Noncommutative Geometry and GMP-type algebra for three-dimensional quantum Hall fluids of extended objects

Giandomenico Palumbo*

CFisUC, Department of Physics, University of Coimbra, Rua Larga, 3004-516 Coimbra, Portugal

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We develop a geometric framework for three-dimensional quantum Hall fluids of extended objects, namely quasistrings, in the presence of a strong three-form background field associated with a bundle gerbe. In the strong-field regime, fast internal dynamics is frozen and the low-energy kinematics is governed by generalized guiding-center variables consisting of vectorial and tensorial coordinates. The resulting reduced bracket is generally shape dependent; however, after projection onto an assumed isotropic gapped internal sector, these guiding-center variables obey a universal noncommutative geometry giving rise to a three-dimensional Girvin-MacDonald-Platzman (GMP)-type algebra for projected density operators. Moreover, we identify a compatible correspondence between this projected algebra and the canonical quantization of a topological $\mathrm{BF}+\mathrm{BB}$ theory at the level of suitably smeared collective operators, with its level taken to be associated with the Dixmier-Douady invariant. Our results provide a kinematic starting point for investigating incompressible quantum Hall-type phases and their geometric and topological features in three spatial dimensions.

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I. INTRODUCTION

The quantum Hall effect provides a paradigmatic example of how strong magnetic fields fundamentally reorganize the kinematics of quantum matter. In two dimensions, projection onto the lowest Landau level leads to a noncommutative geometry of guiding-center coordinates [1], whose algebraic structure underlies the Girvin-MacDonald-Platzman (GMP) algebra of projected densities and the emergence of incompressible quantum fluids [2–5]. Extending these ideas to three spatial dimensions has proven challenging. For point particles and band models, a uniform (Abelian) magnetic field does not lead to a fully three-dimensional Landau-level quantization. Higher-dimensional generalizations of the GMP algebra introduce either some vortex-density operators in addition to the usual charge density operators [6] in time-reversal-invariant topological fluids or nonassociative structures for charge density operators via Nambu brackets [7–10] whose quantization remains unclear [11]. A key observation is that higher-form gauge fields couple naturally to *extended objects*, suggesting that a consistent three-dimensional generalization of quantum Hall physics should involve degrees of freedom beyond point particles.

In this work, we develop a continuum framework for three-dimensional incompressible quantum Hall fluids of closed quasistrings in the presence of a strong three-form background field H, namely the curvature tensor of a Kalb-Ramond gauge field [12] associated to bundle gerbes [13,14]. We show that, in direct analogy with the two-dimensional case, the strong-field limit freezes fast internal dynamics, here corresponding to rapid rotations of oriented area elements, while leaving slow collective degrees of freedom (the oriented area swept by the string, analogous to angular momentum for point particles). The resulting low-energy kinematics is governed by generalized guiding-center variables consisting of vectorial coordinates x^i and antisymmetric tensorial coordinates X^{ij} [15,16].

The strong three-form field induces constraints that reduce the phase space in a manner analogous to lowest-Landau-level projection in two dimensions [17,18]. More precisely, in the strong-field regime the corresponding constraint kernel possesses a null direction along the string tangent associated with worldsheet reparametrizations, so the reduced guiding-center brackets are naturally defined on the physical subspace of reparametrization-invariant observables. This *projection* eliminates fast internal modes and defines a reduced guiding-center sector whose quantization, implemented via Dirac brackets [19], yields a tensorial noncommutative geometry that generalizes the guiding-center structure of the two-dimensional quantum Hall fluids. After projection onto an isotropic internal

*Contact author: giandomenico.palumbo@gmail.com

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sector, this reduced algebra takes a universal form. We then construct projected density operators and show that they satisfy a generalized sine algebra [20], namely a generalized GMP-type algebra.

In the long-wavelength limit, this algebra reduces to the Lie algebra of volume-preserving diffeomorphisms, identifying incompressibility as the organizing principle of the three-dimensional quantum Hall fluid. These results are conceptually complementary to the noncommutative strings arising at the boundary of open membranes in strong three-form backgrounds, as discussed in the M-theory literature [21–23].

Moreover, we show that a bulk topological quantum field theory, known as BF + BB theory [15,24] and characterized by a dimensionless level identified with the Dixmier-Douady invariant [13], provides a compatible effective topological description whose canonically quantized collective operator algebra corresponds to the projected three-dimensional guiding-center algebra.

Our work establishes a direct analogy between two- and three-dimensional quantum Hall physics, providing a unified and physically grounded description of incompressible three-dimensional fluids of extended objects and opens the way to systematic studies of their collective dynamics and responses.

II. GENERALIZED GUIDING CENTERS AND GMP ALGEBRA

The GMP algebra plays a central role in the theory of the two-dimensional quantum Hall effect (QHE): it encodes the noncommutative geometry of guiding, namely center coordinates induced by a strong and uniform magnetic field, and controls the structure of density fluctuations, collective modes, and fractionalized phases. In its most basic form, the GMP algebra arises from the projection of particle coordinates onto a Landau level, where the canonical coordinates cease to commute and acquire a universal commutator proportional to the magnetic length squared. The projected density operators then satisfy a closed "sine algebra" [20], whose long-wavelength limit reproduces the area-preserving diffeomorphism algebra of an incompressible quantum fluid. The purpose of this paper is to develop and physically justify a three-dimensional analog of the GMP algebra appropriate for *quantum Hall fluids in the presence of a strong background three-form field* H in three spatial dimensions. This situation may be viewed as a higher-form generalization of the QHE: while in 2D the magnetic field is a closed two-form $F = \mathrm{d}A$ coupling to pointlike particles, in 3D the natural background field is a closed three-form $H = \mathrm{d}B$ (a Kalb-Ramond field strength) [12], which couples to extended objects and induces a fundamentally different geometric structure. Our main claim is that, in this setting, the correct kinematic variables are not only vectorial coordinates x^i, but also *antisymmetric tensorial coordinates* $X^{ij} = -X^{ji}$ [15,16] and that their mutual noncommutativity gives rise to a well-defined three-dimensional GMP-type algebra. In two spatial dimensions, the appearance of noncommutative coordinates is most transparently understood semiclassically. For a charged particle in a strong, uniform magnetic field $\mathcal{B}$, the cyclotron motion freezes and the guiding-center coordinates satisfy

$$[x^i, x^j] = i\ell_B^2, \epsilon^{ij}, \tag{1}$$

with $\ell_B = 1/\sqrt{e\mathcal{B}}$ the magnetic length (where we have set $\hbar = 1$). Projected density operators $\rho(\mathbf{q}) = e^{iq_i x^i}$ (with $\mathbf{q}$ the momenta) then obey the GMP algebra [2]

$$[\rho(\mathbf{q}), \rho(\mathbf{q}')] = -2i \sin\left(\frac{\ell_B^2}{2} q_i q_j \varepsilon^{ij}\right) \rho(\mathbf{q} + \mathbf{q}'), \tag{2}$$

whose structure constants are fixed entirely by the symplectic form defined by $\mathcal{B}$. In three spatial dimensions, a closed two-form is no longer the natural top-degree background field. Instead, the analog of a "magnetic field" is given by

$$H = \frac{1}{3!} H_{ijk} \mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k,$$
$$H_{ijk} = \partial_i B_{jk} + \partial_j B_{ki} + \partial_k B_{ij}, \tag{3}$$

which defines a volume form rather than an area form. Such three-form fields and corresponding Kalb-Ramond fields B appear naturally in several contexts: string and membrane theory [12], gravity [25], spin foams [26], interacting topological insulators [27,28], fermion-fermion dualities [29,30], non-Abelian higher-rank symmetries [31], and hydrodynamic descriptions of incompressible polar fluids [6]. Crucially, a three-form does not define a symplectic structure on ordinary coordinates x^i; therefore a direct generalization of the 2D associative guiding-center algebra is impossible if one insists on pointlike coordinates alone. In this work, we consider a three-dimensional quantum fluid whose fundamental constituents are closed quasi-strings. In fact, it is well known that such extended objects are the natural degrees of freedom that couple minimally to the two-form gauge potential B. This setup is nothing but the higher-form analog of a charged quantum fluid coupled to an external magnetic field in two dimensions. The fluid is assumed to be

(i) homogeneous, isotropic, and incompressible at long wavelengths;
(ii) subject to a strong and uniform three-form background field, i.e. $H_{ijk} = H\varepsilon_{ijk}$;
(iii) dominated by collective degrees of freedom rather than microscopic string excitations.

Our first goal is to identify the appropriate *projected kinematic variables* that survive in the strong-field limit. For this reason, let us consider the dynamics of a single closed string parametrized by $\sigma \in [0, 2\pi)$ and embedded in

three-dimensional space via the worldsheet coordinates $X^i(\tau,\sigma)$, such that its action is given by [32]

$$S = S_{\rm kin} + q\int_{\Sigma_2} B = \frac{m}{2}\int d\tau d\sigma \partial_\tau X^i \partial_\tau X_i + \frac{e}{2}\int d\tau d\sigma B_{ij}(X)\partial_\alpha X^i \partial_\beta X^j \varepsilon^{\alpha\beta}, \tag{4}$$

where $S_{\rm kin}$ includes tension and inertial terms, e is the effective charge (notice, we use the same symbol of the electric charge, although it can assume a different value), and $\alpha,\beta = \{\tau,\sigma\}$. Here, $X^i(\tau,\sigma)$ embeds the string worldsheet Σ_2 into space and the second term in the action is the Kalb-Ramond coupling. In a uniform three-form background, variation of (4) with respect to X^i yields the following force density:

$$f_i(\sigma) = eH\varepsilon_{ijk}\partial_\tau X^j \partial_\sigma X^k, \tag{5}$$

which is the direct analog of the Lorentz force $f_i = e\mathcal{B}\varepsilon_{ij}\dot{x}^j$ acting on a point particle in two dimensions. This force generates a *torque* proportional to the oriented area spanned by the string, i.e.

$$\tau_{ij} \sim eH\varepsilon_{ijk}\dot{X}^k, \tag{6}$$

which induces rigid rotational motion of the area element. Thus, in a uniform three-form background, the dominant response of aclosed string is *rotation of the area it spans*, rather than circular motion of a point particle. This rotational dynamics is the higher-form analog of cyclotron motion. Thus, in the regime

$$|H| \gg \mathcal{E}, \tag{7}$$

where $\mathcal{E}$ denotes all other energy scales (string tension, interactions, gradients), the dynamics separates into two classes:

(i) *Fast modes*: internal rotational motion of the string area, with characteristic frequency set by $|H|$.
(ii) *Slow modes*: collective motion of the string's center and orientation.

This separation of scales is the direct analog of the decomposition of particle motion into fast cyclotron orbits and slow guiding-center motion in the two-dimensional quantum Hall effect. For our purposes, it is sufficient to calculate the canonical momentum conjugate of $X^i(\sigma)$ from Eq. (4) that reads

$$\Pi_i(\sigma) = \frac{\delta L}{\delta\partial_\tau X^i} = m\partial_\tau X_i + eB_{ij}(X)\partial_\sigma X^j, \tag{8}$$

and take the strong-field limit $m \to 0$, which is the precise analog of the lowest-Landau-level projection in the two-dimensional quantum Hall problem [17,18]. In fact, in two dimensions, lowest Landau level projection occurs when $\omega_c \gg \mathcal{E}$ (i.e. with respect to all other energies in the system). Here, $|H| \gg \mathcal{E}$ plays the analogous role. The limit $m \to 0$ is taken after rescaling to dimensionless variables where eH sets the dominant energy scale, analogous to setting the cyclotron energy as the reference scale in two dimensions. In this limit, the momentum is no longer an independent variable but becomes a constraint, i.e.

$$\phi_i(\sigma) \equiv \Pi_i(\sigma) - eB_{ij}(X)\partial_\sigma X^j \approx 0. \tag{9}$$

These constraints are singular on the full local deformation space, since the corresponding kernel possesses a null direction along the string tangent, reflecting worldsheet reparametrization invariance. Accordingly, the reduced Dirac brackets relevant for the guiding-center sector must be understood on the physical subspace of reparametrization-invariant observables [19].

The fundamental equal-time Poisson brackets are

$$\{X^i(\sigma),\Pi_j(\sigma')\} = \delta^i_j\delta(\sigma-\sigma'), \quad \{X^i,X^j\} = \{\Pi_i,\Pi_j\} = 0. \tag{10}$$

We now compute the Poisson brackets between constraints

$$\{\phi_i(\sigma),\phi_j(\sigma')\} = -eH_{ijk}\partial_\sigma X^k\delta(\sigma-\sigma') - 2eB_{ij}(X(\sigma))\partial_\sigma\delta(\sigma-\sigma'), \tag{11}$$

and remind the reader that the Dirac brackets between two functionals A, B are defined as [19]

$$\{A,B\}_D = \{A,B\} - \iint d\sigma d\sigma' \{A,\phi_i(\sigma)\}C^{-1}_{ij}(\sigma,\sigma') \times \{\phi_j(\sigma'),B\}, \tag{12}$$

where $C_{ij}(\sigma,\sigma') \equiv \{\phi_i(\sigma),\phi_j(\sigma')\}$.

Restricted to the transverse physical subspace, one may use the local distributional pseudoinverse

$$C^{-1}_{ij}(\sigma,\sigma') = \frac{1}{eH}\varepsilon_{ijk}\frac{\partial_\sigma X^k}{|\partial_\sigma X|^2}\delta(\sigma-\sigma'), \tag{13}$$

whose normalization is fixed by the oriented area of the quasistring (see Appendix A for more details). Its defining property is that it reproduces the transverse projector

$$P^{ij} = \delta^{ij} - t^i t^j, \qquad t^i = \frac{\partial_\sigma X^i}{|\partial_\sigma X|} \tag{14}$$

rather than the full identity. Thus, Eq. (13) is a transverse pseudoinverse of the constraint kernel, which is sufficient for the reduced brackets of reparametrization-invariant observables. After quotienting the tangential reparametrization null direction, the reduced constraints are second

class on the transverse physical subspace, so the transverse pseudoinverse in Eq. (13) is the appropriate object entering the reduced Dirac bracket for the reparametrization-invariant observables considered here.

Notice that the second term on the left-hand side of Eq. (11) does not contribute to the Dirac brackets of the collective observables considered here, since we restrict to a uniform H-field and to reparametrization-invariant, σ-integrated quantities. In this case, and in a linear gauge for B_{ij} compatible with constant $H = dB$, the term proportional to $\partial_\sigma \delta$ reduces to a total derivative along the closed string and gives no bulk contribution.

We can now define the collective (guiding-center) variables, given by

$$x^m \equiv \frac{1}{2\pi} \oint d\sigma X^m(\sigma), \tag{15}$$

$$X^{ij} \equiv \frac{1}{2} \oint d\sigma (X^i \partial_\sigma X^j - X^j \partial_\sigma X^i). \tag{16}$$

The only nonvanishing Dirac bracket is (see Appendix A for more details)

$$\{x^m, X^{ij}\}_D = - \iint d\sigma d\sigma' \{x^m, \phi_k(\sigma)\} C^{-1}_{kl}(\sigma, \sigma') \times \{\phi_l(\sigma'), X^{ij}\}, \tag{17}$$

and by employing

$$\{x^m, \phi_k(\sigma)\} = \frac{1}{2\pi} \delta^m_k \tag{18}$$

and

$$\{\phi_l(\sigma), X^{ij}\} = \delta^i_l \partial_\sigma X^j - \delta^j_l \partial_\sigma X^i, \tag{19}$$

we obtain

$$\{x^m, X^{ij}\}_D = -\frac{1}{2\pi} \oint d\sigma \frac{1}{eH} \varepsilon^{m\ell p} \frac{\partial_\sigma X^p(\sigma)}{|\partial_\sigma X(\sigma)|^2} \times (\delta^i{}_\ell \partial_\sigma X^j(\sigma) - \delta^j{}_\ell \partial_\sigma X^i(\sigma)). \tag{20}$$

At this stage, the Dirac bracket in Eq. (20) still retains explicit information about the local tangent distribution along the loop through the combination $\partial_\sigma X^i / |\partial_\sigma X|^2$. As such, it depends on the detailed internal configuration of the quasistring and is not yet universal at the operator level. However, in the strong-field regime $|H| \gg E$, the internal degrees of freedom of the closed string undergo rapid rotational dynamics with a characteristic frequency set by $|H|$, while collective guiding-center motion occurs at much lower energies. Upon quantization, this separation of scales implies that the physical low-energy Hilbert space factorizes into a fast internal sector and a slow collective sector, in direct analogy with the separation between cyclotron and guiding-center degrees of freedom in the two-dimensional quantum Hall effect. The relevant guiding-center algebra is therefore defined as an operator identity within the projected internal ground-state sector.

Denoting by $|0_{\rm int}\rangle$ the rotationally invariant ground state of the frozen internal modes, we define the effective guiding-center Dirac brackets through expectation values of the corresponding operators in this sector. Rotational invariance of $|0_{\rm int}\rangle$ then enforces

$$\left\langle 0_{\rm int} \left| \frac{1}{2\pi} \oint d\sigma \frac{\partial_\sigma X^i(\sigma) \partial_\sigma X^j(\sigma)}{|\partial_\sigma X(\sigma)|^2} \right| 0_{\rm int} \right\rangle = \frac{c_{\rm iso}}{3} \delta^{ij}, \tag{21}$$

where $c_{\rm iso}$ is a dimensionless normalization coefficient that can be absorbed into the definition of the tensorial coordinates. This relation reflects the absence of any preferred direction in the internal ground state and is the direct higher-form analog of the isotropy of cyclotron motion in the lowest Landau level. Although we assume an SO(3)-invariant gapped internal sector, this assumption should not be viewed as a property of a generic closed string configuration.

By employing Eq. (21), and working for convenience in arc-length gauge $|\partial_\sigma X| = 1$, the projected Dirac bracket reduces to

$$\langle 0_{\rm int} | \{x^m, X^{ij}\}_D | 0_{\rm int} \rangle = c_{\rm iso} \ell_H^3 \varepsilon^{mij}, \tag{22}$$

where we have introduced the characteristic volume scale

$$\ell_H^3 \equiv \frac{1}{eH}. \tag{23}$$

Thus, the exact strong-field reduced bracket is shape dependent, while a universal form emerges only after projection onto the isotropic internal sector.

Upon absorbing the coefficient $c_{\rm iso}$ into the definition of the tensorial coordinates, the projected Dirac brackets are promoted to commutators, $\{A, B\}_D \to -i[A, B]$, yielding the universal generalized guiding-center algebra

$$[x^m, X^{ij}] = i\ell_H^3 \varepsilon^{mij}, \qquad [x^i, x^j] = [X^{ij}, X^{kl}] = 0, \tag{24}$$

which holds as an operator identity within the low-energy projected Hilbert space. Moreover, we emphasize that shape-dependent corrections correspond to excitations of the internal sector and are separated from the guiding-center physics by an energy gap of order $|H|$. The above algebra is therefore universal within the isotropic projected sector of frozen internal modes.

Furthermore, Eq. (24) expresses the fact that translations of the guiding center and deformations of the oriented area do not commute. In a quantum fluid composed of many such extended objects, density fluctuations are naturally described by operators built from the projected variables.

We introduce the following generators:

$$\rho(\mathbf{q}) = e^{iq_m x^m}, \qquad \Upsilon(\mathbf{Q}) = e^{iQ_{ij}X^{ij}}, \tag{25}$$

which probe, respectively, center-of-mass density and oriented-area density.Here, q_m and Q_{ij} are the conjugate momenta associated to x^m and X^{ij}, respectively [15].

By employing the Baker-Campbell-Hausdorff formula and the algebra (24) (see Appendix B for more details), one finds

$$[\rho(\mathbf{q}), \Upsilon(\mathbf{Q})] = -2i \sin\left(\frac{\ell_H^3}{2} q_m Q_{ij} \epsilon^{mij}\right) e^{iq_m x^m + iQ_{ij}X^{ij}}. \tag{26}$$

Equation (26) is not closed within the smaller set of operators $\{\rho(\mathbf{q}), \Upsilon(\mathbf{Q})\}$ taken separately. Rather, it is closed in the enlarged operator algebra generated by mixed exponentials of the form $e^{iq_m x^m + iQ_{ij}X^{ij}}$. In this sense, Eq. (26) defines a generalized GMP-type sine algebra for projected scalar and tensor densities.

In the long-wavelength limit, it reduces to the quantum algebra of volume-preserving diffeomorphisms, reflecting the incompressibility of the underlying quantum fluid as we will show in the next section.

III. VOLUME-PRESERVING DIFFEOMORPHISMS AND TENSORIAL COORDINATES

To describe diffeomorphisms in our framework, we work at the level of a *fluid* of extended objects and in this section we introduce coarse-grained, equal-time operator-valued densities.

These fields represent projected long-wavelength variables describing the projected guiding-center sector of a fluid of extended objects. Accordingly, the local algebra written below is an effective continuum representation of the reduced kinematics, rather than a pointwise microscopic identity derived directly from the single-string action.

We therefore promote the guiding-center variables to spatial fields

$$x^i \to x^i(\mathbf{r}), \qquad X^{ij} \to X^{ij}(\mathbf{r}), \tag{27}$$

and the local algebra consistent with the reduced guiding-center kinematics of Sec. II is

$$\{x^m(\mathbf{r}), X^{ij}(\mathbf{r}')\} = \ell_H^3 \epsilon^{mij} \delta^{(3)}(\mathbf{r} - \mathbf{r}'),$$
$$\{x^i(\mathbf{r}), x^j(\mathbf{r}')\} = 0, \qquad \{X^{ij}(\mathbf{r}), X^{kl}(\mathbf{r}')\} = 0. \tag{28}$$

The defining kinematic property of an incompressible quantum fluid is the absence of local density fluctuations. In a continuum description, this constraint implies that the allowed long-wavelength deformations of the fluid configuration space are *volume-preserving diffeomorphisms* (VPDs). Infinitesimally, such transformations act on spatial coordinates as

$$x^i \longrightarrow x^i + \delta x^i(\mathbf{r}), \tag{29}$$

with the incompressibility condition

$$\partial_i \delta x^i(\mathbf{r}) = 0. \tag{30}$$

In two spatial dimensions, the above constraint admits a scalar parametrization $\delta x^i = \epsilon^{ij}\partial_j \phi$, leading to the familiar algebra of area-preserving diffeomorphisms generated by a single function ϕ. This structure underlies the long-wavelength limit of the two-dimensional GMP algebra and provides the hydrodynamic description of incompressible quantum Hall fluids [31]. In three dimensions, however, no single scalar potential can generate all the VPD solutions. Instead, incompressible flows require at least two independent functions or, equivalently, an antisymmetric tensor structure. This obstruction is the geometric origin of the appearance of Nambu brackets [7] in classical treatments of three-dimensional incompressible fluids [8]. As we now show, the tensorial guiding-center variables (x^i, X^{ij}) provide a natural and fully associative realization of the VPD algebra without invoking Nambu brackets.

Starting from the coarse-grained guiding-center phase space and corresponding Dirac brackets discussed previously, we now construct the generators of infinitesimal incompressible deformations directly from the tensorial coordinates X^{ij}. Let us consider the functional

$$G[\Lambda] = \int d^3\mathbf{r} \Lambda_{ij}(\mathbf{r}) X^{ij}(\mathbf{r}), \tag{31}$$

where $\Lambda_{ij}(\mathbf{r}) = -\Lambda_{ji}(\mathbf{r})$ is an arbitrary smooth closed two-form, namely $\Lambda_{ij} = \partial_i A_j - \partial_j A_i$ for some vector field A_i locally. Here the coordinate $\mathbf{r}$ labeling the guiding-center fields $x^i(\mathbf{r})$ and $X^{ij}(\mathbf{r})$ plays the role of a label for the collective degrees of freedom, rather than an active spatial coordinate. Accordingly, the antisymmetric fields $\Lambda_{ij}(\mathbf{r})$ entering $G[\Lambda]$ are treated as parameters specifying infinitesimal incompressible deformations, not as geometric two-forms on physical space. Their variation under the transformation generated by $\delta x^m(\mathbf{r})$ is therefore taken to be pure advection, $\delta\Lambda_{ij} = \delta x^m \partial_m \Lambda_{ij}$, rather than the full Lie derivative. With this interpretation, the closure of the generator algebra derived below follows directly from the guiding-center Dirac brackets.

In fact, by employing the normalized projected guiding-center algebra in Eq. (24), or equivalently its coarse-grained version in Eq. (28), the induced variation of the guiding-center coordinate x^m is given by

$$\delta x^m = \{x^m, G[\Lambda]\}_D = \int d^3\mathbf{r}\Lambda_{ij}(\mathbf{r})\{x^m, X^{ij}\}_D = \frac{1}{eH}\varepsilon^{mij}\Lambda_{ij}(\mathbf{r}), \tag{32}$$

and by taking the divergence of the above equation, we find

$$\partial_m \delta x^m = \frac{1}{eH}\varepsilon^{mij}\partial_m\Lambda_{ij} = 0. \tag{33}$$

Therefore, the transformations generated by $G[\Lambda]$ automatically satisfy the incompressibility condition in Eq. (30). This condition does not hold for a generic antisymmetric tensor Λ_{ij}. Instead, it restricts the allowed generators to those for which Λ_{ij} is a closed two-form, which has a clear physical interpretation. Volume-preserving diffeomorphisms are the kinematic symmetry of an incompressible fluid, and only generators that preserve the local volume element correspond to physical deformations. From the guiding-center perspective, the closedness of Λ_{ij} reflects the higher-form gauge redundancy of the tensorial coordinates X^{ij}, which encode oriented area degrees of freedom rather than local shape deformations. Only closed Λ_{ij} generate physical, incompressible motions of the quantum Hall fluid.

We now show that the generators in Eq. (31) realize a closed Lie algebra corresponding to volume-preserving diffeomorphisms. Let us consider two generators $G[\Lambda_1]$ and $G[\Lambda_2]$, such that the variation of $G[\Lambda_2]$ under the transformation generated by $G[\Lambda_1]$ is given by

$$\delta_{\Lambda_1} G[\Lambda_2] = \int d^3\mathbf{r}\delta_{\Lambda_1}\Lambda_{2ij}(\mathbf{r})X^{ij}(\mathbf{r}). \tag{34}$$

By employing the transformation law in Eq. (32),

$$\delta_{\Lambda_1} x^m = \frac{1}{eH}\varepsilon^{mkl}\Lambda_{1kl}(\mathbf{r}), \tag{35}$$

the induced variation of the tensor field $\Lambda_{2ij}(\mathbf{r})$ is

$$\delta_{\Lambda_1}\Lambda_{2ij} = \delta x^m \partial_m \Lambda_{2ij} = \frac{1}{eH}\varepsilon^{mkl}\Lambda_{1kl}\partial_m\Lambda_{2ij}. \tag{36}$$

Since the tensorial guiding-center coordinates themselves do not transform, $\delta_{\Lambda_1} X^{ij} = 0$, we obtain

$$\delta_{\Lambda_1} G[\Lambda_2] = \int d^3\mathbf{r}\frac{1}{eH}\varepsilon^{mkl}\Lambda_{1kl}\partial_m\Lambda_{2ij}X^{ij}. \tag{37}$$

Antisymmetrizing under $1 \leftrightarrow 2$, the Dirac bracket of the generators takes the form

$$\{G[\Lambda_1], G[\Lambda_2]\}_D = \delta_{\Lambda_1} G[\Lambda_2] - \delta_{\Lambda_2} G[\Lambda_1] = G[\Lambda_3], \tag{38}$$

where the composite parameter Λ_3 is given by

$$\Lambda_{3ij} = \frac{1}{eH}(\varepsilon^{mkl}\Lambda_{1kl}\partial_m\Lambda_{2ij} - \varepsilon^{mkl}\Lambda_{2kl}\partial_m\Lambda_{1ij}). \tag{39}$$

Equations (38)–(39) define a closed algebra that realizes the volume-preserving diffeomorphisms in tensorial form. The nontrivial structure arises entirely from the geometric action of the generators on spatially dependent tensor fields, while the underlying Dirac brackets remain binary and associative.

Moreover, we emphasize that the resulting algebra is a representation of volume-preserving deformations acting on the label-dependent guiding-center fields in the coarse-grained projected description, rather than as the geometric algebra of diffeomorphisms of physical space.

This construction should be contrasted with the standard Nambu-bracket formulation of three-dimensional incompressible fluids, where infinitesimal transformations take the form $\delta x^i \sim \{x^i, h_1, h_2\}$. While suggestive, Nambu brackets notoriously lack a canonical associative operator representation [4]. In our present framework, the two scalar functions (h_1, h_2) could be seen as effectively packaged into a single antisymmetric tensor Λ_{ij}, and the resulting algebra is realized entirely within ordinary associative mechanics. Upon quantization, the generators $G[\Lambda]$ become operators whose commutators reproduce the quantum version of the volume-preserving diffeomorphism algebra. The long-wavelength limit of the three-dimensional GMP algebra derived in Sec. II is therefore identified with the quantum algebra of incompressible deformations of a Hall fluid of quasistrings.

IV. TOPOLOGICAL QUANTUM FIELD THEORY AND DD INVARIANT

In this section, we discuss the topological aspects of the three-dimensional quantum Hall fluids of quasistrings and their connection with the noncommutative geometry derived previously. In particular, we will show that the projected guiding-center algebra derived in Sec. II admits a natural correspondence, at the level of canonically quantized collective operators, with a $(3+1)$-dimensional topological quantum field theory, namely the Abelian BF + BB theory [24,33].

Let us start considering its corresponding metric-independent action, which is given by

$$S_{\mathrm{BF}} = \frac{\kappa_{\mathrm{DD}}}{2\pi}\int d^4x\varepsilon^{\mu\nu\rho\sigma}B_{\mu\nu}\partial_\rho A_\sigma + \frac{\kappa_{\mathrm{DD}}}{4\pi}\int d^4x\varepsilon^{\mu\nu\rho\sigma}B_{\mu\nu}B_{\rho\sigma}, \tag{40}$$

where B is the Kalb-Ramond field and A is a vector gauge field.

Here, κ_{DD} denotes the quantized topological level of the BF + BB theory. In the present framework, we identify this level with the Dixmier-Douady (DD) invariant [13] in a

way consistent with the semiclassical tensorial-coordinate construction of Ref. [15].

Moreover, the DD invariant is related to a tensor Berry connection and related bundle gerbe, which is nothing but a momentum-space Kalb-Ramond field. In this sense, our current work provides a dual real-space version of the gerbe structure in momentum-space discussed in Ref. [15] in the context of three-dimensional Hall fluids of quasistrings.

To derive the canonical quantization of the above action, we perform a $(3+1)$ decomposition and by writing spatial indices $i, j, m = 1, 2, 3$, the BF term becomes

$$S_{\mathrm{BF}} = \frac{\kappa_{\mathrm{DD}}}{2\pi}\int dt d^3x \varepsilon^{ijm} B_{ij}\partial_t A_m + (\text{constraints}). \quad (41)$$

Although the BB term contains no time derivatives and therefore does not affect the canonical structure, it is nevertheless essential in order to break time-reversal symmetry in agreement with the symmetry features of quantum Hall fluids. The momentum conjugate to A_m reads

$$\Pi_A^m = \frac{\delta S_{\mathrm{BF}}}{\delta(\partial_t A_m)} = \frac{\kappa_{\mathrm{DD}}}{2\pi}\varepsilon^{mij}B_{ij}, \quad (42)$$

such that the canonical equal-time commutation relation is given by

$$[A_m(\mathbf{r}), B_{ij}(\mathbf{r}')] = \frac{2\pi i}{\kappa_{\mathrm{DD}}}\varepsilon_{mij}\delta^{(3)}(\mathbf{r}-\mathbf{r}'), \quad (43)$$

where we have replaced the expression for Π_A^l from Eq. (42). In order to compare the above expression with the microscopic guiding-center algebra, we must consider smeared operators corresponding to localized excitations [34]. In fact, since $A_m(\mathbf{r})$ and $B_{ij}(\mathbf{r})$ are operator-valued distributions, physical observables are defined via smearing with smooth test functions. Localized excitations correspond to wave packets, and the associated collective operators are obtained by projecting onto these modes. Let $f(\mathbf{r})$ and $g(\mathbf{r})$ be normalized wave packets and define

$$\hat{A}^m = \int d^3r f(\mathbf{r})A^m(\mathbf{r}), \quad \hat{B}^{ij} = \int d^3r g(\mathbf{r})B^{ij}(\mathbf{r}). \quad (44)$$

For this reason, the comparison with the microscopic guiding-center variables is naturally performed only after smearing, such that one matches collective operator algebras rather than unsmeared local operator-valued distributions.

By employing Eq. (43), we obtain

$$[\hat{A}^m, \hat{B}^{ij}] = \frac{2\pi i V_{fg}}{\kappa_{\mathrm{DD}}}\varepsilon^{mij}, \quad (45)$$

where V_{fg} is a volume element related to f and g. By choosing $f(\mathbf{r})$ and $g(\mathbf{r})$ such that

$$\int d^3r f(\mathbf{r})g(\mathbf{r}) = 1, \quad (46)$$

or, equivalently, by absorbing the overlap factor V_{fg} into the normalization of the collective operators, we can express the smeared commutator in a simplified canonical form. A direct comparison with the microscopic three-dimensional guiding-center algebra $[x^m, X^{ij}] = i\ell_H^3\varepsilon^{mij}$ allows us to obtain the following identifications:

$$x^m \leftrightarrow \ell_H\sqrt{\frac{\kappa_{\mathrm{DD}}}{2\pi}}\hat{A}^m, \qquad X^{ij} \leftrightarrow \ell_H^2\sqrt{\frac{\kappa_{\mathrm{DD}}}{2\pi}}\hat{B}^{ij}. \quad (47)$$

This is the direct three-dimensional analog of the well-known situation in the two-dimensional quantum Hall effect, where the guiding-center noncommutativity $[x^i, x^j] = i\ell_B^2\varepsilon^{ij}$ is related to the background magnetic field while the Chern-Simons theory [35,36] gives the algebraic structure with its quantized level related to the first Chern number.

We can now compare the present construction with the hydrodynamic BF description of three-dimensional incompressible polar fluids developed in Ref. [6]. In that work, an extended GMP-type algebra arises from the canonical quantization of a $(3+1)$-dimensional BF theory describing coupled particle and vortex densities. The long-wavelength limit realizes the algebra of VPDs, providing an effective hydrodynamic description of incompressibility. In Ref. [6], the (equal-time) commutation relation in the BF theory is taken with respect to the dual field of B_{ij}, namely $B^k = (1/2)\varepsilon^{kij}B_{ij}$, such that

$$[A^m(\mathbf{r}), B^k(\mathbf{r}')] \propto i\delta^{mk}\delta^{(3)}(\mathbf{r}-\mathbf{r}'), \quad B^k = \frac{1}{2}\varepsilon^{kij}B_{ij}. \quad (48)$$

In contrast, our construction starts from a microscopic strong-field H projection of extended quasistring degrees of freedom. However, a precise connection with the above BF quantization structure becomes manifest in our case upon (Hodge) dualizing the antisymmetric tensor X^{ij}. In three spatial dimensions we define the dual axial-vector variables

$$\tilde{X}^k \equiv \frac{1}{2}\varepsilon^{kij}X_{ij}, \quad (49)$$

which measure the oriented area element perpendicular to direction k. By using the identity $\varepsilon^{mij}\varepsilon_{kij} = 2\delta_k^m$, we obtain

$$[x^m, \tilde{X}^k] = i\ell_H^3\delta^{mk}. \quad (50)$$

Therefore, in the dual formulation, our algebra reduces to a canonical Heisenberg-Weyl algebra and can be directly related to the canonical quantization of the BF theory in Eq. (48). We stress, however, that Eq. (50) describes the algebra of the *projected* (guiding-center) theory. In the strong-field limit, the microscopic kinetic momentum is

TABLE I. Comparison between two-dimensional and three-dimensional quantum Hall fluids (QHFs).

Aspects	2D QHF	3D QHF
Background field	F (2-form)	H (3-form)
Fundamental objects	Point particles	Extended objects (strings)
Guiding centers	x^m	(x^m, X^{ij})
Characteristic scale	$\ell_B = 1/\sqrt{e\mathcal{B}}$	$\ell_H = 1/\sqrt[3]{eH}$
Noncommutative geometry	$[x^i, x^j] = i\ell_B^2 \epsilon^{ij}$	$[x^m, X^{ij}] = i\ell_H^3 \epsilon^{mij}$
Sine/GMP algebra	$-2i \sin(\frac{\ell_B^2}{2} q \wedge q')$	$-2i \sin(\frac{\ell_H^3}{2} q \wedge Q)$
Diffeomorphisms	Area-preserving diffs	Volume-preserving diffs
Geometric structure	Symplectic	Two-plectic
Topological action	Chern-Simons	BF + BB
Topological invariant	Chern number	DD number

eliminated by Dirac reduction, analogously to the lowest-Landau-level projection in two dimensions. The reduced phase space is then spanned solely by (x^m, X^{ij}), and the dual variable $\tilde{X}^k$ becomes the only operator canonically conjugate to x^m. Accordingly,

$$P^k_{\text{eff}} \equiv \tilde{X}^k / \ell_H^3 \tag{51}$$

acts as an effective translation generator in the projected Hilbert space: momentum is not an independent microscopic degree of freedom but emerges geometrically from the area variables induced by the background three-form field. Thus, in the three-dimensional Hall fluids of quasistrings, spatial motion is intrinsically tied to geometric area fluctuations, namely translating the center of mass necessarily reconfigures the internal flux carried by the excitation. In this way, transport properties emerge from extended-object kinematics rather than from conventional particle momentum. From this perspective, the hydrodynamic density-vortex algebra of Ref. [6] can be viewed as an effective dual representation of the extended-object guiding-center kinematics derived here. Although both approaches lead to a similar long-wavelength noncommutative structure, their starting points and physical interpretation are fundamentally different. In contrast to Ref. [6], our noncommutative geometry arises prior to any coarse-graining, directly from the strong-field projection of a microscopic generalized phase space, consistently with the semiclassical formalism introduced in Ref. [15]. The tensorial guiding-center coordinates X^{ij} follow from the intrinsic coupling of quasistrings to a background three-form field and encode genuine geometric degrees of freedom.

This microscopic derivation parallels the role of the magnetic length in two dimensions: just as ℓ_B controls the guiding-center algebra in the quantum Hall effect, the magnetic length ℓ_H fixes the three-dimensional noncommutative structure and its associated GMP algebra. The BF + BB theory and its quantized level κ_{DD} then emerge as the compatible topological sector rather than as a starting assumption. Hence, our framework provides a geometric and microscopic origin for the three-dimensional GMP algebra and recovers the hydrodynamic BF description as its effective long-wavelength limit.

V. CONCLUSIONS AND OUTLOOK

In this work we have developed a continuum framework for three-dimensional quantum Hall physics based on quantum fluids of extended objects in the presence of a strong background three-form field $H = dB$. Treating the strong-field regime via Dirac brackets and projecting onto σ-integrated observables, we identified a set of guiding-center variables consisting of vectorial coordinates x^m and antisymmetric tensorial coordinates X^{ij}. We showed that, after strong-field reduction and projection onto an isotropic gapped internal sector, these variables obey a closed, associative noncommutative algebra in the projected guiding-center Hilbert space, providing a three-dimensional generalization of the GMP framework appropriate to extended objects. The background three-form H, which defines a two-plectic structure [23,37] at the microscopic level, plays a central role in generating the reduced guiding-center algebra. While formally analogous to the lowest-Landau-level construction in two-dimensional quantum Hall systems, the present framework is distinguished by the appearance of tensorial guiding-center degrees of freedom reflecting the extended nature of the underlying excitations and by the role of the isotropic projected internal sector.

Our results establish an analogy between two- and three-dimensional quantum Hall physics: in both cases, strong background fields freeze fast degrees of freedom, leading to noncommutative guiding-center coordinates and GMP-type algebras that encode incompressibility and collective dynamics. In three dimensions, some of the relevant kinematic variables are tensorial and reflect the coupling of the system to higher-form gauge fields and extended excitations (quasistrings).

Finally, we have shown that the microscopic projected guiding-center algebra admits a consistent correspondence

with the canonical commutator algebra of a topological BF + BB theory after suitable smearing and normalization of collective operators, establishing a precise bridge between the compatible topological bulk description and the noncommutative kinematics of the quantum Hall fluid.

All the main aspects analyzed in this work are summarized in Table I. The present work is intentionally kinematic in nature, while dynamical realizations and microscopic lattice models are left for future investigations. For instance, a clear and precise understanding of a possible charge fractionalization carried by quasistrings in three dimensions beyond an effective topological field theory description (see for instance Refs. [38,39] about fractional quantum Hall effect for membranes) would be highly desirable. Moreover, an important future direction would be the identification of three-dimensional GMP algebra and its geometric consequences in lattice systems and band models. In particular, it would also be highly desirable to realize this algebra in (interacting) three-dimensional topological insulators in symmetry class AIII, where nearly flat topological bands can be characterized by the DD invariant and tensor Berry connections associated with momentum-space bundle gerbes [40–43]. Along this direction, it has been already shown the emergence of a BF theory in interacting topological insulators [27] although the charge fractionalization was not analyzed. Such interacting lattice systems provide a promising platform to emulate the physics of extended objects discussed in this work by using purely electronic degrees of freedom. In this context, the three-dimensional GMP algebra may control density correlations and propagating collective modes in topological bands, in close analogy with the role played by the GMP algebra in fractional Chern insulators [44]. More broadly, the framework developed here opens new avenues for exploring higher-dimensional incompressible quantum fluids, higher-form gauge responses, and the interplay between topology and noncommutative geometry. We expect that further connections with band topology, interacting topological phases, and engineered quantum systems will continue to shed light on the physical realization and consequences of higher-form quantum Hall phenomena.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A

For a generic reparametrization-invariant observable $\mathcal{O} = \oint d\sigma \mathcal{O}[X(\sigma)]$, terms proportional to $\partial_\sigma \delta(\sigma - \sigma')$ contribute as

$$\oint d\sigma B_{ij}(X(\sigma))\partial_\sigma \delta(\sigma-\sigma') = -\oint d\sigma \partial_\sigma B_{ij}(X(\sigma))\delta(\sigma-\sigma'), \tag{A1}$$

where integration by parts produces no boundary term for a closed string. For a uniform background $H = dB$ and a linear gauge choice for B_{ij}, $\partial_\sigma B_{ij}(X(\sigma)) \propto H_{ijk}\partial_\sigma X^k$, which does not contribute to Dirac brackets of σ-integrated observables. This is the sense in which the term proportional to $\partial_\sigma \delta(\sigma - \sigma')$ in Eq. (11) does not affect the reduced guiding-center brackets discussed in the main text.

In the uniform strong-field (projected) regime of Sec. II, the leading contribution to the constraint kernel can be written in the local form

$$C_{ij}(\sigma,\sigma') = -eH\varepsilon_{ijk}\partial_\sigma X^k(\sigma)\delta(\sigma-\sigma'), \tag{A2}$$

where we have kept only the leading local term relevant for the reduced Dirac brackets of reparametrization-invariant observables. At each value of σ, the 3×3 matrix $C_{ij}(\sigma)$ has a null vector along the tangent direction $\partial_\sigma X^i(\sigma)$:

$$C_{ij}(\sigma)X'^j(\sigma) = -eH\varepsilon_{ijk}X'^k(\sigma)X'^j(\sigma) = 0. \tag{A3}$$

Hence C is rank-2 and does not admit a genuine inverse on the full three-component space. This degeneracy is expected and reflects the redundancy associated with σ reparametrizations: infinitesimal variations $\delta X^i(\sigma) \propto \partial_\sigma X^i(\sigma)$ correspond to relabeling the worldsheet coordinate rather than to a physical deformation of the embedded loop. Therefore, the inverse entering the reduced Dirac construction must be understood as a transverse pseudoinverse defined on the physical subspace orthogonal to the string tangent.

To define an inverse on the physical subspace, introduce the unit tangent and the transverse projector

$$t^i(\sigma) \equiv \frac{\partial_\sigma X^i(\sigma)}{|\partial_\sigma X(\sigma)|}, \qquad P^i{}_j(\sigma) \equiv \delta^i{}_j - t^i(\sigma)t_j(\sigma), \tag{A4}$$

which projects onto the plane orthogonal to $\partial_\sigma X^i(\sigma)$. Equivalently, in index-lowered notation one may write

$$P_{ij}(\sigma) = \delta_{ij} - t_i(\sigma)t_j(\sigma). \tag{A5}$$

A convenient *transverse inverse* (Moore-Penrose-type pseudoinverse) for C is the local distributional kernel

$$C^{-1}_{ij}(\sigma,\sigma') \equiv \frac{1}{eH}\varepsilon_{ijk}\frac{\partial_\sigma X^k(\sigma)}{|\partial_\sigma X(\sigma)|^2}\delta(\sigma-\sigma'). \tag{A6}$$

Its defining property is that it inverts C on the transverse subspace, i.e. it reproduces the projector P rather than the full identity:

$$\begin{aligned}
&\int d\sigma'' C_{ik}(\sigma,\sigma'')C^{-1}_{kj}(\sigma'',\sigma') \\
&= [-eH\varepsilon_{ik\ell}\partial_\sigma X^\ell(\sigma)]\left[\frac{1}{eH}\varepsilon_{kjm}\frac{\partial_\sigma X^m(\sigma)}{|\partial_\sigma X(\sigma)|^2}\right]\delta(\sigma-\sigma') \\
&= -\varepsilon_{ik\ell}\varepsilon_{kjm}\frac{\partial_\sigma X^\ell \partial_\sigma X^m}{|\partial_\sigma X|^2}\delta(\sigma-\sigma') \\
&= \left(\delta_{ij}-\frac{\partial_\sigma X_i \partial_\sigma X_j}{|\partial_\sigma X|^2}\right)\delta(\sigma-\sigma') \\
&= P_{ij}(\sigma)\delta(\sigma-\sigma'). \qquad \text{(A7)}
\end{aligned}$$

In the last step we used the standard identity $\varepsilon_{ik\ell}\varepsilon_{kjm} = -\delta_{ij}\delta_{\ell m} + \delta_{im}\delta_{\ell j}$. Equation (A7) shows that C^{-1} coincides with the usual inverse on the subspace orthogonal to $\partial_\sigma X$, while it annihilates the tangential null direction. Thus, C^{-1} is not an inverse on the full local deformation space, but precisely the transverse pseudoinverse needed for the reduced guiding-center brackets.

Consequently, Dirac brackets computed with C^{-1} are unambiguous for reparametrization-invariant observables, whereas quantities sensitive to the tangential sector require a choice of worldsheet gauge, equivalently a completion of the pseudoinverse away from the transverse subspace. For the collective observables considered in the main text, namely the σ-integrated variables x^m and X^{ij}, all such tangential ambiguities drop out identically.

For completeness, we explicitly verify that the remaining projected Dirac brackets among the guiding-center variables vanish identically. We first consider $\{x^i, x^j\}_D$. The canonical Poisson bracket vanishes, $\{x^i, x^j\} = 0$, since x^i is linear in $X^i(\sigma)$. The Dirac correction term involves

$$\{x^i, \phi_k(\sigma)\} = \frac{1}{2\pi}\delta^i_k \qquad \text{(A8)}$$

and the inverse constraint matrix $C^{-1}_{ij}(\sigma,\sigma')$. Using the explicit expression for C^{-1}, one finds

$$\{x^i, x^j\}_D = -\frac{1}{(2\pi)^2}\frac{1}{eH}\varepsilon_{ijk}\oint d\sigma \frac{\partial_\sigma X^k(\sigma)}{|\partial_\sigma X(\sigma)|^2}. \qquad \text{(A9)}$$

Since the string is closed, $\oint d\sigma \partial_\sigma X^k = 0$, and therefore

$$\{x^i, x^j\}_D = 0. \qquad \text{(A10)}$$

We next evaluate $\{X^{ij}, X^{kl}\}_D$. The canonical bracket vanishes identically, $\{X^{ij}, X^{kl}\} = 0$. The relevant brackets with the constraints are

$$\{\phi_m(\sigma), X^{ij}\} = \delta^i_m \partial_\sigma X^j - \delta^j_m \partial_\sigma X^i. \qquad \text{(A11)}$$

Substituting into the Dirac-bracket definition yields

$$\begin{aligned}
\{X^{ij}, X^{kl}\}_D = \frac{1}{eH}\oint d\sigma \varepsilon_{mnp}\frac{\partial_\sigma X^p}{|\partial_\sigma X|^2}(\delta^i_m\partial_\sigma X^j - \delta^j_m \partial_\sigma X^i) \\
\times (\delta^k_n \partial_\sigma X^l - \delta^l_n \partial_\sigma X^k). \qquad \text{(A12)}
\end{aligned}$$

Each term contains a totally antisymmetric contraction of three identical tangent vectors $\partial_\sigma X$, which vanishes identically. Hence,

$$\{X^{ij}, X^{kl}\}_D = 0. \qquad \text{(A13)}$$

These results confirm that the only nonvanishing projected Dirac brackets are the mixed brackets $\{x^m, X^{ij}\}_D$. Their explicit projected universal form, however, emerges only after restricting to the isotropic internal sector discussed in the main text.

APPENDIX B

In this Appendix we summarize the algebraic conventions and operator definitions underlying the projected guiding-center formulation used in the main text. Our goal is to derive the mixed commutator between scalar and tensor density operators and to clarify in what sense the resulting algebra is closed.

We work on a three-dimensional flat space and Latin indices $i, j, k, \ldots$ run over spatial components $\{x, y, z\}$. We use the Levi-Civita symbol ε^{ijk} with $\varepsilon^{123} = +1$. The basic operators are given by

(i) vector coordinates x^i and their scalar projected densities

$$\rho(\mathbf{q}) \coloneqq e^{iq_m x^m}, \qquad q_m \in \mathbb{R}; \qquad \text{(B1)}$$

(ii) antisymmetric tensor coordinates $X^{ij} = -X^{ji}$ and tensor projected densities

$$\Upsilon(\mathbf{Q}) \coloneqq e^{iQ_{ij}X^{ij}}, \qquad Q_{ij} = -Q_{ji}. \qquad \text{(B2)}$$

The goal of this Appendix is to compute the mixed commutator $[\rho(\mathbf{q}), \Upsilon(\mathbf{Q})]$, starting from the following projected noncommutative geometry:

$$\begin{aligned}
[x^i, x^j] = [X^{ij}, X^{kl}] = 0, \\
[x^m, X^{ij}] = i\ell_H^3 \varepsilon^{mij}. \qquad \text{(B3)}
\end{aligned}$$

The last relation defines the magnetic volume ℓ_H^3 of the projected guiding-center algebra. Note that Eq. (B3) is taken to be *central*, namely a c-number relation commuting with both x and X, thereby enabling the Baker-Campbell-Hausdorff simplifications below. This is precisely the projected algebra obtained in the isotropic internal sector discussed in Sec. II.

Let us now define the two generator operators

$$A \coloneqq iq_m x^m, \qquad B \coloneqq iQ_{ij}X^{ij}. \tag{B4}$$

We can then compute the commutator of the generators associated with the scalar and tensor projected densities:

$$\begin{aligned}[A,B] &= [iq_m x^m, iQ_{ij}X^{ij}] \\ &= -q_m Q_{ij}[x^m, X^{ij}] \\ &= -q_m Q_{ij}(i\ell_H^3 \varepsilon^{mij}).\end{aligned} \tag{B5}$$

The corresponding Baker-Campbell-Hausdorff identity is given by

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]}, \qquad e^B e^A = e^{A+B-\frac{1}{2}[A,B]}, \tag{B6}$$

such that

$$\begin{aligned}[e^A, e^B] &= e^{A+B}\left(e^{\frac{1}{2}[A,B]} - e^{-\frac{1}{2}[A,B]}\right) \\ &= 2e^{A+B}\sinh\left(\frac{1}{2}[A,B]\right).\end{aligned} \tag{B7}$$

By substituting Eq. (B5), we obtain

$$\begin{aligned}\sinh\left(\frac{1}{2}[A,B]\right) &= \sinh\left(-\frac{i\ell_H^3}{2}q_m Q_{ij}\varepsilon^{mij}\right) \\ &= -i\sin\left(\frac{\ell_H^3}{2}q_m Q_{ij}\varepsilon^{mij}\right).\end{aligned} \tag{B8}$$

Therefore the exact mixed commutator reads

$$[\rho(\mathbf{q}), \Upsilon(\mathbf{Q})] = -2i\sin\left(\frac{\ell_H^3}{2}q_m Q_{ij}\varepsilon^{mij}\right)e^{iq_m x^m + iQ_{ij}X^{ij}}. \tag{B9}$$

We stress that the right-hand side is not, in general, of the form $\rho(\mathbf{q}')$ or $\Upsilon(\mathbf{Q}')$ separately. Rather, it belongs to the enlarged Weyl-type operator algebra generated by mixed exponentials $e^{iq_m x^m + iQ_{ij}X^{ij}}$. In this precise sense, the generalized GMP-type sine algebra is closed only in the enlarged mixed operator algebra.

For small arguments, namely in the long-wavelength regime $\ell_H^3|q||Q| \ll 1$, we can expand the sine to linear order and obtain

$$[\rho(\mathbf{q}), \Upsilon(\mathbf{Q})] \approx -i\ell_H^3 q_m Q_{ij}\varepsilon^{mij}e^{iq_m x^m + iQ_{ij}X^{ij}} + \cdots. \tag{B10}$$

The leading term in this expansion is the analog in three dimensions of the linearized GMP term in the two-dimensional case. In particular, the structure $q_m Q_{ij}\varepsilon^{mij}$ reflects the underlying tensorial pairing between vectorial and antisymmetric momenta induced by the projected guiding-center algebra. At long wavelengths, this is precisely the structure that underlies the volume-preserving diffeomorphism algebra discussed in Sec. III. This confirms that the noncommutative geometry derived in the main text consistently reproduces the expected projected long-wavelength limit of the three-dimensional generalized GMP algebra.

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