

Nonaxisymmetric (2 + 1) black holes with Dym boundary conditions

Miguel Pino^{1,2,*} and Francisco Reyes^{1,†}

¹*Departamento de Física, Universidad de Santiago de Chile, Avenida Víctor Jara 3493, Santiago, Chile*

²*Center for Interdisciplinary Research in Astrophysics and Space Exploration (CIRAS), Universidad de Santiago de Chile, Avenida Libertador Bernardo O'Higgins 3363, Santiago, Chile*



(Received 8 September 2026; accepted 15 September 2026; published 9 October 2026)

We present the spectrum of stationary black hole solutions associated with the first nontrivial member of the Dym family of boundary conditions. The analysis combines the holonomy conditions provided by the Chern-Simons formulation of AdS₃ gravity with the periodic solutions of the stationary Dym equation. The resulting spacetimes are characterized by two positive real parameters, two positive integers, and a relative phase, generically describing stationary, nonaxisymmetric black holes.

DOI: [10.1103/dncn-nn5r](https://doi.org/10.1103/dncn-nn5r)

I. INTRODUCTION

Motivated by the study of gravitational dualities beyond the standard AdS/CFT correspondence, a variety of boundary conditions for the gravitational field have been explored in the literature [1–18]. Three-dimensional gravity with a negative cosmological constant has proven to be a particularly fruitful arena for such exploration, owing to its relative simplicity. Several proposals [19–39] have extended the well-known Brown-Henneaux boundary conditions [40], for which the asymptotic symmetries span two copies of the Virasoro algebra with central charge $c = 3\ell/2G$, where ℓ represents the AdS radius and G is Newton's constant. One approach to modifying these boundary conditions is to exploit the freedom in choosing the boundary behavior of the lapse and shift functions [41,42]. This path is most naturally expressed in the Chern-Simons formulation [43,44], where the dynamical fields are two $\mathfrak{sl}(2, \mathbb{R})$ connections $\mathcal{A}^\pm$. The radial coordinate ρ can be removed by a suitable gauge choice,

$$\mathcal{A}^\pm = b_\pm^{-1} a^\pm b_\pm + b_\pm^{-1} db_\pm, \quad (1.1)$$

where $b_\pm$ are taken to capture the entire radial dependence of the fields. The boundary conditions are then given by [45]

$$a_\phi^\pm = L_{\pm 1} - \frac{12\pi}{c} \mathcal{L}^\pm L_{\mp 1}, \quad (1.2a)$$

$$a_t^\pm = \pm \frac{1}{\ell} \left(\mu^\pm a_\phi^\pm \mp \partial_\phi \mu^\pm L_0 + \frac{1}{2} \partial_\phi^2 \mu^\pm L_{\mp 1} \right), \quad (1.2b)$$

$$a_\rho^\pm = 0, \quad (1.2c)$$

where t and ϕ stand for the temporal and angular coordinates, respectively, while the generators $\{L_1, L_{-1}, L_0\}$ span $\mathfrak{sl}(2, \mathbb{R})$. The functions $\mathcal{L}^\pm(t, \phi)$ represent the dynamical modes, and $\mu^\pm(t, \phi)$ encode the boundary behavior of the lapse and shift functions. In this formulation, the zero-curvature condition, equivalent to Einstein's equations, reduces to

$$\pm \ell \partial_t \mathcal{L}^\pm = 2\mathcal{L}^\pm \partial_\phi \mu^\pm + \mu^\pm \partial_\phi \mathcal{L}^\pm - \frac{c}{24\pi} \partial_\phi^3 \mu^\pm. \quad (1.3)$$

The $\mu^\pm$ can be chosen freely, provided that the resulting boundary conditions are consistent. One such possibility is to choose $\mu^\pm$ as suitably constructed functionals of $\mathcal{L}^\pm$, for which the evolution equations (1.3) reduce to members of integrable hierarchies [31,33,36]. Implications and extensions of these integrable cases have been analyzed in [35,37,46–52].

A particular family of functionals for $\mu^\pm$, introduced in [36], yields the Dym hierarchy of integrable equations [53] as the evolution equations. The construction is based on an expansion of $\mu^\pm$ in negative powers of the Brown-Henneaux central charge,

$$\mu^\pm = 1 + \sum_{n=1}^N \left(\frac{24\pi}{c} \right)^{N-n+1} \frac{\delta H_n^\pm}{\delta \mathcal{L}^\pm}, \quad (1.4)$$

where $H_n^\pm$ denotes the conserved charges of the Dym hierarchy (see Appendix A), up to an arbitrary positive integer N .

*Contact author: miguel.pino.r@usach.cl

†Contact author: francisco.reyes.z@usach.cl

Published by the American Physical Society under the terms of the [Creative Commons Attribution 4.0 International](https://creativecommons.org/licenses/by/4.0/) license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Funded by SCOAP³.

An important requirement for any admissible boundary condition is that it must accommodate physically meaningful solutions, such as black holes. As shown in [36], configurations with constant $\mathcal{L}^\pm$ correspond to black hole spacetimes, extending the well-known BTZ geometry [54,55]. The present work aims to characterize the spectrum of stationary black hole solutions compatible with the Dym boundary conditions.

Stationary nonaxisymmetric black holes in three dimensions, often referred to as black flowers, are comparatively scarce. In several known examples, the angular dependence is described by arbitrary periodic functions, or the gravitational field equations reduce, for a suitable ansatz, to linear differential equations [32,56,57]. Configurations whose angular profile is instead determined by a nonlinear stationary integrable equation are considerably less common, with notable examples arising in connection with integrable hierarchies [31,35,49]. The Dym boundary conditions provide a new realization of this latter mechanism: the angular dependence of the geometry is constrained by the nonlinear stationary Dym equation, and the existence of smooth black hole solutions requires finding nontrivial periodic solutions of this equation.

Our analysis is carried out in the Chern-Simons formulation, where black holes can be defined in the Euclidean framework [42,58,59]. In particular, we impose two conditions:

- (1) The Euclidean manifold must have the topology of a solid torus.
- (2) The corresponding solution to the Dym equation must satisfy periodic boundary conditions in the angular coordinate ϕ .

We demonstrate the existence of solutions to the $N = 1$ Dym equation fulfilling these requirements, representing nonaxisymmetric black hole spacetimes.

The paper is organized as follows. In Sec. II we derive the conditions that the functions $\mathcal{L}^\pm$ must satisfy in order for the corresponding Euclidean manifold to describe a solid torus. Section III is devoted to the construction of periodic solutions to the Dym equation. In Sec. IV we study the metric properties of the resulting black hole. We conclude with remarks regarding the higher-order equations in the Dym family.

II. HOLONOMIES

Following [42,58,59], we define a Euclidean black hole as a manifold with the topology of a solid torus. An advantage of the Chern-Simons formalism is that it allows to probe the topology of the manifold using holonomies Hol_γ , where γ is a closed curve in the manifold. Hence, for a black hole configuration, the holonomy along the temporal cycle should be trivial, while nontrivial in the angular cycle. In what follows, these two conditions are translated into requirements for the functions $\mathcal{L}^\pm$.

First, consider the angular holonomy

$$\text{Hol}_\phi^\pm = \mathcal{P} \exp \left[\int_0^{2\pi} \mathcal{A}_\phi^\pm d\phi \right]. \quad (2.1)$$

Since the connection is flat on shell, the trace of the holonomy is invariant under continuous deformations of the curve that preserve its homotopy class. Therefore, a sufficient condition for the angular cycle to have nontrivial holonomy is

$$\text{Tr}[\text{Hol}_\phi^\pm] \neq 2. \quad (2.2)$$

We can find a simple sufficient condition ensuring the above relation, even without knowing the explicit form of the solutions for $\mathcal{L}$. Consider only the $+$ sector, since analogous arguments apply to the $-$ sector. We proceed to calculate the holonomy by means of the associated linear problem [60,61]

$$\frac{dU(\phi)}{d\phi} = a_\phi^+ U(\phi), \quad U(0) = \mathbb{I}. \quad (2.3)$$

The auxiliary connection a_ϕ^+ can be used instead of $\mathcal{A}_\phi^+$, since the two are related by a gauge transformation and the trace of the holonomy is gauge invariant. Once the solution to (2.3) is found, the angular holonomy is, by definition,

$$\text{Hol}_\phi^+ = U(2\pi). \quad (2.4)$$

We now analyze the solution for the matrix $U(\phi)$ through the corresponding differential equations. For this purpose, assume that it takes the form

$$U(\phi) = \begin{bmatrix} u_1(\phi) & v_1(\phi) \\ u_2(\phi) & v_2(\phi) \end{bmatrix}. \quad (2.5)$$

The linear problem (2.3) implies the identifications $u_2 = u_1'$ and $v_2 = v_1'$, together with the following differential equations¹:

$$u_1'' = \frac{12\pi}{c} \mathcal{L} u_1, \quad v_1'' = \frac{12\pi}{c} \mathcal{L} v_1. \quad (2.7)$$

Furthermore, the initial condition $U(0) = \mathbb{I}$ implies

¹For concreteness, we use the representation

$$L_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad L_0 = \begin{bmatrix} -1/2 & 0 \\ 0 & 1/2 \end{bmatrix}, \quad L_{-1} = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}. \quad (2.6)$$

$$\begin{aligned} u_1(0) &= 1, & v_1(0) &= 0, \\ u'_1(0) &= 0, & v'_1(0) &= 1. \end{aligned} \quad (2.8)$$

If $\mathcal{L}(\phi) > 0$ for all ϕ , then the differential equations (2.7), together with the initial conditions (2.8), imply that u_1 , v_1 , u'_1 , and v'_1 are strictly increasing functions. In particular,

$$u_1(2\pi) > 1, \quad v'_1(2\pi) > 1. \quad (2.9)$$

Therefore,

$$\text{Tr}[\text{Hol}_\phi^+] = u_1(2\pi) + v'_1(2\pi) > 2. \quad (2.10)$$

Hence, the condition $\mathcal{L}(\phi) > 0$ is sufficient to ensure that the angular holonomy belongs to the hyperbolic conjugacy class of $SL(2, \mathbb{R})$ associated with black hole solutions [62].

Now consider the holonomy along the thermal cycle. In the Euclidean formulation, where for stationary configurations $\mathcal{A}_\tau^\pm = i\mathcal{A}_\tau^\pm$, the relevant holonomy is

$$\text{Hol}_\tau^\pm = \mathcal{P} \exp \left[\int_0^{\beta^\pm} \mathcal{A}_\tau^\pm d\tau \right], \quad (2.11)$$

where the period $\beta^\pm$ stands for the inverse temperature. For stationary configurations the holonomy is straightforwardly evaluated. Notably, the choice

$$\beta^\pm = \frac{2\pi\ell}{\sqrt{\frac{48\pi}{c} \mathcal{L}^\pm \mu^{\pm 2} + (\partial_\phi \mu^\pm)^2 - 2\mu^\pm \partial_\phi^2 \mu^\pm}} \quad (2.12)$$

renders the holonomy trivial $\text{Hol}_\tau^\pm = -\mathbb{I}$. Although it seems to be a function of the angle ϕ , $\beta^\pm$ is actually a constant since

$$\partial_\phi \beta^\pm = -\frac{48\pi^2 \ell}{c} \frac{\mu^\pm (2\mathcal{L}^\pm \partial_\phi \mu^\pm + \partial_\phi \mathcal{L}^\pm \mu^\pm - \frac{c}{24\pi} \partial_\phi^3 \mu^\pm)}{(\frac{48\pi}{c} \mathcal{L}^\pm \mu^{\pm 2} + (\partial_\phi \mu^\pm)^2 - 2\mu^\pm \partial_\phi^2 \mu^\pm)^{3/2}}, \quad (2.13)$$

whose numerator is proportional to the stationary equation of motion (1.3), and then vanishes for time-independent on-shell configurations. Hence, for *any* stationary solution of Eq. (1.3), there is a constant $\beta^\pm$ that trivializes the temporal holonomy. Since in general it might be a complex number, the triviality of the temporal holonomy translates into the simple condition of a real and positive $\beta^\pm$ given by (2.12).

In summary, the condition $\mathcal{L}^\pm(\phi) > 0$ for all values of ϕ , and the existence of a real and positive $\beta^\pm$, ensure that the topology of the manifold corresponds to a solid torus, thereby characterizing black hole configurations. The periodic solutions constructed in the next section fulfill both conditions.

III. PERIODIC SOLUTIONS

Another necessary condition to describe a stationary black hole is that the corresponding solution of the Dym equations must obey periodic boundary conditions in the angular coordinate ϕ . Although explicit solutions to Dym equations are scarce [63–73], analytical insight reinforced by numerical methods is sufficient for the purposes of this work. We outline the construction of periodic and time-independent solutions.

For simplicity, we will focus on the case $N = 1$, for which (1.3) becomes the stationary Dym equation

$$\partial_\phi \mathcal{L}^\pm - \partial_\phi^3 \left(\frac{1}{2\sqrt{\mathcal{L}^\pm}} \right) = 0, \quad (3.1)$$

where the functions $\mathcal{L}^\pm$ depend solely on the angular coordinate $\mathcal{L}^\pm(\phi)$. We drop the $\pm$ signs in the following expressions and will reinstate them when necessary.

By first integrating the equation, then multiplying by the term $\partial_\phi \left(\frac{1}{2\sqrt{\mathcal{L}}} \right)$, and then integrating once again, the equation readily becomes

$$\sqrt{\mathcal{L}} + \left[\partial_\phi \left(\frac{1}{2\sqrt{\mathcal{L}}} \right) \right]^2 = 2B - \frac{A}{\sqrt{\mathcal{L}}}, \quad (3.2)$$

where A and B are integration constants. As a byproduct of such a procedure, the following identities are obtained:

$$(\partial_\phi \mathcal{L})^2 = 16\mathcal{L}^{5/2}(2B\sqrt{\mathcal{L}} - \mathcal{L} - A), \quad (3.3a)$$

$$\partial_\phi^2 \mathcal{L} = 4\mathcal{L}^{3/2}(12B\sqrt{\mathcal{L}} - 7\mathcal{L} - 5A), \quad (3.3b)$$

which will prove to be useful later on. At this point, it is helpful to perform the field redefinition

$$\mathcal{L} = \frac{1}{u^2}, \quad (3.4)$$

by which Eq. (3.2) becomes

$$u'^2 = f(u), \quad (3.5)$$

where the function $f(u)$ is given by

$$f(u) = -\frac{4A}{u}(u - u_-)(u - u_+), \quad u_\pm = \frac{B \pm \sqrt{B^2 - A}}{A}. \quad (3.6)$$

A plot of the function $f(u)$ can be found in Fig. 1.

Further integration of (3.5) is straightforward by separation of variables

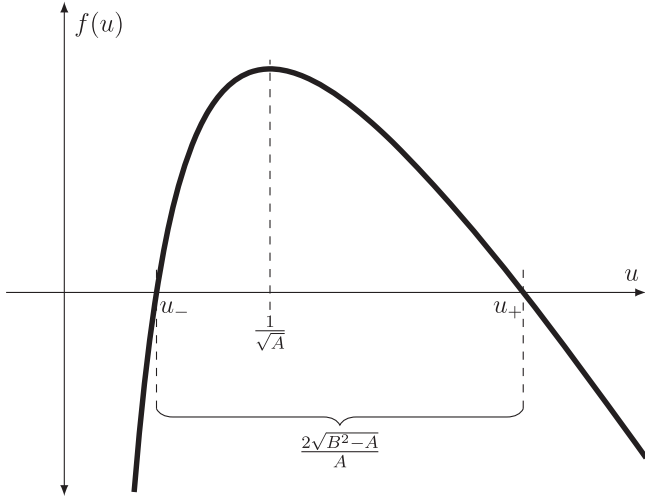


FIG. 1. Plot of the function $f(u)$, showing two turning points $u_{\pm}$ that surround the region where $u^2 > 0$, which is necessary for periodic solutions to exist. In this example $A > 0$ and $B > \sqrt{A}$.

$$\int \frac{du}{\pm \sqrt{8B - \frac{4}{u} - 4Au}} = \phi + \phi_0, \quad (3.7)$$

rendering the third integration constant ϕ_0 , which corresponds to a phase. Although not particularly enlightening, the integral of the left-hand side can be evaluated explicitly, resulting in combinations of incomplete elliptic integrals of the first and second kind [64], thus obscuring the possibility of inverting and expressing u as a function of ϕ . Furthermore, the explicit evaluation of the integral does not clarify whether the solution obeys periodic boundary conditions.

The necessary condition to obtain real periodic solutions can be better understood from Eq. (3.5): the function $f(u)$ should be positive in a range between two real positive zeros $u_{\pm}$, assuring that the function $u(\phi)$ oscillates between two positive turning points. These conditions are met by imposing $A > 0$ and $B > \sqrt{A}$ on the integration constants

(see Fig. 1). Once these conditions are fulfilled, a simple numerical integration of (3.5) renders the periodic solution for $u(\phi)$ shown in Fig. 2(a). The function $\mathcal{L}(\phi)$ can then be obtained from the relation (3.4) [see Fig. 2(b)].

Intricate relations among the parameters are a staple characteristic of periodic solutions in nonlinear integrable equations, and the solution presented here is not an exception. For example, the inflection point located at $u = \frac{1}{\sqrt{A}}$ gives the characteristic “height” of the oscillation. On the other hand, the “amplitude,” i.e., the difference between the turning points, is given by $u_+ - u_- = \frac{2\sqrt{B^2 - A}}{A}$. Moreover, the period T of the solution can be obtained from (3.7), yielding

$$T = 2 \int_{u_-}^{u_+} \frac{du}{\sqrt{8B - \frac{4}{u} - 4Au}}. \quad (3.8)$$

Since we are interested in periodic solutions in the range $0 < \phi < 2\pi$, the period is restricted to $T = \frac{2\pi}{m}$, where m is a positive integer representing the “mode” of the solution. Hence, for a chosen value of m , the parameters A and B are related to each other since the integral in (3.8) must evaluate to a period $2\pi/m$. Hence, the constant B can be obtained from (3.8) once A and m are chosen [see Fig. 3(b)].

Although not evident from the relation (3.8), for a fixed value of A , both the magnitude of B and the amplitude decrease as the mode m increases, as illustrated in Fig. 3(a). However, arbitrarily large values for m are not supported for a given A , as can be seen in Fig. 3(b): for a given value of A (or B), only a finite number of modes can be reached. Equivalently, attaining a periodic solution with a given mode m requires the parameters A and B to exceed corresponding threshold values, as shown in Fig. 3(b).

After finding the periodic solutions, we can evaluate the Dym conserved charges H_n shown in Appendix A. For this purpose, relations (3.3a) and (3.3b) prove useful, since they allow the higher derivatives of $\mathcal{L}(\phi)$ to be reduced to

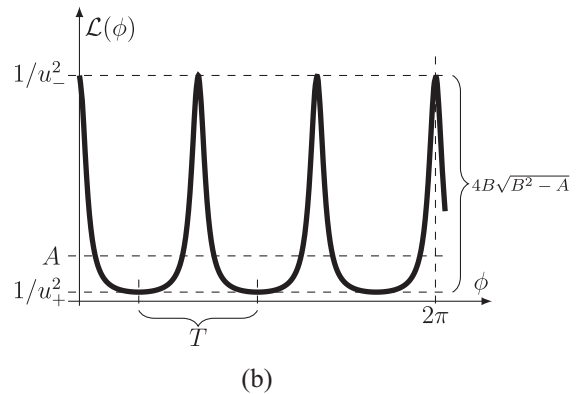
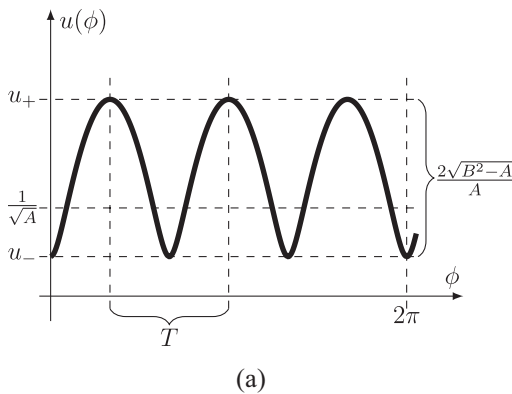


FIG. 2. Numerical integration of equation (3.5). In this example, $A = 2$ and $B = 1.8945$, thus rendering a periodic solution with $T = \frac{2\pi}{3}$. (a) Integration of the function $u(\phi)$. (b) Integration of the function $\mathcal{L}(\phi)$.

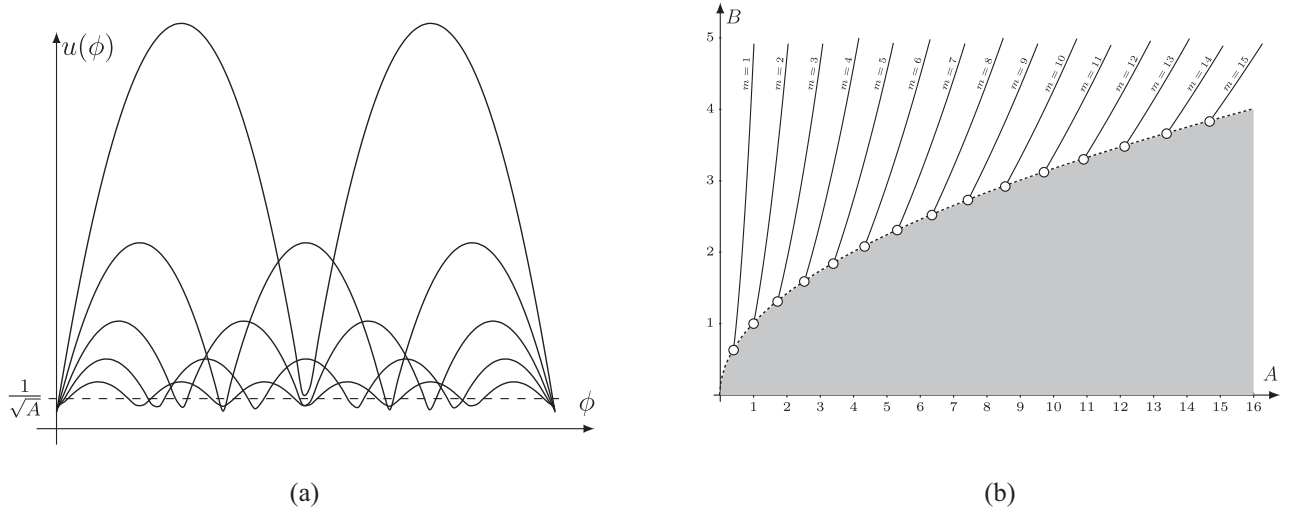


FIG. 3. Exploring the parameter space of stationary periodic solutions of the Dym equation. As can be seen in the plot on the right, for $A = 5$, the maximum attainable mode is $m = 6$, which is the highest displayed in the plot on the left. The limiting values $A_{\min}^{(m)}$ and $B_{\min}^{(m)}$ delimiting the parameter region supporting the m th mode (represented by small white circles in the plot on the right) are given by $(B_{\min}^{(m)})^2 = A_{\min}^{(m)} = (\frac{m}{2})^{4/3}$, which follows by evaluating the period integral (3.8) in the limit $B \rightarrow \sqrt{A}$. (a) Plots of the function $u(\phi)$ for $A = 5$ and various mode values $m = \{2, 3, 4, 5, 6\}$. (b) Parameter space for stationary periodic solutions. Values for A and B within the shaded area, below the dotted parabola $A = B^2$, are not allowed. The loci where the period corresponds to $2\pi/m$ are shown.

expressions involving only the function itself and the parameters A and B . The resulting expressions are remarkably simple, depending only on the first two Dym charges H_0 and H_1 , defined by

$$H_0 = \int_0^{2\pi} \mathcal{L}(\phi) d\phi, \quad H_1 = \int_0^{2\pi} \sqrt{\mathcal{L}(\phi)} d\phi, \quad (3.9)$$

and exhibit a recursive structure reminiscent of the recursion relation (A1),

$$H_2 = 2BH_1 - 2H_0, \quad (3.10a)$$

$$H_3 = \frac{2}{3}(A - 3B^2)H_1 + \frac{4}{3}BH_0, \quad (3.10b)$$

$$H_4 = \frac{4}{5}(-3AB + 5B^3)H_1 + \frac{4}{5}(A - 3B^2)H_0, \quad (3.10c)$$

$$H_5 = \frac{2}{7}(-3A^2 + 30AB^2 - 35B^4)H_1 + \frac{8}{7}(-3AB + 5B^3)H_0, \quad (3.10d)$$

$$H_6 = \frac{4}{9}(15A^2B - 70AB^3 + 63B^5)H_1 + \frac{4}{9}(-3A^2 + 30AB^2 - 35B^4)H_0, \quad (3.10e)$$

$$H_7 = \frac{4}{11}(5A^3 - 105A^2B^2 + 315AB^4 - 231B^6)H_1 + \frac{8}{11}(15A^2B - 70AB^3 + 63B^5)H_0, \quad (3.10f)$$

etc. Surprisingly, when setting $A = 1$, the coefficients in parentheses correspond to the Legendre polynomials evaluated on B .

To finalize this section, we show that the periodic solutions indeed fulfill the conditions specified at the end of the previous section, which are necessary to describe black hole spacetimes.

First, let us examine the condition $\mathcal{L}(\phi) > 0$ required for a nontrivial angular holonomy. This is trivially satisfied within the family of periodic solutions identified here, as the function $\mathcal{L}(\phi)$ oscillates between two positive values, without ever reaching zero.

Regarding the temporal holonomy, the inverse temperature β can be evaluated from Eq. (2.12). As we are addressing the first nontrivial case of the Dym boundary conditions with $N = 1$, the function μ corresponds to

$$\mu = 1 + \frac{12\pi}{c} \frac{1}{\sqrt{\mathcal{L}(\phi)}}. \quad (3.11)$$

The inverse temperature can then be reduced to a constant by virtue of the conditions (3.3a) and (3.3b), yielding

$$\beta = \sqrt{\frac{c}{12\pi}} \frac{\ell\pi}{\sqrt{A + \frac{12\pi}{c}(2B + \frac{12\pi}{c})}}. \quad (3.12)$$

Given the constraints $A > 0$ and $B > \sqrt{A}$, β is guaranteed to be real and positive. Therefore, the temporal holonomy is indeed trivial within the periodic solution space.

IV. METRIC FORMALISM

Since we are dealing with two copies of the construction described in the previous section, the full parameter space of the black hole solution comprises three real numbers and two integers.

On the one hand, we have the two positive real numbers $A^\pm$ and the two integers $m^\pm$. As explained above, for a fixed value of $A^\pm$ only finitely many values of $m^\pm$ are allowed [see Fig. 3(b)]. These two parameters uniquely determine the corresponding values of $B^\pm$ through Eq. (3.8).

Regarding the phases $\phi_0^\pm$ introduced in (3.7), only one of them can be set to zero by a rotation of the reference frame. Hence, the physically relevant parameter is the phase difference $\Delta\phi \equiv \phi_0^+ - \phi_0^-$, which is constrained to lie within $0 \leq \Delta\phi < 2\pi$.

Once these five parameters are fixed, physical quantities of the black hole configuration can be determined [74]. For instance, the black hole mass M is given by

$$M = \frac{1}{\ell} \left(H_0^+ + H_0^- + \frac{24\pi}{c} (H_1^+ + H_1^-) \right), \quad (4.1)$$

which is obtained as the boundary term necessary to obtain a functionally differentiable Hamiltonian [75].

Although ∂_ϕ is not a Killing vector for the solution, the angular momentum of the black hole can still be obtained from the boundary term of the generator associated with the corresponding asymptotic symmetry [36]. Its value is

$$J = H_0^+ - H_0^-. \quad (4.2)$$

The inverse temperatures of both sectors $\beta^\pm$ can be computed from (3.12), while the total black hole inverse temperature is given by $\beta = (\beta^+ + \beta^-)/2$.

The metric field is constructed from the connections $\mathcal{A}^\pm$ (1.1) by the relation

$$g_{\mu\nu} = \frac{\ell^2}{2} \langle (\mathcal{A}_\mu^+ - \mathcal{A}_\mu^-), (\mathcal{A}_\nu^+ - \mathcal{A}_\nu^-) \rangle, \quad (4.3)$$

where $\langle, \rangle$ denotes the invariant bilinear form of $\mathfrak{sl}(2, \mathbb{R})$. The radial dependence is restored by choosing the group element $b_\pm = \exp[\pm \log(\frac{\ell}{c}) L_0]$ in (1.1). The nonvanishing components of the metric then read

$$g_{tt} = -\frac{\rho^2}{\ell^2} \left(1 + \frac{12\pi}{c\sqrt{\mathcal{L}^+}} \right) \left(1 + \frac{12\pi}{c\sqrt{\mathcal{L}^-}} \right) + \frac{6\pi}{c} \left[2(A^+ + A^-) + \frac{48\pi}{c}(B^+ + B^-) \right. \\ \left. + \frac{576\pi^2}{c^2} - \frac{3\pi}{c} \frac{\mathcal{L}^{+'} \mathcal{L}^{-'}}{\mathcal{L}^{+3/2} \mathcal{L}^{-3/2}} \right] - \frac{144\pi^2 \ell^2}{c^2 \rho^2} \left(A^+ + \frac{12\pi}{c} \sqrt{\mathcal{L}^+} \right) \left(A^- + \frac{12\pi}{c} \sqrt{\mathcal{L}^-} \right), \quad (4.4a)$$

$$g_{t\phi} = \frac{3\pi\ell}{c\rho} \left(\frac{\mathcal{L}^{+'}}{\mathcal{L}^{+3/2}} - \frac{\mathcal{L}^{-'}}{\mathcal{L}^{-3/2}} \right), \quad (4.4b)$$

$$g_{t\phi} = \frac{6\pi\rho^2}{c\ell} \left(\frac{1}{\sqrt{\mathcal{L}^+}} - \frac{1}{\sqrt{\mathcal{L}^-}} \right) + \frac{6\pi\ell}{c} \left[(\mathcal{L}^+ + A^+) - (\mathcal{L}^- + A^-) + \frac{24\pi}{c} (\sqrt{\mathcal{L}^+} - \sqrt{\mathcal{L}^-}) \right] \\ + \frac{72\pi^2 \ell^3}{c^2 \rho^2} \left[\left(A^+ + \frac{12\pi}{c} \sqrt{\mathcal{L}^+} \right) \mathcal{L}^- - \left(A^- + \frac{12\pi}{c} \sqrt{\mathcal{L}^-} \right) \mathcal{L}^+ \right], \quad (4.4c)$$

$$g_{\rho\rho} = \frac{\ell^2}{\rho^2}, \quad (4.4d)$$

$$g_{\phi\phi} = \rho^2 + \frac{12\pi\ell^2}{c} (\mathcal{L}^+ + \mathcal{L}^-) + \frac{144\pi^2 \ell^4}{c^2 \rho^2} \mathcal{L}^+ \mathcal{L}^-. \quad (4.4e)$$

In the above expressions, primes denote derivatives with respect to ϕ . The relations (3.3a) and (3.3b) are helpful to express the metric in the desired form, which involves the functions $\mathcal{L}^\pm(\phi)$ and the parameters $A^\pm$ and $B^\pm$. We intentionally leave the first derivatives appearing in (4.4a) and (4.4b) unchanged, to avoid introducing $\pm$ signs.

Note that the above metric does not satisfy the standard Brown-Henneaux boundary conditions [40]. Instead, it lies within a broader class of asymptotic behaviors [30, 76–79], which generalizes the Fefferman-Graham expansion in three dimensions [41, 80–82].

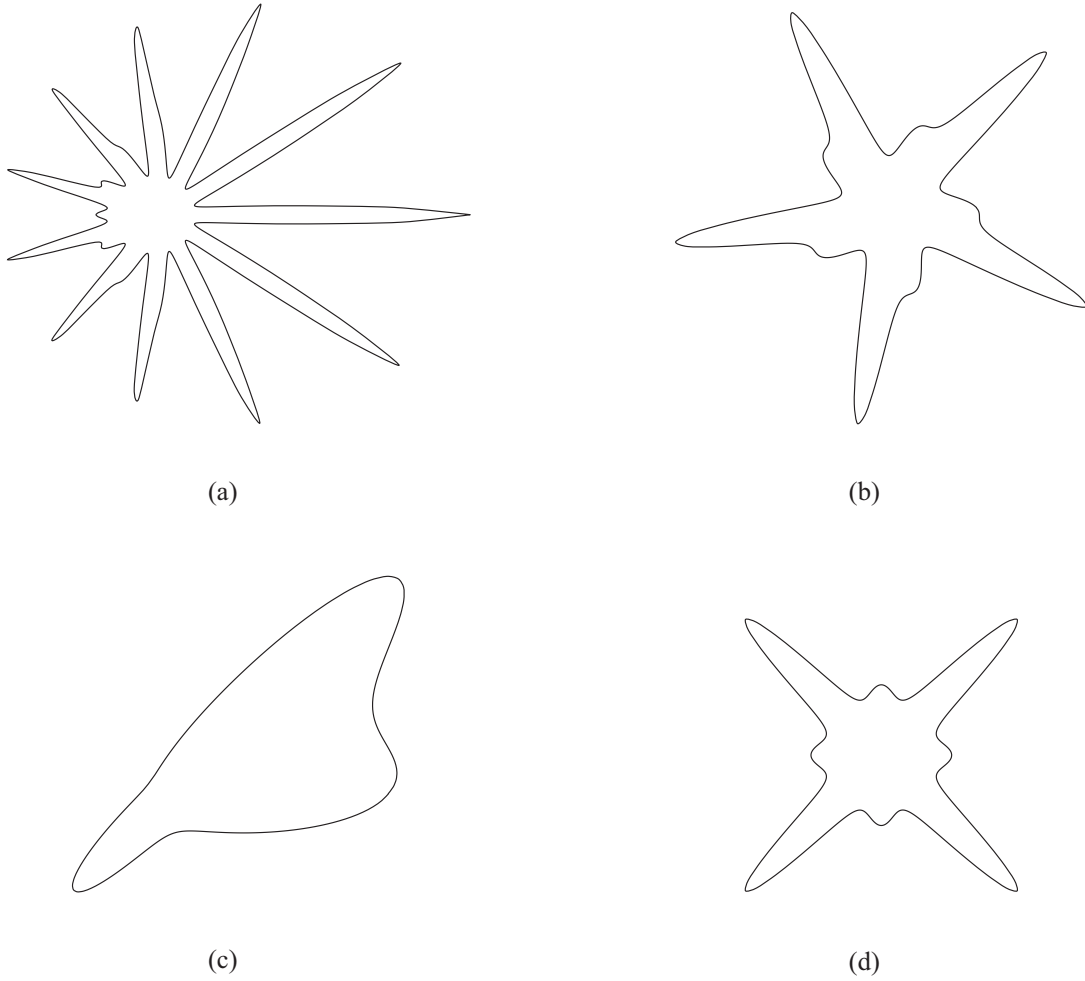


FIG. 4. Examples of surfaces at fixed $\rho = \rho_0 \ll \ell$, illustrating the nonaxisymmetric features of the spacetime. The plots are not to scale with respect to each other. (a) $(A^+, A^-) = (11.2, 11.2)$, $(m^+, m^-) = (12, 11)$, implying $(B^+, B^-) = (3.53, 4.40)$, and $\Delta\phi = 0$. Invariant under dihedral group D_{12} . (b) $(A^+, A^-) = (3.50, 4.00)$, $(m^+, m^-) = (5, 5)$, implying $(B^+, B^-) = (1.99, 2.74)$, and $\Delta\phi = \pi/4$. Invariant under cyclic group C_5 . (c) $(A^+, A^-) = (0.5, 1.2)$, $(m^+, m^-) = (1, 2)$, implying $(B^+, B^-) = (1.09, 1.55)$, and $\Delta\phi = \pi/4$. No discrete symmetry group. (d) $(A^+, A^-) = (2.60, 3.00)$, $(m^+, m^-) = (4, 4)$, implying $(B^+, B^-) = (1.72, 2.42)$, and $\Delta\phi = \pi/4$. Invariant under dihedral group D_4 .

When evaluated in the above metric, Einstein's equations are solved by virtue of Dym equation (3.1) and the relations (3.3a) and (3.3b).

To visualize the nonaxisymmetric features of the spacetime, a useful approach is to examine surfaces at fixed time and radial coordinate, $\rho = \rho_0$. For simplicity, we focus on surfaces close to the origin, i.e., with $\rho_0 \ll \ell$. The induced geometry on such a surface is approximately given by

$$dl \approx \frac{12\pi\ell^2}{c\rho_0} \sqrt{\mathcal{L}^+(\phi)\mathcal{L}^-(\phi)} d\phi. \quad (4.5)$$

These surfaces correspond to closed curves which wobble around the origin. Several examples for different parameter choices are illustrated in Fig. 4. By appropriately tuning the parameters, the resulting curves can display discrete rotational symmetries, reflective symmetries, or a combination of both.

The absence of axisymmetry in the present solutions renders the identification of the black hole horizon significantly more delicate. Stationary nonaxisymmetric black holes are comparatively rare in the literature, mainly due to four-dimensional uniqueness theorems [83–92]. Explicit examples are typically found in higher or lower dimensions, or with different asymptotic behavior [31,32,35,57,93–98].

In our case, the spacetime admits the exact Killing vector field ∂_t , which allows us to identify the hypersurface $\rho = \rho_K(\phi)$ where its norm vanishes. Likewise, the hypersurface where the lapse function $N = 1/\sqrt{-g^{tt}}$ vanishes can be written as $\rho = \rho_N(\phi)$. The explicit forms of the functions $\rho_K(\phi)$ and $\rho_N(\phi)$, though straightforward to derive from (4.4), are not particularly illuminating and are therefore omitted. In general, these two hypersurfaces do not coincide, and neither provides by itself an immediate identification of the event horizon.

These difficulties are alleviated when we consider a simpler sector of the solution space. Let $A^+ = A^- \equiv A$, $B^+ = B^- \equiv B$ and $\mathcal{L}^+ = \mathcal{L}^- \equiv \mathcal{L}$. The metric then reduces to the diagonal form

$$ds^2 = -\left(\frac{12\pi}{c}\right)^2 \left[\frac{\rho}{\ell} \left(\frac{c}{12\pi} + \frac{1}{\sqrt{\mathcal{L}}} \right) - \frac{\ell}{\rho} \left(A + \frac{12\pi}{c} \sqrt{\mathcal{L}} \right) \right]^2 dt^2 + \frac{\ell^2}{\rho^2} d\rho^2 + \left(\rho + \frac{12\pi\ell^2}{c\rho} \mathcal{L} \right)^2 d\phi^2. \quad (4.6)$$

In this sector, the angular momentum (4.2) vanishes, while the remaining Dym charges remain nonzero. Despite this simplification, the spacetime remains nonaxisymmetric, since $\mathcal{L}$ retains a nontrivial dependence on the angular coordinate ϕ , and consequently ∂_ϕ is not an exact Killing vector.

For the diagonal metric, the two hypersurfaces introduced above coincide and yield the same hypersurface,

$$\frac{\rho^2}{\ell^2} = \frac{12\pi}{c} \frac{(A + \frac{12\pi}{c} \sqrt{\mathcal{L}})}{(1 + \frac{12\pi}{c} \frac{1}{\sqrt{\mathcal{L}}})}. \quad (4.7)$$

Thus, in this particular sector, $\rho_K(\phi) = \rho_N(\phi) \equiv \rho_H(\phi)$.

The identification of (4.7) as the black hole horizon can be tested through two independent thermodynamic consistency checks. First, its surface gravity can be obtained directly from (4.6) by the usual relation

$$\kappa_H^2 = h^{ij} \partial_i N \partial_j N|_{\rho=\rho_H(\phi)}, \quad (4.8)$$

where h_{ij} denotes the metric on a constant-time spatial slice. Making use of (3.3), the right-hand side becomes independent of ϕ and yields

$$\kappa_H = \frac{2}{\ell} \sqrt{\frac{12\pi}{c} \left[A + \frac{12\pi}{c} \left(2B + \frac{12\pi}{c} \right) \right]}, \quad (4.9)$$

thus related to the inverse temperature β in (3.12) through

$$\kappa_H = \frac{2\pi}{\beta}, \quad (4.10)$$

as expected for a regular nonextremal Killing horizon.

A second independent check follows from the entropy. Since the horizon is described by $\rho = \rho_H(\phi)$, its length corresponds to

$$\mathcal{A}_H = \int_0^{2\pi} d\phi \sqrt{g_{\phi\phi}|_{\rho_H} + g_{\rho\rho}|_{\rho_H} (\rho'_H)^2}. \quad (4.11)$$

In three spacetime dimensions, $\mathcal{A}_H$ plays the role of the horizon area. As shown in Appendix B, the boundary term of the Euclidean Chern-Simons action evaluated at the horizon gives

$$S_{CS} = \frac{\mathcal{A}_H}{4G}. \quad (4.12)$$

Thus, the entropy obtained independently from the Chern-Simons formulation (B25) coincides exactly with the Bekenstein-Hawking entropy associated with the geometric length of the same curve (4.7).

The agreement between the Lorentzian surface gravity and the Euclidean temperature, together with the agreement between the Chern-Simons entropy and the geometric horizon length, provides two independent consistency checks supporting the identification of (4.7) as the black hole Killing horizon in the diagonal sector.

For the general nondiagonal solution, $\rho_K(\phi)$ and $\rho_N(\phi)$ no longer coincide, and the identification of the corresponding horizon remains more involved. A detailed analysis of this case, together with the global causal structure of the spacetime and the relation between the Killing and event horizons, will be addressed in subsequent work.

V. FINAL COMMENTS

Having characterized the black hole spectrum for the simplest nontrivial element of the Dym family of boundary conditions, we now sketch a method for carrying out the periodic solution analysis for the higher-order equations in the Dym hierarchy.

The stationary equation for an arbitrary integer N takes the form

$$\partial_\phi \mathcal{L}^\pm - \partial_\phi^3 \left(\frac{\delta H_N^\pm}{\delta \mathcal{L}^\pm} \right) = 0, \quad (5.1)$$

where the Dym conserved charge $H_N^\pm$ is constructed via the recursive procedure detailed in Appendix A. In analogy with the $N = 1$ case described above, this equation admits a first integration, yielding

$$\mathcal{L}^\pm - \partial_\phi^2 \left(\frac{\delta H_N^\pm}{\delta \mathcal{L}^\pm} \right) = A^\pm, \quad (5.2)$$

where the integration constant $A^\pm$ plays the same role as in (3.2).

To proceed with further integrations, we exploit the integrable structure of the hierarchy. As shown in Appendix C, multiplying the integrated equation by $\partial_\phi \left(\frac{\delta H_M^\pm}{\delta \mathcal{L}^\pm} \right)$, where $H_M^\pm$ is any of the Dym conserved quantities, transforms the equation into a total derivative, affording a further integration and introducing new integration constants. Once enough integrations have been performed for different values of M , finding relations equivalent to those in (3.3a) and (3.3b) becomes an algebraic goal, although very involved and not simple to achieve. This path toward finding black holes for the higher members of the Dym family of boundary conditions will be pursued elsewhere.

It is worth noting that not all the integration constants emerging from the described procedure are independent,

since the differential equation (5.1) is of order $2N + 1$; two of them correspond to $A^\pm$ and the phase of the periodic solution, while the remaining $2N - 1$ arise via the described integrations. Accordingly, the above procedure can only be applied for values of M from 1 to $2N - 1$ before becoming redundant. This observation can be simply illustrated in the $N = 1$ case, where (5.2) becomes

$$\mathcal{L}^\pm - \partial_\phi^2 \left(\frac{1}{2\sqrt{\mathcal{L}^\pm}} \right) = A^\pm. \quad (5.3)$$

Multiplying by $\partial_\phi \left(\frac{\delta H^\pm}{\delta \mathcal{L}^\pm} \right)$ yields (3.2) and introduces the constant $B^\pm$. Multiplying instead by $\partial_\phi \left(\frac{\delta H^\pm}{\delta \mathcal{L}^\pm} \right)$ and manipulating according to the steps described in the Appendix C introduces a new constant $C^\pm$

$$\left(1 - \frac{\delta H_2^\pm}{\delta \mathcal{L}^\pm} \right) \left(\partial_\phi^2 \frac{\delta H_1^\pm}{\delta \mathcal{L}^\pm} - \mathcal{L}^\pm \frac{\delta H_2^\pm}{\delta \mathcal{L}^\pm} \right) = A^\pm \frac{\delta H_2^\pm}{\delta \mathcal{L}^\pm} + C^\pm. \quad (5.4)$$

However, the conditions (3.3a) and (3.3b) imply that $C^\pm = B^{\pm 2} - A^\pm$, so $C^\pm$ is not an independent constant. A similar observation can be made for higher values of M .

ACKNOWLEDGMENTS

The authors express their gratitude to Marcela Cárdenas, Francisco Correa, Hernán A. González, Kristiansen Lara,

and Ricardo Troncoso for their insightful comments. This work has received support from FONDECYT Grant No. 1231810 and ANID Beca Doctorado Nacional No. 21232318. We also acknowledge Centro de Estudios Científicos (CECs) at Universidad San Sebastián for their hospitality during part of this work.

DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: DYM CONSERVED CHARGES $H_n^\pm$

As explained in [99], the conserved quantities of the Dym hierarchy can be obtained by means of the recursion relation

$$\mathcal{D}^\pm \left(\frac{\delta H_{n+1}^\pm}{\delta \mathcal{L}^\pm} \right) = \mathcal{E} \left(\frac{\delta H_n^\pm}{\delta \mathcal{L}^\pm} \right), \quad (A1)$$

where the Hamiltonian operators are given by $\mathcal{D}^\pm = 2\mathcal{L}^\pm \partial_\phi + \partial_\phi \mathcal{L}^\pm$ and $\mathcal{E} = \partial_\phi^3$, and considering as a starting point the simplest conserved quantity $H_0^\pm = \int d\phi \mathcal{L}^\pm$. In what follows, the prime denotes derivatives with respect to ϕ . Introducing the notation $H_n^\pm = \int d\phi \mathcal{H}_n^\pm$, the first five charge densities are as follows:

$$\begin{aligned} \mathcal{H}_1^\pm &= \sqrt{\mathcal{L}^\pm}, \\ \mathcal{H}_2^\pm &= -\frac{1}{2\sqrt{\mathcal{L}^\pm}} \left(\frac{5}{8} \frac{\mathcal{L}^{\pm/2}}{\mathcal{L}^{\pm 2}} - \frac{1}{2} \frac{\mathcal{L}^{\pm''}}{\mathcal{L}^\pm} \right), \\ \mathcal{H}_3^\pm &= -\frac{1}{2\sqrt{\mathcal{L}^\pm}} \left(\frac{385}{256} \frac{\mathcal{L}^{\pm/4}}{\mathcal{L}^{\pm 5}} - \frac{77}{32} \frac{\mathcal{L}^{\pm/2} \mathcal{L}^{\pm''}}{\mathcal{L}^{\pm 4}} + \frac{7}{16} \frac{\mathcal{L}^{\pm/2}}{\mathcal{L}^{\pm 3}} + \frac{7}{12} \frac{\mathcal{L}^{\pm'} \mathcal{L}^{\pm'''}}{\mathcal{L}^{\pm 3}} - \frac{1}{12} \frac{\mathcal{L}^{\pm(4)}}{\mathcal{L}^{\pm 2}} \right), \\ \mathcal{H}_4^\pm &= -\frac{1}{2\sqrt{\mathcal{L}^\pm}} \left(\frac{85085}{4096} \frac{\mathcal{L}^{\pm/6}}{\mathcal{L}^{\pm 8}} - \frac{51051}{1024} \frac{\mathcal{L}^{\pm/4} \mathcal{L}^{\pm''}}{\mathcal{L}^{\pm 7}} + \frac{35607}{1280} \frac{\mathcal{L}^{\pm/2} \mathcal{L}^{\pm/2}}{\mathcal{L}^{\pm 6}} - \frac{671}{320} \frac{\mathcal{L}^{\pm/3}}{\mathcal{L}^{\pm 5}} \right. \\ &\quad \left. + \frac{429}{32} \frac{\mathcal{L}^{\pm/3} \mathcal{L}^{\pm'''}}{\mathcal{L}^{\pm 6}} - \frac{1419}{160} \frac{\mathcal{L}^{\pm'} \mathcal{L}^{\pm''} \mathcal{L}^{\pm'''}}{\mathcal{L}^{\pm 5}} + \frac{69}{160} \frac{\mathcal{L}^{\pm/2}}{\mathcal{L}^{\pm 4}} - \frac{165}{64} \frac{\mathcal{L}^{\pm/2} \mathcal{L}^{\pm(4)}}{\mathcal{L}^{\pm 5}} \right. \\ &\quad \left. + \frac{57}{80} \frac{\mathcal{L}^{\pm''} \mathcal{L}^{\pm(4)}}{\mathcal{L}^{\pm 4}} + \frac{27}{80} \frac{\mathcal{L}^{\pm'} \mathcal{L}^{\pm(5)}}{\mathcal{L}^{\pm 4}} - \frac{1}{40} \frac{\mathcal{L}^{\pm(6)}}{\mathcal{L}^{\pm 3}} \right), \\ \mathcal{H}_5^\pm &= -\frac{1}{2\sqrt{\mathcal{L}^\pm}} \left(\frac{185910725}{262144} \frac{\mathcal{L}^{\pm/8}}{\mathcal{L}^{\pm 11}} - \frac{37182145}{16384} \frac{\mathcal{L}^{\pm''} \mathcal{L}^{\pm/6}}{\mathcal{L}^{\pm 10}} + \frac{9006855}{14336} \frac{\mathcal{L}^{\pm'''}}{\mathcal{L}^{\pm 9}} \right. \\ &\quad \left. + \frac{123740331}{57344} \frac{\mathcal{L}^{\pm/2} \mathcal{L}^{\pm/4}}{\mathcal{L}^{\pm 9}} - \frac{1859715}{14336} \frac{\mathcal{L}^{\pm(4)} \mathcal{L}^{\pm/4}}{\mathcal{L}^{\pm 8}} + \frac{36465}{1792} \frac{\mathcal{L}^{\pm(5)} \mathcal{L}^{\pm/3}}{\mathcal{L}^{\pm 7}} \right. \\ &\quad \left. - \frac{3201627}{3584} \frac{\mathcal{L}^{\pm''} \mathcal{L}^{\pm'''}}{\mathcal{L}^{\pm 8}} + \frac{282711}{3584} \frac{\mathcal{L}^{\pm/2} \mathcal{L}^{\pm/2}}{\mathcal{L}^{\pm 7}} + \frac{226083}{1792} \frac{\mathcal{L}^{\pm''} \mathcal{L}^{\pm(4)} \mathcal{L}^{\pm/2}}{\mathcal{L}^{\pm 7}} \right) \end{aligned}$$

$$\begin{aligned}
& -\frac{2145}{896} \frac{\mathcal{L}^{\pm(6)} \mathcal{L}^{\pm/2}}{\mathcal{L}^{\pm 6}} - \frac{629629}{1024} \frac{\mathcal{L}^{\pm//3} \mathcal{L}^{\pm/2}}{\mathcal{L}^{\pm 8}} + \frac{97383}{448} \frac{\mathcal{L}^{\pm//2} \mathcal{L}^{\pm//3} \mathcal{L}^{\pm/}}{\mathcal{L}^{\pm 7}} + \frac{11}{56} \frac{\mathcal{L}^{\pm(7)} \mathcal{L}^{\pm/}}{\mathcal{L}^{\pm 5}} \\
& - \frac{1287}{112} \frac{\mathcal{L}^{\pm//} \mathcal{L}^{\pm(5)} \mathcal{L}^{\pm/}}{\mathcal{L}^{\pm 6}} - \frac{3861}{224} \frac{\mathcal{L}^{\pm//3} \mathcal{L}^{\pm(4)} \mathcal{L}^{\pm/}}{\mathcal{L}^{\pm 6}} + \frac{180323}{7168} \frac{\mathcal{L}^{\pm//4}}{\mathcal{L}^{\pm 7}} + \frac{253}{448} \frac{\mathcal{L}^{\pm(4)2}}{\mathcal{L}^{\pm 5}} \\
& + \frac{209}{224} \frac{\mathcal{L}^{\pm//3} \mathcal{L}^{\pm(5)}}{\mathcal{L}^{\pm 5}} + \frac{121}{224} \frac{\mathcal{L}^{\pm//} \mathcal{L}^{\pm(6)}}{\mathcal{L}^{\pm 5}} - \frac{1}{112} \frac{\mathcal{L}^{\pm(8)}}{\mathcal{L}^{\pm 4}} - \frac{13299}{896} \frac{\mathcal{L}^{\pm//} \mathcal{L}^{\pm//2}}{\mathcal{L}^{\pm 6}} \\
& - \frac{10725}{896} \frac{\mathcal{L}^{\pm//2} \mathcal{L}^{\pm(4)}}{\mathcal{L}^{\pm 6}} \Big). \tag{A2}
\end{aligned}$$

The corresponding functional derivatives are

$$\begin{aligned}
\frac{\delta H_1^\pm}{\delta \mathcal{L}^\pm} &= \frac{1}{2\sqrt{\mathcal{L}^\pm}}, \\
\frac{\delta H_2^\pm}{\delta \mathcal{L}^\pm} &= \frac{1}{2\sqrt{\mathcal{L}^\pm}} \left(\frac{5}{16} \frac{\mathcal{L}^{\pm/2}}{\mathcal{L}^{\pm 3}} - \frac{1}{4} \frac{\mathcal{L}^{\pm//}}{\mathcal{L}^{\pm 2}} \right), \\
\frac{\delta H_3^\pm}{\delta \mathcal{L}^\pm} &= \frac{1}{2\sqrt{\mathcal{L}^\pm}} \left(\frac{1155}{512} \frac{\mathcal{L}^{\pm/4}}{\mathcal{L}^{\pm 6}} - \frac{231}{64} \frac{\mathcal{L}^{\pm/2} \mathcal{L}^{\pm//}}{\mathcal{L}^{\pm 5}} + \frac{21}{32} \frac{\mathcal{L}^{\pm//2}}{\mathcal{L}^{\pm 4}} + \frac{7}{8} \frac{\mathcal{L}^{\pm/} \mathcal{L}^{\pm//3}}{\mathcal{L}^{\pm 4}} - \frac{1}{8} \frac{\mathcal{L}^{\pm(4)}}{\mathcal{L}^{\pm 3}} \right), \\
\frac{\delta H_4^\pm}{\delta \mathcal{L}^\pm} &= \frac{1}{2\sqrt{\mathcal{L}^\pm}} \left(\frac{425425}{8192} \frac{\mathcal{L}^{\pm/6}}{\mathcal{L}^{\pm 9}} - \frac{255255}{2048} \frac{\mathcal{L}^{\pm/4} \mathcal{L}^{\pm//}}{\mathcal{L}^{\pm 8}} + \frac{35607}{512} \frac{\mathcal{L}^{\pm/2} \mathcal{L}^{\pm//2}}{\mathcal{L}^{\pm 7}} - \frac{671}{128} \frac{\mathcal{L}^{\pm//3}}{\mathcal{L}^{\pm 6}} \right. \\
& + \frac{2145}{64} \frac{\mathcal{L}^{\pm/3} \mathcal{L}^{\pm//3}}{\mathcal{L}^{\pm 7}} - \frac{1419}{64} \frac{\mathcal{L}^{\pm/} \mathcal{L}^{\pm//} \mathcal{L}^{\pm//3}}{\mathcal{L}^{\pm 6}} + \frac{69}{64} \frac{\mathcal{L}^{\pm//2}}{\mathcal{L}^{\pm 5}} - \frac{825}{128} \frac{\mathcal{L}^{\pm/2} \mathcal{L}^{\pm(4)}}{\mathcal{L}^{\pm 6}} \\
& \left. + \frac{57}{32} \frac{\mathcal{L}^{\pm//} \mathcal{L}^{\pm(4)}}{\mathcal{L}^{\pm 5}} + \frac{27}{32} \frac{\mathcal{L}^{\pm/} \mathcal{L}^{\pm(5)}}{\mathcal{L}^{\pm 5}} - \frac{1}{16} \frac{\mathcal{L}^{\pm(6)}}{\mathcal{L}^{\pm 4}} \right), \\
\frac{\delta H_5^\pm}{\delta \mathcal{L}^\pm} &= \frac{1}{2\sqrt{\mathcal{L}^\pm}} \left(\frac{1301375075}{524288} \frac{\mathcal{L}^{\pm/8}}{\mathcal{L}^{\pm 12}} - \frac{260275015}{32768} \frac{\mathcal{L}^{\pm//} \mathcal{L}^{\pm/6}}{\mathcal{L}^{\pm 11}} + \frac{9006855}{4096} \frac{\mathcal{L}^{\pm//3} \mathcal{L}^{\pm/5}}{\mathcal{L}^{\pm 10}} \right. \\
& + \frac{123740331}{16384} \frac{\mathcal{L}^{\pm//2} \mathcal{L}^{\pm/4}}{\mathcal{L}^{\pm 10}} - \frac{1859715}{4096} \frac{\mathcal{L}^{\pm(4)} \mathcal{L}^{\pm/4}}{\mathcal{L}^{\pm 9}} + \frac{36465}{512} \frac{\mathcal{L}^{\pm(5)} \mathcal{L}^{\pm/3}}{\mathcal{L}^{\pm 8}} \\
& - \frac{3201627}{1024} \frac{\mathcal{L}^{\pm//} \mathcal{L}^{\pm//3} \mathcal{L}^{\pm/3}}{\mathcal{L}^{\pm 9}} + \frac{282711}{1024} \frac{\mathcal{L}^{\pm//2} \mathcal{L}^{\pm/2}}{\mathcal{L}^{\pm 8}} + \frac{226083}{512} \frac{\mathcal{L}^{\pm//} \mathcal{L}^{\pm(4)} \mathcal{L}^{\pm/2}}{\mathcal{L}^{\pm 8}} \\
& - \frac{2145}{256} \frac{\mathcal{L}^{\pm(6)} \mathcal{L}^{\pm/2}}{\mathcal{L}^{\pm 7}} - \frac{4407403}{2048} \frac{\mathcal{L}^{\pm//3} \mathcal{L}^{\pm/2}}{\mathcal{L}^{\pm 9}} + \frac{97383}{128} \frac{\mathcal{L}^{\pm//2} \mathcal{L}^{\pm//3} \mathcal{L}^{\pm/}}{\mathcal{L}^{\pm 8}} \\
& + \frac{11}{16} \frac{\mathcal{L}^{\pm(7)} \mathcal{L}^{\pm/}}{\mathcal{L}^{\pm 6}} - \frac{1287}{32} \frac{\mathcal{L}^{\pm//} \mathcal{L}^{\pm(5)} \mathcal{L}^{\pm/}}{\mathcal{L}^{\pm 7}} - \frac{3861}{64} \frac{\mathcal{L}^{\pm//3} \mathcal{L}^{\pm(4)} \mathcal{L}^{\pm/}}{\mathcal{L}^{\pm 7}} \\
& + \frac{180323}{2048} \frac{\mathcal{L}^{\pm//4}}{\mathcal{L}^{\pm 8}} + \frac{253}{128} \frac{\mathcal{L}^{\pm(4)2}}{\mathcal{L}^{\pm 6}} + \frac{209}{64} \frac{\mathcal{L}^{\pm//3} \mathcal{L}^{\pm(5)}}{\mathcal{L}^{\pm 6}} + \frac{121}{64} \frac{\mathcal{L}^{\pm//} \mathcal{L}^{\pm(6)}}{\mathcal{L}^{\pm 6}} \\
& \left. - \frac{1}{32} \frac{\mathcal{L}^{\pm(8)}}{\mathcal{L}^{\pm 5}} - \frac{13299}{256} \frac{\mathcal{L}^{\pm//} \mathcal{L}^{\pm//2}}{\mathcal{L}^{\pm 7}} - \frac{10725}{256} \frac{\mathcal{L}^{\pm//2} \mathcal{L}^{\pm(4)}}{\mathcal{L}^{\pm 7}} \right). \tag{A3}
\end{aligned}$$

Note that these satisfy the following identity [100]:

$$H_n^\pm = \frac{2}{3-2n} \int d\phi \frac{\delta H_n^\pm}{\delta \mathcal{L}^\pm} \mathcal{L}^\pm. \tag{A4}$$

APPENDIX B: EUCLIDEAN FORMULATION AND THE COMPLEX DYM EQUATION

In this appendix, we construct the Euclidean counterpart of the boundary conditions introduced in (1.2) and study the corresponding complex Dym equation. To this end, we follow the Euclidean construction described in [42].

First, consider a $\mathfrak{sl}(2, \mathbb{C})$ connection $\mathcal{A}^E$. As in the Lorentzian case, the radial dependence can be removed through a suitable gauge choice,

$$\mathcal{A}^E = b^{-1} a^E b + b^{-1} db, \quad (\text{B1})$$

where b is taken to depend solely on the radial coordinate and to capture the entire radial dependence of the fields. The boundary conditions are then given by

$$a_\phi^E = L_1 - \frac{12\pi}{c} \mathcal{L}_E L_{-1}, \quad (\text{B2a})$$

$$a_\tau^E = -\frac{i}{\ell} \left(\mu_E a_\phi^E - \partial_\phi \mu_E L_0 + \frac{1}{2} \partial_\phi^2 \mu_E L_{-1} \right), \quad (\text{B2b})$$

$$a_\rho^E = 0. \quad (\text{B2c})$$

Here, the complex function $\mathcal{L}_E$ represents the dynamical mode, while μ_E , also complex, will be chosen as a suitable functional of $\mathcal{L}_E$ and its angular derivatives. Since we restrict the analysis to stationary configurations, both $\mathcal{L}_E$ and μ_E are taken to depend solely on ϕ .

The zero-curvature condition reduces to

$$2\mathcal{L}_E \partial_\phi \mu_E + \mu_E \partial_\phi \mathcal{L}_E - \frac{c}{24\pi} \partial_\phi^3 \mu_E = 0. \quad (\text{B3})$$

A procedure analogous to that described in [36] and outlined in the Introduction can then be applied. To recover the first nontrivial member of the Dym hierarchy, we consistently choose

$$\mu_E = 1 + \frac{12\pi}{c} \frac{1}{\sqrt{\mathcal{L}_E}}. \quad (\text{B4})$$

The resulting equation is the stationary complex Dym equation. Its form is identical to that of its real counterpart (3.1), except that the dynamical field is now complex-valued,

$$\partial_\phi \mathcal{L}_E - \partial_\phi^3 \left(\frac{1}{2\sqrt{\mathcal{L}_E}} \right) = 0. \quad (\text{B5})$$

The complex Dym equation can be integrated twice following the same steps as in Sec. III, yielding two complex integration constants, A_E and B_E , and fulfilling the analogous relations

$$(\partial_\phi \mathcal{L}_E)^2 = 16\mathcal{L}_E^{5/2} (2B_E \sqrt{\mathcal{L}_E} - \mathcal{L}_E - A_E), \quad (\text{B6a})$$

$$\partial_\phi^2 \mathcal{L}_E = 4\mathcal{L}_E^{3/2} (12B_E \sqrt{\mathcal{L}_E} - 7\mathcal{L}_E - 5A_E). \quad (\text{B6b})$$

As explained in Sec. II, triviality of the temporal holonomy fixes the Euclidean period β to

$$\beta = \sqrt{\frac{c}{12\pi}} \frac{\ell \pi}{\sqrt{A_E + \frac{12\pi}{c} (2B_E + \frac{12\pi}{c})}}. \quad (\text{B7})$$

For the configuration to define a regular Euclidean geometry, this quantity must in addition be real and positive.

The Euclidean metric components are then recovered by the usual relation

$$g_{\mu\nu}^E = \frac{\ell^2}{2} \langle (\mathcal{A}_\mu^E + \mathcal{A}_\mu^{E\dagger}), (\mathcal{A}_\nu^E + \mathcal{A}_\nu^{E\dagger}) \rangle, \quad (\text{B8})$$

resulting in

$$g_{\tau\tau}^E = \frac{\rho^2}{\ell^2} \left(1 + \frac{12\pi}{c\sqrt{\mathcal{L}_E}} \right) \left(1 + \frac{12\pi}{c\sqrt{\overline{\mathcal{L}_E}}} \right) - \frac{6\pi}{c} \left[2(A_E + \overline{A_E}) + \frac{48\pi}{c} (B_E + \overline{B_E}) + \frac{576\pi^2}{c^2} - \frac{3\pi}{c} \frac{\mathcal{L}_E' \overline{\mathcal{L}_E'}}{\mathcal{L}_E^{3/2} \overline{\mathcal{L}_E}^{3/2}} \right] + \frac{144\pi^2 \ell^2}{c^2 \rho^2} \left(A_E + \frac{12\pi}{c} \sqrt{\mathcal{L}_E} \right) \left(\overline{A_E} + \frac{12\pi}{c} \sqrt{\overline{\mathcal{L}_E}} \right), \quad (\text{B9a})$$

$$g_{\tau\rho}^E = -i \frac{3\pi\ell}{c\rho} \left(\frac{\mathcal{L}_E'}{\mathcal{L}_E^{3/2}} - \frac{\overline{\mathcal{L}_E'}}{\overline{\mathcal{L}_E}^{3/2}} \right), \quad (\text{B9b})$$

$$g_{\phi\phi}^E = -i \frac{6\pi\rho^2}{c\ell} \left(\frac{1}{\sqrt{\mathcal{L}_E}} - \frac{1}{\sqrt{\overline{\mathcal{L}_E}}} \right) - i \frac{6\pi\ell}{c} \left[(\mathcal{L}_E + A_E) - (\overline{\mathcal{L}_E} + \overline{A_E}) + \frac{24\pi}{c} (\sqrt{\mathcal{L}_E} - \sqrt{\overline{\mathcal{L}_E}}) \right] - i \frac{72\pi^2 \ell^3}{c^2 \rho^2} \left[\left(A_E + \frac{12\pi}{c} \sqrt{\mathcal{L}_E} \right) \overline{\mathcal{L}_E} - \left(\overline{A_E} + \frac{12\pi}{c} \sqrt{\overline{\mathcal{L}_E}} \right) \mathcal{L}_E \right], \quad (\text{B9c})$$

$$g_{\rho\rho}^E = \frac{\ell^2}{\rho^2}, \quad (\text{B9d})$$

$$g_{\phi\rho}^E = \rho^2 + \frac{12\pi\ell^2}{c} (\mathcal{L}_E + \overline{\mathcal{L}_E}) + \frac{144\pi^2 \ell^4}{c^2 \rho^2} \mathcal{L}_E \overline{\mathcal{L}_E}, \quad (\text{B9e})$$

where the overline denotes complex conjugation. By construction, the resulting metric is real.

It is useful to note that the Euclidean metric can be formally obtained from the Lorentzian metric (4.4) through the identifications

$$\begin{aligned} \mathcal{L}^+ &\rightarrow \mathcal{L}_E, & \mathcal{L}^- &\rightarrow \overline{\mathcal{L}_E}, & A^+ &\rightarrow A_E, \\ A^- &\rightarrow \overline{A_E}, & B^+ &\rightarrow B_E, & B^- &\rightarrow \overline{B_E}, \end{aligned} \quad (\text{B10})$$

along with $t \rightarrow i\tau$.

We now seek the complete set of Euclidean solutions that can be connected to the Lorentzian solutions described in Sec. IV. Recall that a Lorentzian solution is characterized by two independent periodic functions, $\mathcal{L}^+$ and $\mathcal{L}^-$. Accordingly, its Euclidean counterpart should encode two independent real periods within a single complex solution.

In general, stationary solutions to the complex Dym equation possess two independent periods, although these need not be real. This follows from the fact that periodic solutions are determined by an algebraic curve. Indeed, by introducing

$$\mathcal{L}_E = \frac{1}{u_E^2}, \quad p_E = u_E \partial_\phi u_E, \quad (\text{B11})$$

the integrated equation (B6a) can be written as

$$p_E^2 = -4A_E u_E (u_E - u_E^-)(u_E - u_E^+), \quad (\text{B12})$$

which defines a complex elliptic curve and, consequently, a genus-one Riemann surface.

Finding two real periods therefore amounts to identifying two independent closed cycles on this Riemann surface for which the complex integral

$$T_\gamma = \oint_\gamma \frac{u_E du_E}{p_E} = \oint_\gamma \frac{du_E}{\sqrt{-\frac{4A_E}{u_E} (u_E - u_E^-)(u_E - u_E^+)}} \quad (\text{B13})$$

takes real values. The elliptic curve possesses four ramification points,

$$\{u_E^+, u_E^-, 0, \infty\}, \quad (\text{B14})$$

where $u_E^\pm$ are defined as in (3.6). The associated Riemann surface has the topology of a torus and therefore admits two independent noncontractible cycles. The corresponding periods are, however, generically complex.

Determining the conditions on A_E and B_E for which both periods become real is, in general, a nontrivial problem. Moreover, the formal identification (B10) and comparison with the Lorentzian case suggest that the Euclidean solution should not be fully doubly periodic, but encode the two periodicities within different parts of the complex function.

This imposes an additional restriction and makes a complete characterization considerably more involved. For this reason, we restrict the analysis to a particular class of complex solutions that can be directly related to the Lorentzian configurations.

A simple class of such solutions is obtained by assuming constant phases,

$$A_E = e^{i\alpha} A, \quad B_E = e^{i\gamma} B, \quad \mathcal{L}_E = e^{i\delta} \mathcal{L}, \quad (\text{B15})$$

where A , B , and $\mathcal{L}$ are real. Within this ansatz, the integrated relation (B6a) admits three consistent choices for the phases (α, γ, δ) ,

$$\left(-\frac{\pi}{3}, -\frac{2\pi}{3}, \frac{2\pi}{3}\right), \quad \left(\frac{\pi}{3}, \frac{2\pi}{3}, -\frac{2\pi}{3}\right), \quad (0, 0, 0), \quad (\text{B16})$$

while $\mathcal{L}$ satisfies the real stationary Dym equation (3.3a). Hence, every real periodic solution gives rise to three constant-phase solutions of the complex Dym equation.

The two nontrivial branches are related by complex conjugation and therefore give rise to the same Euclidean spacetime (B9). However, the corresponding Euclidean period β is complex, and hence these branches do not describe a regular real Euclidean geometry with solid-torus topology.

The vanishing-phase scenario remains the only possibility within this family. In this case, the solution is real, $\mathcal{L}_E = \overline{\mathcal{L}_E} \equiv \mathcal{L}$, and the Euclidean metric reduces to

$$\begin{aligned} ds^2 = & \left(\frac{12\pi}{c}\right)^2 \left[\frac{\rho}{\ell} \left(\frac{c}{12\pi} + \frac{1}{\sqrt{\mathcal{L}}} \right) - \frac{\ell}{\rho} \left(A + \frac{12\pi}{c} \sqrt{\mathcal{L}} \right) \right]^2 d\tau^2 \\ & + \frac{\ell^2}{\rho^2} d\rho^2 + \left(\rho + \frac{12\pi\ell^2}{c\rho} \mathcal{L} \right)^2 d\phi^2. \end{aligned} \quad (\text{B17})$$

This is precisely the Euclidean counterpart of the Lorentzian spacetime with vanishing angular momentum described at the end of Sec. IV.

Having constructed the corresponding real and regular Euclidean geometry, the entropy can be evaluated from the Chern-Simons action. In the Euclidean formulation, the entropy is obtained from the boundary term that must be added to the action to make it differentiable. In the Hamiltonian formulation, the variation of such a boundary term is

$$\delta B^E = -\frac{c}{12\pi} \int d\tau d^2 x \epsilon^{ij} \partial_i \langle \mathcal{A}_\tau^E \delta \mathcal{A}_j^E \rangle. \quad (\text{B18})$$

This receives contributions from the boundary at infinity and from the horizon. In the present case, the latter is described by

$$\rho = \rho_H(\phi). \quad (\text{B19})$$

Since the radial coordinate ranges from $\rho_H(\phi)$ to infinity, Stokes's theorem gives

$$\begin{aligned} \delta B^E = & -\frac{c}{12\pi} \int d\tau d\phi \langle \mathcal{A}_\tau^E \delta \mathcal{A}_\phi^E \rangle \Big|_{\rho=\infty} \\ & + \frac{c}{12\pi} \int d\tau d\phi \langle \mathcal{A}_\tau^E (\delta \mathcal{A}_\phi^E + \rho'_H \delta \mathcal{A}_\rho^E) \rangle \Big|_{\rho=\rho_H(\phi)}. \end{aligned} \quad (\text{B20})$$

The first term contains the contributions from the asymptotic charges, while the second gives the horizon contribution. The horizon corresponds to the hypersurface $\rho = \rho_H(\phi)$, and therefore $d\rho = \rho'_H(\phi)d\phi$. Hence, restricting the connection $\mathcal{A}^E = \mathcal{A}_\tau^E d\tau + \mathcal{A}_\rho^E d\rho + \mathcal{A}_\phi^E d\phi$ to the horizon gives

$$\mathcal{A}^E|_{\rho=\rho_H} = \mathcal{A}_\tau^E d\tau + (\mathcal{A}_\phi^E + \rho'_H \mathcal{A}_\rho^E) d\phi. \quad (\text{B21})$$

Thus, the combination appearing in the horizon term is simply the component of the connection tangent to the horizon.

Equivalently, one may introduce the horizon-adapted radial coordinate

$$r = \rho - \rho_H(\phi), \quad (\text{B22})$$

for which the horizon is located at the fixed position $r = 0$. In this coordinate system, the variation is performed keeping the position of the inner boundary fixed, so that $\rho_H(\phi)$ is not independently varied.

On the other hand, Euclidean regularity fixes the holonomy of the connection along the contractible temporal cycle. Using a Euclidean-time coordinate with fixed period, and choosing a gauge in which a fixed representative of this holonomy is kept fixed at the horizon [42], one may impose

$$\delta \mathcal{A}_\tau^E|_{\rho=\rho_H} = 0. \quad (\text{B23})$$

The horizon contribution is then integrable. Recalling that the Euclidean Einstein-Hilbert action is recovered from the $\mathfrak{sl}(2, \mathbb{C})$ Chern-Simons action as $I_{\text{EH}} = i(I_{\text{CS}}[\mathcal{A}^E] - I_{\text{CS}}[\mathcal{A}^{E^\dagger}])$, then the entropy is given by

$$S = -\frac{c}{6\pi} \text{Im} \int d\tau d\phi \langle \mathcal{A}_\tau^E (\mathcal{A}_\phi^E + \rho'_H \mathcal{A}_\rho^E) \rangle \Big|_{\rho=\rho_H(\phi)}. \quad (\text{B24})$$

Using the horizon defined in (4.7), this gives

$$\begin{aligned} S = & 2\pi \left(\frac{c}{12\pi} \right)^{3/2} \sqrt{A + \frac{12\pi}{c} \left(2B + \frac{12\pi}{c} \right)} \\ & \times \int_0^{2\pi} d\phi \frac{(A + \frac{24\pi}{c} \sqrt{\mathcal{L}} + \mathcal{L}) \sqrt{\mathcal{L}}}{(1 + \frac{c}{12\pi} \sqrt{\mathcal{L}})(A + \frac{12\pi}{c} \sqrt{\mathcal{L}})} \end{aligned} \quad (\text{B25})$$

which agrees with one quarter of the horizon length (4.11), as expected from the Bekenstein-Hawking formula.

APPENDIX C: INTEGRATION OF THE N th STATIONARY DYM EQUATION

The objective of this appendix is to demonstrate that Eq. (5.2) becomes a total derivative and can be readily integrated when multiplied by appropriate terms. In what follows, we drop the $\pm$ signs. We begin by multiplying by the term $\partial_\phi(\frac{\delta H_M}{\delta \mathcal{L}})$, yielding,

$$\mathcal{L} \partial_\phi \left(\frac{\delta H_M}{\delta \mathcal{L}} \right) - \partial_\phi^2 \left(\frac{\delta H_N}{\delta \mathcal{L}} \right) \partial_\phi \left(\frac{\delta H_M}{\delta \mathcal{L}} \right) = A \partial_\phi \left(\frac{\delta H_M}{\delta \mathcal{L}} \right). \quad (\text{C1})$$

First, the term on the right-hand side of the equation is a total derivative since A is a constant.

Next, consider the recursion relation (A1), which, after some straightforward manipulation, can be expressed as

$$\mathcal{L} \partial_\phi \left(\frac{\delta H_{n+1}}{\delta \mathcal{L}} \right) = \partial_\phi \left[\partial_\phi^2 \left(\frac{\delta H_n}{\delta \mathcal{L}} \right) - \mathcal{L} \left(\frac{\delta H_{n+1}}{\delta \mathcal{L}} \right) \right]. \quad (\text{C2})$$

Thus, the first term on the left-hand side of Eq. (C1) also yields a total derivative.

The proof that the second term on the left-hand side is a total derivative follows the same logic as demonstrating that the Dym conserved quantities H_n span an Abelian algebra under both Poisson brackets defined by the operators $\mathcal{D}$ and $\mathcal{E}$ (see [101]). First, note that both operators are antisymmetric, in the sense that

$$f\mathcal{D}g + g\mathcal{D}f = \partial_\phi(2\mathcal{L}fg), \quad (\text{C3})$$

$$f\mathcal{E}g + g\mathcal{E}f = \partial_\phi(f\partial_\phi^2 g - \partial_\phi f \partial_\phi g + g\partial_\phi^2 f), \quad (\text{C4})$$

for any functions f and g . These results, along with the recursion relation (A1), lead to the identity

$$\begin{aligned} \mathcal{E} \left(\frac{\delta H_N}{\delta \mathcal{L}} \right) \left(\frac{\delta H_M}{\delta \mathcal{L}} \right) = & \mathcal{E} \left(\frac{\delta H_{N+1}}{\delta \mathcal{L}} \right) \left(\frac{\delta H_{M-1}}{\delta \mathcal{L}} \right) \\ & + \text{boundary terms.} \end{aligned} \quad (\text{C5})$$

Assuming $M > N$, without loss of generality, this identity is applied recursively $M - N$ times, implying

$$\mathcal{E}\left(\frac{\delta H_N}{\delta \mathcal{L}}\right)\left(\frac{\delta H_M}{\delta \mathcal{L}}\right) = \mathcal{E}\left(\frac{\delta H_M}{\delta \mathcal{L}}\right)\left(\frac{\delta H_N}{\delta \mathcal{L}}\right) + \text{boundary terms.} \quad (\text{C6})$$

Combining this result with the antisymmetry relation (C4), it follows that $\mathcal{E}(\frac{\delta H_N}{\delta \mathcal{L}})(\frac{\delta H_M}{\delta \mathcal{L}})$ is a boundary term. Since the second term on the left-hand side of equation (C1) differs only by a boundary term, it follows that this term is also a boundary term. Thus, the complete Eq. (C1) is a total derivative and can be readily integrated.

-
- [1] Glenn Barnich and Cedric Troessaert, Aspects of the BMS/CFT correspondence, *J. High Energy Phys.* **05** (2010) 062.
 - [2] Monica Guica, Thomas Hartman, Wei Song, and Andrew Strominger, The Kerr/CFT correspondence, *Phys. Rev. D* **80**, 124008 (2009).
 - [3] D. T. Son, Toward an AdS/cold atoms correspondence: A geometric realization of the Schrodinger symmetry, *Phys. Rev. D* **78**, 046003 (2008).
 - [4] Sean A. Hartnoll, Andrew Lucas, and Subir Sachdev, *Holographic Quantum Matter* (MIT Press, Cambridge, MA, 2018).
 - [5] Stephen W. Hawking, Malcolm J. Perry, and Andrew Strominger, Superrotation charge and supertranslation hair on black holes, *J. High Energy Phys.* **05** (2017) 161.
 - [6] Stephen W. Hawking, Malcolm J. Perry, and Andrew Strominger, Soft hair on black holes, *Phys. Rev. Lett.* **116**, 231301 (2016).
 - [7] Laura Donnay, Gaston Giribet, Hernan A. Gonzalez, and Miguel Pino, Supertranslations and superrotations at the black hole horizon, *Phys. Rev. Lett.* **116**, 091101 (2016).
 - [8] Laura Donnay, Gaston Giribet, Hernán A. González, and Miguel Pino, Extended symmetries at the black hole horizon, *J. High Energy Phys.* **09** (2016) 100.
 - [9] Gaston Giribet and Luciano Montecchio, Colored black holes and Kac-Moody algebra, *Phys. Rev. D* **105**, 064006 (2022).
 - [10] Gaston Giribet, Juan Launagaray, and Pedro Schmied, Probing the near-horizon geometry of black rings, *Phys. Rev. D* **108**, 024061 (2023).
 - [11] Alex Kehagias and Antonio Riotto, BMS in cosmology, *J. Cosmol. Astropart. Phys.* **05** (2016) 059.
 - [12] Béatrice Bonga and Kartik Prabhu, BMS-like symmetries in cosmology, *Phys. Rev. D* **102**, 104043 (2020).
 - [13] Martin Enriquez-Rojo and Till Heckelbacher, Asymptotic symmetries in spatially flat FRW spacetimes, *Phys. Rev. D* **103**, 064009 (2021).
 - [14] Martin Enriquez-Rojo and Till Heckelbacher, Holography and black holes in asymptotically flat FLRW spacetimes, *Phys. Rev. D* **103**, 104035 (2021).
 - [15] Sabrina Pasterski, A chapter on celestial holography, [arXiv:2310.04932](https://arxiv.org/abs/2310.04932).
 - [16] Laura Donnay, Celestial holography: An asymptotic symmetry perspective, *Phys. Rep.* **1073**, 1 (2024).
 - [17] Aneesh B. Prema, Geoffrey Compère, Leonardo Pipo de Gioia, Igor Mol, and Bianca Swidler, Celestial holography: Lectures on asymptotic symmetries, *SciPost Phys. Lect. Notes* **47**, 1 (2022).
 - [18] Cédric Troessaert, The BMS4 algebra at spatial infinity, *Classical Quantum Gravity* **35**, 074003 (2018).
 - [19] Marc Henneaux, Cristian Martinez, Ricardo Troncoso, and Jorge Zanelli, Black holes and asymptotics of $2 + 1$ gravity coupled to a scalar field, *Phys. Rev. D* **65**, 104007 (2002).
 - [20] Yan Liu and Ya-Wen Sun, Consistent boundary conditions for new massive gravity in AdS3, *J. High Energy Phys.* **05** (2009) 039.
 - [21] Marc Henneaux, Cristian Martinez, and Ricardo Troncoso, Asymptotically warped anti-de Sitter spacetimes in topologically massive gravity, *Phys. Rev. D* **84**, 124016 (2011).
 - [22] Marc Henneaux, Cristian Martinez, and Ricardo Troncoso, More on asymptotically anti-de Sitter spaces in topologically massive gravity, *Phys. Rev. D* **82**, 064038 (2010).
 - [23] Geoffrey Compère, Wei Song, and Andrew Strominger, New boundary conditions for AdS3, *J. High Energy Phys.* **05** (2013) 152.
 - [24] Cédric Troessaert, Enhanced asymptotic symmetry algebra of AdS3, *J. High Energy Phys.* **08** (2013) 044.
 - [25] Steven G. Avery, Rohan R. Poojary, and Nemani V. Suryanarayana, An $sl(2, \mathbb{R})$ current algebra from AdS3 gravity, *J. High Energy Phys.* **01** (2014) 144.
 - [26] Hamid Afshar, Stéphane Detournay, Daniel Grumiller, and Blagoje Oblak, Near- horizon geometry and warped conformal symmetry, *J. High Energy Phys.* **03** (2016) 187.
 - [27] Daniel Grumiller and Max Riegler, Most general AdS3 boundary conditions, *J. High Energy Phys.* **10** (2016) 023.
 - [28] C. E. Valcárcel, New boundary conditions for (extended) AdS3 supergravity, *Classical Quantum Gravity* **36**, 065002 (2019).
 - [29] Marc Henneaux, Wout Merbis, and Arash Ranjbar, Asymptotic dynamics of AdS3 gravity with two asymptotic regions, *J. High Energy Phys.* **03** (2020) 064.
 - [30] Francesco Alessio, Glenn Barnich, Luca Ciambelli, Pujian Mao, and Romain Ruzziconi, Weyl charges in asymptotically locally AdS3 spacetimes, *Phys. Rev. D* **103**, 046003 (2021).
 - [31] Emilio Ojeda and Alfredo Pérez, Boundary conditions for general relativity in three-dimensional spacetimes, integrable systems and the KdV/mKdV hierarchies, *J. High Energy Phys.* **08** (2019) 079.
 - [32] Hamid Afshar, Stéphane Detournay, Daniel Grumiller, Wout Merbis, Alfredo Perez, David Tempo, and Ricardo

- Troncoso, Soft Heisenberg hair on black holes in three dimensions, *Phys. Rev. D* **93**, 101503 (2016).
- [33] Alfredo Pérez, David Tempo, and Ricardo Troncoso, Boundary conditions for general relativity on AdS3 and the KdV hierarchy, *J. High Energy Phys.* **06** (2016) 103.
- [34] Oscar Fuentealba, Hernán A. González, Alfredo Pérez, David Tempo, and Ricardo Troncoso, Superconformal Bondi-Metzner-Sachs algebra in three dimensions, *Phys. Rev. Lett.* **126**, 091602 (2021).
- [35] Marcela Cárdenas, Francisco Correa, and Miguel Pino, Integrable black hole dynamics in the asymptotic structure of AdS3, *Phys. Rev. D* **114**, 026010 (2026).
- [36] Kristiansen Lara, Miguel Pino, and Francisco Reyes, 1/c deformations of AdS3 boundary conditions and the Dym hierarchy, *J. High Energy Phys.* **11** (2024) 042.
- [37] Marcela Cárdenas, Francisco Correa, Kristiansen Lara, and Miguel Pino, Integrable systems and spacetime dynamics, *Phys. Rev. Lett.* **127**, 161601 (2021).
- [38] H. T. Özer and Aytül Filiz, $N = (2, 2)$ extended $sl(3 - 2)$ Chern-Simons AdS3 supergravity with new boundaries, *Eur. Phys. J. C* **82**, 472 (2022).
- [39] H. T. Özer and Aytül Filiz, Exploring new boundary conditions for $\mathcal{N} = (1, 1)$ extended higher-spin AdS3 supergravity, *Eur. Phys. J. C* **80**, 1072 (2020).
- [40] J. David Brown and M. Henneaux, Central charges in the canonical realization of asymptotic symmetries: An example from three-dimensional gravity, *Commun. Math. Phys.* **104**, 207 (1986).
- [41] Maximo Banados and Rodrigo Caro, Holographic ward identities: Examples from 2 + 1 gravity, *J. High Energy Phys.* **12** (2004) 036.
- [42] Claudio Bunster, Marc Henneaux, Alfredo Perez, David Tempo, and Ricardo Troncoso, Generalized black holes in three-dimensional spacetime, *J. High Energy Phys.* **05** (2014) 031.
- [43] A. Achucarro and P. K. Townsend, A Chern-Simons Action for three-dimensional anti-De Sitter supergravity theories, *Phys. Lett. B* **180**, 89 (1986).
- [44] Edward Witten, (2 + 1)-dimensional gravity as an exactly soluble system, *Nucl. Phys. B* **311**, 46 (1988).
- [45] Oliver Coussaert, Marc Henneaux, and Peter van Driel, The asymptotic dynamics of three-dimensional Einstein gravity with a negative cosmological constant, *Classical Quantum Gravity* **12**, 2961 (1995).
- [46] Oscar Fuentealba, Javier Matulich, Alfredo Pérez, Miguel Pino, Pablo Rodríguez, David Tempo, and Ricardo Troncoso, Integrable systems with BMS3 poisson structure and the dynamics of locally flat spacetimes, *J. High Energy Phys.* **01** (2018) 148.
- [47] Hernán A. González, Javier Matulich, Miguel Pino, and Ricardo Troncoso, Revisiting the asymptotic dynamics of general relativity on AdS3, *J. High Energy Phys.* **12** (2018) 115.
- [48] Emilio Ojeda and Alfredo Pérez, Integrable systems and the boundary dynamics of higher spin gravity on AdS3, *J. High Energy Phys.* **11** (2020) 089.
- [49] Anatoly Dymarsky and Sotaro Sugishita, KdV-charged black holes, *J. High Energy Phys.* **05** (2020) 041.
- [50] Marcela Cárdenas, KdV conformal symmetry breaking in nearly AdS2, *J. High Energy Phys.* **10** (2024) 052.
- [51] Cristián Erices, Miguel Riquelme, and Pablo Rodríguez, BTZ black hole with Korteweg-de Vries-type boundary conditions: Thermodynamics revisited, *Phys. Rev. D* **100**, 126026 (2019).
- [52] Hamed Adami and Anousheh Latifi, Integrability in three-dimensional gravity: Eigenfunction-Forced KdV flows, *J. High Energy Phys.* **07** (2026) 120.
- [53] Martin Kruskal, Nonlinear wave equations, *Dynamical Systems, Theory and Applications: Battelle Seattle 1974 Rencontres*, Edited by J. Moser (Springer Berlin Heidelberg, Berlin, Heidelberg, 1975), pp. 310–354.
- [54] Maximo Banados, Claudio Teitelboim, and Jorge Zanelli, The black hole in three-dimensional space-time, *Phys. Rev. Lett.* **69**, 1849 (1992).
- [55] Maximo Banados, Marc Henneaux, Claudio Teitelboim, and Jorge Zanelli, Geometry of the (2 + 1) black hole, *Phys. Rev. D* **48**, 1506 (1993); **88**, 069902(E) (2013).
- [56] Cédric Troessaert, David Tempo, and Ricardo Troncoso, Asymptotically flat black holes and gravitational waves in three-dimensional massive gravity, [arXiv:1512.09046](https://arxiv.org/abs/1512.09046).
- [57] Gökhan Alkaç, Ercan Kilicarslan, and Bayram Tekin, Asymptotically flat black holes in 2 + 1 dimensions, *Phys. Rev. D* **93**, 084003 (2016).
- [58] Steven Carlip and Claudio Teitelboim, Aspects of black hole quantum mechanics and thermodynamics in 2 + 1 dimensions, *Phys. Rev. D* **51**, 622 (1995).
- [59] Maximo Banados, Thorsten Brotz, and Miguel E. Ortiz, Boundary dynamics and the statistical mechanics of the (2 + 1)-dimensional black hole, *Nucl. Phys. B* **545**, 340 (1999).
- [60] J. Balog, L. Feher, and L. Palla, Coadjoint orbits of the Virasoro algebra and the global Liouville equation, *Int. J. Mod. Phys. A* **13**, 315 (1998).
- [61] M. M. Sheikh-Jabbari and Hossein Yavartanoo, On quantization of AdS3 gravity I: Semi-classical analysis, *J. High Energy Phys.* **07** (2014) 104.
- [62] Emil J. Martinec, Conformal field theory, geometry, and entropy, [arXiv:hep-th/9809021](https://arxiv.org/abs/hep-th/9809021).
- [63] B Fuchssteiner, T Schulze, and S Carillo, Explicit solutions for the Harry Dym equation, *J. Phys. A* **25**, 223 (1992).
- [64] Reza Mokhtari, Exact solutions of the Harry Dym equation, *Commun. Theor. Phys.* **55**, 204 (2011).
- [65] Miki Wadati, Yoshi H. Ichikawa, and Toru Shimizu, Cusp soliton of a new integrable nonlinear evolution equation, *Prog. Theor. Phys.* **64**, 1959 (1980).
- [66] D. P. Novikov, Algebraic-geometric solutions of the Harry Dym equation, *Siberian mathematical Journal* **40**, 136 (1999).
- [67] A. Roy Chowdhury and Rebecca Mukherjee, Elliptic solutions, recursion operators and complete Lie-Backlund symmetry for the Harry-Dym equation, *Phys. Scr.* **29**, 293 (1984).
- [68] L. A. Bordag, Periodical solutions of the Harry Dym equation and related problems, Report No. LEIPZIG-16-95, 1995, https://www.researchgate.net/publication/254465555_Periodical_solutions_of_the_Harry_Dym_equation_and_related_problems.
- [69] O. González-Gaxiola, J. Ruiz de Chávez, and Sunday O. Edeki, Iterative method for constructing analytical solutions to the Harry-Dym initial value problem, *Int. J. Appl. Math.* **31**, 627 (2018).

- [70] Yu Xiao and Engui Fan, Long time behavior and soliton solution for the Harry Dym equation, *J. Math. Anal. Appl.* **480**, 123248 (2019).
- [71] M. A. Assabaai and O. F. Mukherij, Exact solutions of the Harry Dym Equation using Lie group method, *Univ. Aden J. Nat. Appl. Sci.* **24**, 481 (2020).
- [72] Hongyu Li and Jian Xu, On the double-pole and two-soliton solutions of the Harry Dym equation, *Appl. Math. Lett.* **104**, 106276 (2020).
- [73] Zhu Li, Algebro-Geometric solutions of the Harry Dym hierarchy, *Int. J. Nonlinear Sci. Numer. Simul.* **18**, 129 (2017).
- [74] Maximo Banados, Global charges in Chern-Simons field theory and the $(2 + 1)$ black hole, *Phys. Rev. D* **52**, 5816 (1996).
- [75] Tullio Regge and Claudio Teitelboim, Role of surface integrals in the Hamiltonian formulation of general relativity, *Ann. Phys. (N.Y.)* **88**, 286 (1974).
- [76] Luca Ciambelli and Robert G. Leigh, Weyl Connections and their role in holography, *Phys. Rev. D* **101**, 086020 (2020).
- [77] Weizhen Jia and Manthos Karydas, Obstruction tensors in Weyl geometry and holographic Weyl anomaly, *Phys. Rev. D* **104**, 126031 (2021).
- [78] Luca Ciambelli, Arnaud Delfante, Romain Ruzziconi, and Céline Zwikel, Symmetries and charges in Weyl-Fefferman-Graham gauge, *Phys. Rev. D* **108**, 126003 (2023).
- [79] Gabriel Arenas-Henriquez, Felipe Diaz, and David Rivera-Betancour, Generalized Fefferman-Graham gauge and boundary Weyl structures, *J. High Energy Phys.* **02** (2025) 007.
- [80] Charles Fefferman and C. Robin Graham, *The Ambient Metric*, Annals of Mathematics Studies (Princeton University Press, Princeton, NJ, 2011), [10.1515/9781400840588](https://doi.org/10.1515/9781400840588).
- [81] Charles Fefferman and C Robin Graham, *Élie Cartan et les mathématiques d'aujourd'hui – Lyon*, Astérisque (Société Mathématique de France, 1985).
- [82] Kostas Skenderis and Sergey N. Solodukhin, Quantum effective action from the AdS/CFT correspondence, *Phys. Lett. B* **472**, 316 (2000).
- [83] Werner Israel, Event horizons in static vacuum spacetimes, *Phys. Rev.* **164**, 1776 (1967).
- [84] Werner Israel, Event horizons in static electrovac spacetimes, *Commun. Math. Phys.* **8**, 245 (1968).
- [85] Pawel O. Mazur, Proof of uniqueness of the Kerr-Newman black hole solution, *J. Phys. A* **15**, 3173 (1982).
- [86] Paweł Oskar Mazur, Black hole uniqueness from a hidden symmetry of Einstein's gravity, *Gen. Relativ. Gravit.* **16**, 211 (1984).
- [87] B. Carter, Axisymmetric black hole has only two degrees of freedom, *Phys. Rev. Lett.* **26**, 331 (1971).
- [88] Stephen W. Hawking, Black holes in general relativity, *Commun. Math. Phys.* **25**, 152 (1972).
- [89] Stephen W. Hawking, *Black Holes (Les Astres Occlus)* (Gordon and Breach, New York, 1973).
- [90] James M. Bardeen, Brandon Carter, and Stephen W. Hawking, The four laws of black hole mechanics, *Commun. Math. Phys.* **31**, 161 (1973).
- [91] D. C. Robinson, Uniqueness of the Kerr Black Hole, *Phys. Rev. Lett.* **34**, 905 (1975).
- [92] Stephen W. Hawking and George F.R. Ellis, *The Large Scale Structure of Space-Time* (Cambridge University Press, Cambridge, England, 2023).
- [93] Roberto Emparan and Harvey S. Reall, A rotating black ring solution in five dimensions, *Phys. Rev. Lett.* **88**, 101101 (2002).
- [94] Glenn Barnich, Cédric Troessaert, David Tempo, and Ricardo Troncoso, Asymptotically locally flat spacetimes and dynamical nonspherically-symmetric black holes in three dimensions, *Phys. Rev. D* **93**, 084001 (2016).
- [95] Oscar J. C. Dias, Gary T. Horowitz, and Jorge E. Santos, Black holes with only one Killing field, *J. High Energy Phys.* **07** (2011) 115.
- [96] M. M. Sheikh-Jabbari and H. Yavartanoo, Horizon fluffs: Near horizon soft hairs as microstates of generic AdS3 black holes, *Phys. Rev. D* **95**, 044007 (2017).
- [97] Sean Stotyn, Miok Park, Paul McGrath, and Robert B. Mann, Black holes and boson stars with one killing field in arbitrary odd dimensions, *Phys. Rev. D* **85**, 044036 (2012).
- [98] Maximo Banados, Three-dimensional quantum geometry and black holes, *AIP Conf. Proc.* **484**, 147 (1999).
- [99] Franco Magri, A simple model of the integrable Hamiltonian equation, *J. Math. Phys. (N.Y.)* **19**, 1156 (1978).
- [100] Gui-Zhang Tu, The trace identity, a powerful tool for constructing the Hamiltonian structure of integrable systems, *J. Math. Phys. (N.Y.)* **30**, 330 (1989).
- [101] P.J. Olver., Applications of lie groups to differential equations, *Graduate Texts in Mathematics* (Springer, New York, 2000).