

# Study of $B_c \rightarrow \chi_{cJ}(P,V)$ decays in the improved perturbative QCD formalism

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Motivated by the recent LHCb measurements of the ratios between the branching ratios (BRs),  $R_{\chi_{c2}/J/\psi} \equiv \text{BR}(B_c^+ \rightarrow \chi_{c2}\pi^+)/\text{BR}(B_c^+ \rightarrow J/\psi\pi^+)$  and  $R_{\chi_{c1}/\chi_{c2}} \equiv \text{BR}(B_c^+ \rightarrow \chi_{c1}\pi^+)/\text{BR}(B_c^+ \rightarrow \chi_{c2}\pi^+)$ , we analyze the decays  $B_c^+ \rightarrow \chi_{cJ}(P,V)^+$  in the improved perturbative QCD (iPQCD) formalism, where  $\chi_{cJ}$  denotes the  $p$ -wave charmonia with  $J = 0, 1, 2$ , and  $P(V)$  denotes the pseudoscalars  $\pi$  and  $K$  (the vectors  $\rho$  and  $K^*$ ). Our results  $R_{\chi_{c2}/J/\psi} = 0.32 \pm 0.05$  and  $R_{\chi_{c1}/\chi_{c2}} = 0.11 \pm 0.01$  at leading order in the strong coupling  $\alpha_s$  agree well with the data within uncertainties. The result  $R_{\chi_{c0}/J/\psi} \equiv \text{BR}(B_c^+ \rightarrow \chi_{c0}\pi^+)/\text{BR}(B_c^+ \rightarrow J/\psi\pi^+) = 0.17 \pm 0.02$  is lower than  $1.41^{+0.50}_{-0.45}$  inferred by the  $B_c^+ \rightarrow \chi_{c0}\pi^+$  measurement. The large longitudinal polarization fractions imply that these components dominate the  $B_c^+ \rightarrow \chi_{c1,c2}\rho^+$  BRs. A peculiar constructive (destructive) interference between the twist-2 and twist-3 contributions is identified in the  $B_c^+ \rightarrow \chi_{c1}\rho^+$  ( $B_c^+ \rightarrow \chi_{c1}\pi^+$ ) mode, but not in the corresponding ones with  $\chi_{c0,c2}$  mesons. Inputting the BRs of the strong decays  $\chi_{c0,c2} \rightarrow \pi^+\pi^-/K^+K^-$  and  $\chi_{cJ} \rightarrow \pi^+\pi^- (\pi^+\pi^-/K^+K^-)$  reported by the BESIII Collaboration, we obtain those of  $B_c^+ \rightarrow \chi_{cJ}(\rho, \pi)^+$  followed by secondary decay chains under the narrow-width approximation. All the above iPQCD predictions can be confronted by more precise experiments in the future to deepen our understanding of QCD dynamics in  $B_c \rightarrow \chi_{cJ}$  transitions.

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## I. INTRODUCTION

A  $B_c$  meson is the ground state of a unique meson family consisting of two different heavy flavors, i.e.,  $b$  and  $c$  quarks [1,2]. It is also the only meson whose constituent decays compete with each other. Undoubtedly, the rich and complicated dynamics in  $B_c$  meson decays, being disparate from those in charmonia and bottomonia, deserves thorough studies. On the experimental side, the LHCb has accomplished the first upgrade of its detector and resumed running since 2022 [3]. A huge amount of data with superior quality will offer a precious opportunity for analyzing the production and decay of  $B_c$  mesons and for promoting the  $B_c$  physics into a precision era. On the

theoretical side, the dramatically distinct predictions for  $B_c$  meson decays in the literature indicate that our understanding of the involved dynamics is still far from complete. More endeavors need to be devoted to this field. The present work will concentrate on the decays  $B_c^+ \rightarrow \chi_{cJ}(P,V)^+$ , where  $P(V)$  denotes the pseudoscalars  $\pi$  and  $K$  (the vectors  $\rho$  and  $K^*$ ). The  $p$ -wave charmonia  $\chi_{cJ}$  with  $J = 0, 1, 2$  represent the scalar  $\chi_{c0}(1P)$ , the axial-vector  $\chi_{c1}(1P)$ , and the tensor  $\chi_{c2}(1P)$ .

Recently, the  $B_c^+ \rightarrow \chi_{c2}\pi^+$  decay was observed for the first time by the LHCb with a significance exceeding seven standard deviations. The relative ratio between the  $B_c^+ \rightarrow \chi_{c2}\pi^+$  and  $B_c^+ \rightarrow J/\psi\pi^+$  branching ratios (BRs) was measured as [4]

$$R_{\chi_{c2}/J/\psi}^{\text{Exp}} \equiv \frac{\text{BR}(B_c^+ \rightarrow \chi_{c2}\pi^+)}{\text{BR}(B_c^+ \rightarrow J/\psi\pi^+)} = 0.37 \pm 0.06 \pm 0.02 \pm 0.01, \quad (1)$$

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where the first uncertainty is statistical, the second is systematic, and the third is associated with the  $\chi_{c2} \rightarrow J/\psi\gamma$  BR. As to the decay  $B_c^+ \rightarrow \chi_{c1}\pi^+$ , the LHCb set the upper limit of the ratio between the  $B_c^+ \rightarrow \chi_{c1}\pi^+$  and  $B_c^+ \rightarrow \chi_{c2}\pi^+$  BRs at the 90% confidence level [4],

$$R_{\chi_{c1}/\chi_{c2}}^{\text{Exp}} \equiv \frac{\text{BR}(B_c^+ \rightarrow \chi_{c1}\pi^+)}{\text{BR}(B_c^+ \rightarrow \chi_{c2}\pi^+)} < 0.49. \quad (2)$$

The LHCb also reported the evidence for the  $B_c^+ \rightarrow \chi_{c0}\pi^+$  mode at a significance of four standard deviations [5],

$$\frac{\sigma(B_c^+)}{\sigma(B^+)} \times \text{BR}(B_c^+ \rightarrow \chi_{c0}\pi^+) = (9.8_{-3.0}^{+3.4}(\text{stat}) \pm 0.8(\text{syst})) \times 10^{-6}, \quad (3)$$

with the  $B_c^+$  and  $B^+$  meson production cross sections,  $\sigma(B_c^+)$  and  $\sigma(B^+)$ , respectively. If the ratio  $\sigma(B_c^+)/\sigma(B^+)$  is determined definitely, the absolute  $B_c^+ \rightarrow \chi_{c0}\pi^+$  BR can be deduced, and vice versa.

The decays  $B_c^+ \rightarrow \chi_{cJ}\pi^+$  with the angular momenta  $J = 0, 1, 2$  have been investigated in various theoretical frameworks [6–16], and different predictions for their BRs and relative ratios are presented in the Tables I–III. It has been a consensus that the two larger  $B_c^+ \rightarrow (\chi_{c0,c2})\pi^+$  BRs

are of similar order of magnitude, and the  $B_c^+ \rightarrow \chi_{c1}\pi^+$  BR is lower. Despite this qualitative observation, the tables exhibit obvious quantitative discrepancies among the predictions. Big spreads of the results for  $\text{BR}(B_c^+ \rightarrow \chi_{c2}\pi^+)$  and for  $R_{\chi_{c2}/J/\psi}$  over three orders of magnitude are seen in Table I. Only the two ratios  $R_{\chi_{c2}/J/\psi}$  from [9,10] are roughly compatible with the recent LHCb data, though the predicted  $\text{BR}(B_c^+ \rightarrow \chi_{c2}\pi^+)$  in Refs. [6,9–12] are close to each other. A broad range of  $\text{BR}(B_c^+ \rightarrow \chi_{c1}\pi^+)$  also exists in Table II. Nevertheless, all  $R_{\chi_{c1}/\chi_{c2}}$  except the one in [11] meet the upper bound set by the LHCb. Almost all the predicted  $\text{BR}(B_c^+ \rightarrow \chi_{c0}\pi^+)$  in Table III except the one in [16] are of  $\mathcal{O}(10^{-4})$  or higher. These BRs vary significantly, and, in particular, the value from the NRQCD factorization [15] manifests huge next-to-leading order corrections. The notable disparity summarized above strongly suggests the necessity of further explorations into this challenging subject in alternative approaches.

We shall analyze the  $B_c^+ \rightarrow \chi_{cJ}(P, V)^+$  decays in the improved perturbative QCD (iPQCD) formalism, which incorporates the charm quark mass effects into the newly established Sudakov resummation [19–21], besides keeping them in the hard decay kernels as done in the conventional PQCD calculations [22]. This framework with the consistent treatment of the charm quark effects at leading

TABLE I. The  $B_c^+ \rightarrow \chi_{c2}\pi^+$  BRs and the ratios between the  $B_c^+ \rightarrow \chi_{c2}\pi^+$  and  $B_c^+ \rightarrow J/\psi\pi^+$  BRs from various theoretical approaches and the data.

|                                                                                                                            | Data [4]        | [6]  | [7]  | [8]   | [9]  | [10] | [11] | [12]                   | [13]                   | [14] <sup>a</sup>      | [15] <sup>b</sup>      | [16] <sup>c</sup>         | [17]                                          |
|----------------------------------------------------------------------------------------------------------------------------|-----------------|------|------|-------|------|------|------|------------------------|------------------------|------------------------|------------------------|---------------------------|-----------------------------------------------|
| $10^4 \cdot \text{BR}(B_c^+ \rightarrow \chi_{c2}\pi^+)$                                                                   | ...             | 2.5  | 79.0 | 0.75  | 4.6  | 2.2  | 3.8  | $2.1 \pm 0.5$          | $40.0_{-9.4}^{+10.0}$  | $43.7_{-15.2}^{+22.3}$ | $240_{-110}^{+310}$    | $0.37_{-0.12}^{+0.14}$    | $130.3_{-5.1}^{+31.7} (16.6_{-1.2}^{+4.2})$   |
| $R_{\chi_{c2}/J/\psi} \equiv \frac{\text{BR}(B_c^+ \rightarrow \chi_{c2}\pi^+)}{\text{BR}(B_c^+ \rightarrow J/\psi\pi^+)}$ | $0.37 \pm 0.06$ | 0.14 | 1.37 | 0.069 | 0.27 | 0.29 | 0.62 | $0.19_{-0.03}^{+0.03}$ | $1.72_{-0.13}^{+0.06}$ | $1.50_{-0.32}^{+0.48}$ | $8.25_{-2.86}^{+8.22}$ | $0.049_{-0.015}^{+0.014}$ | $5.21_{-0.00}^{+1.38} (6.15_{-0.21}^{+1.55})$ |

<sup>a</sup>These results are from the leading-order nonrelativistic QCD (NRQCD) with relativistic corrections.

<sup>b</sup>These results are from the next-to-leading-order NRQCD. The corresponding leading-order BR is given by  $(9.7_{-4.5}^{+14.0}) \times 10^{-3}$ .

<sup>c</sup>The values in the parentheses were obtained without Coulomb-like corrections in the same framework based on three-point QCD sum rules.

TABLE II. Same as Table I but for the decay  $B_c^+ \rightarrow \chi_{c1}\pi^+$ .

|                                                                                                                                  | Data [4] | [6]  | [7]  | [9]  | [10]  | [11] | [12]                   | [13]                   | [14]                   | [15]                   | [17]                                          | [18]                   |
|----------------------------------------------------------------------------------------------------------------------------------|----------|------|------|------|-------|------|------------------------|------------------------|------------------------|------------------------|-----------------------------------------------|------------------------|
| $10^4 \cdot \text{BR}(B_c^+ \rightarrow \chi_{c1}\pi^+)$                                                                         | ...      | 0.7  | 0.89 | 0.68 | 0.014 | 2.0  | $0.21 \pm 0.02$        | $5.1_{-1.2}^{+1.1}$    | $0.64_{-0.16}^{+0.11}$ | $2.3_{-1.0}^{+2.8}$    | $0.70_{-0.26}^{+0.25} (0.11_{-0.04}^{+0.05})$ | 1.3                    |
| $R_{\chi_{c1}/\chi_{c2}} \equiv \frac{\text{BR}(B_c^+ \rightarrow \chi_{c1}\pi^+)}{\text{BR}(B_c^+ \rightarrow \chi_{c2}\pi^+)}$ | $< 0.49$ | 0.28 | 0.01 | 0.15 | 0.01  | 0.53 | $0.10_{-0.01}^{+0.02}$ | $0.13_{-0.01}^{+0.00}$ | 0.02                   | 0.01                   | 0.005(0.007)                                  | ...                    |
| $R_{\chi_{c1}/J/\psi} \equiv \frac{\text{BR}(B_c^+ \rightarrow \chi_{c1}\pi^+)}{\text{BR}(B_c^+ \rightarrow J/\psi\pi^+)}$       | ...      | 0.04 | 0.02 | 0.04 | 0.002 | 0.33 | 0.02                   | $0.22_{-0.02}^{+0.01}$ | 0.02                   | $0.08_{-0.03}^{+0.07}$ | $0.03_{-0.01}^{+0.00} (0.04_{-0.00}^{+0.01})$ | $0.07_{-0.01}^{+0.01}$ |

TABLE III. Same as Table I but for the decay  $B_c^+ \rightarrow \chi_{c0}\pi^+$ .

|                                                                                                                                  | Data | [6]  | [7]  | [9]  | [10] | [11] | [12]                   | [13]                   | [14]                   | [15]                    | [16]                   | [17]                                          | [18]                |
|----------------------------------------------------------------------------------------------------------------------------------|------|------|------|------|------|------|------------------------|------------------------|------------------------|-------------------------|------------------------|-----------------------------------------------|---------------------|
| $10^4 \cdot \text{BR}(B_c^+ \rightarrow \chi_{c0}\pi^+)$                                                                         | ...  | 2.8  | 98.0 | 5.5  | 2.6  | 2.1  | $3.1 \pm 0.4$          | $16.0_{-3.7}^{+3.6}$   | $64.7_{-13.2}^{+18.6}$ | $280_{-130}^{+380}$     | $0.24_{-0.08}^{+0.09}$ | $19.6_{-3.0}^{+2.1} (2.4_{-0.4}^{+0.5})$      | $6.6_{-0.7}^{+0.8}$ |
| $R_{\chi_{c0}/\chi_{c2}} \equiv \frac{\text{BR}(B_c^+ \rightarrow \chi_{c0}\pi^+)}{\text{BR}(B_c^+ \rightarrow \chi_{c2}\pi^+)}$ | ...  | 1.12 | 1.24 | 1.20 | 1.18 | 0.55 | $1.48_{-0.13}^{+0.21}$ | $0.40_{-0.00}^{+0.01}$ | $1.48_{-0.22}^{+0.33}$ | $1.17_{-0.02}^{+0.03}$  | 0.658                  | $0.15_{-0.02}^{+0.00} (0.14_{-0.01}^{+0.00})$ | ...                 |
| $R_{\chi_{c0}/J/\psi} \equiv \frac{\text{BR}(B_c^+ \rightarrow \chi_{c0}\pi^+)}{\text{BR}(B_c^+ \rightarrow J/\psi\pi^+)}$       | ...  | 0.16 | 1.70 | 0.32 | 0.34 | 0.34 | $0.28 \pm 0.01$        | $0.69_{-0.07}^{+0.03}$ | $2.22_{-0.08}^{+0.27}$ | $9.62_{-3.40}^{+10.14}$ | 0.032                  | $0.78_{-0.08}^{+0.09} (0.89_{-0.06}^{+0.11})$ | 0.34                |

order in the strong coupling  $\alpha_s$  is thus expected to refine the previous PQCD studies on hadronic  $B_c$  meson decays. It has been demonstrated that the iPQCD predictions for the  $B_c^+ \rightarrow J/\psi M^+$  modes, which are dominated by factorizable emission contributions, agree well with the available data [19]. It will be shown that the relative ratios among the  $B_c^+ \rightarrow (J/\psi, \chi_{c1,c2})\pi^+$  BRs from the iPQCD also match the LHCb measurements. The longitudinal polarization fractions for the  $B_c^+ \rightarrow \chi_{c1,c2}\rho^+$  decays are computed, and found to be nearly unity. Inputting the BRs of the strong decays  $\chi_{c0,c2} \rightarrow \pi^+\pi^-/K^+K^-$  [23] and  $\chi_{cJ} \rightarrow \pi^+\pi^-(\pi^+\pi^-/K^+K^-)$  [24,25], we estimate those of multi-body  $B_c$  meson decays via charmonium resonances under the narrow-width approximation. This work justifies the reliability of our theoretical setup for  $B_c$  meson decays, explores the involved QCD dynamics, and helps probe the inner structure of charmonium states.

The rest of the paper is organized as follows. We present the iPQCD factorization formulas in Sec. II, which convolute the perturbative  $b$  quark decay kernels with the nonperturbative wave functions for the initial and final states in parton transverse momenta [20]. Numerical outcomes and related phenomenological insights are discussed and elaborated in Sec. III. Section IV contains the summary.

## II. FORMALISM

The weak effective Hamiltonian  $H_{\text{eff}}$  for the decays  $B_c^+ \rightarrow \chi_{cJ}(P, V)^+$  is given by [26]

$$H_{\text{eff}} = \frac{G_F}{\sqrt{2}} \left\{ V_{cb}^* V_{uq} [C_1(\mu) O_1(\mu) + C_2(\mu) O_2(\mu)] \right\} + \text{H.c.}, \quad (4)$$

with the Fermi constant  $G_F = 1.16639 \times 10^{-5} \text{ GeV}^{-2}$ , the Cabibbo-Kobayashi-Maskawa (CKM) matrix elements  $V_{cb}$  and  $V_{uq}$  [27,28],  $q$  being the light quark  $d$  or  $s$ , and the local four-quark tree operators

$$\begin{aligned} O_1 &= \bar{q}_\alpha \gamma_\mu (1 - \gamma_5) u_\beta \bar{c}_\beta \gamma_\mu (1 - \gamma_5) b_\alpha, \\ O_2 &= \bar{q}_\alpha \gamma_\mu (1 - \gamma_5) u_\alpha \bar{c}_\beta \gamma_\mu (1 - \gamma_5) b_\beta. \end{aligned} \quad (5)$$

The Wilson coefficients  $C_i(\mu)$  sum the large logarithms  $\ln(m_W/\mu)$  to all orders in the strong coupling  $\alpha_s$ , where  $m_W$  is the  $W$  boson mass and  $\mu$  is the renormalization scale.

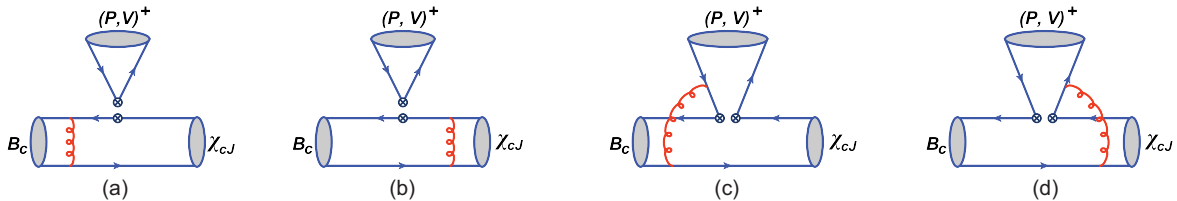


FIG. 1. Leading-order Feynman diagrams for the decays  $B_c^+ \rightarrow \chi_{cJ}(P, V)^+$  in the iPQCD framework, with (a), (b) and (c), (d) contributing to the factorizable and nonfactorizable emission amplitudes correspondingly.

We will not address  $CP$  violation here due to the absence of penguin operators.

Figure 1 depicts the leading-order Feynman diagrams for the  $B_c^+ \rightarrow \chi_{cJ}(P, V)^+$  decays in the iPQCD framework, whose evaluation is similar to that of  $B_c^+ \rightarrow J/\psi M^+$ . The factorization of the  $B_c^+ \rightarrow \chi_{cJ}(P, V)^+$  decay amplitudes is formulated, according to [19], as

$$\begin{aligned} A(B_c^+ \rightarrow \chi_{cJ}(P, V)^+) &\equiv \langle \chi_{cJ}(P, V)^+ | H_{\text{eff}} | B_c^+ \rangle \\ &\sim \int dx_1 dx_2 dx_3 b_1 db_1 b_2 db_2 b_3 db_3 \\ &\quad \cdot \text{Tr} [C(t) \Phi_{B_c}(x_1, b_1) \Phi_{P,V}(x_2, b_2) \Phi_{\chi_{cJ}} \\ &\quad \times (x_3, b_3) H(x_i, b_i, t) e^{-S(t)}], \end{aligned} \quad (6)$$

where  $x_i$  are the longitudinal quark momentum fractions, the variables  $b_i$  are conjugate to the quark transverse momenta  $k_{iT}$ ,  $t$  is set to the largest running energy scale involved in the hard kernel  $H$ ,  $\text{Tr}$  is the abbreviation of the traces over Dirac and color indices, and the wave functions  $\Phi$  describe the quark momentum distributions within mesons. The Sudakov factor  $e^{-S(t)}$ , whose explicit expression was derived in [19–21,29], strongly suppresses the long-distance (large- $b$ ) contributions [29,30]. We emphasize that the present formalism has included the charm quark mass effects into the Sudakov factor, compared to the conventional PQCD approach to the  $B_c^+ \rightarrow \chi_{cJ}(P, V)^+$  decays [13].

We specify the meson distribution amplitudes needed for implementing the factorization formula in Eq. (6). The knowledge about the  $\chi_{cJ}$  meson distribution amplitudes  $\Phi_{\chi_{cJ}}(x)$ , whose dependence on the intrinsic  $b$  dependence has been ignored, is still quite limited. We employ the models in Refs. [13,31,32]. The twist-2 distribution amplitude  $\phi_{\chi_{c0}}^v(x)$  and the twist-3 one  $\phi_{\chi_{c0}}^s(x)$  of the scalar  $\chi_{c0}$  are defined via

$$\Phi_{\chi_{c0}}(x) = \frac{i}{\sqrt{2N_c}} \{ \mathcal{P} \phi_{\chi_{c0}}^v(x) + m_{\chi_{c0}} \phi_{\chi_{c0}}^s(x) \}, \quad (7)$$

with the  $\chi_{c0}$  meson momentum  $P$  and mass  $m_{\chi_{c0}}$ , and the charm quark momentum fraction  $x$ . They are assumed to take the asymptotic forms

$$\phi_{\chi_{c0}}^v(x) = 53.74 \frac{f_{\chi_{c0}}}{2\sqrt{2N_c}} [x(1-x)(1-2x)] \mathcal{C}(x), \quad (8)$$

$$\phi_{\chi_{c0}}^s(x) = 2.12 \frac{f_{\chi_{c0}}}{2\sqrt{2N_c}} \mathcal{C}(x), \quad (9)$$

with the decay constant  $f_{\chi_{c0}}$ . The function  $\mathcal{C}(x)$  is extracted from the  $p$ -wave Schrödinger states in a Coulomb potential (details can be found in the appendix of Ref. [13]),

$$\mathcal{C}(x) = \{[x(1-x)(1-4x(1-x))^3]^{1/2} / [1-4x(1-x)(1-v^2/4)]^2\}^{1-v^2}, \quad (10)$$

where  $v^2 = 0.3$  [32] stands for small relativistic corrections to the Coulomb wave functions [33]. The distribution amplitudes  $\phi_{\chi_{c0}}^v(x)$  and  $\phi_{\chi_{c0}}^s(x)$  are antisymmetric and symmetric, respectively, to satisfy the charge conjugation invariance of scalars [34,35].

Note that the numerical coefficients of the charmonium distribution amplitudes in Eqs. (8) and (9) are chosen to meet the normalization conditions,

$$\int_0^1 dx \phi_{\text{sym}}(x) = \frac{f}{2\sqrt{2N_c}}, \quad (11)$$

for the symmetric distribution amplitudes, and

$$\int_0^1 dx \phi_{\text{asym}}(x)(1-2x) = \frac{f}{2\sqrt{2N_c}}, \quad (12)$$

for the antisymmetric ones, where  $f$  denotes a decay constant. Such numerical factors appear, because the charmonium distribution amplitudes are not expanded in terms of the Gegenbauer polynomials like light meson distribution amplitudes.

For the axial-vector  $\chi_{c1}$ , the twist-2 distribution amplitude  $\phi_{\chi_{c1}}^{L(T)}(x)$  and the twist-3 distribution amplitude  $\phi_{\chi_{c1}}^{t(v)}(x)$  associated with the longitudinal (transverse) polarization are defined via

$$\Phi_{\chi_{c1}}^L(x) = \frac{1}{\sqrt{2N_c}} \gamma_5 \{m_{\chi_{c1}} \epsilon_{\chi_{c1}}^L \phi_{\chi_{c1}}^L(x) + \epsilon_{\chi_{c1}}^L \not{P} \phi_{\chi_{c1}}^t(x)\}, \quad (13)$$

$$\Phi_{\chi_{c1}}^T(x) = \frac{1}{\sqrt{2N_c}} \gamma_5 \{m_{\chi_{c1}} \epsilon_{\chi_{c1}}^T \phi_{\chi_{c1}}^v(x) + \epsilon_{\chi_{c1}}^T \not{P} \phi_{\chi_{c1}}^T(x)\}, \quad (14)$$

where  $m_{\chi_{c1}}$  is the  $\chi_{c1}$  meson mass, and  $\epsilon^L$  ( $\epsilon^T$ ) denotes the longitudinal (transverse) polarization vector of  $\chi_{c1}$ . These distribution amplitudes are written as

$$\phi_{\chi_{c1}}^L(x) = 12.60 \frac{f_{\chi_{c1}}}{2\sqrt{2N_c}} [x(1-x)] \mathcal{C}(x), \quad (15)$$

$$\phi_{\chi_{c1}}^T(x) = 53.74 \frac{f_{\chi_{c1}}^\perp}{2\sqrt{2N_c}} [x(1-x)(1-2x)] \mathcal{C}(x), \quad (16)$$

$$\phi_{\chi_{c1}}^t(x) = 23.16 \frac{f_{\chi_{c1}}^\perp}{2\sqrt{2N_c}} [(1-2x)(1-6x+6x^2)] \mathcal{C}(x), \quad (17)$$

$$\phi_{\chi_{c1}}^v(x) = 1.59 \frac{f_{\chi_{c1}}}{2\sqrt{2N_c}} [1 + (1-2x)^2] \mathcal{C}(x), \quad (18)$$

with the decay constants  $f_{\chi_{c1}}$  and  $f_{\chi_{c1}}^\perp$ .

A similar decomposition applies to the distribution amplitudes for longitudinally and transversely polarized tensors  $\chi_{c2}$ ,

$$\Phi_{\chi_{c2}}^L(x) = \frac{1}{\sqrt{2N_c}} \{m_{\chi_{c2}} \epsilon_{\chi_{c2}}^L \phi_{\chi_{c2}}^L(x) + \epsilon_{\chi_{c2}}^L \not{P} \phi_{\chi_{c2}}^t(x)\}, \quad (19)$$

$$\Phi_{\chi_{c2}}^T(x) = \frac{1}{\sqrt{2N_c}} \{m_{\chi_{c2}} \epsilon_{\chi_{c2}}^T \phi_{\chi_{c2}}^v(x) + \epsilon_{\chi_{c2}}^T \not{P} \phi_{\chi_{c2}}^T(x)\}, \quad (20)$$

respectively, with the twist-2 components

$$\phi_{\chi_{c2}}^L(x) = 53.74 \frac{f_{\chi_{c2}}}{2\sqrt{2N_c}} [x(1-x)(1-2x)] \mathcal{C}(x), \quad (21)$$

$$\phi_{\chi_{c2}}^T(x) = 53.74 \frac{f_{\chi_{c2}}^\perp}{2\sqrt{2N_c}} [x(1-x)(1-2x)] \mathcal{C}(x), \quad (22)$$

and the twist-3 ones

$$\phi_{\chi_{c2}}^t(x) = 23.16 \frac{f_{\chi_{c2}}^\perp}{2\sqrt{2N_c}} \times [(1-2x)(1-6x+6x^2)] \mathcal{C}(x), \quad (23)$$

$$\phi_{\chi_{c2}}^v(x) = 12.44 \frac{f_{\chi_{c2}}}{2\sqrt{2N_c}} [(1-2x)^3] \mathcal{C}(x), \quad (24)$$

$m_{\chi_{c2}}$  being the  $\chi_{c2}$  meson mass, and  $f_{\chi_{c2}}$  and  $f_{\chi_{c2}}^\perp$  being the decay constants.

The  $B_c$  meson wave function  $\Phi_{B_c}(x, b)$  and the distribution amplitudes  $\Phi_{P,V}(x)$  of the light pseudoscalar and vector mesons have been collected in [19], which are not shown for simplicity. For the inputs, such as the decay constants, the shape parameter  $\beta_{B_c}$  in  $\Phi_{B_c}(x, b)$  and the Gegenbauer moments in  $\Phi_{P,V}(x)$  refer to [19]. The related measurements, as well as lattice QCD or QCD sum rule calculations, are expected to enhance the accuracy of the above nonperturbative quantities.

We construct the  $B_c^+ \rightarrow \chi_{cJ}(P, V)^+$  decay amplitudes in Eq. (6) by convoluting the hard kernels  $H(x_i, b_i, t)$  from Fig. 1 with the aforementioned meson distribution amplitudes. The factorization formulas read

$$A = V_{cb}^* V_{uq} \left[ \left( \frac{1}{3} C_1 + C_2 \right) F_e \cdot f_P + C_1 M_e \right] \quad (25)$$

for  $B_c^+ \rightarrow \chi_{cJ}P^+$  and  $B_c^+ \rightarrow \chi_{c0}V^+$ , and

$$A^h = V_{cb}^* V_{uq} \left[ \left( \frac{1}{3} C_1 + C_2 \right) F_e^h \cdot f_V + C_1 M_e^h \right] \quad (26)$$

for  $B_c^+ \rightarrow \chi_{c1,c2}V^+$  with the vector meson polarizations  $h = L, N, T$ , where  $f_{P(V)}$  represents the decay constant of the light pseudoscalar (vector) meson. The expressions of the factorizable (nonfactorizable) emission amplitudes  $F_e^{(h)}$  ( $M_e^{(h)}$ ) induced by the  $(V-A)(V-A)$  operators in Eq. (5) have been provided in Eqs. (32)–(48) of [13]. The corresponding BR is then given, in the rest frame of a heavy  $B_c$  meson by

$$\begin{aligned} \text{BR}(B_c^+ \rightarrow \chi_{cJ}(P, V)^+) & \equiv \tau_{B_c} \cdot \Gamma(B_c^+ \rightarrow \chi_{cJ}(P, V)^+) \\ & = \tau_{B_c} \cdot \frac{G_F^2 |\mathbf{P}_c|}{16\pi m_{B_c}^2} |A(B_c^+ \rightarrow \chi_{cJ}(P, V)^+)|^2, \end{aligned} \quad (27)$$

where  $\tau_{B_c} = 0.507$  ps is the  $B_c$  meson lifetime,  $\Gamma$  is the decay width, and  $|\mathbf{P}_c| \equiv |\mathbf{P}_{P(V)}| = |\mathbf{P}_{\chi_{cJ}}| = \sqrt{\lambda(m_{B_c}^2, m_{P(V)}^2, m_{\chi_{cJ}}^2)} / (2m_{B_c})$  is the momentum of either the  $P(V)$  or  $\chi_{cJ}$  meson in the final state, with the  $B_c$  ( $P(V)$ ) meson mass  $m_{B_c}$  ( $m_{P(V)}$ ) and the Källén function  $\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz$  [24].

### III. NUMERICAL RESULTS AND DISCUSSIONS

To proceed with the numerical study, we adopt the following QCD scale and meson masses (in units of GeV) [24,30]:

$$\begin{aligned} \Lambda_{\overline{\text{MS}}}^{(f=4)} &= 0.250, & m_W &= 80.41, \\ m_{B_c} &= 6.275, & m_{\chi_{c2}} &= 3.556, \\ m_{\chi_{c1}} &= 3.511, & m_{\chi_{c0}} &= 3.415, \\ m_b &= 4.8, & m_c &= 1.5, \end{aligned} \quad (28)$$

and the CKM matrix elements [24]

$$\begin{aligned} |V_{cb}| &= 0.04182_{-0.00074}^{+0.00085}, \\ |V_{ud}| &= 0.97435 \pm 0.00016, \\ |V_{us}| &= 0.22500 \pm 0.00067. \end{aligned} \quad (29)$$

The  $\chi_{cJ}$  meson decay constants at the scale  $\mu = m_c$  are set to (in units of GeV) [36]

$$\begin{aligned} f_{\chi_{c0}} &= 0.093 \pm 0.009, \\ f_{\chi_{c1}} &= 0.185 \pm 0.019, \\ f_{\chi_{c1}}^\perp &= 0.090 \pm 0.009, \\ f_{\chi_{c2}} &= 0.181 \pm 0.018, \\ f_{\chi_{c2}}^\perp &= 0.131 \pm 0.013, \end{aligned} \quad (30)$$

for which 10% variations of the central values have been assigned for assessing the potential theoretical errors. As stated in [20], the charmonium distribution amplitudes are defined at the charm quark mass scale in the iPQCD framework, so are the associated decay constants. Note that the decay constants in Ref. [36] were derived at the scale  $m_c = 1.628$  GeV, while the charm quark mass is chosen as  $m_c = 1.5$  GeV in Eq. (28). Therefore, the renormalization-group evolution [37]

$$f_{c\bar{c}}(\mu) = f_{c\bar{c}}(\mu_0) \left( \frac{\alpha_s(\mu)}{\alpha_s(\mu_0)} \right)^{\gamma/b_0}, \quad (31)$$

with the coefficient  $b_0 = 11 - 2n_f/3$ ,  $n_f$  being the active flavor number, should be taken into account. The anomalous dimensions take  $\gamma = 8C_F/3$  for  $f_{\chi_{c0}}$  and  $f_{\chi_{c2}}$  with the color factor  $C_F = 4/3$ ,  $\gamma = 3C_F$  for  $f_{\chi_{c1}}^\perp$  and  $f_{\chi_{c2}}^\perp$ , and  $\gamma = 0$  for  $f_{\chi_{c1}}$  [37]. Equation (30) represents the results of the evolution from 1.628 GeV down to 1.5 GeV.

#### A. $B_c^+ \rightarrow \chi_{cJ}P^+$

We present the iPQCD predictions for the BRs and ratios associated with the decays  $B_c^+ \rightarrow \chi_{cJ}(P, V)^+$ . The  $CP$ -averaged  $B_c^+ \rightarrow \chi_{c2}P^+$  BRs are written as

$$\text{BR}(B_c^+ \rightarrow \chi_{c2}\pi^+) = 3.75_{-0.80}^{+1.07} (\beta_{B_c})_{-0.80}^{+0.91} (f_{c\bar{c}})_{-0.00}^{+0.01} (a_\pi)_{-0.13}^{+0.16} (V_{cb}) \times 10^{-4}, \quad (32)$$

$$\text{BR}(B_c^+ \rightarrow \chi_{c2}K^+) = 2.75_{-0.57}^{+0.74} (\beta_{B_c})_{-0.59}^{+0.66} (f_{c\bar{c}})_{-0.21}^{+0.22} (a_K)_{-0.10}^{+0.11} (V_{cb}) \times 10^{-5}, \quad (33)$$

where the errors mainly arise from the shape parameter  $\beta_{B_c}$  and the less constrained decay constants  $f_{\chi_{c2}}$  and  $f_{\chi_{c2}}^\perp$ . As mentioned in the Introduction, the decays  $B_c^+ \rightarrow \chi_{c2}P^+$  have been investigated extensively in the literature [6–17]

with dramatically different outcomes. Our prediction for  $\text{BR}(B_c^+ \rightarrow \chi_{c2}\pi^+)$  is consistent with those from Refs. [9,11] in Table I within uncertainties. The value in Eq. (32) is lower than those in the conventional PQCD approach by an

order of magnitude [13] and in the leading-order NRQCD factorization by a factor of about 25 [15]. Owing to the unclear production cross section for  $B_c$  mesons, one has to consider relative ratios between BRs of  $B_c$  meson decays. As shown in Eq. (1), the  $B_c^+ \rightarrow \chi_{c2}\pi^+$  channel was observed in terms of its BR relative to the  $B_c^+ \rightarrow J/\psi\pi^+$  one. Combined with the  $B_c^+ \rightarrow J/\psi\pi^+$  BR in the iPQCD

formalism [19], the ratio is given by

$$R_{\chi_{c2}/J/\psi} = 0.32 \pm 0.05, \quad (34)$$

where errors from various sources have been added in quadrature. It is compatible with the data in Eq. (1), and also with the predictions in [9,10].

We obtain the  $B_c^+ \rightarrow \chi_{c0}P^+$  and  $B_c^+ \rightarrow \chi_{c1}P^+$  BRs,

$$\text{BR}(B_c^+ \rightarrow \chi_{c0}\pi^+) = 2.02_{-0.41}^{+0.57}(\beta_{B_c})_{-0.37}^{+0.41}(f_{c\bar{c}})_{-0.00}^{+0.00}(a_\pi)_{-0.07}^{+0.08}(V_{ub}) \times 10^{-4}, \quad (35)$$

$$\text{BR}(B_c^+ \rightarrow \chi_{c0}K^+) = 1.47_{-0.29}^{+0.40}(\beta_{B_c})_{-0.27}^{+0.30}(f_{c\bar{c}})_{-0.11}^{+0.12}(a_K)_{-0.05}^{+0.06}(V_{cb}) \times 10^{-5}, \quad (36)$$

and

$$\text{BR}(B_c^+ \rightarrow \chi_{c1}\pi^+) = 4.16_{-0.62}^{+0.68}(\beta_{B_c})_{-1.12}^{+1.26}(f_{c\bar{c}})_{-0.02}^{+0.00}(a_\pi)_{-0.14}^{+0.17}(V_{cb}) \times 10^{-5}, \quad (37)$$

$$\text{BR}(B_c^+ \rightarrow \chi_{c1}K^+) = 2.65_{-0.36}^{+0.37}(\beta_{B_c})_{-0.74}^{+0.83}(f_{c\bar{c}})_{-0.59}^{+0.67}(a_K)_{-0.10}^{+0.10}(V_{cb}) \times 10^{-6}, \quad (38)$$

respectively. It is seen from Table III that the iPQCD result for  $\text{BR}(B_c^+ \rightarrow \chi_{c0}\pi^+)$  matches those in Refs. [10,11] and in [17] without Coulomb-like corrections. The decay  $B_c^+ \rightarrow \chi_{c1}\pi^+$  has not been observed, and the upper bound of its BR has been imposed by the LHCb as in Eq. (2). Our predicted  $\text{BR}(B_c^+ \rightarrow \chi_{c1}\pi^+)$  agrees with those from Refs. [14,17] with Coulomb-like corrections in Table II. We also compute the ratio between the  $B_c^+ \rightarrow \chi_{c1}\pi^+$  and  $B_c^+ \rightarrow \chi_{c2}\pi^+$  BRs,

$$R_{\chi_{c1}/\chi_{c2}} = 0.11_{-0.01}^{+0.01}, \quad (39)$$

which satisfies Eq. (2). It has been pointed out that most of the predictions for this ratio, spanning a wide range as listed in Table II, respect Eq. (2). More precise data will help discriminate the different theoretical approaches.

The evidence of the  $B_c^+ \rightarrow \chi_{c0}\pi^+$  mode was reported in association with the ratio  $\sigma(B_c^+)/\sigma(B^+)$  of the  $B_c^+$  and  $B^+$  production cross sections in Eq. (3). Note that these cross sections include contributions from excited states. The same ratio  $\sigma(B_c^+)/\sigma(B^+)$  appears in the measurements by the LHCb [38,39],

$$R_{c/u}^{\text{LHCb}} \equiv \frac{\sigma(B_c^+)}{\sigma(B^+)} \cdot \frac{\text{BR}(B_c^+ \rightarrow J/\psi\pi^+)}{\text{BR}(B^+ \rightarrow J/\psi K^+)} = \begin{cases} (0.61 \pm 0.12)\% & (p_T > 4 \text{ GeV}, 2.5 < \eta < 4.5) \\ (0.698 \pm 0.023)\% & (4 < p_T < 20 \text{ GeV}, 2.5 < \eta < 4.5), \\ (0.683 \pm 0.020)\% & (0 < p_T < 20 \text{ GeV}, 2.0 < y < 4.5) \end{cases} \quad (40)$$

and by the CMS and ATLAS collaborations [40,41],

$$R_{c/u}^{\text{CMS}} \equiv \frac{\sigma(B_c^+)}{\sigma(B^+)} \cdot \frac{\text{BR}(B_c^+ \rightarrow J/\psi\pi^+)}{\text{BR}(B^+ \rightarrow J/\psi K^+)} = (0.48 \pm 0.08)\% \quad (p_T > 15 \text{ GeV}, |y| < 1.6), \quad (41)$$

$$R_{c/u}^{\text{ATLAS}} \equiv \frac{\sigma(B_c^+)}{\sigma(B^+)} \cdot \frac{\text{BR}(B_c^+ \rightarrow J/\psi\pi^+)}{\text{BR}(B^+ \rightarrow J/\psi K^+)} = (0.34_{-0.05}^{+0.07})\% \quad (p_T > 13 \text{ GeV}, |y| < 2.3), \quad (42)$$

with the bottom-meson transverse momentum  $p_T$  and (pseudo)rapidity ( $\eta$ )y.

Given the measured  $\text{BR}(B^+ \rightarrow J/\psi K^+) = (1.020 \pm 0.019) \times 10^{-3}$  [24], we extract  $\frac{\sigma(B_c^+)}{\sigma(B^+)} \times \text{BR}(B_c^+ \rightarrow J/\psi \pi^+) = (6.97 \pm 0.13) \times 10^{-6}$  from Eq. (40) in the fiducial region corresponding to the transverse momentum  $0 < p_T < 20$  GeV and the rapidity  $2.0 < y < 4.5$ . The LHCb data  $\frac{\sigma(B_c^+)}{\sigma(B^+)} \times \text{BR}(B_c^+ \rightarrow \chi_{c0} \pi^+) = (9.8^{+3.5}_{-3.1}) \times 10^{-6}$  from the same fiducial region in Eq. (3) then yields<sup>1</sup>

$$R_{\chi_{c0}/J/\psi}^{\text{Exp}} \equiv \frac{\text{BR}(B_c^+ \rightarrow \chi_{c0} \pi^+)}{\text{BR}(B_c^+ \rightarrow J/\psi \pi^+)} = \frac{\frac{\sigma(B_c^+)}{\sigma(B^+)} \times \text{BR}(B_c^+ \rightarrow \chi_{c0} \pi^+)}{\frac{\sigma(B_c^+)}{\sigma(B^+)} \times \text{BR}(B_c^+ \rightarrow J/\psi \pi^+)} = 1.41^{+0.50}_{-0.45}. \quad (43)$$

Our prediction for the above ratio

$$R_{\chi_{c0}/J/\psi} = 0.17^{+0.02}_{-0.02}, \quad (44)$$

seems to deviate from Eq. (43). On one hand, the data suffer large uncertainties. On the other hand, the nonperturbative inputs related to a  $\chi_{c0}$  meson are still poorly known. If the decay constant  $f_{\chi_{c0}}$  takes a value around 0.3 GeV [17,32,42], the  $B_c^+ \rightarrow \chi_{c0} \pi^+$  BR in Eq. (44) can be greatly amplified by a factor of 10, and the consistency with the data will be improved.

Analogous to Eqs. (39) and (44), we consider the ratios,

$$R_{\chi_{c1}/J/\psi} \equiv \frac{\text{BR}(B_c^+ \rightarrow \chi_{c1} \pi^+)}{\text{BR}(B_c^+ \rightarrow J/\psi \pi^+)} = (3.56^{+0.81}_{-0.82}) \times 10^{-2}, \quad (45)$$

which agrees with the predictions from Refs. [6,9,17] in Table II, and

$$R_{\chi_{c0}/\chi_{c2}} \equiv \frac{\text{BR}(B_c^+ \rightarrow \chi_{c0} \pi^+)}{\text{BR}(B_c^+ \rightarrow \chi_{c2} \pi^+)} = 0.54^{+0.02}_{-0.02}, \quad (46)$$

$$R_{\chi_{c1}/\chi_{c0}} \equiv \frac{\text{BR}(B_c^+ \rightarrow \chi_{c1} \pi^+)}{\text{BR}(B_c^+ \rightarrow \chi_{c0} \pi^+)} = 0.21^{+0.01}_{-0.04}, \quad (47)$$

which await future experimental scrutiny.

The ratio between the  $B_c^+ \rightarrow (c\bar{c})K^+$  and  $B_c^+ \rightarrow (c\bar{c})\pi^+$  BRs, governed by the factorizable emission amplitudes [19],

is simplified into

$$R_{K/\pi} \equiv \frac{\text{BR}(B_c^+ \rightarrow (c\bar{c})K^+)}{\text{BR}(B_c^+ \rightarrow (c\bar{c})\pi^+)} \approx \frac{f_K^2}{f_\pi^2} \cdot \frac{|V_{us}|^2}{|V_{ud}|^2} = (7.95^{+0.05}_{-0.01}) \times 10^{-2}, \quad (48)$$

with the kaon (pion) decay constant  $f_K$  ( $f_\pi$ ). The observed departures from Eq. (48) signify the nonfactorizable contributions in the considered channels. To be specific, we present the ratios between the  $B_c^+ \rightarrow \chi_{cJ}K^+$  and  $B_c^+ \rightarrow \chi_{cJ}\pi^+$  BRs,

$$R_{K/\pi}^{\chi_{c2}} \equiv \frac{\text{BR}(B_c^+ \rightarrow \chi_{c2} K^+)}{\text{BR}(B_c^+ \rightarrow \chi_{c2} \pi^+)} = (7.33^{+0.57}_{-0.57}) \times 10^{-2}, \quad (49)$$

$$R_{K/\pi}^{\chi_{c0}} \equiv \frac{\text{BR}(B_c^+ \rightarrow \chi_{c0} K^+)}{\text{BR}(B_c^+ \rightarrow \chi_{c0} \pi^+)} = (7.28^{+0.68}_{-0.56}) \times 10^{-2}, \quad (50)$$

$$R_{K/\pi}^{\chi_{c1}} \equiv \frac{\text{BR}(B_c^+ \rightarrow \chi_{c1} K^+)}{\text{BR}(B_c^+ \rightarrow \chi_{c1} \pi^+)} = (6.37^{+1.61}_{-1.40}) \times 10^{-2}, \quad (51)$$

where the uncertainties are dominated by the variation of the Gegenbauer moment  $a_1^K$  in the twist-2 kaon distribution amplitude. Though Eqs. (49)–(51) basically align with the naive expectation in Eq. (48), they reveal a dependence of the nonfactorizable emission contributions on the  $\chi_{cJ}$  final states.

The BESIII Collaboration reported their latest measured BRs of the strong decays  $\chi_{c0,c2} \rightarrow \pi^+ \pi^-$  and  $K^+ K^-$  [23],

$$\begin{aligned} \text{BR}(\chi_{c0} \rightarrow \pi^+ \pi^-) &= (0.606 \pm 0.015)\%, \\ \text{BR}(\chi_{c0} \rightarrow K^+ K^-) &= (0.636 \pm 0.015)\%, \end{aligned} \quad (52)$$

$$\begin{aligned} \text{BR}(\chi_{c2} \rightarrow \pi^+ \pi^-) &= (0.161 \pm 0.005)\%, \\ \text{BR}(\chi_{c2} \rightarrow K^+ K^-) &= (0.122 \pm 0.004)\%, \end{aligned} \quad (53)$$

where errors from various sources have been added in quadrature. With these data, one can deduce BRs of multibody  $B_c$  meson decays via charmonium resonances under the narrow-width approximation. The  $B_c^+ \rightarrow \pi^+ \chi_{c0,c2} (\rightarrow \pi^+ \pi^- / K^+ K^-)$  BRs are then found to be

$$\text{BR}(B_c^+ \rightarrow \chi_{c0} (\rightarrow \pi^+ \pi^-) \pi^+) \equiv \text{BR}(B_c^+ \rightarrow \chi_{c0} \pi^+) \cdot \text{BR}(\chi_{c0} \rightarrow \pi^+ \pi^-) = (1.22^{+0.43}_{-0.34}) \times 10^{-6}, \quad (54)$$

$$\text{BR}(B_c^+ \rightarrow \chi_{c0} (\rightarrow K^+ K^-) \pi^+) \equiv \text{BR}(B_c^+ \rightarrow \chi_{c0} \pi^+) \cdot \text{BR}(\chi_{c0} \rightarrow K^+ K^-) = (1.28^{+0.45}_{-0.36}) \times 10^{-6}, \quad (55)$$

<sup>1</sup>To derive the error  $\delta_X$  associated with a quantity  $X = u/v$ , we follow the formula  $\delta_X^2 = (u^2 \delta_v^2 + v^2 \delta_u^2)/v^4$ , where  $\delta_u$  ( $\delta_v$ ) represents the uncertainty of  $u$  ( $v$ ).

$$\text{BR}(B_c^+ \rightarrow \chi_{c2}(\rightarrow \pi^+\pi^-)\pi^+) \equiv \text{BR}(B_c^+ \rightarrow \chi_{c2}\pi^+) \cdot \text{BR}(\chi_{c2} \rightarrow \pi^+\pi^-) = (0.60_{-0.18}^{+0.23}) \times 10^{-6}, \quad (56)$$

$$\text{BR}(B_c^+ \rightarrow \chi_{c2}(\rightarrow K^+K^-)\pi^+) \equiv \text{BR}(B_c^+ \rightarrow \chi_{c2}\pi^+) \cdot \text{BR}(\chi_{c2} \rightarrow K^+K^-) = (0.46_{-0.14}^{+0.17}) \times 10^{-6}. \quad (57)$$

The above results suggest that the  $B_c^+ \rightarrow \chi_{c0,c2}(\rightarrow \pi^+\pi^-)\pi^+$  and  $B_c^+ \rightarrow \chi_{c0}(\rightarrow K^+K^-)\pi^+$  decays with larger BRs are observable in the near future. Employing the data in Eq. (3) and the predictions in Eqs. (35), (55), we attain the ratio

$$\frac{\sigma(B_c^+)}{\sigma(B^+)} \cdot \text{BR}(B_c^+ \rightarrow \chi_{c0}(\rightarrow K^+K^-)\pi^+) = (4.98_{-2.47}^{+3.05}) \times 10^{-8}, \quad (58)$$

in line with the measured one  $(8.0_{-3.8}^{+4.4}) \times 10^{-8}$  within large uncertainties [5]. More precise predictions and data are required for a serious comparison.

The BESIII Collaboration also reported the improved measurements of the decays  $\chi_{cJ} \rightarrow 2(\pi^+\pi^-)$  [25]. The  $\chi_{cJ} \rightarrow \pi^+\pi^-K^+K^-$  BRs are available in Ref. [24]. The inputs of these four-body strong decays,

$$\text{BR}(\chi_{c0} \rightarrow 2(\pi^+\pi^-)) = (2.127 \pm 0.101)\%, \quad \text{BR}(\chi_{c0} \rightarrow \pi^+\pi^-K^+K^-) = (1.82 \pm 0.16)\%, \quad (59)$$

$$\text{BR}(\chi_{c1} \rightarrow 2(\pi^+\pi^-)) = (0.685 \pm 0.031)\%, \quad \text{BR}(\chi_{c1} \rightarrow \pi^+\pi^-K^+K^-) = (0.45 \pm 0.10)\%, \quad (60)$$

$$\text{BR}(\chi_{c2} \rightarrow 2(\pi^+\pi^-)) = (1.153 \pm 0.063)\%, \quad \text{BR}(\chi_{c2} \rightarrow \pi^+\pi^-K^+K^-) = (0.83 \pm 0.11)\%, \quad (61)$$

lead to the BRs of the five-body decays  $B_c^+ \rightarrow \pi^+\chi_{cJ}[\rightarrow \pi^+\pi^-(\pi^+\pi^-/K^+K^-)]$ ,

$$\text{BR}(B_c^+ \rightarrow \chi_{c0}(\rightarrow \pi^+\pi^-\pi^+\pi^-)\pi^+) \equiv \text{BR}(B_c^+ \rightarrow \chi_{c0}\pi^+) \cdot \text{BR}(\chi_{c0} \rightarrow 2(\pi^+\pi^-)) = (4.30_{-1.21}^{+1.52}) \times 10^{-6}, \quad (62)$$

$$\text{BR}(B_c^+ \rightarrow \chi_{c0}(\rightarrow \pi^+\pi^-K^+K^-)\pi^+) \equiv \text{BR}(B_c^+ \rightarrow \chi_{c0}\pi^+) \cdot \text{BR}(\chi_{c0} \rightarrow \pi^+\pi^-K^+K^-) = (3.68_{-1.07}^{+1.33}) \times 10^{-6}, \quad (63)$$

$$\text{BR}(B_c^+ \rightarrow \chi_{c1}(\rightarrow \pi^+\pi^-\pi^+\pi^-)\pi^+) \equiv \text{BR}(B_c^+ \rightarrow \chi_{c1}\pi^+) \cdot \text{BR}(\chi_{c1} \rightarrow 2(\pi^+\pi^-)) = (2.85_{-0.89}^{+0.99}) \times 10^{-7}, \quad (64)$$

$$\text{BR}(B_c^+ \rightarrow \chi_{c1}(\rightarrow \pi^+\pi^-K^+K^-)\pi^+) \equiv \text{BR}(B_c^+ \rightarrow \chi_{c1}\pi^+) \cdot \text{BR}(\chi_{c1} \rightarrow \pi^+\pi^-K^+K^-) = (1.87_{-0.71}^{+0.77}) \times 10^{-7}, \quad (65)$$

$$\text{BR}(B_c^+ \rightarrow \chi_{c2}(\rightarrow \pi^+\pi^-\pi^+\pi^-)\pi^+) \equiv \text{BR}(B_c^+ \rightarrow \chi_{c2}\pi^+) \cdot \text{BR}(\chi_{c2} \rightarrow 2(\pi^+\pi^-)) = (4.32_{-1.34}^{+1.64}) \times 10^{-6}, \quad (66)$$

$$\text{BR}(B_c^+ \rightarrow \chi_{c2}(\rightarrow \pi^+\pi^-K^+K^-)\pi^+) \equiv \text{BR}(B_c^+ \rightarrow \chi_{c2}\pi^+) \cdot \text{BR}(\chi_{c2} \rightarrow \pi^+\pi^-K^+K^-) = (3.11_{-1.03}^{+1.24}) \times 10^{-6}, \quad (67)$$

among which those of  $\mathcal{O}(10^{-6})$  could be probed promisingly.

### B. $B_c^+ \rightarrow \chi_{cJ}V^+$

We then come to the decays  $B_c^+ \rightarrow \chi_{cJ}V^+$ . The  $B_c^+ \rightarrow \chi_{c0}V^+$  BRs are predicted, in the iPQCD framework, as

$$\text{BR}(B_c^+ \rightarrow \chi_{c0}\rho^+) = 2.97_{-0.59}^{+0.83}(\beta_{B_c})_{-0.63}^{+0.75}(f_{c\bar{c}})_{-0.11}^{+0.12}(a_\rho)_{-0.10}^{+0.12}(V_{cb}) \times 10^{-4}, \quad (68)$$

$$\text{BR}(B_c^+ \rightarrow \chi_{c0}K^{*+}) = 2.00_{-0.40}^{+0.56}(\beta_{B_c})_{-0.42}^{+0.47}(f_{c\bar{c}})_{-0.10}^{+0.12}(a_{K^*})_{-0.07}^{+0.09}(V_{cb}) \times 10^{-5}, \quad (69)$$

to which only longitudinally polarized  $V^+$  meson distribution amplitudes contribute. It explains why  $\text{BR}(B_c^+ \rightarrow \chi_{c0}V^+)$  are in the same order as  $\text{BR}(B_c^+ \rightarrow \chi_{c0}P^+)$ . Our prediction for  $\text{BR}(B_c^+ \rightarrow \chi_{c0}\rho^+)$  is close to that from Ref. [11] in Table IV. The predicted  $\text{BR}(B_c^+ \rightarrow \chi_{c0}K^{*+})$  is close to those from Refs. [10,17] without the Coulomb-like corrections. The values in Eqs. (68) and (69) are smaller than those in Refs. [7,9,13,18] by one or even two orders of magnitude. Future measurements of these two modes will discriminate the different theoretical methods. We present the ratios

$$R_{J/\psi\pi}^{\chi_{c0}\rho} \equiv \frac{\text{BR}(B_c^+ \rightarrow \chi_{c0}\rho^+)}{\text{BR}(B_c^+ \rightarrow J/\psi\pi^+)} = 0.25_{-0.04}^{+0.05}, \quad (70)$$

$$R_{K^*/\rho}^{\chi_{c0}} \equiv \frac{\text{BR}(B_c^+ \rightarrow \chi_{c0} K^{*+})}{\text{BR}(B_c^+ \rightarrow \chi_{c0} \rho^+)} = (6.73_{-0.18}^{+0.12}) \times 10^{-2}. \quad (71)$$

The latter, which deviates from the naive expectation  $R_{K^*/\rho} \approx |V_{us}|^2/|V_{ud}|^2 \cdot f_{K^*}^2/f_\rho^2 = (5.75_{-0.16}^{+0.15}) \times 10^{-2}$  relying only on the factorizable emission contributions, implies the relevance of the nonfactorizable contributions to these decays.

Next we inspect the contributions to the  $B_c^+ \rightarrow \chi_{c1,c2} V^+$  decays from the three polarization configurations, whose transversity amplitudes are written as

$$\begin{aligned} \mathcal{A}_L &= \xi m_{B_c}^2 A^L, \\ \mathcal{A}_\parallel &= \xi \sqrt{2} m_{B_c}^2 A^N, \\ \mathcal{A}_\perp &= \xi m_V m_{\chi_{c1,c2}} \sqrt{2(r^2 - 1)} A^T, \end{aligned} \quad (72)$$

with the normalization factor  $\xi = \sqrt{G_F^2 |\mathbf{P}_c| / (16\pi m_{B_c}^2 \Gamma)}$  and the ratio  $r = P_2 \cdot P_3 / (m_V \cdot m_{\chi_{c1,c2}})$ . For their corresponding factorization formulas, refer to [13]. These transversity amplitudes obey the normalization

$$|\mathcal{A}_L|^2 + |\mathcal{A}_\parallel|^2 + |\mathcal{A}_\perp|^2 = 1. \quad (73)$$

The  $CP$ -averaged longitudinal polarization fraction  $f_L$  is defined as

$$f_L \equiv \frac{|\mathcal{A}_L|^2}{|\mathcal{A}_L|^2 + |\mathcal{A}_\parallel|^2 + |\mathcal{A}_\perp|^2}. \quad (74)$$

We predict the  $B_c^+ \rightarrow \chi_{c1,c2} V^+$  BRs

$$\text{BR}(B_c^+ \rightarrow \chi_{c1} \rho^+) = 1.40_{-0.34}^{+0.49} (\beta_{B_c})_{-0.21}^{+0.22} (f_{c\bar{c}})_{-0.00}^{+0.00} (a_\rho)_{-0.05}^{+0.06} (V_{cb}) \times 10^{-3}, \quad (75)$$

$$\text{BR}(B_c^+ \rightarrow \chi_{c1} K^{*+}) = 2.49_{-0.57}^{+0.85} (\beta_{B_c})_{-0.36}^{+0.41} (f_{c\bar{c}})_{-0.05}^{+0.05} (a_{K^*})_{-0.09}^{+0.10} (V_{cb}) \times 10^{-5}, \quad (76)$$

$$\text{BR}(B_c^+ \rightarrow \chi_{c2} \rho^+) = 1.18_{-0.26}^{+0.36} (\beta_{B_c})_{-0.24}^{+0.27} (f_{c\bar{c}})_{-0.00}^{+0.01} (a_\rho)_{-0.04}^{+0.05} (V_{cb}) \times 10^{-3}, \quad (77)$$

$$\text{BR}(B_c^+ \rightarrow \chi_{c2} K^{*+}) = 6.83_{-1.49}^{+2.05} (\beta_{B_c})_{-1.42}^{+1.58} (f_{c\bar{c}})_{-0.09}^{+0.09} (a_{K^*})_{-0.24}^{+0.28} (V_{cb}) \times 10^{-5}, \quad (78)$$

where the uncertainties are mainly caused by the variations of the  $B_c$  meson shape parameter  $\beta_{B_c}$  and the charmonium decay constants. The large  $B_c^+ \rightarrow \chi_{c1,c2} \rho^+$  BRs are measurable at the LHC. We also predict the ratio between the  $B_c^+ \rightarrow \chi_{c1} \rho^+$  and  $B_c^+ \rightarrow J/\psi \pi^+$  BRs,

$$R_{J/\psi\pi}^{\chi_{c1}\rho} \equiv \frac{\text{BR}(B_c^+ \rightarrow \chi_{c1} \rho^+)}{\text{BR}(B_c^+ \rightarrow J/\psi \pi^+)} = 1.20_{-0.12}^{+0.13}, \quad (79)$$

and the ratio between the  $B_c^+ \rightarrow \chi_{c2} \rho^+$  and  $B_c^+ \rightarrow J/\psi \pi^+$  BRs,

$$R_{J/\psi\pi}^{\chi_{c2}\rho} \equiv \frac{\text{BR}(B_c^+ \rightarrow \chi_{c2} \rho^+)}{\text{BR}(B_c^+ \rightarrow J/\psi \pi^+)} = 1.01_{-0.15}^{+0.15}. \quad (80)$$

Compared with the results in Table IV, our  $\text{BR}(B_c^+ \rightarrow \chi_{c1} \rho^+)$  is similar to that in Ref. [13], and our  $\text{BR}(B_c^+ \rightarrow \chi_{c1} K^{*+})$  and  $\text{BR}(B_c^+ \rightarrow \chi_{c2} V^+)$  agree well with those in Refs. [17,18] and [9,11], respectively. All the above predictions in the iPQCD formalism can be tested in future experiments.

TABLE IV.  $B_c^+ \rightarrow \chi_{cJ} V^+$  BRs predicted in the literature.

| Decay modes ( $\Delta S = 0$ )                  | [6] ( $10^{-4}$ ) | [7] ( $10^{-2}$ ) | [9] ( $10^{-3}$ ) | [10] ( $10^{-4}$ ) | [11] ( $10^{-4}$ ) | [12] ( $10^{-4}$ ) | [13] ( $10^{-3}$ )  | [17] ( $10^{-3}$ )                                     | [18] ( $10^{-3}$ )     |
|-------------------------------------------------|-------------------|-------------------|-------------------|--------------------|--------------------|--------------------|---------------------|--------------------------------------------------------|------------------------|
| $\text{BR}(B_c^+ \rightarrow \chi_{c0} \rho^+)$ | 7.2               | 3.3               | 1.3               | 6.7                | 5.8                | 7.6                | $5.8_{-1.4}^{+1.3}$ | $5.16_{-0.87}^{+0.69} (0.64_{-0.13}^{+0.11})$          | $1.69_{-0.19}^{+0.20}$ |
| $\text{BR}(B_c^+ \rightarrow \chi_{c1} \rho^+)$ | 2.9               | 0.46              | 0.29              | 1.0                | 1.5                | 2.3                | $2.8_{-0.5}^{+0.5}$ | $0.52_{-0.13}^{+0.08} (0.068_{-0.018}^{+0.018})$       | $0.43_{-0.01}^{+0.01}$ |
| $\text{BR}(B_c^+ \rightarrow \chi_{c2} \rho^+)$ | 5.1               | 3.2               | 1.2               | 6.5                | 11                 | 5.6                | $16_{-3}^{+4}$      | $35.02_{-1.88}^{+7.56} (4.47_{-0.35}^{+0.98})$         |                        |
| Decay modes ( $\Delta S = 1$ )                  | [6] ( $10^{-6}$ ) | [7]               | [9] ( $10^{-5}$ ) | [10] ( $10^{-5}$ ) | [11] ( $10^{-5}$ ) | [12] ( $10^{-5}$ ) | [13] ( $10^{-4}$ )  | [17] ( $10^{-3}$ )                                     | [18] ( $10^{-5}$ )     |
| $\text{BR}(B_c^+ \rightarrow \chi_{c0} K^{*+})$ | 3.9               |                   | 7.0               | 3.7                | 4.0                | 4.5                | $3.3_{-0.7}^{+0.8}$ | $0.27_{-0.04}^{+0.04} (0.033_{-0.006}^{+0.006})$       | $9.6_{-1.1}^{+1.1}$    |
| $\text{BR}(B_c^+ \rightarrow \chi_{c1} K^{*+})$ | 1.8               |                   | 1.8               | 0.73               | 1.0                | 1.7                | $1.8_{-0.3}^{+0.5}$ | $0.032_{-0.008}^{+0.005} (0.0041_{-0.0010}^{+0.0011})$ | $2.7_{-0.1}^{+0.1}$    |
| $\text{BR}(B_c^+ \rightarrow \chi_{c2} K^{*+})$ | 3.1               |                   | 6.5               | 3.8                | 7.4                | 3.3                | $9.6_{-2.0}^{+2.2}$ | $1.85_{-0.10}^{+0.39} (0.24_{-0.02}^{+0.05})$          |                        |

The longitudinal polarization fractions associated with the  $B_c^+ \rightarrow \chi_{c1,c2} V^+$  decays are given by

$$\begin{aligned} f_L(B_c^+ \rightarrow \chi_{c1} \rho^+) &= (95.0_{-1.1}^{+0.9})\%, \\ f_L(B_c^+ \rightarrow \chi_{c1} K^{*+}) &= (79.7_{-3.4}^{+3.7})\%, \end{aligned} \quad (81)$$

$$\begin{aligned} f_L(B_c^+ \rightarrow \chi_{c2} \rho^+) &= (92.8_{-1.3}^{+1.1})\%, \\ f_L(B_c^+ \rightarrow \chi_{c2} K^{*+}) &= (90.9_{-1.6}^{+1.4})\%, \end{aligned} \quad (82)$$

which indicate the dominance of the longitudinal components in these modes. Our  $f_L(B_c^+ \rightarrow \chi_{c1} \rho^+)$  and  $f_L(B_c^+ \rightarrow \chi_{c1} K^{*+})$  differ slightly from those in the conventional PQCD approach [13], while  $f_L(B_c^+ \rightarrow \chi_{c2} V^+)$  are consistent.

The ratios between the  $B_c^+ \rightarrow \chi_{cJ}(P, V)^+$  BRs read, for  $B_c^+ \rightarrow \chi_{c1,c2} K^{*+}$  and  $B_c^+ \rightarrow \chi_{c1,c2} \rho^+$ ,

$$R_{K^*/\rho}^{\chi_{c1}} \equiv \frac{\text{BR}(B_c^+ \rightarrow \chi_{c1} K^{*+})}{\text{BR}(B_c^+ \rightarrow \chi_{c1} \rho^+)} = (1.78_{-0.04}^{+0.04}) \times 10^{-2}, \quad (83)$$

$$R_{K^*/\rho}^{\chi_{c2}} \equiv \frac{\text{BR}(B_c^+ \rightarrow \chi_{c2} K^{*+})}{\text{BR}(B_c^+ \rightarrow \chi_{c2} \rho^+)} = (5.79_{-0.09}^{+0.03}) \times 10^{-2}. \quad (84)$$

The latter is close to  $R_{K^*/\rho}^{J/\psi}$  in [19], but the former is not. This disparity traces back to the behaviors of the charmonium distribution amplitudes in Eqs. (15)–(18) and (21)–(24): the overall antisymmetric  $\chi_{c2}$  distribution amplitudes result in the analogy to the decays  $B_c^+ \rightarrow J/\psi(\rho, K^*)^+$ , i.e., the similarity between  $R_{K^*/\rho}^{\chi_{c2}}$  and  $R_{K^*/\rho}^{J/\psi}$ . The  $\chi_{c1}$  distribution amplitudes are (anti)symmetric for the (transverse) longitudinal polarization at leading twist, and (anti)symmetric for the (longitudinal) transverse polarization at twist 3.

These properties, producing distinct nonfactorizable emission contributions, differentiate the  $B_c^+ \rightarrow \chi_{c1}(\rho, K^*)^+$  and  $B_c^+ \rightarrow J/\psi(\rho, K^*)^+$  channels.

The combinations of the  $B_c^+ \rightarrow \chi_{cJ}(\pi^+, \rho^+)$  BRs in Eqs. (32), (35), (37), (68), (75), and (77) give rise to the ratios

$$R_{\rho/\pi}^{\chi_{c0}} \equiv \frac{\text{BR}(B_c^+ \rightarrow \chi_{c0} \rho^+)}{\text{BR}(B_c^+ \rightarrow \chi_{c0} \pi^+)} = 1.47_{-0.09}^{+0.06}, \quad (85)$$

$$R_{\rho/\pi}^{\chi_{c1}} \equiv \frac{\text{BR}(B_c^+ \rightarrow \chi_{c1} \rho^+)}{\text{BR}(B_c^+ \rightarrow \chi_{c1} \pi^+)} = 33.65_{-5.28}^{+7.70}. \quad (86)$$

$$R_{\rho/\pi}^{\chi_{c2}} \equiv \frac{\text{BR}(B_c^+ \rightarrow \chi_{c2} \rho^+)}{\text{BR}(B_c^+ \rightarrow \chi_{c2} \pi^+)} = 3.15_{-0.05}^{+0.06}, \quad (87)$$

in a nonuniform pattern. The value of  $R_{\rho/\pi}^{\chi_{c2}}$  matches perfectly the ratio  $3.15_{-0.10}^{+0.09}$  between the  $B_c^+ \rightarrow J/\psi \rho^+$  and  $B_c^+ \rightarrow J/\psi \pi^+$  BRs [19]. The value of  $R_{\rho/\pi}^{\chi_{c1}}$ , in spite of the significant theoretical uncertainties, is surprisingly huge and marks a sharp contrast to  $R_{\rho/\pi}^{J/\psi}$ . A careful check unveils that the contributions from the twist-2 and twist-3  $\chi_{c1}$  meson distribution amplitudes are constructive (destructive) in the  $B_c^+ \rightarrow \chi_{c1} \rho^+$  ( $B_c^+ \rightarrow \chi_{c1} \pi^+$ ) mode. Such peculiar interferences do not appear in the  $\chi_{c0,c2}$  involved decays. It is encouraged to examine the iPQCD predictions and to deepen our understanding of the intricate dynamics in  $B_c \rightarrow \chi_{cJ}$  transitions by measuring the above ratios.

At last, adopting the BRs of the  $\chi_{cJ}$  strong decays in Eqs. (52), (53) and (59)–(61), we get the BRs of  $B_c^+ \rightarrow \chi_{cJ} \rho^+$  with secondary decay chains under the narrow-width approximation,

$$\text{BR}(B_c^+ \rightarrow \chi_{c0}(\rightarrow \pi^+ \pi^-) \rho^+) \equiv \text{BR}(B_c^+ \rightarrow \chi_{c0} \rho^+) \cdot \text{BR}(\chi_{c0} \rightarrow \pi^+ \pi^-) = (1.80_{-0.54}^{+0.69}) \times 10^{-6}, \quad (88)$$

$$\text{BR}(B_c^+ \rightarrow \chi_{c0}(\rightarrow K^+ K^-) \rho^+) \equiv \text{BR}(B_c^+ \rightarrow \chi_{c0} \rho^+) \cdot \text{BR}(\chi_{c0} \rightarrow K^+ K^-) = (1.89_{-0.56}^{+0.72}) \times 10^{-6}, \quad (89)$$

$$\text{BR}(B_c^+ \rightarrow \chi_{c2}(\rightarrow \pi^+ \pi^-) \rho^+) \equiv \text{BR}(B_c^+ \rightarrow \chi_{c2} \rho^+) \cdot \text{BR}(\chi_{c2} \rightarrow \pi^+ \pi^-) = (1.90_{-0.58}^{+0.74}) \times 10^{-6}, \quad (90)$$

$$\text{BR}(B_c^+ \rightarrow \chi_{c2}(\rightarrow K^+ K^-) \rho^+) \equiv \text{BR}(B_c^+ \rightarrow \chi_{c2} \rho^+) \cdot \text{BR}(\chi_{c2} \rightarrow K^+ K^-) = (1.44_{-0.44}^{+0.56}) \times 10^{-6}, \quad (91)$$

and

$$\text{BR}(B_c^+ \rightarrow \chi_{c0}(\rightarrow \pi^+ \pi^- \pi^+ \pi^-) \rho^+) \equiv \text{BR}(B_c^+ \rightarrow \chi_{c0} \rho^+) \cdot \text{BR}(\chi_{c0} \rightarrow 2(\pi^+ \pi^-)) = (6.32_{-1.90}^{+2.42}) \times 10^{-6}, \quad (92)$$

$$\text{BR}(B_c^+ \rightarrow \chi_{c0}(\rightarrow \pi^+ \pi^- K^+ K^-) \rho^+) \equiv \text{BR}(B_c^+ \rightarrow \chi_{c0} \rho^+) \cdot \text{BR}(\chi_{c0} \rightarrow \pi^+ \pi^- K^+ K^-) = (5.41_{-1.67}^{+2.11}) \times 10^{-6}, \quad (93)$$

$$\text{BR}(B_c^+ \rightarrow \chi_{c1}(\rightarrow \pi^+ \pi^- \pi^+ \pi^-) \rho^+) \equiv \text{BR}(B_c^+ \rightarrow \chi_{c1} \rho^+) \cdot \text{BR}(\chi_{c1} \rightarrow 2(\pi^+ \pi^-)) = (0.96_{-0.28}^{+0.37}) \times 10^{-5}, \quad (94)$$

$$\text{BR}(B_c^+ \rightarrow \chi_{c1}[\rightarrow \pi^+ \pi^- K^+ K^-] \rho^+) \equiv \text{BR}(B_c^+ \rightarrow \chi_{c1} \rho^+) \cdot \text{BR}(\chi_{c1} \rightarrow \pi^+ \pi^- K^+ K^-) = (6.30_{-2.28}^{+2.80}) \times 10^{-6}, \quad (95)$$

$$\text{BR}(B_c^+ \rightarrow \chi_{c2}(\rightarrow \pi^+ \pi^- \pi^+ \pi^-) \rho^+) \equiv \text{BR}(B_c^+ \rightarrow \chi_{c2} \rho^+) \cdot \text{BR}(\chi_{c2} \rightarrow 2(\pi^+ \pi^-)) = (1.36_{-0.42}^{+0.54}) \times 10^{-5}, \quad (96)$$

$$\text{BR}(B_c^+ \rightarrow \chi_{c2}(\rightarrow \pi^+ \pi^- K^+ K^-) \rho^+) \equiv \text{BR}(B_c^+ \rightarrow \chi_{c2} \rho^+) \cdot \text{BR}(\chi_{c2} \rightarrow \pi^+ \pi^- K^+ K^-) = (0.98_{-0.33}^{+0.40}) \times 10^{-5}. \quad (97)$$

These predictions provide a useful reference for experimental search for multibody  $B_c$  meson decays.

#### IV. SUMMARY

We have studied the  $B_c$ -meson decays into  $\chi_{cJ}$  plus light pseudoscalar or vector mesons in the leading-order iPQCD approach. The  $B_c^+ \rightarrow \chi_{cJ}(P, V)^+$  BRs and the longitudinal polarization fractions of the  $B_c^+ \rightarrow \chi_{c1,c2} V^+$  decays were evaluated. In view of the unclear  $B_c$  meson production cross sections, we defined and predicted the ratios between the  $B_c^+ \rightarrow \chi_{cJ}(P, V)^+$  and  $B_c^+ \rightarrow J/\psi \pi^+$  BRs for experimental verifications. In summary, we found that the predicted ratios  $R_{\chi_{c2}/J/\psi} = 0.32 \pm 0.05$  and  $R_{\chi_{c1}/\chi_{c2}} = 0.11 \pm 0.01$  coincide with the current observations and upper limit reported by the LHCb, and the ratios  $R_{\chi_{c0}/J/\psi} = 0.17 \pm 0.02$  and  $R_{\chi_{c1}/J/\psi} = (3.56_{-0.82}^{+0.81}) \times 10^{-2}$  can be confronted by future measurements. The different ratios  $R_{\rho/\pi}^{\chi_{cJ}}$  were obtained:  $R_{\rho/\pi}^{\chi_{c2}} = 3.15_{-0.05}^{+0.06}$  matches perfectly  $R_{\rho/\pi}^{J/\psi}$ , but  $R_{\rho/\pi}^{\chi_{c0}} = 1.47_{-0.09}^{+0.06}$  ( $R_{\rho/\pi}^{\chi_{c1}} = 33.65_{-5.28}^{+7.70}$ ) is a bit lower (surprisingly higher) than  $R_{\rho/\pi}^{J/\psi}$ . This nonuniform pattern is attributed to the distinct interplays between the twist-2 and twist-3 contributions in these decays. The predicted ratios  $R_{K/\pi}^{\chi_{cJ}}$  and  $R_{K^*/\rho}^{\chi_{c2}}$  meet the naive expectation that they are governed by factorizable emission contributions. The larger

(smaller)  $R_{K^*/\rho}^{\chi_{c0}}$  ( $R_{K^*/\rho}^{\chi_{c1}}$ ), being evidently disparate from the naive anticipation, suggests the varying involved non-factorizable emission contributions. Inputting the measured  $\text{BR}(\chi_{c0,c2} \rightarrow \pi^+ \pi^- / K^+ K^-)$  and  $\text{BR}(\chi_{cJ} \rightarrow \pi^+ \pi^- (\pi^+ \pi^- / K^+ K^-))$ , the BRs of  $B_c^+ \rightarrow \chi_{cJ}(\pi, \rho)^+$  with secondary decay chains were estimated under the narrow-width approximation, which serve as useful references for their experimental search.

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#### DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

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