

Anisotropic linear waves and breakdown of the momentum expansion in spin magnetohydrodynamics

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We formulate spin magnetohydrodynamics (MHD) by including the magnetic flux and total angular momentum conservation laws in the hydrodynamic framework. To specify the local angular momentum conservation, we choose the totally antisymmetric spin current. The entropy-current analysis allows for ten dissipative first-order transport coefficients, including anisotropic spin relaxation rates and the conversion rate between a vorticity (shear) to a symmetric stress (antisymmetric torque), as well as anisotropic viscosities and resistivities. By employing the linear-mode analysis, we solve the first-order spin MHD equations to determine the dispersion relations with the complete information of anisotropy retained. Our analytic solutions indicate that the small-momentum expansion is spoiled by blow up of the higher-order terms when the angle between the momentum and the magnetic field approaches the right angle. This also reveals the existence of another expansion parameter, and, in light of it, we provide solutions in an alternative series expression beyond the critical angle. We confirm that these two series expansions work well in the appropriate angle ranges, as compared with numerical results. Building on our findings regarding the breakdown of the small-momentum expansion in first-order theory, we proceed to discussing how these first-order solutions are modified when we include the relaxation dynamics for dissipative modes with the Israel-Stewart framework. We find that, due to the presence of the critical behavior in the first-order solutions, there remains a diffusive window even after the relaxation dynamics is introduced.

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I. INTRODUCTION

In the realm of nuclear physics, the study of spin polarization in relativistic nucleus-nucleus collisions has unveiled a fascinating exploration domain. High-energy collisions generate vortical fields within the quark-gluon plasma (QGP) through the initial orbital angular momentum, leading to the global spin polarization of Λ and $\bar{\Lambda}$ hyperons and the alignment of vector mesons via spin-orbit couplings [1–6]. The polarization of hyperons was observed in experiments by STAR at RHIC and ALICE at the LHC [7–11], while the alignment of vector mesons was also observed in recent years [12,13], representing a pivotal

moment in understanding the QCD matter under extreme conditions.

Theoretical investigations have also been devoted to developing relativistic frameworks for spin dynamics in QGP. Early theoretical projections suggested that the spin polarization of particles would be affected by the thermal vorticity tensor at the freeze-out surface, intricately associated with the fluid's angular velocity and acceleration. Various theoretical models are adopted to include the thermal vorticity and have successfully described the global polarization using the modified Cooper-Frye formula [4,14–18]. Nevertheless, the experimental results on the local Λ spin polarization pose a spin sign puzzle that cannot be solely explained by thermal effects [19–21], i.e., theoretical calculations give the opposite azimuthal angle dependence of Λ polarization to the experimental results. This deviation may be attributed to the nonequilibrium evolution of spin degrees of freedom, inspiring theoretical research on hydrodynamic descriptions that include spin effects. Substantial theoretical studies have been conducted on the relativistic spin hydrodynamics based on the entropy analysis or the second law of thermodynamics [22–34],

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quantum kinetic theories for fermions [35–54], holographic approaches [55–57], effective action approach [58,59], and the statistical density operator methods [26,27,60–64]. Different from hydrodynamic modes, spin density is not a strictly conserved charge since the spatial rotational symmetry only guarantees the conservation of the total angular momentum, including the orbital part. Therefore, there is not an *a priori* justification of spin hydrodynamics as a (quasi-)hydrodynamics, which should be instead justified *posteriori* by evaluating the spin relaxation rate. Evaluations of the spin relaxation rate have been addressed recently in Refs. [48,49,63,65,66] for relativistic heavy ion collisions. Also, the shear-induced polarization [67] has been proposed as a source triggering the local spin polarization. Including them in hydrodynamic simulations seems to qualitatively resolve the sign issue [68,69] (see also Refs. [70,71]).

Besides the experimental observation of the spin polarization, strong magnetic fields are expected to be generated in the noncentral collisions (see, e.g., Refs. [72–75] for reviews and references therein). These magnetic fields are thought to be transient, as they will fade out rapidly in the heavy-ion collision, but may result in some important phenomena like chiral magnetic effect [76–79] and chiral magnetic wave [80]. Recently, there have been developments in the numerical simulation of relativistic magnetohydrodynamics (MHD) [81–85]. Furthermore, the strong magnetic field may also have a significant impact on the spin polarization. Unlike rotational effects, a magnetic field induces opposite spin polarization of particles and antiparticles, as they carry opposite charges. Though there is still no experimental certainty, the deviations in the polarization of Λ and $\bar{\Lambda}$ in heavy-ion collisions have drawn a lot of attention recently [8], which may be accounted for by the effect of the magnetic field. Recently, magnetic field effects were discussed as sources of not only the splitting between Λ and $\bar{\Lambda}$ [86–88] but also the local spin polarization [89]. The interplay between the magnetic fields and spin dynamics in the QGP draws considerable interest.

In this study, we explore formulation and linear-mode analysis of the relativistic spin MHD, extending our study on the first-order MHD [90]. We plug spin dynamics into relativistic MHD that has been formulated in recent years with anisotropic viscosities and resistivities [63,75,91–97] (see Ref. [98] for a review). We will find anisotropic rotational viscosities and a new transport coefficient that converts the vorticity to a symmetric stress and the shear to an antisymmetric torque. We also solve the system of hydrodynamic equations for the linear perturbation in the vicinity of an equilibrium configuration. Note that, while the linear-mode analyses have been applied to MHD [90,91,96,99] and spin hydrodynamics [22,27,30,31,33,63], the linear waves in the spin MHD have not been addressed, which is exactly what we will

present in this work. We also note that spin MHD was recently discussed with the kinetic theory approach [100,101]. In this paper, we also introduce relaxation dynamics with the Israel-Stewart (IS) framework. This offers an application of the IS framework to anisotropic systems. We will find that there remains a diffusive window, of which the boundary is specified by a direction of momentum as well as its magnitude.

Notably, when solving the linearized equations, we retain the full anisotropy of the solutions parametrized by the general angle θ between the momentum and magnetic field, which is one of the important properties inherent in a magnetized medium. Relying on the anisotropic analytical solutions, we find that the momentum expansion breaks down when θ approaches $\pi/2$, because diverging powers of $1/\cos\theta$ emerge in the higher-order terms in the momentum expansion. This issue stems from noncommutative limits of small values of the momentum and the cosine. In light of this, we provide an alternative solution that covers the regime where the momentum expansion breaks down. We confirm these two series expressions work well in the appropriate regimes by comparing them with the numerical results. Our study may have broader implications for solving the perturbation problems involving multiple small parameters.

In Sec. II, we provide the derivation details of the hydrodynamic equations for spin MHD. In Sec. III, we focus on the linear-mode analysis near an equilibrium state. We elaborate on the solutions for the linear waves in Sec. IV. In this part, a pair of the Alfvén waves and two pairs of the magneto-sonic waves are found in the MHD sector, while three damping modes are found in the spin sector. We provide a remedy for the breakdown of the momentum expansion. A comparison with our previous work on MHD [90] is also given. We give a summary in Sec. V.

In this paper, we use the mostly plus metric convention $\eta^{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ and the completely antisymmetric tensor with the convention $\epsilon^{0123} = +1$. Then, the fluid velocity u^μ is normalized as $u^\mu u_\mu = -1$. We define the projection operator $\Delta^{\mu\nu} = \eta^{\mu\nu} + u^\mu u^\nu$ such that $u_\mu \Delta^{\mu\nu} = 0$.

II. FORMULATION OF SPIN MAGNETOHYDRODYNAMICS

A. Conservation laws

Our investigation commences by introducing a finite angular momentum density into the framework of magnetohydrodynamics. We focus on neutral systems in the absence of vector or axial charges. The conservation laws deduced from the translational, rotational, and one-form symmetries read [91,98,102]

$$\partial_\mu \Theta^{\mu\nu} = 0, \quad \partial_\mu J^{\mu\alpha\beta} = 0, \quad \partial_\mu \tilde{F}^{\mu\nu} = 0, \quad (1)$$

where $\Theta^{\mu\nu}$, $J^{\mu\alpha\beta}$, and $\tilde{F}^{\mu\nu}$ are the energy-momentum tensor, the angular momentum tensor, and the dual electromagnetic field strength tensor, respectively.

The angular momentum tensor $J^{\mu\alpha\beta}$ can be decomposed into the contributions of the spin current $\Sigma^{\mu\alpha\beta}$ and the orbital angular momentum as

$$J^{\mu\alpha\beta} = \Sigma^{\mu\alpha\beta} + x^\alpha \Theta^{\mu\beta} - x^\beta \Theta^{\mu\alpha}. \quad (2)$$

Then, the total angular momentum conservation in Eq. (1) is cast into another form,

$$\partial_\mu \Sigma^{\mu\alpha\beta} = -2\Theta^{[\alpha\beta]}. \quad (3)$$

This equation indicates that there is no symmetry that guarantees the conservation of spin in relativistic systems. The antisymmetric part of the energy-momentum tensor quantifies the torque exerting on a fluid cell that possesses a nonzero average spin. Since the spin density is not a conserved quantity, spin hydrodynamics should be understood as an extended hydrodynamic framework with a quasihydrodynamic mode as previously formulated in the literature [22,23,29,63]. The temporal components of these currents provide the density of corresponding conserved charges

$$e = u_\mu u_\nu \Theta^{\mu\nu}, \quad S^{\alpha\beta} = -u_\mu \Sigma^{\mu\alpha\beta}, \quad B^\mu = -\tilde{F}^{\mu\nu} u_\nu. \quad (4)$$

Note that there is an ambiguity of the pseudogauge choice where the local conservation laws of the energy-momentum tensor and the angular momentum tensor are preserved under the simultaneous shifts of $\Theta^{\mu\nu}$ and $J^{\mu\alpha\beta}$ transformations [103–105] (see, e.g., Refs. [22,23,25,62] for recent discussions). Because of this ambiguity, the separation between the spin and orbital angular momentum is not unique and depends on a choice of the pseudogauge. In the end, the definitions of the spin and orbital angular momentum may obey the measurement processes. We assume that $\Sigma^{\mu\alpha\beta}$ is a totally antisymmetric tensor. In this pseudogauge, we find an identity $u_\mu S^{\mu\nu} = 0$ from Eq. (4), indicating that the spin density only has three spatial components. Accordingly, three components of Eq. (3) are not dynamical equations, but serve as constraint equations [29],

$$u_\alpha \Theta^{[\alpha\beta]} = -\frac{1}{2} u_\alpha \partial_\mu \Sigma^{\mu\alpha\beta}. \quad (5)$$

As a consequence, the temporal component of $\Theta^{[\alpha\beta]}$ is determined by the spatial derivative of the spin density.

Next, the conserved quantities (4) are supposed to satisfy the first law of thermodynamics,

$$Ts = e + p - \omega_{\mu\nu} S^{\mu\nu} - H_\mu B^\mu, \quad (6a)$$

$$Td s = de - \omega_{\mu\nu} dS^{\mu\nu} - H_\mu dB^\mu, \quad (6b)$$

$$TDs = De - \omega_{\mu\nu} DS^{\mu\nu} - H_\mu DB^\mu. \quad (6c)$$

The temporal derivative D is defined as $u^\mu \partial_\mu$. We introduced the spin potential $\omega_{\mu\nu}$ and the magnetic field H_μ as the Lagrange multipliers for the spin density $S^{\mu\nu}$ and the magnetic flux B^μ , respectively. We assume the linear relations between the conserved charges and the Lagrange multipliers, i.e., $H^\mu = B^\mu/\mu_m$ and $\omega^{\mu\nu} = S^{\mu\nu}/\chi$, with the magnetic permeability μ_m and the spin susceptibility χ being constants in spacetime.

To close the equations of motion, one needs to express the spatial components of the conserved currents by the conserved charges. This is implemented on the basis of a derivative expansion. Up to the first order in derivative, one can write the conserved currents as

$$\begin{aligned} \Theta^{\mu\nu} &= e u^\mu u^\nu + p_\parallel b^\mu b^\nu + p_\perp \Xi^{\mu\nu} + \Theta_{(1)}^{\mu\nu}, \\ \Sigma^{\mu\alpha\beta} &= u^\mu S^{\alpha\beta} - u^\alpha S^{\mu\beta} + u^\beta S^{\mu\alpha} + \Sigma_{(1)}^{\mu\alpha\beta}, \\ \tilde{F}^{\mu\nu} &= B^\mu u^\nu - B^\nu u^\mu + \tilde{F}_{(1)}^{\mu\nu}, \end{aligned} \quad (7)$$

where we introduced a unit vector $b^\mu = B^\mu/\sqrt{B^\nu B_\nu}$ and the projection operator $\Xi^{\mu\nu} = \Delta^{\mu\nu} - b^\mu b^\nu$ such that $b_\mu \Xi^{\mu\nu} = 0 = u_\mu \Xi^{\mu\nu}$. The unknown parameters $p_{\parallel,\perp}$ are identified with thermodynamic pressure below. The above explicitly written tensor structures exhaust possible tensor structures respecting the discrete symmetries in the zeroth order in derivative. The last terms on the right-hand side of Eq. (7) with the subscript (1) denote the first-order corrections that will be constrained by the entropy-current analysis below.

B. Entropy-current analysis

The entropy-current analysis provides constraints on the possible first-order tensor structures in Eq. (7). Up to the first order, the entropy current is given in the form $s^\mu = s u^\mu + s_{(1)}^\mu$, where $s_{(1)}^\mu \sim \mathcal{O}(\partial)$. By making use of the first law of thermodynamics (6) and the equations of motion (1) with Eq. (7), one finds that

$$\begin{aligned} \partial_\mu s^\mu &= s \partial_\mu u^\mu + Ds + \partial_\mu s_{(1)}^\mu \\ &= \beta(Ts - e - p_\perp + \omega_{\alpha\beta} S^{\alpha\beta} + H_\mu B^\mu) \partial_\mu u^\mu \\ &\quad - \beta[(p_\parallel - p_\perp) b^\mu b^\nu + B b^\mu H^\nu + 2S^{\mu\alpha} \omega^\nu{}_\alpha] \partial_\mu u_\nu \\ &\quad - \Theta_{(1)}^{\mu\nu} (\partial_\mu \beta_\nu - 2\beta \omega_{\mu\nu}) - \Sigma_{(1)}^{\mu\alpha\beta} \partial_\mu (\beta \omega_{\alpha\beta}) + \tilde{F}_{(1)}^{\mu\nu} \partial_\mu (\beta H_\nu) \\ &\quad + \partial_\mu (s_{(1)}^\mu + \beta u_\nu \Theta_{(1)}^{\mu\nu} + \beta \omega_{\alpha\beta} \Sigma_{(1)}^{\mu\alpha\beta} - \beta H_\nu \tilde{F}_{(1)}^{\mu\nu}), \end{aligned} \quad (8)$$

where we used $u_\mu \omega^{\mu\nu} = 0 = u_\mu S^{\mu\nu}$ in the totally antisymmetric pseudogauge mentioned above. In a local equilibrium state where $\Theta_{(1)}^{\mu\nu} (\partial_\mu \beta_\nu - 2\beta \omega_{\mu\nu}) \rightarrow 0$, the spin potential is determined by the thermal vorticity $\beta \omega_{\mu\nu} = \frac{1}{2} \partial_{[\mu} \beta_{\nu]}$. Therefore, in a weak vorticity $\partial_{[\mu} \beta_{\nu]} \sim \mathcal{O}(\partial^1)$, the magnitude of the spin potential is $\beta \omega_{\mu\nu} \sim \partial_{[\mu} \beta_{\nu]} \sim \mathcal{O}(\partial^1)$.

Accordingly, we drop the term proportional to $\partial_\mu \Sigma_{(1)}^{\mu\alpha\beta}$, which is a third-order correction. When $S^{\mu\nu} \sim \omega^{\mu\nu}$, the term proportional to $S^{\mu\alpha} \omega^\nu_\alpha$ is also a third-order correction and can be dropped. Otherwise, this term provides a pressure anisotropy induced by spin.

At the leading order in derivative, the entropy production should vanish. According to this condition, we find that

$$Ts = \epsilon + p_\perp - \omega_{\alpha\beta} S^{\alpha\beta} - H_\mu B^\mu, \quad (9a)$$

$$(p_\parallel - p_\perp) b^\nu + H^\nu B = 0. \quad (9b)$$

One can identify $p_\perp$ with the thermodynamic pressure in Eq. (6) according to the first condition. The second condition shows that the pressure becomes anisotropic in the presence of a magnetic field.

For the first-order corrections to be semipositive definite, we require that individual contributions of distinct thermodynamic forces take semipositive values, i.e.,

$$-\Theta_{(1)}^{\mu\nu} (\partial_\mu \beta_\nu - 2\beta \omega_{\mu\nu}) \geq 0, \quad (10a)$$

$$\tilde{F}_{(1)}^{\mu\nu} \partial_\mu (\beta H_\nu) \geq 0. \quad (10b)$$

To satisfy the inequality, the energy-momentum tensor should be decomposed as

$$\Theta_{(1)}^{\mu\nu} = 2h^{(\mu} u^{\nu)} + 2q^{[\mu} u^{\nu]} - T \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} \eta^{\mu\nu\rho\sigma} & \xi^{\mu\nu\rho\sigma} \\ \xi^{\mu\nu\rho\sigma} & \gamma^{\mu\nu\rho\sigma} \end{pmatrix} \begin{pmatrix} \partial_{[\rho} \beta_{\sigma]} \\ \partial_{[\rho} \beta_{\sigma]} - 2\beta \omega_{\rho\sigma} \end{pmatrix}. \quad (11)$$

The temporal components with u^μ are the conventional heat current h^μ and the boost heat current q^μ introduced in Ref. [22]. We choose to work with the Landau frame, so that the heat current is vanishing $h^\mu = 0$. Also, plugging the expansion (11) into the constraint equation (5), one finds that the boost heat current is completely determined by the spin density as

$$q^\mu = -\frac{1}{2} u_\alpha \partial_\mu \Sigma_{(0)}^{\mu\alpha\beta} = -\frac{1}{2} \Delta_\rho^\beta \Delta_\alpha^\gamma \partial_\gamma S^{\alpha\rho}. \quad (12)$$

In our power counting $S^{\mu\nu} \sim \mathcal{O}(\partial^1)$, the boost heat current turns out to be the second-order quantity $q^\mu \sim \mathcal{O}(\partial^2)$. Therefore, it can be dropped in the current working accuracy. Under those conditions, one can take

$$h^\mu = 0 = q^\mu. \quad (13)$$

In Eq. (11), the viscous tensors are presented in the matrix form. While the symmetric component $\eta^{\mu\nu\rho\sigma}$ provides the viscosities in MHD [91,98,102], the antisymmetric part $\gamma^{\mu\nu\rho\sigma}$ provides a torque when there is a deviation between rotations of a fluid cell and the surrounding

fluid [22]. The existence of the cross terms $\xi^{\mu\nu\rho\sigma}$ and $\xi'^{\mu\nu\rho\sigma}$ was pointed out in Ref. [29]. Onsager's reciprocal relation states that $\eta^{\mu\nu\rho\sigma}(b^\mu) = \eta^{\rho\sigma\mu\nu}(-b^\mu)$, $\gamma^{\mu\nu\rho\sigma}(b^\mu) = \gamma^{\rho\sigma\mu\nu}(-b^\mu)$, $\xi^{\mu\nu\rho\sigma}(b^\mu) = \xi^{\rho\sigma\mu\nu}(-b^\mu)$, and $\xi'^{\mu\nu\rho\sigma}(b^\mu) = \xi'^{\rho\sigma\mu\nu}(-b^\mu)$. These vectors and tensors are all transverse to u_μ .

The general structures of the viscous tensors are constructed as

$$\begin{aligned} \eta^{\mu\nu\rho\sigma} = & \begin{pmatrix} b^\mu b^\nu & \Xi^{\mu\nu} \end{pmatrix} \begin{pmatrix} \zeta_\parallel & \zeta_\times \\ \zeta'_\times & \zeta_\perp \end{pmatrix} \begin{pmatrix} b^\rho b^\sigma \\ \Xi^{\rho\sigma} \end{pmatrix} \\ & + 2\eta_\parallel (b^\mu \Xi^{\nu(\rho} b^{\sigma)} + b^\nu \Xi^{\mu(\rho} b^{\sigma)}) \\ & + \eta_\perp (\Xi^{\mu\rho} \Xi^{\nu\sigma} + \Xi^{\mu\sigma} \Xi^{\nu\rho} - \Xi^{\mu\nu} \Xi^{\rho\sigma}), \end{aligned} \quad (14a)$$

$$\begin{aligned} \gamma^{\mu\nu\rho\sigma} = & \gamma_\perp (\Xi^{\mu\rho} \Xi^{\nu\sigma} - \Xi^{\mu\sigma} \Xi^{\nu\rho}) \\ & - 2\gamma_\parallel (b^\mu \Xi^{\nu[\rho} b^{\sigma]} - b^\nu \Xi^{\mu[\rho} b^{\sigma]}), \end{aligned} \quad (14b)$$

$$\begin{aligned} \xi^{\mu\nu\rho\sigma} = & 2\xi_\parallel (b^\mu \Xi^{\nu[\rho} b^{\sigma]} + b^\nu \Xi^{\mu[\rho} b^{\sigma]}) \\ & + b^\mu \Xi^{\nu(\rho} b^{\sigma)} - b^\nu \Xi^{\mu(\rho} b^{\sigma)}, \end{aligned} \quad (14c)$$

where $\zeta_\times = \zeta'_\times$ by virtue of Onsager's reciprocal relation. These viscous tensors are analogs of those in the presence of a strong vorticity [29]. The transport coefficients, $\gamma_\perp, \gamma_\parallel, \xi_\parallel$, in the antisymmetric part control the spin relaxation rate in Eq. (3).¹ Therefore, spin density persists near, *albeit not strictly in*, an equilibrium and the current framework works as a quasihydrodynamics when these transport coefficients are small enough. The symmetric viscous tensor $\eta^{\mu\nu\rho\sigma}$ reduces to the familiar viscous tensor in the isotropic limit $\zeta_\parallel = \zeta_\perp = \zeta_\times$ and $\eta_\parallel = \eta_\perp$. The antisymmetric viscous tensor $\gamma^{\mu\nu\rho\sigma}$ also reduces to that without a magnetic field [22] in the isotropic limit $\gamma_\parallel = \gamma_\perp$. The cross term $\xi^{\mu\nu\rho\sigma}$ does not exist without b^μ . Therefore, the most general tensor structure in Eq. (14) works for an arbitrary magnitude of a magnetic field.

As for the dual field strength tensor, the first-order correction should be proportional to the thermodynamic force as

$$\tilde{F}_{(1)}^{\mu\nu} = -T \rho^{\mu\nu\rho\sigma} \partial_{[\rho} (\beta H_{\sigma]}). \quad (15)$$

To ensure Eq. (10) is satisfied, the general form of the fourth-rank tensor $\rho^{\mu\nu\rho\sigma}$ can be constructed as [91,98,102]

$$\rho^{\mu\nu\rho\sigma} = -2\rho_\perp (b^\mu \Xi^{\nu[\rho} b^{\sigma]} - b^\nu \Xi^{\mu[\rho} b^{\sigma]}) + 2\rho_\parallel \Xi^{\mu[\rho} \Xi^{\sigma]\nu}. \quad (16)$$

This tensor provides the constitutive equation for an induced electric field and is a resistivity tensor [91,93,98]. There is no Hall term in neutral systems.

¹One can confirm this in the solutions for the linear-mode analysis in a later section [see, e.g., Eqs. (47) and (49)].

The semipositivity of the entropy current requires that all the transport coefficients be semipositive except for $\zeta_\times$ and $\xi_\parallel$ in the off-diagonal components of matrices. These transport coefficients are instead constrained by semipositivity of the matrices, i.e., the eigenvalues. The coefficients should satisfy the inequalities

$$\begin{aligned} \zeta_\perp &\geq 0, & \zeta_\parallel &\geq 0, & \zeta_\parallel \zeta_\perp &\geq \zeta_\times^2, & \eta_\parallel &\geq 0, & \eta_\perp &\geq 0, \\ \gamma_\parallel &\geq 0, & \gamma_\perp &\geq 0, & \gamma_\parallel \eta_\parallel &\geq \xi_\parallel^2, \\ \rho_\perp &\geq 0, & \rho_\parallel &\geq 0. \end{aligned} \quad (17)$$

III. LINEAR-MODE ANALYSIS

In this section, we derive the linearized first-order hydrodynamic equations for small perturbations near an equilibrium state. This is often called the linear-mode analysis. We apply small perturbations on top of the equilibrium values $u^\mu = (1, 0, 0, 0)$, $B^\mu = (0, 0, 0, B)$, and $S^{\mu\nu} = 0$. Here, without loss of generality, we took the direction of the magnetic field along the z axis in equilibrium. The conserved charges are displaced from their equilibrium values as

$$\begin{aligned} e &\rightarrow e + \delta e(x), & u^\mu &\rightarrow u^\mu + \delta u^\mu(x), \\ B^\mu &\rightarrow B^\mu + \delta B^\mu(x), & S^{\mu\nu} &\rightarrow S^{\mu\nu} + \delta S^{\mu\nu}(x). \end{aligned} \quad (18)$$

We will linearize the hydrodynamic equations with respect to these perturbations. Since $S^{\mu\nu}$ has only three spatial components due to our pseudogauge choice, it is more convenient to extract the spin density as a spatial vector,

$$\sigma^\mu = -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}u_\nu S_{\rho\sigma}. \quad (19)$$

Thus, the last decomposition in Eq. (18) is alternatively written as $\sigma^\mu = 0 + \delta\sigma^\mu$.

For simplicity, we assume that the contributions of the matter and magnetic components to the equilibrium energy density and pressure can be separated as

$$p_\perp = P + \frac{B^2}{2\mu_m}, \quad e = \epsilon + \frac{B^2}{2\mu_m}, \quad (20)$$

where P and ϵ are the equilibrium energy density and pressure from the matter contribution.

A. Equations for the energy-momentum tensor

The conservation law of the energy-momentum tensor in Eq. (1) can be projected as

$$u_\nu \partial_\mu \Theta^{\mu\nu} = 0, \quad \Xi_\nu^\rho \partial_\mu \Theta^{\mu\nu} = 0, \quad b_\nu \partial_\mu \Theta^{\mu\nu} = 0. \quad (21)$$

Plugging Eq. (7) into the above and focusing on the linear order in perturbations, one arrives at the equations

$$0 = \partial_0 \delta\epsilon + \frac{B}{\mu_m} \partial_0 \delta B + h \partial_{\perp\mu} \delta u_\perp^\mu + \left(h - \frac{B^2}{\mu_m}\right) \partial_z \delta u_z, \quad (22a)$$

$$\begin{aligned} 0 = & h \partial_0 \delta u_x + c_s^2 \partial_x \delta\epsilon - \frac{B}{\mu_m} \partial_z \delta B_x + \frac{B}{\mu_m} \partial_x \delta B_z \\ & - [(\zeta_\perp + \eta_\perp) \partial_x^2 + (\eta_\perp + \gamma_\perp) \partial_y^2 + (\eta_\parallel + \gamma_\parallel - 2\xi_\parallel) \partial_z^2] \delta u_x \\ & - (\zeta_\perp - \gamma_\perp) \partial_x \partial_y \delta u_y - (\zeta_\times + \eta_\parallel - \gamma_\parallel) \partial_z \partial_x \delta u_z \\ & - \frac{4}{\chi} (\gamma_\parallel - \xi_\parallel) \partial_z \delta\sigma_y + \frac{4}{\chi} \gamma_\perp \partial_y \delta\sigma_z, \end{aligned} \quad (22b)$$

$$\begin{aligned} 0 = & h \partial_0 \delta u_y + c_s^2 \partial_y \delta\epsilon - \frac{B}{\mu_m} \partial_z \delta B_y + \frac{B}{\mu_m} \partial_y \delta B_z \\ & - [(\zeta_\perp + \eta_\perp) \partial_y^2 + (\eta_\perp + \gamma_\perp) \partial_x^2 + (\eta_\parallel + \gamma_\parallel - 2\xi_\parallel) \partial_z^2] \delta u_y \\ & - (\zeta_\perp - \gamma_\perp) \partial_x \partial_y \delta u_x - (\zeta_\times + \eta_\parallel - \gamma_\parallel) \partial_z \partial_y \delta u_z \\ & + \frac{4}{\chi} (\gamma_\parallel - \xi_\parallel) \partial_z \delta\sigma_x - \frac{4}{\chi} \gamma_\perp \partial_x \delta\sigma_z, \end{aligned} \quad (22c)$$

$$\begin{aligned} 0 = & c_s^2 \partial_z \delta\epsilon + \left(h - \frac{B^2}{\mu_m}\right) \partial_0 \delta u_z \\ & - (\zeta_\times + \eta_\parallel - \gamma_\parallel) \partial_z (\partial_x \delta u_x + \partial_y \delta u_y) \\ & - (\zeta_\parallel \partial_z^2 + (\eta_\parallel + \gamma_\parallel + 2\xi_\parallel) (\partial_x^2 + \partial_y^2)) \delta u_z \\ & + \frac{4}{\chi} (\xi_\parallel + \gamma_\parallel) (\partial_x \delta\sigma_y - \partial_y \delta\sigma_x), \end{aligned} \quad (22d)$$

where we defined $\delta u_\perp^\mu = (0, \delta u_x, \delta u_y, 0)$ and the enthalpy

$$h = e + p = \epsilon + P + B^2/\mu_m. \quad (23)$$

Here, $c_s^2 = \delta P/\delta\epsilon$ is the squared sound velocity.

B. Equations for the magnetic field

The equations for $\tilde{F}^{\mu\nu}$ can be projected and linearized in the same manner. The projected conservation law reads

$$u_\nu \partial_\mu \tilde{F}^{\mu\nu} = 0, \quad \Xi_\nu^\rho \partial_\mu \tilde{F}^{\mu\nu} = 0, \quad b_\nu \partial_\mu \tilde{F}^{\mu\nu} = 0. \quad (24)$$

Note that the first equation contains neither a time derivative nor derivative corrections. This equation is the Gauss law constraint, as explicitly shown below. Note also that the set of equations in (24) involves a redundancy due to an identity

$$0 = \partial_\mu \partial_\nu \tilde{F}^{\mu\nu} = (\Xi_{\alpha\beta} - u_\alpha u_\beta + b_\alpha b_\beta) \partial^\alpha \partial_\mu \tilde{F}^{\mu\beta}. \quad (25)$$

The left identity is satisfied by any antisymmetric tensor. The right identity serves as a sum rule constraint for the set

of equations in (24), as explicitly confirmed below. Therefore, we are left with two independent dynamical equations and the corresponding two spatial components of the magnetic field that satisfies $u_\mu B^\mu = 0$ and $\partial_\mu \delta B^\mu = 0$. The former equality reads $B\delta u_z - \delta B^0 = 0$, which is used to obtain the following equations.

The explicit forms of the linearized equations are obtained as

$$0 = \partial_t \delta B^i, \quad (26a)$$

$$0 = B\partial_z \delta u_x - \partial_0 \delta B_x - \rho'_\perp T[\partial_z \partial_x \delta(\beta B_z) - \partial_z^2 \delta(\beta B_x)] + \rho'_\parallel T[\partial_\perp^2 \delta(\beta B_x) - \partial_x \partial_\perp \delta(\beta B_\perp)], \quad (26b)$$

$$0 = B\partial_z \delta u_y - \partial_0 \delta B_y - \rho'_\perp T[\partial_z \partial_y \delta(\beta B_z) - \partial_z^2 \delta(\beta B_y)] + \rho'_\parallel T[\partial_\perp^2 \delta(\beta B_y) - \partial_y \partial_\perp \delta(\beta B_\perp)], \quad (26c)$$

$$0 = -B\partial_\perp \delta u_\perp^\mu - \partial_0 \delta B_z + \rho'_\perp T[\partial_\perp^2 \delta(\beta B_z) - \partial_z \partial_\perp \delta(\beta B_\perp)], \quad (26d)$$

where we defined

$$\rho'_\parallel = \frac{\rho_\parallel}{\mu_m}, \quad \rho'_\perp = \frac{\rho_\perp}{\mu_m}. \quad (27)$$

The first equation is nothing but the Gauss law and one can also explicitly confirm the sum rule (25). Thus, there are only two independent dynamical equations in the above.

The derivative of $\delta\beta$ can be expressed with that of $\delta\epsilon$ with the help of a relation obtained from the thermodynamic relation Eq. (6), that is,

$$\delta\beta = -\frac{c_s^2 \beta}{h - B^2/\mu_m} \delta\epsilon. \quad (28)$$

C. Equations for the spin angular momentum

As mentioned around Eq. (19), the spin density only has three spatial degrees of freedom in the totally antisymmetric pseudogauge. The corresponding three equations are obtained by the projections

$$\begin{aligned} \Xi_\alpha^1 b_\beta (\partial_\mu \Sigma^{\mu\alpha\beta} + 2\Theta^{[\alpha\beta]}) &= 0, \\ \Xi_\alpha^2 b_\beta (\partial_\mu \Sigma^{\mu\alpha\beta} + 2\Theta^{[\alpha\beta]}) &= 0, \\ \Xi_\alpha^1 \Xi_\beta^2 (\partial_\mu \Sigma^{\mu\alpha\beta} + 2\Theta^{[\alpha\beta]}) &= 0. \end{aligned} \quad (29)$$

One can obtain the linearized equations

$$0 = (\Gamma_\parallel + \partial_0) \delta\sigma_x + 2[\gamma_\parallel (\partial_y \delta u_z - \partial_z \delta u_y) + \xi_\parallel (\partial_y \delta u_z + \partial_z \delta u_y)], \quad (30a)$$

$$0 = (\Gamma_\parallel + \partial_0) \delta\sigma_y + 2[\gamma_\parallel (\partial_z \delta u_x - \partial_x \delta u_z) - \xi_\parallel (\partial_z \delta u_x + \partial_x \delta u_z)], \quad (30b)$$

$$0 = (\Gamma_\perp + \partial_0) \delta\sigma_z + 2[\gamma_\perp (\partial_x \delta u_y - \partial_y \delta u_x)], \quad (30c)$$

where we defined

$$\Gamma_\parallel = \frac{8\gamma_\parallel}{\chi}, \quad \Gamma_\perp = \frac{8\gamma_\perp}{\chi}. \quad (31)$$

D. Summary of linearized equations

We have obtained the linearized equations for spin MHD, and it is useful to note that the equations obtained above can be divided into two groups. One of them contains the variables $(\delta u_y, \delta B_y, \delta\sigma_x, \delta\sigma_z)$, while the other contains $(\delta\epsilon, \delta u_x, \delta u_z, \delta B_x, \delta\sigma_y)$. The former group is summarized as

$$\begin{aligned} 0 &= h\partial_0 \delta u_y - \frac{h}{B} v_A^2 \partial_z \delta B_y \\ &\quad - [(\eta_\perp + \gamma_\perp) \partial_x^2 + (\eta_\parallel + \gamma_\parallel - 2\xi_\parallel) \partial_z^2] \delta u_y \\ &\quad - \frac{1}{2} \Gamma_\perp \partial_x \delta\sigma_z + \frac{4}{\chi} (\gamma_\parallel - \xi_\parallel) \partial_z \delta\sigma_x, \end{aligned} \quad (32a)$$

$$0 = B\partial_z \delta u_y - \partial_0 \delta B_y + \rho'_\perp \partial_z^2 \delta B_y + \rho'_\parallel \partial_x^2 \delta B_y, \quad (32b)$$

$$0 = (\Gamma_\parallel + \partial_0) \delta\sigma_x - 2(\gamma_\parallel - \xi_\parallel) \partial_z \delta u_y, \quad (32c)$$

$$0 = (\Gamma_\perp + \partial_0) \delta\sigma_z + 2\gamma_\perp \partial_x \delta u_y. \quad (32d)$$

We assume a single-mode solution for the flow perturbation

$$\delta \mathbf{u}(t, x, z) = \delta \tilde{\mathbf{u}}(\omega, k_\perp, k_\parallel) e^{-i\omega t + ik_\perp x + ik_\parallel z}, \quad (33)$$

and similar forms for other perturbations. Then, these equations can be summarized in an algebraic matrix form

$$0 = M_1 (\delta \tilde{\mathbf{u}}, \delta \tilde{B}_y, \delta \tilde{\sigma}_x, \delta \tilde{\sigma}_z)^T, \quad (34)$$

where M_1 is the coefficient matrix. On the other hand, the latter group reads

$$0 = \partial_0 \delta\epsilon + \frac{h}{B} v_A^2 \partial_0 \delta B + h\partial_x \delta u_x + h(1 - v_A^2) \partial_z \delta u_z, \quad (35a)$$

$$\begin{aligned} 0 &= h\partial_0 \delta u_x + c_s^2 \partial_x \delta\epsilon - \frac{h}{B} v_A^2 (\partial_z \delta B_x - \partial_x \delta B_z) \\ &\quad - [(\zeta_\perp + \eta_\perp) \partial_x^2 + (\eta_\parallel + \gamma_\parallel - 2\xi_\parallel) \partial_z^2] \delta u_x \\ &\quad - (\zeta_x + \eta_\parallel - \gamma_\parallel) \partial_z \partial_x \delta u_z - \frac{4}{\chi} (\gamma_\parallel - \xi_\parallel) \partial_z \delta\sigma_y, \end{aligned} \quad (35b)$$

$$0 = c_s^2 \partial_z \delta \epsilon + h(1 - v_A^2) \partial_0 \delta u_z - (\zeta_x + \eta_{\parallel} - \gamma_{\parallel}) \partial_z \partial_x \delta u_x - (\zeta_{\parallel} \partial_z^2 + (\eta_{\parallel} + \gamma_{\parallel} + 2\xi_{\parallel}) \partial_x^2) \delta u_z + \frac{4}{\chi} (\xi_{\parallel} + \gamma_{\parallel}) \partial_x \delta \sigma_y, \quad (35c)$$

$$0 = B \partial_z \delta u_x - \partial_0 \delta B_x - \rho'_{\perp} \left[\partial_z \partial_x \delta B_z - \frac{c_s^2}{1 - v_A^2} \frac{B}{h} \partial_z \partial_x \delta \epsilon - \partial_z^2 \delta B_x \right], \quad (35d)$$

$$0 = (\Gamma_{\parallel} + \partial_0) \delta \sigma_y + 2[(\gamma_{\parallel} - \xi_{\parallel}) \partial_z \delta u_x - (\gamma_{\parallel} + \xi_{\parallel}) \partial_x \delta u_z]. \quad (35e)$$

Here, we defined the so-called Alfvén wave velocity

$$v_A = \frac{B}{\sqrt{\mu_m h}}. \quad (36)$$

Without loss of generality, we have set the transverse coordinate system in such a way that the derivatives along the y direction vanish. Then, one obtains a matrix equation

$$0 = M_2 (\delta \tilde{\epsilon}, \delta \tilde{u}_x, \delta \tilde{u}_z, \delta \tilde{B}_x, \delta \tilde{\sigma}_y)^T, \quad (37)$$

where the M_2 is the coefficient matrix of these five equations. For nontrivial solutions, we require that

$$\det(M_1) = 0 \quad \text{and} \quad \det(M_2) = 0. \quad (38)$$

Corresponding to the number of above equations, we will find nine modes. While six of them are modifications of the conventional gapless MHD modes, the remaining three are gapped spin modes in the totally antisymmetric pseudogauge.

IV. SOLUTIONS FOR THE LINEARIZED EQUATIONS

In this section, we seek solutions for the linearized equations obtained in Sec. III. Because of the anisotropy induced by a magnetic field, the solutions depend on the angle θ between the momentum $\vec{k}$ and the magnetic field in equilibrium. Accordingly, we introduce $k_z = k \cos \theta$ and $k_x = k \sin \theta$. It is not difficult to find the solutions at specific angles $\theta = 0, \pi/2$, as shown in Appendix A. However, it is challenging to find solutions at an arbitrary angle. We first show the first-order solutions in the small- k expansion for an arbitrary angle. We use the method developed in Ref. [90] and obtain the solutions up to the k^2 order, as we are considering the first-order hydrodynamics.

Next, we compare the solutions in the small- k expansion and the solutions at specific angles (without any expansion). We point out that the small- k expansion breaks down when the angle approaches the perpendicular direction to

the magnetic field, i.e., $\theta = \pi/2$, because the n th-order term blows up as $\sim 1/\cos^n \theta$. In this regime, we show an alternative series expression for a small value of $\cos \theta$. Furthermore, we confirm that these two series expressions agree with numerical solutions in appropriate angle regimes. Remarkably, there is a critical angle in between these regimes, where some propagating modes change to purely dissipative modes.

A. Leading-order solutions

To illustrate the crucial steps for the method developed in Ref. [90], we begin with the leading-order solutions. Let us take Eq. (34), which is a quartic equation in ω . Up to the k^1 order, the secular equation from $\det(M_1) = 0$ should be factorized as

$$\det(M_1) \propto (\omega - \tilde{v}_1 k)(\omega - \tilde{v}_2 k)(\omega + i\tilde{\Gamma}_1)(\omega + i\tilde{\Gamma}_2). \quad (39)$$

This is because one should have two gapless modes and two gapped modes. The latter stems from the nonconservation equation for spin (3).

By comparing the coefficients on the left- and right-hand sides of Eq. (39) on an order-by-order basis in k , one can determine $\tilde{v}_{1,2}$ and $\tilde{\Gamma}_{1,2}$. At the leading order, one can easily get

$$\tilde{v}_{1,2} = \pm v_A \cos \theta, \quad (40a)$$

$$\tilde{\Gamma}_1 = \Gamma_{\parallel}, \quad \tilde{\Gamma}_2 = \Gamma_{\perp}. \quad (40b)$$

We find that $0 \leq v_A \leq 1$ when the enthalpy is given, as in Eq. (23).

One can apply the same method to Eq. (37). Since there is only one spin variable in Eq. (37), one should have one gapped mode in the set of solutions. Therefore, the leading-order secular equation should be factorized as

$$\det(M_2) \propto (\omega - \tilde{v}'_1)(\omega - \tilde{v}'_2)(\omega - \tilde{v}'_3)(\omega - \tilde{v}'_4)(\omega + i\tilde{\Gamma}). \quad (41)$$

We then obtain the leading-order solutions

$$\tilde{v}'_{1,2} = \pm v_1 k, \quad \tilde{v}'_{3,4} = \pm v_2 k, \quad (42a)$$

$$\tilde{\Gamma} = \Gamma_{\parallel}, \quad (42b)$$

where we defined

$$v_{1,2} = \frac{1}{\sqrt{2}} \left(V^2 \pm \sqrt{V^4 - 4v_A^2 c_s^2 \cos^2 \theta} \right)^{\frac{1}{2}}, \quad (43a)$$

$$V = \sqrt{v_A^2 + c_s^2 (1 - v_A^2 \sin^2 \theta)}. \quad (43b)$$

Here, we get four propagating modes, which are known as the magneto-sonic waves. When the sign in Eq. (43a) takes plus, they are called the fast modes. In the other case, they are called the slow modes. We find that the leading-order solutions are exactly the same as the case without spin (see, e.g., Refs. [90,91,99] and references therein). It can be shown that $0 \leq v_{1,2} \leq 1$, assuming that $0 \leq c_s \leq 1$ and $0 \leq v_A \leq 1$ (see Ref. [90]).

B. Next-to-leading-order solutions

Now, we step forward to the k^2 order. As we are discussing solutions within the first-order hydrodynamics, the terms beyond the k^2 order are uncertain. Therefore even though we get the expansion to the k^3 or even higher orders, we cannot judge if it is a better solution than the expansion up to the k^2 order. In all, since higher-order terms are not under control, one should stop at the k^2 order.

It is straightforward to extend the discussion in Sec. IV A to the next-to-leading order. We can now assume the factorization for Eq. (34) up to the k^2 order as

$$\det(M_1) \propto (\omega - v_A k_{\parallel} + i\tilde{\omega}_1 k^2)(\omega + v_A k_{\parallel} + i\tilde{\omega}_2 k^2) \times (\omega + i\Gamma_{\parallel} + i\tilde{\omega}_3 k^2)(\omega + i\Gamma_{\perp} + i\tilde{\omega}_4 k^2), \quad (44)$$

where $\tilde{\omega}_i$ should be independent of k . By comparing $\det(M_1)$ with the above factorized form order-by-order in k , one can determine $\tilde{\omega}_i$. Getting rid of the irrelevant higher-order terms allows us to efficiently identify the corrections,

$$\tilde{\omega}_1 = \tilde{\omega}_2 = \frac{1}{2h}[(\tilde{\eta}_{\parallel} + h\rho'_{\perp})\cos^2\theta + (\eta_{\perp} + h\rho'_{\parallel})\sin^2\theta], \quad (45a)$$

$$\tilde{\omega}_3 = \frac{(\gamma_{\parallel} - \xi_{\parallel})^2}{h\gamma_{\parallel}}\cos^2\theta, \quad (45b)$$

$$\tilde{\omega}_4 = \frac{\gamma_{\perp}}{h}\sin^2\theta, \quad (45c)$$

where we defined

$$\tilde{\eta}_{\parallel} = \eta_{\parallel} - \frac{\xi_{\parallel}^2}{\gamma_{\parallel}}. \quad (46)$$

Then, the first-order solutions can be summarized as

$$\omega = \pm v_A k \cos\theta - i\tilde{\omega}_{1,2} k^2, \quad (47a)$$

$$\omega = -i\Gamma_{\parallel} - i\tilde{\omega}_3 k^2, \quad (47b)$$

$$\omega = -i\Gamma_{\perp} - i\tilde{\omega}_4 k^2. \quad (47c)$$

Next, let us discuss the k^2 -order solutions for Eq. (37). We assume the factorization of $\det(M_2)$ as

$$\det(M_2) \propto (\omega - v_1 k + i\tilde{\omega}'_1 k^2)(\omega + v_1 k + i\tilde{\omega}'_2 k^2) \times (\omega - v_2 k + i\tilde{\omega}'_3 k^2)(\omega + v_2 k + i\tilde{\omega}'_4 k^2) \times (\omega + i\Gamma_{\parallel} + i\tilde{\omega}'_5 k^2). \quad (48)$$

Again, by comparing this ansatz with the determinant on an order-by-order basis, one can obtain $\tilde{\omega}'_i$ and ω_5 . We find that $\tilde{\omega}'_1 = \tilde{\omega}'_2$ and that $\tilde{\omega}'_3 = \tilde{\omega}'_4$. This indicates that a pair of waves propagating in opposite directions with the same speed acquire the same damping rate at the k^2 order. The solutions are summarized as

$$\omega = \pm v_1 k + ik^2 \frac{W_1 - W_2 v_1^2}{2(v_1^2 - v_2^2)}, \quad (49a)$$

$$\omega = \pm v_2 k - ik^2 \frac{W_1 - W_2 v_2^2}{2(v_1^2 - v_2^2)}, \quad (49b)$$

$$\omega = -i\Gamma_{\parallel} - ik^2(W_3 - W_2), \quad (49c)$$

where the explicit forms of W_i are found to be

$$W_1 = \frac{1}{h} \left[\tilde{\eta}_{\parallel} \left(c_s^2 \cos^2(2\theta) + \frac{v_A^2 \sin^2\theta}{1 - v_A^2} \right) + h\rho'_{\perp} c_s^2 \frac{1 - v_A^2 \cos^2\theta}{1 - v_A^2} + \zeta_{\parallel} \frac{v_A^2}{1 - v_A^2} \cos^2\theta + (\zeta_{\parallel} + \zeta_{\perp} - 2\zeta_{\times} + \eta_{\perp}) c_s^2 \sin^2\theta \cos^2\theta \right], \quad (50a)$$

$$W_2 = \frac{1}{h} \left[\tilde{\eta}_{\parallel} \frac{1 - v_A^2 \cos^2\theta}{1 - v_A^2} + \zeta_{\parallel} \frac{\cos^2\theta}{1 - v_A^2} + (\zeta_{\perp} + \eta_{\perp}) \sin^2\theta + h\rho'_{\perp} \left(1 + \frac{v_A^2 c_s^2}{1 - v_A^2} \sin^2\theta \right) \right], \quad (50b)$$

$$W_3 = \frac{1}{h} \left[\cos^2\theta (\eta_{\parallel} + \gamma_{\parallel} - 2\xi_{\parallel}) + \frac{\sin^2\theta}{1 - v_A^2} (\eta_{\parallel} + \gamma_{\parallel} + 2\xi_{\parallel}) + \zeta_{\parallel} \frac{\cos^2\theta}{1 - v_A^2} + (\zeta_{\perp} + \eta_{\perp}) \sin^2\theta + h\rho'_{\perp} \left(1 + \frac{v_A^2 c_s^2}{1 - v_A^2} \sin^2\theta \right) \right]. \quad (50c)$$

The above solutions in the small- k expansion are one of the main results in this paper. However, we will further see that the above series expressions are not valid when $\cos\theta$ becomes small in the next subsection. Before going into this point, we give a couple of comments on the above solutions.

One can compare the above results with the first-order solutions without spin [90]. First, the propagating modes

in the k expansion, which are shown in Eqs. (47a), (49a), and (49b), have the same forms as the damping Alfvén and magneto-sonic waves up to the replacement of the shear viscosity $\eta_{\parallel}$ by $\tilde{\eta}_{\parallel}$ defined in Eq. (46). In other words, only when there is a nonzero cross rotational viscosity $\xi_{\parallel}$ the propagating modes are modified by dynamics of angular momentum. In Eq. (11), the cross rotational viscosity converts the antisymmetric perturbation to the symmetric stress, and modifies the Alfvén and sound modes. The other three modes in Eqs. (47b), (47c), and (49c) are damping spin modes. The damping factor remains in the zero-momentum limit, and spin density does not persist in the strict equilibrium limit because spin angular momentum is not a conserved quantity by itself [22].

One can also confirm that all the eigenmodes obtained in Eqs. (47) and (49) are damping in time according to the inequalities on transport coefficients in Eq. (17) from the entropy-current analysis. It is clear that all the $\tilde{\omega}_i$ in Eq. (45) are semipositive definite for any θ , indicating that all the modes in Eq. (47) are damping in the k^2 order. It is less obvious that the other set of solutions (49) are also all damping in the k^2 order. In Ref. [90] without spin, this has been shown for the first two modes for $\eta_{\parallel} \geq 0$. Replacing $\eta_{\parallel}$ by $\tilde{\eta}_{\parallel}$, we find that these two modes are still damping. One can show that the third mode is also damping. Its coefficient of the k^2 order reads

$$W_3 - W_2 = \frac{1}{h(1 - v_A^2)\gamma_{\parallel}} \left[(1 - v_A^2 \cos^2 \theta)(\gamma_{\parallel}^2 + \xi_{\parallel}^2) + 2\{\sin^2 \theta - (1 - v_A^2)\cos^2 \theta\}\xi_{\parallel}\gamma_{\parallel} \right]. \quad (51)$$

Since we have $4(1 - v_A^2 \cos^2 \theta)^2 - 2^2\{\sin^2 \theta - (1 - v_A^2)\cos^2 \theta\}^2 = 4(1 - v_A^2)\sin^2(2\theta) \geq 0$, we find that

$$W_3 - W_2 \geq \frac{1}{h(1 - v_A^2)\gamma_{\parallel}} (1 - v_A^2 \cos^2 \theta) \left[(\gamma_{\parallel}^2 + \xi_{\parallel}^2) - 2\xi_{\parallel}\gamma_{\parallel} \right] \geq 0. \quad (52)$$

Thus, we can conclude that all the solutions in Eq. (49) are also damping. Then, all the nine modes in spin MHD are damping within the current linear-mode analysis in the rest frame.

C. Breakdown of the small-momentum expansion

We have obtained the solutions for an arbitrary angle θ up to the k^2 order in Sec. IV B. Here, however, we point out that the small- k expansion breaks down depending on the angle θ , and thus provide an alternative method of expansion in such an angle regime. One can realize the breakdown of the small- k expansion by comparing the solutions in Eqs. (47) and (49) with the solutions at specific angles shown in Appendix A. The latter solutions can be obtained without any expansion.

When $\theta = \pi/2$, the solution in (47a) does not agree with any of the solutions in the limit $\theta = \pi/2$.² Therefore, one should investigate the behavior of the solution near $\theta = \pi/2$ more closely. As discussed below, the exact solution (without the small- k expansion) has a nonanalytic boundary at $k = k_c$, where k_c depends on θ . The k expansion gets invalid when k goes beyond this boundary. The critical momentum k_c approaches zero when $\cos \theta$ approaches zero. Accordingly, the Taylor expansion for k breaks down when $\cos \theta$ approaches zero. This nonanalytic property arises from that of a square root. Since waves propagating in opposite directions appear in pairs, the secular equation is an equation for ω^2 , and its solution for ω is in general given by a square root.

This singular behavior was recently discussed in MHD without spin by the present authors [90]. Here, we recapitulate the discussions therein. The simplest mode, which nevertheless involves the singularity, is the Alfvén waves

$$\omega = \pm \sqrt{v_A^2 k^2 \cos^2 \theta - \frac{1}{4}(\tilde{\rho} - \tilde{\eta})^2 k^4} - \frac{i}{2}(\tilde{\rho} + \tilde{\eta})k^2, \quad (53)$$

where $\tilde{\rho} = \rho'_{\perp} \cos^2 \theta + \rho'_{\parallel} \sin^2 \theta$ and $\tilde{\eta} = \frac{1}{h}(\eta_{\parallel} \cos^2 \theta + \eta_{\perp} \sin^2 \theta)$. This is the solution without any expansion. The square root in Eq. (53) becomes a pure imaginary number when k grows larger than $2v_A |\frac{\cos \theta}{\tilde{\rho} - \tilde{\eta}}|$. Namely, the dispersion relation loses the real part, turning to a purely diffusive mode. Clearly, this behavior is different from that of the solution in the small- k expansion that always has the Alfvén velocity as the group velocity. In this regime, where $k > 2v_A |\frac{\cos \theta}{\tilde{\rho} - \tilde{\eta}}|$, one can instead expand Eq. (53) with $\cos \theta$ near $\cos \theta = 0$. This alternative expansion is valid in the regime where the square root is imaginary.

We find the same issue in spin MHD. The two Alfvén modes shown in Eq. (45a) are effective only when $\cos \theta$ is large. When $\cos \theta$ gets smaller than a certain value, the real parts of the dispersion relations merge to zero and the imaginary parts split into two distinct ones. We confirm these merging and splitting behaviors with numerical plots in Fig. 1. The dotted line shows the small- k expansion, while the red solid line shows the cosine expansion. We find a good agreement with the blue solid line that shows the solution without any expansion. In Appendix B, we obtain the cosine expansion as

$$\omega = -\frac{i\eta_{\perp} k^2}{h} + \Omega_1 \cos^2 \theta, \quad -i\rho'_{\parallel} k^2 + \Omega_2 \cos^2 \theta. \quad (54)$$

²Consistent cases can be confirmed as follows. For the set of solutions in (47) containing the Alfvén modes, the solutions in (47) are consistent with those in Eqs. (A2) and (A3) when $\theta = 0$. When $\theta = \pi/2$, the solutions (47b) and (47c) are still consistent with the one in Eq. (A7d) and the first one in Eq. (A7b).

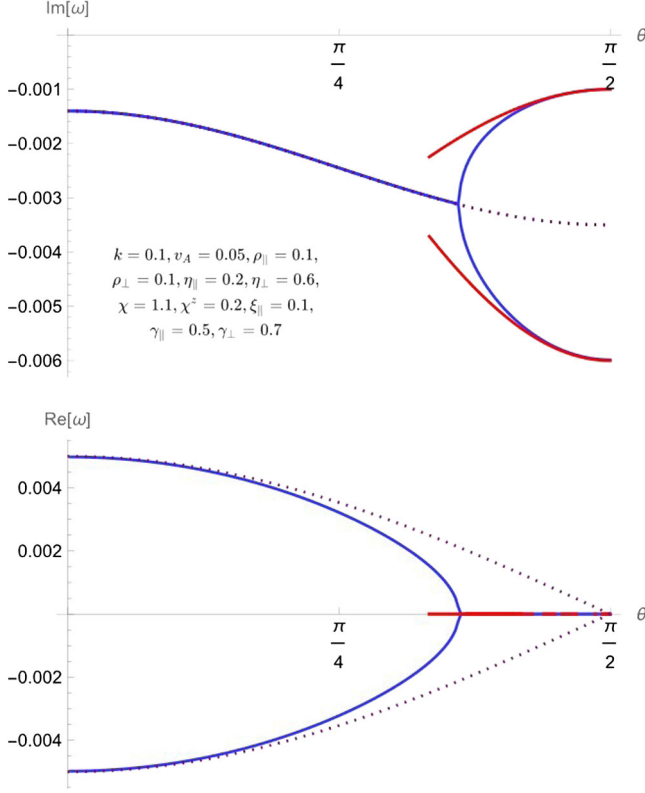


FIG. 1. The dispersion relations for the Alfvén modes. Blue curves show the “exact solution” without any expansion. Dotted curves show the small- k expansion in Eq. (45a). Red curves show the small-cosine expansion in Eq. (54).

The explicit forms of Ω_i are somewhat involved, and are shown in Appendix B together with their derivations.

The same issue also exists in the quintic equation containing the magneto-sonic modes. The limit $\theta \rightarrow \pi/2$ for Eq. (49) can be readily obtained as

$$\omega_{1,\pm} \rightarrow \pm v_f k - \frac{ik^2}{2} \left(\frac{v_A^2}{v_f^2} (1 - c_s^2)^2 \rho'_\perp + \frac{\eta_\perp + \zeta_\perp}{h} \right), \quad (55a)$$

$$\omega_{2,\pm} \rightarrow -\frac{ik^2}{2h(1 - v_A^2)} \left(\tilde{\eta}_\parallel + \frac{c_s^2}{v_f^2} h \rho'_\perp \right), \quad (55b)$$

$$\omega \rightarrow -i\Gamma_\parallel - \frac{ik^2}{(1 - v_A^2)h} \frac{(\gamma_\parallel + \xi_\parallel)^2}{\gamma_\parallel}, \quad (55c)$$

where $v_f^2 = c_s^2 + v_A^2(1 - c_s^2)$. However, we notice that the limiting behavior of the two slow magneto-sonic modes (55b) disagrees with Eq. (A5) and the second solution in Eq. (A7a) that are obtained without any expansion. This disagreement also stems from the existence of a nonanalytic boundary in the $k - \theta$ plane, as in the case of the Alfvén wave. The correct result at this limit is obtained in Eq. (A2) by taking $\theta = \pi/2$ without the k expansion.

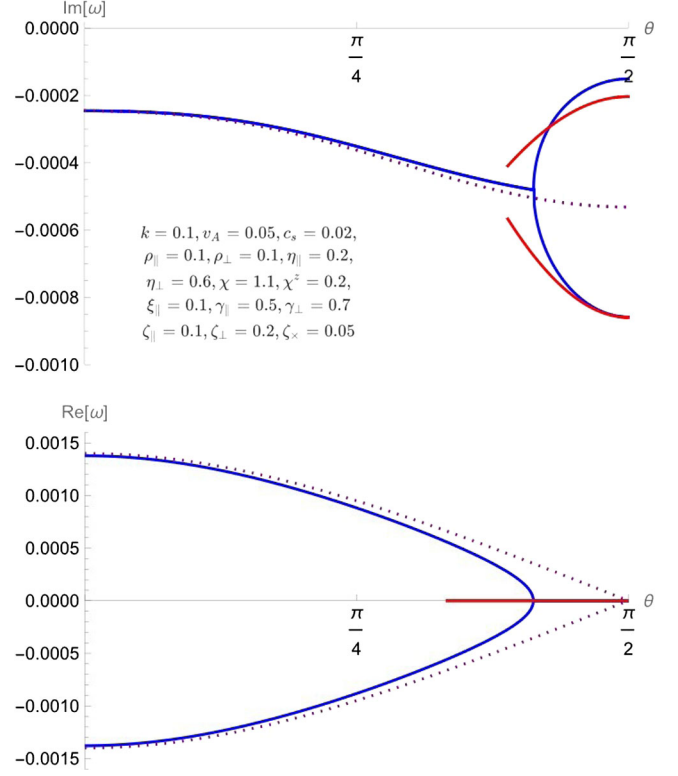


FIG. 2. The dispersion relations for the slow magneto-sonic modes. The blue curves represent the numerical solutions of the quartic equation. Dotted curves show the small- k expansion in Eq. (49b). Red curves show the small-cosine expansion in Eq. (56).

In Fig. 2, we compare the k expansion in (49) to the numerical solutions. The former and latter are shown by the dotted and blue solid lines, respectively. While we find a good agreement between them when $\cos \theta$ is large, they do not at all agree with each other near $\theta = \pi/2$. Instead, one can organize the cosine expansion

$$\omega = -\frac{ik^2 \tilde{\eta}_\parallel}{h(1 - v_A^2)} + \Omega'_1 \cos^2 \theta, \\ -\frac{ic_s^2 k^2 \rho'_\perp}{(1 - v_A^2)v_f^2} + \Omega'_2 \cos^2 \theta. \quad (56)$$

The explicit forms of Ω'_i are shown in Appendix B. These series expressions, shown by the red solid lines, well reproduce the solution in blue. These solutions for Ω'_i explicitly depend on the rotational viscosities even when the cross rotational viscosity $\xi_\parallel$ is vanishing. Whereas the magneto-sonic modes (49b) depend on the rotational viscosity only via $\tilde{\eta}_\parallel$, they get explicit dependences on the rotational viscosities after turning into the purely diffusive modes (56) near $\theta = \pi/2$.

These solutions in Eqs. (54) and (56) are all expected to be stable, as seen in Figs. 1 and 2. According to the

solutions in Appendix A 2, one can easily confirm that all the modes are damping at $\theta = \pi/2$. As we decrease θ , the imaginary parts of the solutions approach those of the Alfvén and magneto-sonic modes that have the definite signs, as shown in Sec. IV B. Therefore, the solutions in Eqs. (54) and (56) are stable (unless there arise non-monotonic behaviors with respect to θ).

D. Extension to the Israel-Stewart equations

In the first-order hydrodynamics, diffusive modes typically lead to violation of causality, as diffusion equations imply instantaneous signal propagation. Since a prescription for this issue was provided by Israel and Stewart [106–108], causality has been one of the important topics in relativistic hydrodynamics and is still a hot topic for discussion (see, e.g., Refs. [109–115] for recent developments).

Our first-order solutions, shown in Figs. 1 and 2, contain nonpropagating regimes near the vertical limit, i.e., $\theta \rightarrow \pi/2$. We investigate these diffusive modes based on the IS equations by introducing a finite relaxation time τ . We will find that, near the vertical limit, there remain diffusive modes in the IS equations in a low-momentum region, while they are modified to become propagating modes in the higher-momentum region. Therefore, the IS-modified dispersion relation exhibits a diffusive window in the intermediate momentum region, as shown below.

We focus on the Alfvén modes to demonstrate how the diffusive modes, emerging near the vertical limit, are modified in the IS equations. For simplicity, we assume that the transport coefficients take isotropic values, i.e., $\eta := \eta_{\parallel} = \eta_{\perp}$ and $\rho := \rho_{\parallel} = \rho_{\perp}$, since this simplification does not affect a conclusion regarding the presence of residual diffusive modes in the IS equations at the qualitative level. We also suppress the spin sector for the same reason. Then, the IS equations for the Alfvén modes read

$$\partial_0 \delta B_y = B \partial_z \delta u_y + \partial_x \delta \tilde{\mathcal{F}}_{xy} - \partial_z \delta \tilde{\mathcal{F}}_{yz}, \quad (57a)$$

$$\tau \partial_0 \delta \tilde{\mathcal{F}}_{xy} + \delta \tilde{\mathcal{F}}_{xy} = \frac{1}{\mu_m} \rho \partial_x \delta B_y, \quad (57b)$$

$$\tau \partial_0 \delta \tilde{\mathcal{F}}_{yz} + \delta \tilde{\mathcal{F}}_{yz} = -\frac{1}{\mu_m} \rho \partial_z \delta B_y, \quad (57c)$$

$$h \partial_0 \delta u^y = B \partial_z \delta B_y + \partial_x \delta \pi_{xy} + \partial_z \delta \pi_{yz}, \quad (57d)$$

$$\tau \partial_0 \delta \pi_{xy} + \delta \pi_{xy} = \eta \partial_x \delta u_y, \quad (57e)$$

$$\tau \partial_0 \delta \pi_{yz} + \delta \pi_{yz} = \eta \partial_z \delta u_y, \quad (57f)$$

where we denote the dynamical variables in the IS equations as π_{ij} and $\tilde{\mathcal{F}}_{ij}$ and the corresponding relaxation times as τ_R and τ_B , respectively. We take both relaxation times to be the same value τ . When $\tau \rightarrow 0$, these equations

go back the first-order equations (32) discussed in the previous sections (without spin).

We first provide analytic solutions at the vertical limit, which are obtained as

$$\omega = -\frac{ih \pm \sqrt{4\eta h k^2 \tau - h^2}}{2h\tau}, \quad -\frac{i\mu_m \pm \sqrt{4\mu_m \rho k^2 \tau - \mu_m^2}}{2\mu_m \tau}. \quad (58)$$

The other two equations for $\delta \tilde{\mathcal{F}}_{yz}$ and $\delta \pi_{yz}$ are decoupled in this limit. Expanding these solutions for a small k , one finds two diffusive modes and two gapped modes,

$$\omega = -i\frac{\eta}{h}k^2, \quad -\frac{i}{\tau} + i\frac{\eta}{h}k^2, \quad (59a)$$

$$\omega = -i\frac{\rho}{\mu_m}k^2, \quad -\frac{i}{\tau} + i\frac{\rho}{\mu_m}k^2. \quad (59b)$$

Two of these modes, which depend on τ , are gapped modes introduced by the IS relaxation equations. The other two gapless diffusive modes are independent of the relaxation time at the leading order in k and reduce to the Alfvén modes in the first-order solutions in Fig. 1 when $\tau \rightarrow 0$. Thus, these modes are identified as the “IS-modified Alfvén modes.” In Panel 1 of Fig. 3, blue curves show the IS-modified Alfvén modes, while orange curves show the Alfvén modes in the first-order solutions at $\tau = 0$. We take $\eta h^{-3/4} = 0.2$, $\rho h^{1/4} = 0.2$, $\tau h^{1/4} = 1$, and $v_A = 0.7$, where dimensionful quantities are rescaled by h in Eq. (23).

The small- k expansion (59b) shows that the IS-modified Alfvén modes remain diffusive modes at the vertical limit even after the IS relaxation equations are introduced. This behavior is in clear contrast to the typical behavior of dispersion relations from the IS equations, where diffusive modes in the first-order equations are modified to become propagating modes in a small-momentum region down to vanishing k (see, e.g., Ref. [116]). As we increase k , the square root arguments in the full solution (58) turn into positive values at the critical momenta,

$$k_c^{\text{IS}} = \sqrt{\frac{h}{4\eta\tau}}, \quad \sqrt{\frac{\mu_m}{4\rho\tau}}. \quad (60)$$

At each of these critical momenta, a diffusive mode merges with one of the gapped relaxation modes to form a pair of propagating modes, as shown with the blue curves in the same figure (Panel 1 of Fig. 3).

On top of the above observations at the vertical limit, we examine the angle dependence of the dispersion relations. In Fig. 4, diffusive and propagating regimes are shown by the unshaded and shaded regions, respectively, as a phase diagram with respect to θ and k . The numbered cross sections at different angles correspond to the dispersion

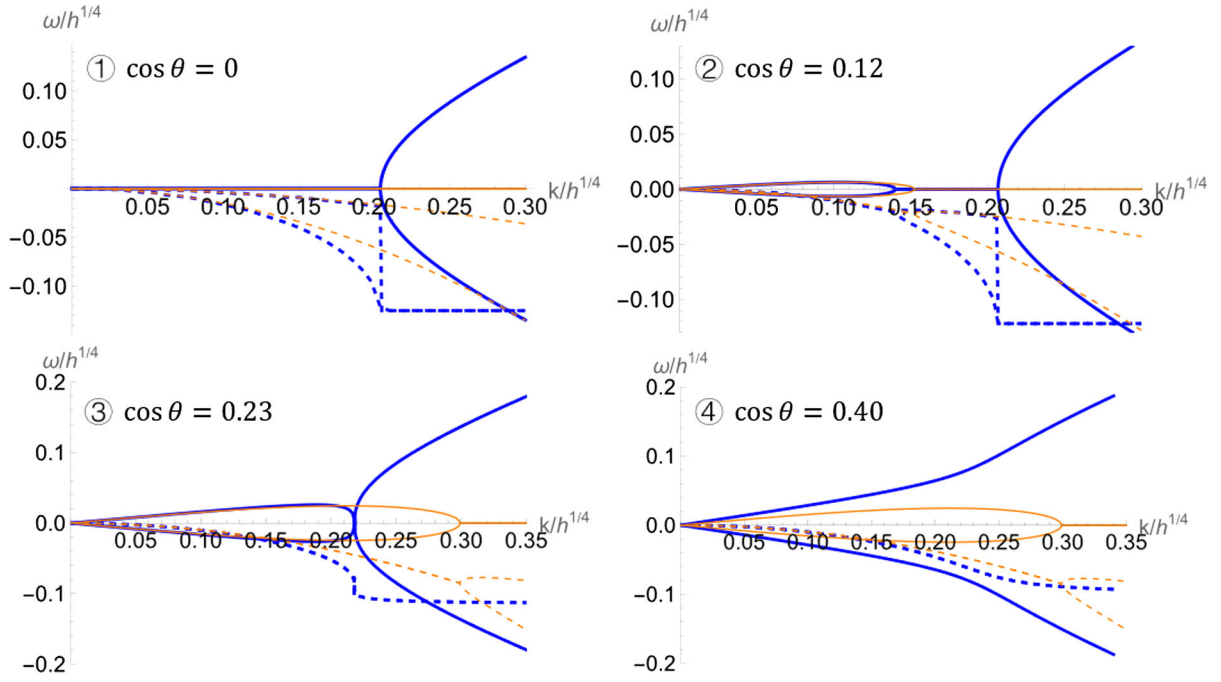


FIG. 3. Dispersion relations of the IS-modified Alfvén modes (blue curves) and the first-order Alfvén modes (orange curves) at fixed angles. Solid and dotted curves show the real and imaginary parts of the dispersion relations. See also Fig. 4 for general angles.

relations shown in Fig. 3. We take the same parameters in Figs. 3 and 4.

First, let us examine the low-momentum regions. In case of the first-order MHD discussed in the previous sections, we have the propagating Alfvén waves, as we decrease

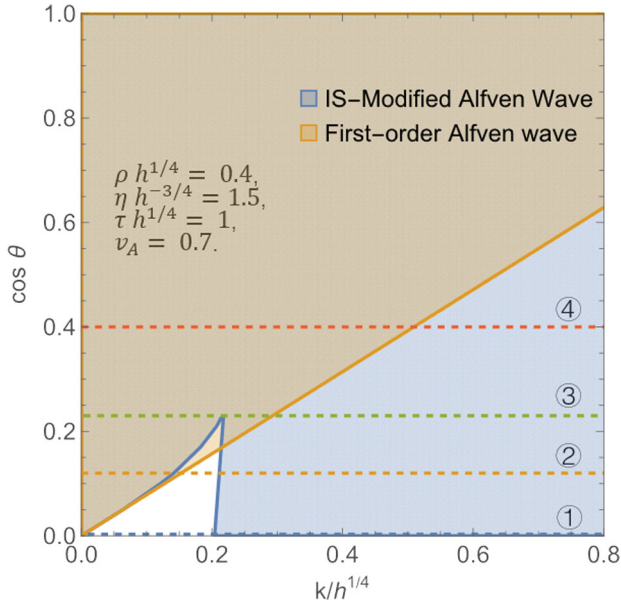


FIG. 4. Unshaded and shaded regions show diffusive and propagating regimes, respectively, in the IS-modified Alfvén modes (blue shade) and the first-order Alfvén modes (orange shade). The IS-modified Alfvén modes retain a diffusive window in the unshaded “shark fin” region.

the angle from the vertical limit, which corresponds to the region shaded in dark orange. However, the small-momentum expansion breaks down at the critical momentum $k_c(\theta)$ discussed in Sec. IV C. The Alfvén modes in the first-order MHD become diffusive above $k_c(\theta)$. As shown in Fig. 4, the IS-modified Alfvén modes basically succeed these properties, though the critical momentum is slightly modified, as we find a slight deviation between the orange and blue boundaries. These propagating modes are also confirmed in the low-momentum regions in Panels 2, 3, and 4 of Fig. 3.

In the high-momentum region, diffusive modes are modified by the relaxation dynamics introduced in the IS equations in general. Indeed, as already discussed above at the vertical limit, the diffusive regime in the first-order solutions turns into the propagating regime. This modification is found in all the angles in Fig. 3 in the high-momentum regions. Recall, however, that this modification is effective only above the other critical momentum k_c^{IS} defined in Eq. (60) that is controlled by the relaxation time τ and depends on θ in general.

It is, therefore, interesting to focus on a momentum window defined by the two critical momenta, $k_c(\theta) \leq k \leq k_c^{\text{IS}}(\theta)$. Since the critical momenta depend on the angle θ , the window is closed when $k_c(\theta) = k_c^{\text{IS}}(\theta)$. At this angle, the low- and high-momentum propagating regimes touch each other (see Panel 3 in Fig. 3). Above this angle (closer to the vertical limit), a diffusive window persists even after a finite relaxation time is introduced with the IS equations. This regime corresponds to the unshaded “shark fin” region

in Fig. 4. The presence of the diffusive window in an intermediate-momentum region deserves further study in future works.

V. CONCLUSION

In this paper, we formulated the first-order spin MHD. The magnetic field breaks the rotational symmetry, and gives rise to anisotropic spin damping rates in terms of the rotational viscosities in the antisymmetric part of the energy-momentum tensor (11). The energy-momentum tensor also acquires a novel cross term that converts a vorticity into the symmetric stress and a shear into the antisymmetric torque. This effect arises only in the presence of a zeroth-order vector in derivative, that is, a magnetic field in case of spin MHD and a vorticity in case of gyrohydrodynamics [29].

We then performed the linear-mode analysis to get the nine dispersion relations in Sec. III. In the absence of spin degrees of freedom, three pairs among them are known as the Alfvén waves and the fast and slow magneto-sonic waves. We found that the presence of spin damping rates modifies the shear viscosity in a particular combination (46). There are three remaining modes in the totally antisymmetric pseudogauge, which are damping spin modes. They are quasihydrodynamic modes that have nonvanishing damping rates even at a vanishing momentum. That is, spin is not a conserved quantity by itself and is converted into the orbital angular momentum until the spin potential and the thermal vorticity match each other. We also proved the linear stability of these modes in the rest frame by extending our previous analysis for MHD [90].

We stress that the above analytic solutions are obtained for an arbitrary angle θ in the small-momentum expansion. However, these solutions indicate that the small-momentum expansion is spoiled due to the blow up of the higher-order terms when the angle θ approaches the right angle where the linear waves propagate nearly perpendicular to an equilibrium magnetic field. We showed that there is a critical angle where the two competing expansion parameters, the momentum k and $\cos \theta$, have similar magnitudes. Since the critical angle depends on the momentum, the solution has a nonanalytic boundary in the $k - \theta$ plane. This boundary defines the critical momentum $k_c(\theta)$ when we consider the dispersion relations as functions of k at a fixed θ . On the other side of the boundary, we provided the alternative solution in the small-cosine expansion.

We confirmed that the momentum and cosine expansions well reproduce the numerical solutions in the appropriate domains separated by the boundary. We point out that this nonanalytic structure is a prevalent issue in anisotropic systems.

We also included the relaxation dynamics for the electric resistivity and shear viscosity with the IS framework. These higher-order corrections modify the dispersion relations in a high-momentum region. In particular, the purely diffusive regime, which was found in the first-order solutions above the critical momentum $k_c(\theta)$, turns into a propagating regime due to the relaxation dynamics. The boundary between the diffusive and propagating regimes defines another critical momentum $k_c^{\text{IS}}(\theta)$. We found that there remains a diffusive window between the lower critical momentum k_c and the upper critical momentum k_c^{IS} , as shown in Fig. 4.

We would like to give a future outlook. To address the splitting between Λ and $\bar{\Lambda}$ spin polarization in the framework of spin MHD, it is important to include a finite chemical potential since a net spin polarization is not induced in a charge neutral fluid. Then, one may need to include a finite electric current and an electric field accordingly. They are quasihydrodynamic modes and require an extended framework. Also, it is interesting to investigate the spin relaxation to a nonzero thermal vorticity. One needs to include a zeroth-order vorticity into the constitutive equations (cf. Ref. [29]) and perform the linear-mode analysis near a vortical equilibrium configuration. Last but not least, extensions to the Israel-Stewart-type frameworks will be useful for numerical implementation toward applications to relativistic heavy ion collisions.

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: SOLUTIONS AT TWO SPECIFIC ANGLES

In this appendix, we focus on the solutions at two specific angles at $k_{\perp} = 0$ and $k_{\parallel} = 0$.

1. Eigenmodes at $k_\perp = 0$

When $k_\perp = 0$, the Gauss law constraint in Eq. (26) indicates that $\delta B_z = -k_\perp \delta B_x / k_\parallel = 0$, so that one can get rid of the last equation in Eq. (26). Notice also that the linearized equations are further factorized into block-diagonal forms,

$$0 = \left(\begin{array}{ccc|cc} Bk_\parallel & \omega + i\rho'_\perp k_\parallel^2 & 0 & & \\ h\omega + i(\eta_\parallel - 2\xi_\parallel + \gamma_\parallel)k_\parallel^2 & \frac{h}{B}v_A^2 k_\parallel & \frac{4}{\chi}(\gamma_\parallel - \xi_\parallel)k_\parallel & & \\ -2(\gamma_\parallel - \xi_\parallel)k_\parallel & 0 & \omega + i\Gamma_\parallel & & \\ \hline & \omega & -h(1 - v_A^2)k_\parallel & & \\ & -c_s^2 k_\parallel & h(1 - v_A^2)\omega + i\zeta'_\parallel k_\parallel^2 & & \end{array} \right) \begin{pmatrix} \delta u_x \\ \delta B_x \\ \delta \sigma_y \\ \delta \epsilon \\ \delta u_z \end{pmatrix}, \quad (\text{A1a})$$

$$0 = \left(\begin{array}{ccc|cc} Bk_\parallel & \omega + i\rho'_\perp k_\parallel^2 & 0 & & \\ h\omega + i(\eta_\parallel - 2\xi_\parallel + \gamma_\parallel)k_\parallel^2 & \frac{h}{B}v_A^2 k_\parallel & -\frac{4}{\chi}(\gamma_\parallel - \xi_\parallel)k_\parallel & & \\ 2(\gamma_\parallel - \xi_\parallel)k_\parallel & 0 & \omega + i\Gamma_\parallel & & \\ \hline & \omega + i\Gamma_\perp & & & \end{array} \right) \begin{pmatrix} \delta u_y \\ \delta B_y \\ \delta \sigma_x \\ \delta \sigma_z \end{pmatrix}. \quad (\text{A1b})$$

By solving the secular equations, one can straightforwardly obtain the dispersion relations of the eigenmodes. One will find degenerate eigenmodes for $\delta u_x, \delta B_x, \delta \sigma_y$ and $\delta u_y, \delta B_y, \delta \sigma_x$ because of the rotational symmetry around the direction of a magnetic field. The degeneracy manifests itself in the determinants for the upper block-diagonal parts of the two matrices that result in the same cubic equations.³ Up to the order of $k_\parallel^2$, the three dispersion relations are obtained as

$$\omega = -i\Gamma_\parallel - i\frac{(\gamma_\parallel - \xi_\parallel)^2}{h\gamma_\parallel}k_\parallel^2, \\ \pm v_A k_\parallel - \frac{i}{2}\left(\rho'_\perp + \frac{\eta_\parallel \gamma_\parallel - \xi_\parallel^2}{h\gamma_\parallel}\right)k_\parallel^2. \quad (\text{A2})$$

Notice that these modes are always damped out due to the first-order transport coefficients that satisfy the thermodynamic constraints (17). The spin mode is a gapped and intrinsically damping mode at the hydrodynamic limit $k_\parallel \rightarrow 0$, because of $\Gamma_\parallel$ from the nonvanishing right-hand side in Eq. (3). The other two modes propagate in opposite directions with the Alfvén wave velocity v_A , while they are

damped out due to the dissipative transport coefficients. All those modes have transverse polarizations with respect to the momentum.

The remaining blocks in the above two matrices yield three dispersion relations,

$$\omega = -i\Gamma_\perp, \quad (\text{A3a})$$

$$\omega = \pm c_s k_\parallel - \frac{i}{2}\zeta'_\parallel k_\parallel^2, \quad (\text{A3b})$$

where $\zeta'_\parallel = \frac{\zeta_\parallel}{h(1-v_A^2)}$. Again, the spin mode is damped out in the hydrodynamic limit with $\Gamma_\perp$ from the nonvanishing right-hand side in Eq. (3). The other two modes are sound waves that are damped out due to one of the bulk viscosities. All these modes have longitudinal polarizations with respect to the momentum.

2. Eigenmodes at $k_\parallel = 0$

Next, we discuss the case at $k_\parallel = 0$. In this case, we have $\delta B_x = -k_\parallel \delta B_z / k_\perp = 0$. The remaining nonvanishing elements of the matrices are block-diagonalized as

$$0 = \left(\begin{array}{ccc|cc} -i\frac{c_s^2 \rho'_\perp}{1-v_A^2} \frac{B}{h} k_\perp^2 & -k_\perp B & \omega + i\rho'_\perp k_\perp^2 & & \\ \omega & -hk_\perp & \frac{h}{B}v_A^2 \omega & & \\ -c_s^2 k_\perp & h\omega + i(\zeta_\perp + \eta_\perp)k_\perp^2 & -\frac{h}{B}v_A^2 k_\perp & & \\ \hline & h(1 - v_A^2)\omega + i(\eta_\parallel + 2\xi_\parallel + \gamma_\parallel)k_\perp^2 & -\frac{4}{\chi}(\xi_\parallel + \gamma_\parallel)k_\perp & & \\ & 2(\gamma_\parallel + \xi_\parallel)k_\perp & \omega + i\Gamma_\parallel & & \end{array} \right) \begin{pmatrix} \delta \epsilon \\ \delta u_x \\ \delta B_z \\ \delta u_z \\ \delta \sigma_y \end{pmatrix}, \quad (\text{A4a})$$

³Whereas there are sign differences between the coefficients in front of $\delta \sigma_x$ and $\delta \sigma_y$, they cancel out in the determinants, as immediately confirmed with the Laplace expansion.

$$0 = \left(\begin{array}{cc|c} h\omega + i(\eta_{\perp} + \gamma_{\perp})k_{\perp}^2 & \frac{4}{\chi}\gamma_{\perp}k_{\perp} & \\ -2\gamma_{\perp}k_{\perp} & \omega + i\Gamma_{\perp} & \\ \hline & \omega + i\rho'_{\parallel}k_{\perp}^2 & \\ & \omega + i\Gamma_{\parallel} & \end{array} \right) \begin{pmatrix} \delta u_y \\ \delta \sigma_z \\ \delta B_y \\ \delta \sigma_x \end{pmatrix}. \quad (\text{A4b})$$

The 3×3 block-diagonal part yields the three eigenmodes,

$$\omega = -i \frac{c_s^2}{v_f^2} \frac{\rho'_{\perp}}{1 - v_A^2} k_{\perp}^2, \quad (\text{A5})$$

$$\pm v_f k_{\perp} - \frac{i}{2} \left(\frac{v_A^2}{v_f^2} (1 - c_s^2)^2 \rho'_{\perp} + \frac{\eta_{\perp} + \zeta_{\perp}}{h} \right) k_{\perp}^2, \quad (\text{A6})$$

where $v_f^2 = c_s^2 + v_A^2(1 - c_s^2)$. The two propagating modes are the fast magneto-sonic waves with the velocity v_f . While there are two fast and two slow magneto-sonic waves in a general momentum direction, the two slow waves vanish when $k_{\parallel} = 0$ (see, e.g., Refs. [93,99,117]). The remaining blocks yield six modes,

$$\omega = -i\Gamma_{\parallel} - i \frac{(\gamma_{\parallel} + \xi_{\parallel})^2}{h(1 - v_A^2)\gamma_{\parallel}} k_{\perp}^2, \\ - i \frac{\tilde{\eta}_{\parallel}}{h(1 - v_A^2)} k_{\perp}^2, \quad (\text{A7a})$$

$$\omega = -i\Gamma_{\perp} - i \frac{\gamma_{\perp}}{h} k_{\perp}^2, \quad -i \frac{\eta_{\perp}}{h} k_{\perp}^2, \quad (\text{A7b})$$

$$\omega = -i\rho'_{\parallel}k_{\perp}^2, \quad (\text{A7c})$$

$$\omega = -i\Gamma_{\parallel}, \quad (\text{A7d})$$

where $\tilde{\eta}_{\parallel}$ is defined in Eq. (46). The first four modes stem from the mixing between the spin and shear modes. According to the thermodynamic constraints (17), all the imaginary parts are negative, indicating that all those modes are purely diffusive modes.

APPENDIX B: EXPLICIT FORMS OF THE COSINE EXPANSION

Here, starting from the solutions in the limit of $\theta = \pi/2$, we provide the explicit forms of the cosine expansion. For the quartic equation (A4b), we have already got the solution in Eq. (A7) as

$$\begin{aligned} \tilde{\omega}_1 &= -\frac{i\eta_{\perp}k^2}{h}, \\ \tilde{\omega}_2 &= -i\rho'_{\parallel}k^2, \\ \tilde{\omega}_3 &= -i\Gamma_{\perp} - \frac{i\gamma_{\perp}k^2}{h}, \\ \tilde{\omega}_4 &= -i\Gamma_{\parallel}. \end{aligned} \quad (\text{B1})$$

The determinant of coefficient matrix M_1 should be factorized as

$$\det(M_1) \propto (\omega - \tilde{\omega}_1 - \omega_1 \cos^2 \theta)(\omega - \tilde{\omega}_2 - \omega_2 \cos^2 \theta) \\ \times (\omega - \tilde{\omega}_3 - \omega_3 \cos^2 \theta)(\omega - \tilde{\omega}_4 - \omega_4 \cos^2 \theta). \quad (\text{B2})$$

By contracting the terms order-by-order in the cosine factor, we conclude that

$$\omega_i = \tilde{\omega}_i - \frac{(\tilde{\omega}_i^3 X_3 + \tilde{\omega}_i^2 X_2 + \tilde{\omega}_i X_1 + X_0)}{\prod_{j \neq i} (\tilde{\omega}_i - \tilde{\omega}_j)} \cos^2 \theta. \quad (\text{B3})$$

The coefficients X_i are obtained as

$$X_3 = \frac{k^2}{h} \Gamma_{\perp} \Gamma_{\parallel} \left(h v_A^2 + k^2 (\tilde{\eta}_{\parallel} \rho'_{\parallel} + \eta_{\perp} (\rho'_{\perp} - 2\rho'_{\parallel})) \right), \quad (\text{B4a})$$

$$X_2 = -\frac{ik^2}{h} \left[(\Gamma_{\parallel} + \Gamma_{\perp}) h v_A^2 - \Gamma_{\parallel} \Gamma_{\perp} (-\tilde{\eta}_{\parallel} + \eta_{\perp} + h\rho'_{\parallel} - h\rho'_{\perp}) \right] \\ - \frac{ik^4}{h} \left[\Gamma_{\parallel} (\rho'_{\parallel} \tilde{\eta}_{\parallel} + (\rho'_{\perp} - 2\rho'_{\parallel})(\gamma_{\perp} + \eta_{\perp})) \right. \\ \left. + \Gamma_{\perp} (\rho'_{\parallel} (\gamma_{\parallel} + \eta_{\parallel} - 2\eta_{\perp} - 2\xi_{\parallel}) + \eta_{\perp} \rho'_{\perp}) \right], \quad (\text{B4b})$$

$$X_1 = \frac{k^2}{h} \left[-h v_A^2 + \Gamma_{\parallel} (\gamma_{\perp} - \tilde{\eta}_{\parallel} + \eta_{\perp} + h\rho'_{\parallel} - h\rho'_{\perp}) \right. \\ \left. - \Gamma_{\perp} (\gamma_{\parallel} + \eta_{\parallel} - \eta_{\perp} - h\rho'_{\parallel} + h\rho'_{\perp} - 2\xi_{\parallel}) \right] \quad (\text{B4c})$$

$$- \frac{k^4}{h} \left[(\rho'_{\parallel} (\gamma_{\parallel} + \eta_{\parallel} - 2\xi_{\parallel}) + (\rho'_{\perp} - 2\rho'_{\parallel})(\gamma_{\perp} + \eta_{\perp})) \right], \\ X_0 = -\frac{ik^2}{h} \left(\gamma_{\perp} - \gamma_{\parallel} - \eta_{\parallel} + \eta_{\perp} + 2\xi_{\parallel} + h(\rho'_{\parallel} - \rho'_{\perp}) \right). \quad (\text{B4d})$$

As already discussed, the $\mathcal{O}(k^2)$ terms are outside of our interest, as they are beyond the first-order spin MHD. Expanding Eq. (B3) up to the k^2 order, one gets the parameters Ω_i for Eq. (54) as

$$\Omega_1 = \frac{ihv_A^2}{\eta_\perp - h\rho'_\parallel} + ik^2 \left(\frac{1}{h}(\eta_\perp - \tilde{\eta}_\parallel) + \frac{v_A^2 \chi}{8h\rho'_\parallel - 8\eta_\perp} \right), \quad (\text{B5})$$

$$\Omega_2 = -\frac{ihv_A^2}{\eta_\perp - h\rho'_\parallel} + ik^2 \left(\rho'_\parallel - \rho'_\perp - \frac{v_A^2 \chi}{8h\rho'_\parallel - 8\eta_\perp} \right). \quad (\text{B6})$$

In the same way, we obtain Ω_i for Eq. (56),

$$\begin{aligned} \Omega'_1 = & \frac{ic_s^2 hv_A^2(1-v_A^2)}{c_s^2 h\rho_\perp - v_f^2 \tilde{\eta}_\parallel} + \frac{ik^2}{8\gamma_\parallel^2 h^3(1-v_A^2)v_f^2(v_f^2 \tilde{\eta}_\parallel - c_s^2 h\rho_\perp)} \left[c_s^2 v_A^2 \left(2 \left(\frac{\rho_\perp}{v_f^2} \gamma_\parallel (1-c_s^2)^2 h^2 v_A^2 (1-v_A^2) \right. \right. \right. \\ & \left. \left. \left. - 2h\gamma_\parallel \tilde{\eta}_\parallel + \gamma_\parallel h(1-v_A^2)(\zeta_\perp + \eta_\perp) \right)^2 + h^3(1-v_A^2)v_f^2 \chi((\gamma_\parallel + \xi_\parallel)^2 - \gamma_\parallel \tilde{\eta}_\parallel) \right) \right. \\ & - h^2 v_f^2 (8\gamma_\parallel^2 c_s^2 h\rho_\perp (\zeta_\parallel - (4-3v_A^2)\tilde{\eta}_\parallel + (1-v_A^2)^2(\zeta_\perp + \eta_\perp) - 2\zeta_\times(1-v_A^2)) \\ & - \gamma_\parallel \tilde{\eta}_\parallel (c_s^2(-hv_A^2(1-v_A^2)\chi + 8\gamma_\parallel(1-v_A^2)(\zeta_\parallel + \zeta_\perp - 2\zeta_\times + \eta_\perp)) \\ & \left. \left. - 8(c_s^2(4-3v_A^2) + v_A^2)\gamma_\parallel \tilde{\eta}_\parallel + 8\gamma_\parallel \zeta_\parallel v_A^2) \right] \right], \quad (\text{B7}) \end{aligned}$$

and

$$\begin{aligned} \Omega'_2 = & -\frac{i\gamma c_s^2 hv_A^2(1-v_A^2)}{\gamma c_s^2 h\rho_\perp - v_f^2 \gamma_\parallel \tilde{\eta}_\parallel} - \frac{ic_s^2 k^2}{8h(1-v_A^2)v_f^2 \gamma_\parallel (v_f^2 \tilde{\eta}_\parallel - c_s^2 h\rho_\perp)} \left[v_A^2 \left(2\gamma_\parallel \left((1-v_A^2) \frac{(1-c_s^2)^2 h\rho_\perp v_A^2}{v_f^2} \right. \right. \right. \\ & \left. \left. \left. + (1-v_A^2)(\zeta_\perp + \eta_\perp) - \frac{2c_s^2 h\rho_\perp}{v_f^2} \right)^2 + h(1-v_A^2)\chi \left(c_s^2(-h\rho_\perp) + \frac{v_f^2(\gamma_\parallel + \xi_\parallel)^2}{\gamma_\parallel} \right) \right) \right. \\ & - h\rho_\perp \left(\frac{8\gamma_\parallel c_s^4 h\rho_\perp v_A^2}{v_f^2} - 8v_A^2 \gamma_\parallel (\zeta_\parallel - \tilde{\eta}_\parallel) + c_s^2 [hv_A^2(8\gamma_\parallel \rho_\perp + (v_A^2-1)\chi) + 8(v_A^2-1)(\gamma_\parallel(\zeta_\parallel + \zeta_\perp - 2\zeta_\times - 4\tilde{\eta}_\parallel + \eta_\perp))] \right. \\ & \left. \left. + 8v_f^2 \gamma_\parallel (\zeta_\parallel + (\eta_\perp + \zeta_\perp)(v_A^2-1)^2 + 2\zeta_\times(v_A^2-1) + 2\tilde{\eta}_\parallel(v_A^2-2)) \right) \right] \right]. \quad (\text{B8}) \end{aligned}$$

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