levi-Civita conventions), and we have included their expression in the ancillary file of this Letter for completeness.

Recall from (10) that the radial action exhibits an (asymptotic) symmetry if its Dirac bracket with the corresponding conserved quantity vanishes. Since available results hold only up to a fixed order in the spins, conservation is likewise expected only at that order. More specifically, for a conserved quantity $\mathcal{Q}$ from the previous section and a radial action $I_r^{n,m}$ including terms up to $\mathcal{O}(s_1^n s_2^m)$, conservation requires

$$\{\mathcal{Q}, I_r^{n,m}\}_{\text{DB}} = 0 + \mathcal{O}(s_1^{n+1} s_2^m) + \mathcal{O}(s_1^n s_2^{m+1}). \quad (16)$$

The Dirac brackets do not mix loop orders but do mix spin orders, so all spin terms up to the desired order must be included in $I_r^{n,m}$. Moreover, because the brackets involve explicit factors of $1/m_i$, they also mix powers in the self-force expansion. In fact, in the probe limit $m_2 \rightarrow \infty$, one must therefore use the probe limit of the radial action and discard any terms in (6) proportional to $1/m_2$. The results for conservation at different spin and PM orders are summarized in the Supplemental Material [41] for additional technical details in Table 1, but there are some general features that we wish to mention. First, in the probe limit $m_2 \rightarrow \infty$,

- (1) The $\mathcal{O}(G^\infty s_1^1 s_2^\infty)$ radial action commutes with all conserved quantities.
- (2) The 1PM radial action commutes with all of the conserved quantities to all orders in spin.
- (3) The 2PM and 3PM radial action commutes with all conserved quantities through $\mathcal{O}(s^4)$; likewise for the 4PM radial action through $\mathcal{O}(s^3)$ and the 5PM radial action through $\mathcal{O}(s^2)$.

Beyond the probe limit, the conservation laws continue to hold for the spin-orbit sector—that is, for terms of order $\mathcal{O}(s_1^0 s_2^1)$ or $\mathcal{O}(s_1^1 s_2^0)$ —since these couplings are universal. However, once we go beyond the spin-orbit level and include, for example, $\mathcal{O}(s_1^1 s_2^1)$ contributions, the conservation laws are generically broken. We also discuss further checks of conservation laws for terms beyond $\mathcal{O}(s_1^n s_2^m)$ for $n + m \leq 4$ and including nonminimal couplings in the Supplemental Material [41].

Before discussing asymptotic integrability we mention another symmetry of the spinning black hole scattering known as *spin-shift symmetry*. The spinning four-point amplitude exhibits this symmetry through quartic order in spin up to two-loop order [10,11,14,29]. Hence the amplitude is invariant under the shift $s_i^\mu \rightarrow s_i^\mu + \xi q^\mu$ for any ξ with a formal counting of $\mathcal{O}(1/|q|^2)$. This statement is realized at the linearized level through the action of a differential operator $q^\mu \frac{\partial}{\partial s_i^\mu}$. Thus, in position space, spin-shift symmetry becomes

$$\hat{\mathcal{O}}_{\text{SS}} I_r(\{v_i, s_i, b\}) = 0, \quad \hat{\mathcal{O}}_{\text{SS}} = \Pi_{\mu\nu} \frac{\partial}{\partial b^\mu} \frac{\partial}{\partial s_i^\nu}, \quad (17)$$

where we use the projector $(y^2 - 1)\Pi^{\mu\nu} = (y^2 - 1)\eta^{\mu\nu} + v_1^\mu(v_1^\nu - yv_2^\nu) + v_2^\mu(v_2^\nu - yv_1^\nu)$ since the Fourier transform is performed in the transverse subspace. We have verified that the probe limit radial action in Ref. [30] at $\mathcal{O}(G^\infty s_1^1 s_2^\infty)$ is invariant under such a symmetry, as the radial momentum p_r itself is invariant. We discuss possible interpretations of this symmetry in the Supplemental Material [41].

The integrable sector of Kerr black hole dynamics. Here we wish to define a new asymptotic notion of classical integrability for the scattering dynamics of binary black holes. The spinning radial action I_r depends on $\{v_1^\mu, v_2^\mu, s_1^\mu, s_2^\mu, b^\mu\}$, which define a 20-dimensional phase space. Imposing the on shell conditions $v_i^2 = 1$, the covariant spin supplementary condition $v_i \cdot s_i = 0$, and transversality $b \cdot v_i = 0$ gives six constraints, leaving a 14-dimensional phase space on which the Dirac brackets (6) are defined. Additional conserved quantities can reduce it further.

First, $\langle \Delta s_i^2 \rangle|_{\text{cons}} = 0$ follows from the bracket structure since s_i^2 are Casimirs of the algebra. This yields two constraints and reduces the phase space to 12 dimensions. Translation invariance implies conservation of the center-of-mass momentum, $v_{\text{CM}}^\mu := (m_1 v_1^\mu + m_2 v_2^\mu)/M$ together with $M := m_1 + m_2$, so that $\langle \Delta v_{\text{CM}}^\mu \rangle|_{\text{cons}} = 0$. It is convenient to also introduce the *relative momentum*, $p_{\text{rel}}^\mu := (E_2 v_1^\mu - E_1 v_2^\mu)/M$, which reduces to $(0, \vec{p}_\infty)$ in the center-of-mass frame. As y , E_1 , E_2 , and $|\vec{p}_\infty| \propto \sqrt{y^2 - 1}$ are also conserved under Dirac bracket evolution, the magnitude of $\vec{p}_{\text{rel}}$ is fixed. Writing $(\vec{v}_1, \vec{v}_2)$ in terms of $(\vec{v}_{\text{CM}}, |\vec{p}_\infty| \vec{u}_{\text{rel}})$, where $\vec{u}_{\text{rel}} = \vec{p}_{\text{rel}}/|\vec{p}_\infty|$ is a unit vector, implies four additional reductions, yielding a final phase space of dimension $12 - 4 = 8$.

Therefore we obtain the reduced phase space $\mathcal{H}_{2\text{B}}^{\text{red}} = \{\vec{b}_\perp, \vec{u}_{\text{rel}}, \vec{s}_{1\perp}, \vec{s}_{2\perp}\}$ together with $b_\perp^\mu = \Pi^{\mu\nu} b_\nu$, and $s_{i\perp}^\mu = (\eta^{\mu\nu} - v_i^\mu v_i^\nu) s_{i\nu} / s_i^2$. This dynamical system, with a $2n$ -dimensional reduced phase space equipped with a Dirac bracket structure, is said to be asymptotically integrable in the Liouville sense if there exist n independent phase-space functions F_i —said to be in *involution*—that are all mutually commuting under the Dirac brackets, i.e., $\{F_i, F_j\}_{\text{DB}} = 0$ for all i, j . This notion is *generically weaker* than bulk Liouville integrability: the existence of commuting asymptotic charges does not guarantee that the full equations of motion admit the same number of conserved quantities in involution. As seen in the spinning-probe dynamics in Kerr [30,45–50] and discussed earlier, the obstruction arises from curvature-dependent spin couplings that are absent in the asymptotic on shell regime but are present in the local bulk dynamics, breaking

the correspondence between asymptotic and local conservation laws.

While the reduced phase space is the natural one for the conservative dynamics of spinning black hole binaries, we showed earlier that there exist further conservation laws, at least for a spinning probe particle in Kerr. Using their asymptotic expansion (14), we define $\vec{F}^{\text{BH}} = \{I_r, \tilde{L}, Q, Q_Y\}$ which we use to test integrability of our spinning radial action. In the probe limit, when both black holes are spinning, we have checked that $\{F_i^{\text{BH}}, F_j^{\text{BH}}\}_{\text{DB}} = 0$ for all $i, j = 1, 2, 3, 4$. This relation holds in the probe limit at the same orders in PM and spin where conservation holds in Table 1 found in the Supplemental Material [41]. Beyond the probe limit, when both black holes are spinning, $\tilde{L}$, Q and Q_Y are not in involution, and thus integrability breaks down (albeit at least at 1PM this can be restored; see the Supplemental Material [41]). In the case of one spinning black hole, say, $s_1 \rightarrow 0$ ($s_2 \rightarrow 0$) we find that Q_Y ($\tilde{L}$) is zero, such that the remaining bracket $\{Q, \tilde{L}\}$ ($\{Q, Q_Y\}$) vanishes *without taking the probe limit*. We therefore conclude the following:

- (1) (Conjecture) The probe dynamics of one spinning black hole (BH) in the background of a Kerr BH is asymptotically integrable at all PM orders up to quartic order in spin for the probe; and
- (2) The scattering of one spinless BH and one spinning BH is asymptotically integrable at the 2PM order and quartic order in the BH spin, even beyond the probe limit.

This property can be made manifest by expressing the radial action in terms of the conserved charges $\tilde{L}$, Q and Q_Y . These results are included in the ancillary file.

Bootstrapping the spinning radial action from symmetries. A key consequence of our analysis is that, in the probe limit, integrability and spin-shift symmetry nontrivially constrain the radial action, allowing us to bootstrap it under these assumptions.

Consider a generic expression for the spinning radial action at order G^n , including terms up to $s_1 + s_2 \leq 4$:

$$\mathcal{I}_r^{(n)}(b) = \frac{1}{|b|^{(n-1)(1-2\epsilon)}} \sum_{s_1, s_2} \sum_i \alpha^{(s_1, s_2, i)} \mathcal{J}^{(s_1, s_2, i)}, \quad (18)$$

where $|b|^2 = -b^2$ and $\epsilon = (4 - d)/2$ is the usual dimensional regularization parameter. The integers s_1 and s_2 denote the spin orders of the two bodies, while i labels the independent structures at each order. The coefficients α are arbitrary functions of the kinematic variables, and $\mathcal{J}$ the corresponding spin structures. There are 70 independent spin structures, and thus 70 *a priori* undetermined coefficients in the radial action. The goal is to constrain them by imposing conservation laws and spin-shift symmetry:

- (1) Conservation of the charges Q_Y and Q yields 40 independent constraints;
- (2) Conservation of the azimuthal angular momentum $\tilde{L}$ imposes no further constraints; and
- (3) Spin-shift symmetry imposes 15 constraints.

In total, this reduces the number of free parameters to 15. Remarkably, this exactly matches the independent coefficients appearing in the aligned-spin scattering angle. As a result, knowledge of the aligned-spin scattering angle suffices to determine all 70 coefficients in the general spinning radial action. Notice that this is reminiscent of the tutti-frutti method [51], albeit using symmetries in place of combining different perturbative schemes.

Conclusions and future directions. The study of symmetries in classical gravitational two-body dynamics has been central to the development of analytical methods in general relativity, from self-force theory to the effective one-body framework. In this Letter, we introduced an efficient on shell method, based on the Dirac bracket formalism, to analyze conservation laws in conservative scattering using only the spinning gauge-invariant radial action extracted from PM calculations.

First, we derived the asymptotic expansion of the exact conserved quantities—the energy E , azimuthal angular momentum $\tilde{L}$, Rüdiger invariant Q , and quadrupolar Carter constant Q_Y —for a spinning particle in Kerr space-time, expressing them in terms of kinematic invariants. Second, we proposed a new on shell definition of asymptotic Liouville integrability, inspired by the Hamiltonian case, in which the charges are in involution under the Dirac bracket.

Our results are summarized in Table 1 of the Supplemental Material [41]. We find that $\{E, \tilde{L}, Q, Q_Y\}$ are conserved and in involution up to quartic order in spin for a spinning probe in Kerr, showing that our notion of integrability extends beyond the quadratic-in-spin results of Ref. [23]. The same holds for generic spinning-spinning scattering at 1PM order and spinless-spinning scattering at 2PM order. We also find that asymptotic integrability strongly constrains the bootstrap of generic spinning interactions for the radial action, implying in particular that the dynamics is fully fixed by the aligned-spin case.

We also considered another symmetry, known as spin-shift, previously identified for amplitudes in momentum space. Translating it into position space, we find that it is not naturally interpreted as a standard spacetime symmetry, but rather resembles a (redundant) gauge symmetry, warranting further investigation.

In summary, our work indicates that the Kerr dynamics is more constrained than previously thought, pointing to a richer underlying geometric structure. Important open directions include understanding the implications of quartic-in-spin probe integrability in Kerr for bound motion, possibly through a coordinate and tetrad basis in which the equations of motion become separable [50]; extending these results to

dissipative dynamics by reformulating the self-force evolution equations for $\{\dot{E}, \dot{L}, \dot{Q}, \dot{Q}_Y\}$ in terms of phase-space integrals involving higher-point kernels [25,33]; and exploring applications of our on shell notion of conservation laws and asymptotic integrability beyond gravity.

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Data availability. We provide a computer-readable file, [39], which collects all radial actions considered in our analysis in a consistent notation.

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