

Schwinger-Keldysh effective action for hydrodynamics with approximate symmetries

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We study hydrodynamic theories with approximate symmetries in the recently developed effective action approach on the Schwinger-Keldysh contour. We employ the method of spurious symmetry transformation for small explicit symmetry-breaking parameters to systematically constrain symmetry-breaking effects in the nonequilibrium effective action for hydrodynamics. We apply our method to the hydrodynamic theory of chiral symmetry in quantum chromodynamics at finite temperature and density and its explicit breaking by quark masses. We show that the spurious symmetry and the Kubo-Martin-Schwinger relation dictate that the Ward-Takahashi identity for the axial symmetry, i.e., the partial conservation of axial vector current (PCAC) relation, contains a relaxational term proportional to the axial chemical potential, whose kinetic coefficient is at least of the second order in the quark mass. In the phase where the chiral symmetry is spontaneously broken, and the pseudo-Nambu-Goldstone pions appear as hydrodynamic variables, this relaxation effect is subleading compared to the conventional pion mass term in the PCAC relation, which is of the first order in the quark mass. On the other hand, in the chiral symmetry restored phase, we show that our relaxation term, which is of the second order in the quark mass, becomes the leading contribution to the axial charge relaxation. Therefore, the leading axial charge relaxation mechanism is parametrically different in the quark mass across a chiral phase transition.

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I. INTRODUCTION

Symmetries, even if approximate, provide powerful constraints on both static and dynamic properties of a system. One of the most well-known examples in high-energy physics is the chiral symmetry of quantum chromodynamics (QCD), $SU(2)_L \times SU(2)_R$, which is explicitly broken by small but nonvanishing quark masses m_q , more precisely a quark mass matrix $M = \text{diag}(m_u, m_d)$. In the QCD vacuum, the chiral symmetry is further spontaneously broken by the quark-antiquark condensate $\langle \bar{q}q \rangle \neq 0$ even if there is no explicit breaking, $m_q = 0$. As a consequence, the Nambu-Goldstone (NG) theorem [1–3] dictates the

appearance of pseudo-NG bosons, which are identified as pions.

The low-energy dynamics of pions is described by an effective field theory (EFT) constrained by chiral symmetry and its explicit breaking by small quark masses: the chiral perturbation theory [4–6]. A convenient concept used in its construction is the spurious symmetry transformation acting on the quark mass matrix (see, e.g., Refs. [7,8]), which, together with the original symmetry transformation on dynamical fields, defines the enlarged symmetry of the EFT. Combined with the appropriate power counting, requiring the EFT to be invariant under the enlarged symmetry transformation systematically incorporates the explicit symmetry-breaking effects as perturbations.

At a finite temperature T that is low enough so that $\langle \bar{q}q \rangle \neq 0$, the pions must be included as one of the hydrodynamic variables in the chiral limit $m_q = 0$. In the presence of small quark masses, the chiral symmetry becomes approximate and the pions have finite correlation time and length, which are of the order of pion inverse mass m_π^{-1} . Thus, the validity of such hydrodynamics with pion degrees of freedom is restricted to a corresponding quasi-hydrodynamic or Hydro + regime [9], where the time and

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length scales are sufficiently shorter than m_π^{-1} . Beyond this scale, both the pions and the axial charges, which are inseparable from each other, become averaged out and disappear from the hydrodynamic description.

The hydrodynamic description of pions has been developed for more than two decades, beginning with pioneering work in the chiral and dissipationless limits [10]. Subsequently, it has been extended to include the dissipative and nonzero quark mass effects [11]. More recently, the second law of thermodynamics [12] was used to show a constraint between two kinetic coefficients introduced in Ref. [11]. These works assume the partial conservation of axial current (PCAC) relation [13,14] to the first order in quark mass,

$$\partial_\mu J_{Aa}^\mu = \langle \bar{q}q \rangle m_q \pi^a, \quad (1)$$

where $J_{Aa}^\mu \equiv \bar{q} \gamma^\mu \gamma_5 (\sigma^a/2) q$ is the axial isospin current with σ^a being the Pauli matrices and π^a denotes the pion field. However, the PCAC relation (1) can, in general, be modified by dissipative effects, which have not been studied systematically. This is because the source term [the right-hand side of Eq. (1)] can be seen as a constitutive relation from the viewpoint of hydrodynamics, which can generally be written in terms of the hydrodynamic variables of the system, i.e., J_{Aa}^0 and π^a . It can, for example, contain a linear term proportional to the axial isospin charge density J_{Aa}^0 (or the corresponding chemical potential μ_{Aa}) causing damping of the charge densities.

Recently, there have been theoretical developments to formulate dissipative hydrodynamics based on the EFT on the Schwinger-Keldysh (SK) contour [15,16] (see also Refs. [17–29] for complementary or further works and Refs. [30–35] for extensions to nonequilibrium open systems). These nonequilibrium EFTs are based on the action principle together with the symmetries of a microscopic theory on the SK contour, which provides an efficient framework for systematic construction and analysis of hydrodynamics relying on derivative expansion. In particular, Ref. [36] applied this framework to describe a lattice system with non-Abelian symmetries in the symmetry-restored phase. Moreover, Ref. [37] explored relaxation dynamics of pseudo-NG modes in the symmetry-broken phase at the linearized level of the NG field.

In the nonequilibrium EFTs, the explicit symmetry-breaking terms are introduced by writing down all possible terms that break the symmetry at the lowest order of the dynamical fields (see, e.g., Ref. [37] and Refs. [38,39], for symmetry-broken and symmetry-restored phases, respectively). More specifically, this approach introduces a relaxation (or damping) term in the PCAC relation that is linear in the chemical potential μ_{Aa} corresponding to the axial charge J_{Aa}^0 ,

$$\partial_\mu J_{Aa}^\mu = \langle \bar{q}q \rangle m_q \pi^a - \tilde{\gamma}_A \mu_{Aa}. \quad (2)$$

Here, $\tilde{\gamma}_A$ is a kinetic coefficient responsible for the axial charge damping. However, from the perspective of spurious symmetry transformation, we need a more systematic approach that ensures the invariance of the EFT action under the enlarged symmetry, where the explicit breaking parameter is elevated to a dynamical field that also transforms under the (spurious) symmetry group. Without such a systematic approach, we cannot determine the quark mass dependence of $\tilde{\gamma}_A = \tilde{\gamma}_A(m_q)$, other than the obvious expectation that $\tilde{\gamma}_A$ vanishes when $m_q = 0$. Thus, in order to formulate the EFT for hydrodynamics associated with the approximate chiral symmetry of QCD, we aim to construct the fully nonlinear EFT action for a spontaneously broken non-Abelian symmetry while keeping the spurious symmetry.

The main objective of this work is to apply the idea of spurious symmetry to the construction of an effective action for dissipative hydrodynamics with the approximate chiral symmetry of QCD at finite temperature and density. Based on the spurious chiral symmetry together with the so-called Kubo-Martin-Schwinger (KMS) symmetry that ensures thermal equilibrium satisfying the fluctuation-dissipation theorem, we show that the relaxation term highlighted above should be at least of second order in the quark mass, i.e., the coefficient $\tilde{\gamma}_A$ in Eq. (2) must have the following dependence on the quark mass in lowest order perturbative expansion,

$$\tilde{\gamma}_A = \gamma_A m_q^2. \quad (3)$$

We note that γ_A is nonvanishing in the chiral limit and its dependence on other parameters, such as T , is determined by the details of microscopic theory. This indicates that the PCAC relation is still a good approximation, even in a (quasi)hydrodynamic regime, to the leading order of the quark mass, as

$$\partial_\mu J_{Aa}^\mu = \langle \bar{q}q \rangle m_q \pi^a + O(m_q^2) \text{ (broken phase)}, \quad (4)$$

where the relaxation term introduced in Eq. (2) is systematically a higher-order term at $O(m_q^2)$.

On the other hand, our framework is equally applicable to the symmetry-restored phase, where the same parametric dependence on the quark mass (3) holds for $\tilde{\gamma}_A$. Furthermore, it becomes the leading contribution to the axial charge relaxation because the order parameter $\langle \bar{q}q \rangle$ vanishes in this phase,

$$\partial_\mu J_{Aa}^\mu = -\tilde{\gamma}_A \mu_{Aa} \text{ (restored phase)}. \quad (5)$$

In fact, it has been argued in the weakly coupled regime of high-temperature QCD that $\tilde{\gamma}_A \sim \alpha_s m_q^2 T$ [40] with $\alpha_s = g^2/(4\pi)$ being the QCD coupling constant. Therefore, we conclude that the leading axial charge relaxation mechanism is parametrically different in the quark mass across a chiral transition between the symmetry-broken and symmetry-restored phases.

Our method and the results are general enough to be applicable to any system with approximate internal symmetries, where spurious symmetry transformations can be introduced to the symmetry-breaking parameters. We will discuss the examples with approximate $U(1)$ and $SU(N)$ symmetries, and eventually approximate $SU(N)_L \times SU(N)_R$ chiral symmetry in QCD. In fact, the quadratic dependence of the charge damping rate on the explicit symmetry breaking parameter (3) has been widely observed in other fermion systems without spontaneous symmetry breaking (SSB), such as supernovae and neutron star plasma [41] and electron-positron plasma [42], etc., where the axial charge relaxation is caused by the chirality (or helicity) flipping processes.

The outline of the paper is as follows. In Sec. II, we review the construction of the low-energy effective theory on the SK contour for non-Abelian symmetries. In Sec. III, we start with an approximate $U(1)$ symmetry as a simple warm-up and introduce the spurious symmetry transformation for the explicit symmetry-breaking parameter. In Secs. IV and V, we generalize to the non-Abelian $SU(N)$ and the chiral symmetry of QCD, $SU(N)_L \times SU(N)_R$, respectively. In Sec. VI, we derive the Kubo formula for the kinetic coefficients introduced in this work, and we conclude in Sec. VII.

II. EFFECTIVE ACTION APPROACH TO HYDRODYNAMICS

In this section, we briefly review the recently developed effective action approach to dissipative hydrodynamics, focusing on the case of internal global symmetries without symmetry-breaking terms. Our discussion will be pragmatic, and we refer the reader to Ref. [22] for more conceptual details. We will discuss a non-Abelian unitary symmetry G in full generality, which can be easily applied to both $U(1)$ and $SU(N)_L \times SU(N)_R$ as special cases.

We define dynamical variables of the nonequilibrium EFT as unitary group fields, $U_1(x)$ and $U_2(x) \in G$, defined in the two (forward and backward) SK contours, labeled by the indices 1 and 2, respectively. We follow the convention in Ref. [36] that the symmetry transformations $V_1(x)$ and $V_2(x) \in G$ on both contours, which are elevated to local transformations, act on these fields to the right;

$$U \rightarrow UV^{-1} = UV^\dagger, \quad (6)$$

where we omitted the contour labels for simplicity of presentation.

Moreover, we introduced the external gauge fields $A_{\mu 1}$ and $A_{\mu 2}$ on the two contours, which we will use to find the symmetry currents with the variation of the action with respect to them. Their transformation rule is given by

$$A_\mu \rightarrow VA_\mu V^\dagger + iV\partial_\mu V^\dagger. \quad (7)$$

The useful gauge-invariant combination is

$$B_\mu = iUD_\mu U^\dagger = UA_\mu U^\dagger + iU\partial_\mu U^\dagger, \quad (8)$$

where we defined the covariant derivative as

$$D_\mu U^\dagger = (\partial_\mu - iA_\mu)U^\dagger. \quad (9)$$

In constructing the effective action for dissipative hydrodynamics based on the underlying quantum theory, we take the classical limit $\hbar \rightarrow 0$ in the SK formalism. In this limit, we assume that the variables in the two contours differ by $\mathcal{O}(\hbar)$, and the timescale of interest is much longer than the microscopic timescale of quantum fluctuations of typical energy $k_B T$, i.e., $\tau_q = \hbar/(k_B T)$ (we set the Boltzmann constant $k_B = 1$ in the following). Expanding the full effective action in the SK contours up to linear order in $\hbar$, and taking $\hbar \rightarrow 0$ limit, the path integral measure reduces to a classical action for hydrodynamics,

$$\lim_{\hbar \rightarrow 0} \frac{i}{\hbar} S_{\text{SK}} \rightarrow iS_{\text{hydro}}, \quad (10)$$

where S_{SK} is a microscopic action on the SK path and S_{hydro} is the effective action of dissipative hydrodynamics. As we will see later, S_{hydro} should inherit the classical version of the original KMS symmetry that S_{SK} must possess.

Following Ref. [36], we introduce the variables in the classical action, (U_r, ϕ_a) , as

$$\begin{aligned} U_1 &= U_r e^{\frac{i\hbar}{2}\phi_a} = U_r \left(1 + \frac{i\hbar}{2}\phi_a \right) + \dots, \\ U_2 &= U_r e^{-\frac{i\hbar}{2}\phi_a} = U_r \left(1 - \frac{i\hbar}{2}\phi_a \right) + \dots, \end{aligned} \quad (11)$$

where the ellipsis denotes the higher order terms in $\hbar$. There is an ordering ambiguity between the “classical value” U_r (or “r field”) and the “quantum fluctuation” variable ϕ_a (or “a field”), while it simply redefines the latter by the unitary transformation with U_r . This choice of definition for ϕ_a gives rise to its many simple properties under symmetry transformations. Although the external gauge fields are arbitrary and do not need to follow the above rules of $\hbar$ expansion, we simply define

$$A_1 = A_r + \frac{\hbar}{2}A_a, \quad A_2 = A_r - \frac{\hbar}{2}A_a; \quad (12)$$

otherwise, the $\hbar \rightarrow 0$ limit of the SK action diverges.

This assumption on the external gauge fields then requires us to consider the gauge transformations only of the following type ($v_a^\dagger = v_a$),

$$\begin{aligned} V_1 &= V_r e^{\frac{i\hbar}{2}v_a} = V_r \left(1 + \frac{i\hbar}{2}v_a \right) + \dots, \\ V_2 &= V_r e^{-\frac{i\hbar}{2}v_a} = V_r \left(1 - \frac{i\hbar}{2}v_a \right) + \dots, \end{aligned} \quad (13)$$

under which the hydrodynamic fields and gauge fields transform as follows:

$$\begin{cases} U_r \rightarrow U_r V_r^\dagger, \\ \phi_a \rightarrow V_r(\phi_a - v_a) V_r^\dagger, \end{cases} \quad \text{and} \quad \begin{cases} A_{r\mu} \rightarrow V_r A_{r\mu} V_r^\dagger + i V_r \partial_\mu V_r^\dagger, \\ A_{a\mu} \rightarrow V_r(A_{a\mu} + \partial_\mu v_a - i[A_{r\mu}, v_a]) V_r^\dagger. \end{cases} \quad (14)$$

We also decompose the gauge-invariant B_μ terms as

$$B_{1\mu} = B_{r\mu} + \frac{\hbar}{2} B_{a\mu}, \quad B_{2\mu} = B_{r\mu} - \frac{\hbar}{2} B_{a\mu}, \quad (15)$$

where the gauge-invariant r- and a-type terms are given by

$$\begin{aligned} B_{r\mu} &= i U_r D_\mu U_r^\dagger = U_r A_{r\mu} U_r^\dagger + i U_r \partial_\mu U_r^\dagger, \\ B_{a\mu} &= U_r (D_\mu \phi_a) U_r^\dagger, \end{aligned} \quad (16)$$

with the covariant derivative,

$$D_\mu \phi_a \equiv \partial_\mu \phi_a - i[A_{r\mu}, \phi_a] + A_{a\mu}. \quad (17)$$

The symmetry current in the classical limit and its quantum correction are defined as $J_r^\mu \equiv (J_1^\mu + J_2^\mu)/2$ and $J_a^\mu \equiv (J_1^\mu - J_2^\mu)/\hbar$ with $J_{1,2}^\mu = \delta S_{\text{SK}}/\delta(A_{1,2})_\mu$ on the microscopic action S_{SK} . It can be easily shown that these currents are obtained by taking the variation of the effective action S_{hydro} with respect to the a-type and the r-type gauge fields A_a and A_r , respectively,

$$J_r^\mu = \frac{\delta S_{\text{hydro}}}{\delta A_{a\mu}}, \quad J_a^\mu = \frac{\delta S_{\text{hydro}}}{\delta A_{r\mu}}. \quad (18)$$

Note that the r-type current is obtained by taking the variation with respect to the a-type gauge field and vice versa.

The group fields $U_{1,2}$ represent the state of the system with regard to the symmetry G , which can be parametrized by the NG fields on the SK contours. Roughly speaking, they are “angle variables,” canonically conjugate to the Noether charges, whose corresponding chemical potentials are given by time derivatives of these fields. In the symmetry-restored phase, the origin in the angle variable should not matter because the order parameter measuring such “angle” is vanishing, and only the time derivative, i.e., the charge, gives a physical variable. As the fluid cells of hydrodynamics are locally independent of each other, the effective action should not depend on local redefinitions of the origin in the group fields. This indicates that the description of the symmetry-restored phase with the group fields $U_{1,2}$ possesses a gauge redundancy, which is called shift symmetry;

$$U_{1,2}(x) \rightarrow \Lambda(x) U_{1,2}(x), \quad (19)$$

where $\Lambda(x) \in G$ depends arbitrarily on space x , but not on time (otherwise, it would affect the time derivatives). This effectively makes the time derivatives of U fields, i.e., the

charges, the basic degrees of freedom of the effective action. For the hydrodynamic variables U_r and ϕ_a , the shift symmetry takes the form

$$U_r \rightarrow \Lambda U_r, \quad \phi_a \rightarrow \phi_a, \quad (20)$$

and thus, the G -invariant building blocks behave as

$$B_{a\mu} \rightarrow \Lambda B_{a\mu} \Lambda^\dagger, \quad B_{r\mu} \rightarrow \Lambda B_{r\mu} \Lambda^\dagger + i \delta_{\mu i} \Lambda \partial_i \Lambda^\dagger \quad (i=1,2,3). \quad (21)$$

On the other hand, in a phase where G -symmetry is spontaneously broken down to its subgroup $H \subset G$, the shift symmetry should be reduced to the unbroken part only: $\Lambda(x) \in H$. The effective action for hydrodynamics then is allowed to contain a new term, which makes the G/H part of U fields propagating modes. This is nothing but the NG bosons associated with the SSB. A nice example of this is the chiral symmetry of QCD, where $G = \text{SU}(N)_L \times \text{SU}(N)_R$ and $H = \text{SU}(N)_V$, which we will discuss in detail in Sec. V. We also mention in passing that the gauge-invariant B_r field in the G/H part has been historically called the Stueckelberg field in EFTs for SSB.

The last ingredient we review is the KMS symmetry [15,16,22]. The KMS symmetry is the discrete $\mathbb{Z}_2$ symmetry of the generating functional in the SK formalism, consisting of imaginary-time translation with the inverse temperature $\beta = T^{-1}$ and the time-reversal transformation. It acts on the external gauge fields as the following transformations:

$$A_{1\mu}(t, \mathbf{x}) \rightarrow (-1)^\mu A_{1\mu}^*(-t, \mathbf{x}) = (-1)^\mu A_{1\mu}^T(-t, \mathbf{x}), \quad (22a)$$

$$A_{2\mu}(t, \mathbf{x}) \rightarrow (-1)^\mu A_{2\mu}^*(-t - i\hbar\beta, \mathbf{x}), \quad (22b)$$

where $(-1)^\mu = +1$ for $\mu = t$ and -1 for $\mu = i$, and we used the fact that the A fields are Hermitian matrices, i.e., $A^* = A^T$. We note that the complex conjugate operation originates from the time-reversal transformation.¹ The transpose operation does not matter in the $U(1)$ case, but we keep

¹The complex conjugate operation comes from the time reversal invariance of the microscopic theory under $A_\mu(t) \rightarrow (-1)^\mu A_\mu^*(-t) = (-1)^\mu A_\mu^T(-t)$, which can be explicitly seen in several examples, e.g., Dirac fermions and complex scalar fields. One can also find this transformation rule by noting that the currents transform under time reversal as $j_\mu(t) \rightarrow (-1)^\mu J_\mu^*(-t) = (-1)^\mu J_\mu^T(-t)$. Here, $*$ is a complex (not Hermitian) conjugation acting on the Hermitian matrix fields. The complex conjugate operation was missed in Ref. [36].

them for the general $SU(N)$ case. From now on, we shall omit the spatial argument of functions under the time-reversal operation. Considering that the external field appears in the covariant derivative of U fields, we require the KMS symmetry to act on the dynamical fields U as

$$U_1(t) \rightarrow U_1^*(-t), \quad U_2(t) \rightarrow U_2^*(-t - i\hbar\beta), \quad (23)$$

which enables us to respect the KMS symmetry manifestly in the low-energy EFT.

In the classical limit, the transformation rules of the dynamical and external fields in Eqs. (22) and (23) become²

$$\begin{aligned} A_{r\mu}(t) &\rightarrow (-1)^\mu A_{r\mu}^*(-t), \\ A_{a\mu}(t) &\rightarrow (-1)^\mu [A_{a\mu}^*(-t) + i\beta(\partial_t A_{r\mu}^*)(-t)], \\ U_r(t) &\rightarrow U_r^*(-t), \\ \phi_a(t) &\rightarrow -\phi_a^*(-t) + \beta(U_r^\dagger \partial_t U_r^*)(-t), \end{aligned} \quad (24)$$

$$(25)$$

respectively. Note that these transformations do not preserve the Hermitian nature of $A_{a\mu}$ and ϕ_a fields since the second term comes from the imaginary time $i\beta\hbar$. As a consequence, the KMS transformations for their complex conjugate are not obtained by taking a complex conjugate operation to the above transformations. Instead, following the definition carefully, e.g.,

$$\phi_a^* = \lim_{\hbar \rightarrow 0} \frac{1}{-i\hbar} U_r^\dagger (U_1^* - U_2^*), \quad (26)$$

we derive the KMS transformations for the complex conjugate fields:

$$\begin{aligned} A_{a\mu}^*(t) &\rightarrow (-1)^\mu [A_{a\mu}^*(-t) + i\beta(\partial_t A_{r\mu}^*)(-t)], \\ \phi_a^*(t) &\rightarrow -\phi_a^*(-t) - \beta(U_r^\dagger \partial_t U_r^*)(-t). \end{aligned} \quad (27)$$

From these, it can be checked that the gauge-invariant B fields transform nicely as

$$B_{r\mu}(t) \rightarrow (-1)^\mu B_{r\mu}^*(-t) = (-1)^\mu B_{r\mu}^\dagger(-t), \quad (28)$$

$$B_{a\mu}(t) \rightarrow (-1)^\mu [B_{a\mu}^\dagger(-t) + i\beta(\partial_t B_{r\mu}^\dagger)(-t)], \quad (29)$$

which is ultimately a consequence of the fact that the gauge transformation and the KMS transformation commute.

The effective action respecting all these symmetries is constructed in the gradient expansion of the dynamical variables (U_r, ϕ_a) . It is also required from the unitarity of the microscopic theory that the action satisfies the condition, $S_{\text{hydro}}[\Phi_a \rightarrow -\Phi_a] = -S_{\text{hydro}}^*[\Phi_a]$, where Φ_a generically denotes “a”-type fields. Specifically, terms containing

odd numbers of a-type fields should be real, and terms with even numbers of a-type fields should be purely imaginary. Finally, a stability condition requires $\text{Im}[S_{\text{hydro}}] \geq 0$.

III. A WARM-UP WITH APPROXIMATE $U(1)$ SYMMETRY

Our main goal is to construct the effective action as a power series in a set of explicit symmetry-breaking parameters, denoted by M , which encapsulate the explicit symmetry breaking in the microscopic theory. Having in mind its application to QCD where M is the quark mass matrix, we simplify our discussion by assuming that M forms a simple multiplet under spurious symmetry transformations. In this section, we consider the simplest case of $U(1)$ symmetry with a complex symmetry-breaking parameter M , which is promoted to a spurious complex field of charge -1 .

A. Exact symmetry

At zeroth order in M , i.e., the theory with the exact $U(1)$ symmetry, the effective Lagrangian up to the second order in the derivative expansion is known to be [37]

$$\begin{aligned} \mathcal{L}_{U(1)}^{(2)} &= \chi B_{at} B_{rt} - f_\pi^2 B_{ai} B_{ri} - \chi^2 \kappa B_{at} (\partial_t B_{rt} - i\beta^{-1} B_{at}) \\ &\quad - \lambda B_{ai} (\partial_t B_{ri} - i\beta^{-1} B_{ai}), \end{aligned} \quad (30)$$

where χ is the linearized charge susceptibility,³ f_π the decay constant for the NG boson, λ the conductivity, and κ a dissipative kinetic coefficient that plays an important role when the symmetry is spontaneously broken. Writing the $U(1)$ field explicitly as $U_r = e^{i\phi_r}$, we have the simple expressions for B fields (16) as

$$B_{r\mu} = \partial_\mu \phi_r + A_{r\mu} = D_\mu \phi_r, \quad B_{a\mu} = \partial_\mu \phi_a + A_{a\mu} = D_\mu \phi_a. \quad (31)$$

Let us first consider the symmetry-restored phase where there is no SSB. The shift symmetry (20) forbids the second term, i.e., $f_\pi = 0$, and we see that the action can be written solely in terms of the chemical potential $\mu \equiv B_{rt}$ without ϕ_r explicitly appearing as we emphasized around Eq. (19). The action (30) becomes the theory of stochastic charge diffusion [22,23]. Note that the κ term becomes a higher order term for the counting $\partial_t \sim \nabla^2$ and can be neglected in the present case of the symmetry restored phase. Nevertheless, we shall keep this term for later purposes in describing the symmetry-broken phase.

The equation of motion (EOM) for ϕ_a field reduces to the ordinary stochastic diffusion equation accompanied by the noisy constitutive relation for the charge current

²Notation: $(\partial_t f)(-t) = f'(-t)$, where $f'(t) = \partial_t f(t)$.

³It is straightforward to generalize it to the case where χ is replaced with a general function of B_{rt} .

(using $\partial_t B_{ri} = \partial_t D_i \phi_r = \partial_i D_t \phi_r - E_i = \partial_i B_{ri} - E_i$),

$$\partial_\mu J_r^\mu = 0, \quad J_r^i = \chi \mu - \chi^2 \kappa \partial_i \mu + \xi_\kappa, \quad J_r^i = -\lambda(\partial_i \mu - E_i) + \xi^i, \quad (32)$$

where $A_{a\mu}$ is switched off after computing J_r^μ . We have changed the variable ϕ_a involved in the expression followed by its original definition (18) into the noise terms ξ_κ and ξ^i with the noise correlations⁴:

$$\begin{aligned} \langle \xi_\kappa(x) \xi_\kappa(x') \rangle &= 2\chi^2 \kappa \beta^{-1} \delta^{(4)}(x - x'), \\ \langle \xi^i(x) \xi^j(x') \rangle &= 2\lambda \beta^{-1} \delta^{ij} \delta^{(4)}(x - x'). \end{aligned} \quad (33)$$

The noise correlators have been determined by $\mathcal{O}(\phi_a^2)$ terms in the action (30), or equivalently the coefficients in front of the $\mathcal{O}(\phi_a)$ terms in the original expression of the current. The changing variable enables us to compute certain correlation functions in the later sections without discussing the possible contact terms in the Schwinger-Dyson equations for the (r,a) variables.

On the other hand, the absence of the shift symmetry in the symmetry-broken phase allows $f_\pi \neq 0$, and thus, ϕ_r becomes a propagating gapless mode of linear dispersion relation with a finite velocity,

$$v_s^2 = \frac{f_\pi^2}{\chi}, \quad (34)$$

which describes the NG boson associated with spontaneous U(1) symmetry breaking. Note that the expression for (34) is valid only in the exact symmetry limit described by the effective Lagrangian (30); we will discuss its correction from the symmetry-breaking parameter later [see Eq. (55)].

The EOM yields the conservation equation $\partial_\mu J_r^\mu = 0$ with

$$\begin{aligned} J_r^t &= \chi D_t \phi_r - \chi^2 \kappa \partial_t D_t \phi_r + \xi_\kappa, \\ J_r^i &= -f_\pi^2 D_i \phi_r - \lambda(\partial_i \mu - E_i) + \xi^i, \end{aligned} \quad (35)$$

where ξ_κ and ξ^i are the same thermal noises given in Eq. (33). Equivalently, we can also rewrite these in two coupled equations of (ϕ_r, μ) ,

$$\begin{aligned} D_t \phi_r &= \mu, \\ \chi \partial_t \mu &= f_\pi^2 \partial_i D_i \phi_r + \lambda \partial_i (\partial_i \mu - E_i) + \chi^2 \kappa \partial_t^2 \mu - \partial_t \xi_\kappa - \partial_i \xi^i, \end{aligned} \quad (36)$$

where the first equation is called the Josephson relation. Other ways of expressing the same equations exist,

⁴See, e.g., Ref. [43] for details of changing variables. One can equivalently understand this by identifying Eq. (30) as the effective action in the path-integral formalism of stochastic charge diffusion [44–46].

truncating at the second order in derivatives and using the lowest order EOM, $\chi \partial_t D_i \phi_r = f_\pi^2 \partial_i D_i \phi_r$. For example, we can move the dissipative effect from κ to the Josephson relation, by defining $\tilde{\mu} \equiv J_r^t / \chi = D_t \phi_r - \chi \kappa \partial_t D_t \phi_r + \xi_\kappa / \chi = D_t \phi_r - f_\pi^2 \kappa \partial_i D_i \phi_r + \xi_\kappa / \chi$ and $\tilde{\xi}_\kappa \equiv \xi_\kappa / \chi$, which results in

$$\begin{aligned} D_t \phi_r &= \tilde{\mu} + f_\pi^2 \kappa \partial_i D_i \phi_r - \tilde{\xi}_\kappa, \\ \chi \partial_t \tilde{\mu} &= f_\pi^2 \partial_i D_i \phi_r + \lambda \partial_i (\partial_i \tilde{\mu} - E_i) - \partial_i \xi^i, \end{aligned} \quad (37)$$

where $\langle \tilde{\xi}_\kappa(x) \tilde{\xi}_\kappa(x') \rangle = 2T\kappa \delta^{(4)}(x - x')$. Both κ and λ contribute to the leading diffusive term in the following dispersion relation for the frequency ω and momentum k :

$$\omega(k) = \pm v_s k - \frac{i}{2} \left(\frac{\lambda}{\chi} + f_\pi^2 \kappa \right) k^2 + \mathcal{O}(k^3), \quad (38)$$

with $k \equiv |k|$ and the propagating velocity v_s already defined in Eq. (34).

B. Approximate symmetry

We next consider effects from the explicit symmetry-breaking terms containing the symmetry-breaking parameter M . Although we will eventually set M_1 and M_2 on the two SK contours to their physical value as $M_1 = M_2 = M = \text{const.}$, we will initially treat them as spurious fields that transform under the doubled symmetry. In other words, in constructing the effective action, we consider each of M_1 and M_2 ($M_1 \neq M_2$ in general) as independent spurious fields (functions of space and time), on which the doubled U(1) symmetries act. One sees that this gives an extended notion of the background field in the SK formalism, and thus, M_1 and M_2 are not dynamical fields. Indeed, once the action is constructed respecting the full spurious symmetry, we replace $M_{1,2}$ with those physical values.

We then assume that the microscopic theory remains invariant by assigning a spurious transformation rule to the M fields as

$$M_1 \rightarrow M_1 V_1^\dagger, \quad M_2 \rightarrow M_2 V_2^\dagger, \quad (39)$$

with $V \in \text{SU}(N)$ for latter purpose. Expressing

$$M_1 = M_r \left(1 + \frac{i\hbar}{2} m_a \right) + \dots, \quad M_2 = M_r \left(1 - \frac{i\hbar}{2} m_a \right) + \dots, \quad (40)$$

we have the gauge transformations as [recall Eq. (13)]

$$M_r \rightarrow M_r V_r^\dagger, \quad m_a \rightarrow V_r(m_a - v_a) V_r^\dagger, \quad (41)$$

where it simplifies $m_a \rightarrow m_a - v_a$ for the U(1) case.

The KMS symmetry transformation should also act on the M fields since the theory would be time reversal invariant if the theory without M fields is time reversal invariant. The M fields also transform according to the rules of time reversal transformations. In the phase where the $U(1)$ symmetry is spontaneously broken, we also assume that the time reversal symmetry is not spontaneously broken by the order parameter (see Ref. [32] for the discussion about the KMS symmetry in the time-translational symmetry breaking case). Therefore, we require the action to be KMS invariant. We assume that the time reversal parity of the M fields is $+1$, i.e., the KMS transformation is the same as that for the dynamical U fields [see Eq. (25)]⁵;

$$\begin{aligned} M_r(t) &\rightarrow M_r^*(-t), \\ m_a(t) &\rightarrow -m_a^*(-t) + \beta(M_r^*)^{-1}(\partial_t M_r^*)(-t), \end{aligned} \quad (42)$$

and (note the sign flip in the second term as before for ϕ_a^*)

$$m_a^*(t) \rightarrow -m_a(-t) - \beta M_r^{-1}(\partial_t M_r)(-t). \quad (43)$$

Finally, the shift symmetry does not act on the M fields since it refers to the emergent gauge redundancy of our description of hydrodynamics rather than the symmetry of the microscopic theory. Since we will replace M fields with their physical value at the end, i.e., $M_r = M$, a constant, and $m_a = 0$, we will organize the action in the power series of M_r .

1. The first order in M

At the first order in M_r , there is no action that respects all the identified symmetries, including the shift symmetry. Therefore, if there is no $U(1)$ SSB, the effect from the explicit symmetry breaking starts to appear only in the second order of M_r , which we will discuss in Sec. III B 2.

We can prove the above statement as follows: let us write the first-order term in the effective action as

$$M_r \cdot f(U_r, D_\mu U_r, \phi_a, m_a, \dots). \quad (44)$$

Since M_r has charge -1 under the gauge symmetry V_r , the gauge invariance requires that f factor as a whole has charge $+1$. Consider a homogeneous transformation of shift symmetry Λ . Due to the Abelian nature of $U(1)$, its action to the fields is found to be identical to the $U(1)$ -symmetry transformation with $V_r^{-1} = \Lambda$ and $v_a = 0$. Therefore, we see that f term has charge -1 under the shift symmetry, and transforms as $f \rightarrow f\Lambda^{-1}$. Since M_r does not transform under the shift symmetry, the term in consideration (44) is inconsistent with the shift symmetry. Note that the covariant derivatives of M_r , e.g., $D_\mu M_r = \partial_\mu M_r + iM_r A_{r\mu}$

share the same charge as M_r , and are allowed to appear in the action. Our result equally applies to them as well.

On the other hand, the shift symmetry is absent in the $U(1)$ SSB phase. Thus, we can find the leading terms at $\mathcal{O}(M)$ respecting all other symmetries as

$$\mathcal{L}_M = \sigma(M_1 U_1^* - M_2 U_2^*)/\hbar + \text{c.c.} = -i\sigma\mathcal{M}_a + \text{c.c.}, \quad (45)$$

with a real constant σ . Here, for later purposes, we introduced a gauge-invariant (shift covariant) quantity linear both in M_r and a fields:

$$\mathcal{M}_a \equiv M_r(\phi_a - m_a)U_r^\dagger. \quad (46)$$

We also note that, for the present $U(1)$ case, $U_r^\dagger$ is replaced by U_r^* . Moreover, the KMS invariance is respected up to the total derivative term as⁶

$$\mathcal{L}_M \rightarrow \mathcal{L}_M + i\sigma\beta\partial_t[M_r^*U_r(-t) + M_rU_r^*(-t)]. \quad (47)$$

We will assume that the physical value of $M_r = M$ is real and positive, which is the stability condition for the ϕ_r field, and we can always redefine the origin of U_r to satisfy this condition. As we will shortly see, this term makes the NG boson ϕ_r massive with $m_{\text{NG}}^2 = 2\sigma M/\chi$; the analogy to what happens in QCD is clear: indeed, Eq. (45) corresponds to the leading-order term linear in M in the $\mathcal{O}(p^2)$ chiral Lagrangian [4–8] on the SK contour.

Let us now investigate some consequences of the $\mathcal{O}(M)$ term by adding Eq. (45) to the action with the exact symmetry (30). Firstly, the linearized EOM following from $\mathcal{L}_{U(1)}^{(2)} + \mathcal{L}_M$ can be expressed up to second order in gradients as

$$D_t\phi_r = \tilde{\mu} + \chi\kappa(v_s^2\partial_i D_i\phi_r - m_{\text{NG}}^2\phi_r) - \tilde{\xi}_\kappa, \quad (48a)$$

$$\chi\partial_t\tilde{\mu} = \chi(v_s^2\partial_i D_i\phi_r - m_{\text{NG}}^2\phi_r) + \lambda\partial_i(\partial_t\tilde{\mu} - E_i) - \partial_i\xi_i, \quad (48b)$$

with the noise correlations for $(\tilde{\xi}_\kappa, \xi^i)$ unchanged as before. In writing this, we used the lowest order EOM, $\chi\partial_t D_i\phi_r = f_\pi^2\partial_i D_i\phi_r - \chi m_{\text{NG}}^2\phi_r$, and $\tilde{\mu} \equiv J_r^t/\chi$. This form of the EOM was obtained previously, using the second law of thermodynamics [12], the Poisson bracket method with dissipation [47], or the locality of the response function to an external source for ϕ_r field [37].

By solving the linearized EOM (48), we obtain the following dispersion relation:

$$\omega(k) = \pm\sqrt{v_s^2k^2 + m_{\text{NG}}^2} - \frac{i}{2}\left[\frac{\lambda}{\chi}k^2 + \chi\kappa(v_s^2k^2 + m_{\text{NG}}^2)\right], \quad (49)$$

⁵That is, under the time reversal, we have $M(t) \rightarrow M^*(-t)$. This remains the same even in the non-Abelian case discussed later, where M is matrix valued.

⁶In fact, the KMS transformation acts as $\sigma \rightarrow \sigma^*$. The constant σ is the order parameter of the SSB, and the reality condition of σ is essentially our assumption that the time reversal symmetry is not spontaneously broken by the order parameter.

where we have kept the relativistic form of the real part (first term) with $m_{\text{NG}}^2 = 2\sigma M/\chi$ [it is worth noticing the counting $m_{\text{NG}}^2 = \mathcal{O}(M)$ for later purposes]. We also note that the combination $(v_s^2 \partial_i D_i \phi_r - m_{\text{NG}}^2 \phi_r)$ appearing in the EOM (48) captures a feature in the dispersion relation, i.e., the effect of m_{NG}^2 always appears via the replacement $v_s^2 k^2 \rightarrow v_s^2 k^2 + m_{\text{NG}}^2$. As pointed out in Ref. [11], the term in Eq. (45) also contributes to the damping of the pseudo-NG boson in a definite way.

The nonvanishing damping term in $k \rightarrow 0$ limit describes an exponential decay of the total U(1) charge in the system. This is the leading mechanism of U(1) charge relaxation of the system due to an explicit symmetry breaking, which is of the first order in the symmetry-breaking parameter M (or m_{NG}^2). As the damping rate at $k = 0$ is proportional to $\chi \kappa m_{\text{NG}}^2$ (i.e., $\kappa \sigma M$), it is an effect that arises from the interplay between the kinetic coefficient κ and both the spontaneous and the explicit symmetry breakings, σ and M . Physically, this damping occurs because the charge in the symmetry-broken phase is carried by the pseudo-NG boson, which decays at finite m_{NG} .

To illustrate the effect of the explicit symmetry breaking, it is useful to investigate the current conservation law and its explicit violation by M . The U(1) current, which is obtained by taking variation with respect to gauge fields, $J_r^\mu = \delta S / \delta A_{a\mu}$, is not modified by $\mathcal{L}_M$ as it does not contain gauge fields. The EOM can be rephrased as the conservation law, which is modified at first order in the explicit symmetry breaking M_r ,

$$\partial_\mu J_r^\mu = i\sigma M_r (U_r - U_r^*) \simeq -\chi m_{\text{NG}}^2 \phi_r, \quad (50)$$

where we linearized in ϕ_r in the rightmost side. We note that the nonvanishing right-hand side corresponds to the term appearing in the PCAC relation in QCD. As we have shown, this mechanism or the right-hand side of the above equation is absent in the phase where there is no SSB, and the leading mechanism for charge relaxation in that phase is of the second order in M , as we will show in Sec. III B 2.

When we go to the next order of derivative expansion while keeping the mass expansion to the first order, we find two additional terms that are KMS invariant up to the total derivative,

$$\begin{aligned} & \ell_t [iB_{at} D_t (M_r U_r^*) - (D_t B_{rt}) \mathcal{M}_a] \\ & - \ell_s [iB_{ai} D_i (M_r U_r^*) - (D_i B_{ri}) \mathcal{M}_a] + \text{c.c.}, \end{aligned} \quad (51)$$

which is real and thus contributes to nondissipative effects. In fact, we find corresponding terms in the next-to-leading order $\mathcal{O}(p^4)$ chiral Lagrangian in vacuum [6], which reduces to

$$(D_\mu U D^\mu U^*) M U^* + \text{c.c.} = iB^\mu D_\mu (M U^*) + \text{c.c.}, \quad (52)$$

for the U(1) symmetry case. One can show that (52) is equivalent to (51) when it is put on the SK contour. Note here that the Lorenz symmetry of (52) is broken (to spatial rotational symmetry), which gives two independent parameters, ℓ_t and ℓ_s , for the time and spatial components, respectively.

From Eq. (51), we have corrections to both the current (35) and the conservation law (74),

$$\begin{aligned} \Delta J_r^t &= i\ell_t D_t (M_r U_r^* - M_r^* U_r), \\ \Delta J_r^i &= -i\ell_s D_i (M_r U_r^* - M_r^* U_r), \end{aligned} \quad (53)$$

$$\begin{aligned} \Delta(\partial_\mu J_r^\mu) &= -\ell_t (D_t B_{rt}) (M_r U_r^* + M_r^* U_r) \\ &+ \ell_s (D_i B_{ri}) (M_r U_r^* + M_r^* U_r), \end{aligned} \quad (54)$$

respectively. These lead to nondissipative corrections to the Josephson relation and the pion velocity linear in M_r . As a result, the relation between the velocity, decay constant, and susceptibility in the chiral limit (34) receives $\mathcal{O}(M)$ correction upon choosing the physical value $M_r = M$,

$$v_s^2 = \frac{f_\pi^2 + 4M\ell_s}{\chi + 4M\ell_t}. \quad (55)$$

On the other hand, (51) does not affect the expression for the dispersion relation given by (49) as a function of v_s .

Similar corrections to the velocity of the NG field have been first pointed out in Ref. [48] within the thermodynamics approach based on the second law, and subsequently in Ref. [37] within the nonequilibrium EFT (which called them “pinning effects”). Our expression (51) is fully nonlinear in the NG field and is also invariant under the spurious symmetry transformation (41). We find that these terms already exist in the chiral effective theory in vacuum as higher-order gradient terms as (52) and do not represent new hydrodynamic effects [up to the replacement of (34) by (55)].

2. The second order in M

We next consider the effective action that is second order in M_r .

A useful method to construct a KMS-invariant term is to rely on the $\mathbb{Z}_2$ -nature of the KMS transformation [22]. Under the KMS transformation, we have $\Phi(t) \rightarrow (K\Phi)(-t)$ for a generic field Φ , which defines a map $K: \Phi(t) \rightarrow (K\Phi)(t)$ (note that this map leaves the argument t the same). For example, $KU = U^*$, $K\phi_a = -\phi_a^* + \beta(U_r^\dagger \partial_t U_r^*)$ and $K\phi_a^* = -\phi_a - \beta(U_r^* \partial_t U)$ as we have obtained in Eqs. (25) and (27) (note the sign flip in the second term in $K\phi_a$ and $K\phi_a^*$), and similarly for M fields. From the definition of K , when K acts on a time derivative, it changes the sign of the time derivative, i.e., $K(\partial_t \Phi) = -\partial_t (K\Phi)$. It can be checked easily that $K^2 = 1$, i.e., K is a $\mathbb{Z}_2$ operator.

Then, consider a term in the effective action, $F(t)(KF)(t)$, where F is an arbitrary term constructed from the fields Φ . Under the KMS transformation, this becomes $(KF)(-t)F(-t) = (KF)(t')F(t')$ where $t' = -t$, and the action is KMS invariant with the time direction reversed. In fact, the previous dissipative terms in the action (30) can be written as

$$-i\lambda\beta^{-1}B_{ai}(KB_{ai}) + i\kappa\chi^2\beta^{-1}B_{ai}(KB_{ai}). \quad (56)$$

In constructing the KMS-invariant terms with M fields using the above method, it is crucial to note that the time derivative appearing in the KMS transformations for a-type fields is the ordinary, not gauge covariant, derivative. This means that the above method would result in an expression that is not gauge invariant in general unless F is gauge invariant. For the gauge-invariant F , its ordinary time derivative (equal to the covariant derivative) is consistent with the gauge symmetry. This can also be understood from the fact that the gauge transformation and the KMS transformation commute. Since we are interested in a dissipative term, we consider F that is linear in a-type fields; then $F(KF)$ would contain a term quadratic in a-type fields, which should then be purely imaginary due to the unitarity condition, and hence dissipative. Finally, a quadratic action term in M_r would require us to consider F linear in M_r fields.

The F satisfying these requirements at zeroth order in the derivative is unique;

$$F = c_1\mathcal{M}_a + c_2\mathcal{M}_a^*, \quad (57)$$

where $\mathcal{M}_a$ is given by (46) with $c_{1,2}$ being constants. Using Eqs. (25) and (42) with $U_r^*U_r = 1$, we obtain

$$K\mathcal{M}_a = -\mathcal{M}_a^* - \beta\partial_t(M_r^*U_r), \quad (58)$$

and similarly using (27) and (43), we obtain

$$K\mathcal{M}_a^* = -\mathcal{M}_a + \beta\partial_t(M_rU_r^*). \quad (59)$$

Thus

$$KF = -c_1\mathcal{M}_a^* - c_2\mathcal{M}_a - \beta[c_1\partial_t(M_r^*U_r) - c_2\partial_t(M_rU_r^*)]. \quad (60)$$

It can be shown that for arbitrary $c_{1,2}$, we have

$$F(KF) = i(c_1^2 + c_2^2)L_1 + 2ic_1c_2L_2 - \frac{\beta}{2}(c_1^2 - c_2^2)L_3, \quad (61)$$

where $L_{1,2,3}$ are the KMS-invariant terms that satisfy the unitarity and stability conditions independently,

$$L_1 = i\mathcal{M}_a\mathcal{M}_a^* + \frac{i\beta}{2}[\mathcal{M}_a\partial_t(M_r^*U_r) - \mathcal{M}_a^*\partial_t(M_rU_r^*)], \quad (62)$$

$$L_2 = \frac{i}{2}[\mathcal{M}_a^2 + (\mathcal{M}_a^*)^2] - \frac{i\beta}{2}[\mathcal{M}_a\partial_t(M_rU_r^*) - \mathcal{M}_a^*\partial_t(M_r^*U_r)], \quad (63)$$

$$L_3 = \mathcal{M}_a\partial_t(M_r^*U_r) + \mathcal{M}_a^*\partial_t(M_rU_r^*). \quad (64)$$

Here, pure imaginaries in $L_{1,2}$ for later purposes. Therefore, a general KMS-invariant action consistent with unitarity is an arbitrary linear combination of $L_{1,2,3}$ with positive real coefficients. The L_3 is linear in a-type fields, and hence real and nondissipative. When we use the physical value of $M_r = M$, $m_a = 0$, and also $\phi_a = \phi_a^*$ for U(1), this term vanishes identically,

$$|M|^2\phi_a\partial_t(U_rU_r^*) = 0, \quad (65)$$

where we used the U(1) property, $U_rU_r^* = 1$. We will not further discuss about this term in the following. On the other hand, the $L_{1,2}$ contains the imaginary valued terms quadratic in a fields, and they are dissipative.

Under the shift symmetry under which U has charge +1, L_1 is invariant, but L_2 is not. Therefore, in the case the U(1) symmetry is not spontaneously broken, we impose the invariance under the shift symmetry, L_2 is not allowed in the action, and the action contains the following terms in general, with a kinetic coefficient γ_1 ,

$$\gamma_1\beta^{-1}L_1 = \frac{i\gamma_1}{2}[\mathcal{M}_a\partial_t(M_r^*U_r) - \mathcal{M}_a^*\partial_t(M_rU_r^*)] + i\gamma_1\beta^{-1}\mathcal{M}_a\mathcal{M}_a^*. \quad (66)$$

The first piece, which is linear in a-type fields, contributes to the classical EOM for the U(1) charge density and the second piece represents the corresponding noise according to the fluctuation-dissipation theorem. To understand the physics meaning of these terms, we first note that the ordinary time derivative is equal to the gauge covariant derivative for a gauge-invariant term, so the first piece can be written as

$$\gamma_1\beta^{-1}L_1 = \frac{i\gamma_1}{2}[\mathcal{M}_aD_t(M_r^*U_r^T) - \mathcal{M}_a^*D_t(M_rU_r^*)] + i\gamma_1\beta^{-1}\mathcal{M}_a\mathcal{M}_a^*. \quad (67)$$

Using the Leibniz rule for the covariant derivative and assuming that the covariant derivative of M_r vanishes,⁷ we

⁷This is a reasonable assumption based on the fact that a dynamical M field, e.g., the Higgs field in the Standard Model, would adjust itself to satisfy this condition in a more fundamental theory. A nonvanishing D_tM can act as an external source or sink of the symmetry charge, which we assume is absent.

arrive at

$$\gamma_1 \beta^{-1} L_1 = -\gamma_1 |M_r|^2 \phi_a B_{rt} + i\gamma_1 |M_r|^2 \beta^{-1} \phi_a^2, \quad (68)$$

where we have set $m_a = 0$ and used the fact that $\phi_a^* = \phi_a$. Note also that $B_{rt} = iU_r D_t U_r^*$ for U(1). The EOM from this can be written as a violation of the charge conservation law with stochastic noise variable ξ ,

$$\partial_\mu J_r^\mu = -\gamma_1 |M_r|^2 \mu + \xi, \quad (69)$$

where

$$\langle \xi(x) \xi(x') \rangle = 2\gamma_1 \beta^{-1} |M_r|^2 \delta^{(4)}(x - x'). \quad (70)$$

We recalled the identification of the chemical potential $\mu = B_{rt}$. This is a stochastic charge relaxation equation, which gives a linear relaxation of charges with a relaxation rate $\Gamma_1 = \gamma_1 |M_r|^2 / \chi = \gamma_1 M^2 / \chi$ when $k \rightarrow 0$. Note that $L_{1,2,3}$ terms do not modify the symmetry current, J_r^μ computed by (18), since they do not contain the gauge fields. As (68) is the leading order mechanism of charge relaxation in the phase where the symmetry is not spontaneously broken, we find that the charge relaxation in this phase is of second order in M .

In the SSB phase, both L_1 and L_2 are allowed, and the action contains in general,

$$\gamma_1 \beta^{-1} L_1 + \gamma_2 \beta^{-1} L_2. \quad (71)$$

The L_2 term is explicitly given by

$$\begin{aligned} \gamma_2 \beta^{-1} L_2 = & -\frac{\gamma_2}{2} [(M_r^* U_r)^2 + (U_r^* M_r)^2] \phi_a B_{rt} \\ & + \frac{i}{2} \gamma_2 \beta^{-1} [(M_r^* U_r)^2 + (U_r^* M_r)^2] \phi_a^2. \end{aligned} \quad (72)$$

We note that the stability condition, i.e., the positivity of the imaginary part of the action, imposes a condition, $\gamma_1 > |\gamma_2|$ for arbitrary ϕ_r with $U = e^{i\phi_r}$ and the physical mass $M_r = M$. Linearizing in ϕ_r , we find

$$\gamma_2 \beta^{-1} L_2 = -\gamma_2 M^2 \phi_a B_{rt} + i\gamma_2 \beta^{-1} M^2 \phi_a^2 + \mathcal{O}(\phi_r^2), \quad (73)$$

which takes the same form as the L_1 term, that is, the effect of L_2 is identical to that of L_1 for linearized dynamics of ϕ_r , although they are different at the nonlinear level.

The conservation law, or the EOM, becomes

$$\begin{aligned} \partial_\mu J_r^\mu = & i\sigma M_r (U_r - U_r^*) - \gamma_1 |M_r|^2 B_{rt} \\ & - \frac{\gamma_2}{2} [(M_r^* U_r)^2 + (U_r^* M_r)^2] B_{rt} + \xi, \end{aligned} \quad (74)$$

with the noise correlation

$$\langle \xi(x) \xi(x') \rangle = 2T \left\{ \gamma_1 |M_r|^2 + \frac{\gamma_2}{2} [(M_r^* U_r)^2 + (U_r^* M_r)^2] \right\} \delta^{(4)}(x - x') \simeq 2\gamma T M^2 \delta^{(4)}(x - x'), \quad (75)$$

where $\gamma \equiv \gamma_1 + \gamma_2$ and we have linearized in ϕ_r field and used the physical value $M_r = M$ in the last line. The linearized dispersion relation for ϕ_r , including the effects from $L_{1,2}$ is found to be

$$\omega(k) = \pm \sqrt{v_s^2 k^2 + m_{\text{NG}}^2} - \frac{i}{2} \left[\frac{\lambda k^2}{\chi} + \chi \kappa (v_s^2 k^2 + m_{\text{NG}}^2) + \frac{\chi \gamma m_{\text{NG}}^4}{4\sigma^2} \right]. \quad (76)$$

The damping effect caused by L_1 and L_2 , $\chi \gamma m_{\text{NG}}^4 / (4\sigma^2) = \gamma M^2 / \chi$ is subleading compared to the same effect from $\chi \kappa m_{\text{NG}}^2 = \kappa \sigma M$, which is linear in M as discussed before.

IV. AN EXTENSION TO SU(N)

The extension to the non-Abelian group SU(N) is straightforward without much modification, and our

notations for the discussion of U(1) case in the previous section are already chosen for them to be applicable to SU(N), e.g., our use of ϕ_a^* , as long as we understand these fields as matrix-valued fields taking values in the SU(N) algebra. As $\phi_a^\dagger$ and ϕ_a^* are different for the non-Abelian fields, we will also use in this section the hermitian conjugates $\phi_a^\dagger$ and $m_a^\dagger$, whose KMS transformations are

$$\phi_a^\dagger(t) \rightarrow -\phi_a^\dagger(-t) + \beta U_r^\dagger (\partial_t U_r^*)(-t), \quad m_a^\dagger(t) \rightarrow -m_a^\dagger(-t) + \beta M_r^\dagger [\partial_t (M_r^\dagger)^{-1}](-t). \quad (77)$$

The SU(N) symmetric action up to quadratic in B fields is the same as in the U(1) case (30), with additional trace applied to all terms,

$$\mathcal{L}_{\text{SU}(N)}^{(2)} = 2\text{tr}[\chi B_{ai}B_{ri} - f_\pi^2 B_{ai}B_{ri} - \chi^2 \kappa B_{ai}(\partial_t B_{ri} - i\beta^{-1} B_{ai}) - \lambda B_{ai}(\partial_t B_{ri} - i\beta^{-1} B_{ai})], \quad (78)$$

where we use the fundamental representation with $\text{tr}(t_a t_b) = \delta_{ab}/2$. Each term is invariant under the full shift symmetry, except the second term, which is allowed only when there is no shift symmetry, i.e., the $\text{SU}(N)$ symmetry is spontaneously broken.

More general structures are possible for all terms: different t_a may have different values of χ , f_π^2 , and the kinetic coefficients, κ and λ . For this general case, χ and f_π^2 , etc., become a matrix, depending on the detail of the SSB pattern. Moreover, if the symmetry is partially broken spontaneously, i.e., $\text{SU}(N) \rightarrow H \subset \text{SU}(N)$, then the restricted shift symmetry in H may require other structures, e.g., $2\text{tr}(f_\pi^2 B_{ai} P B_{ri})$, where P is the projection operator to the subspace $\text{SU}(N)/H$ in the group algebra, so that the propagating NG bosons exist only for $\text{SU}(N)/H$. We will not discuss these general possibilities and consider only the simplest case with $H \ni \{\emptyset\}$ in this section. We consider nontrivial H for a specific case of chiral symmetry breaking in the next section.

We define the matrix valued chemical potential as $\mu = U_r^\dagger B_{ri} U_r = i(D_t U_r^\dagger) U_r$, which can also be written in terms of generators as $\mu = \mu_a t_a$, and $\mu_a = 2\text{tr}(t_a \mu)$. Note that we distinguish the subscripts “a” and “r” for the a-type fields and the $\text{SU}(N)$ index introduced here, respectively. Recall that (16) and (17), we take the variations with respect to $A_{a\mu}$ and find the matrix-valued current,

$$\begin{aligned} J_r^t &= \chi \mu - \chi^2 \kappa D_t \mu + \xi_\kappa, \\ J_r^i &= -f_\pi^2 U_r^\dagger B_{ri} U_r - \lambda(D_i \mu - E_i) + \xi^i, \end{aligned} \quad (79)$$

where $D_\mu \mu = \partial_\mu \mu - i[A_{r\mu}, \mu]$ and $E_i = \partial_t A_{ri} - \partial_i A_{rt} - i[A_{rt}, A_{ri}]$, and we used $[D_t, D_i] U_r^\dagger = -iE_i U_r^\dagger$. The noise correlations are

$$\begin{aligned} \langle \xi_\kappa^a(x) \xi_\kappa^b(x') \rangle &= 2\chi^2 \kappa T \delta^{ab} \delta^{(4)}(x - x'), \\ \langle \xi^{ai}(x) \xi^{bj}(x') \rangle &= 2\lambda T \delta^{ab} \delta^{ij} \delta^{(4)}(x - x'), \end{aligned} \quad (80)$$

where $\xi_\kappa = \xi_\kappa^a t_a$ and $\xi^i = \xi^{ia} t_a$. The EOM for the classical field takes the form of the conservation law,

$$D_\mu J_r^\mu \equiv \partial_\mu J_r^\mu - i[A_{r\mu}, J_r^\mu] = 0. \quad (81)$$

Note that the gauge-dependent term in the covariant derivative arises from the gauge-field dependence in $B_{a\mu}$ through (17).

In the phase where the symmetry is not spontaneously broken, we have $f_\pi^2 = 0$, and each μ_a is the independent dynamical degree of freedom, and we have a non-Abelian version of charge diffusion dynamics, where external non-Abelian gauge fields couple different μ_a 's. Readers who are

interested in the case shall refer to Ref. [36]. On the other hand, in the phase where the symmetry is spontaneously broken, we write $U_r = e^{i\phi_r}$ in terms of the NG modes $\phi_r = \phi_r^a t_a$, which become propagating modes. The linearized EOM for ϕ_r^a without external gauge fields and the dispersion relation is identical to the U(1) case for each generator t_a independently, but a non-Abelian external gauge field would couple them together.

Let us now introduce an explicit symmetry-breaking parameter M , which we assume is a matrix transforming in the same representation of the U field as given by (41). The KMS transformation is what is given in the previous section, Eqs. (42) and (43), whereas in the present case, the matrix structure of M is understood. To the first order of M_r , it can be shown by a similar argument in the U(1) case that there is no action that is consistent with the shift symmetry, and the effect of M_r in the symmetry restored phase appears in the second order in M_r .

In the SSB phase, the unique KMS-invariant action at leading derivatives is similar to (45),

$$\mathcal{L}_M = -i\sigma \text{tr} \mathcal{M}_a + \text{c.c.} \simeq -2\sigma M \text{tr}(\phi_a \phi_r), \quad (82)$$

where $\mathcal{M}_a$ is given by (46) and we assume a physical value $M_r = M1$ and linearized the fields in the last expression. The EOM is modified as

$$D_\mu J_r^\mu = -\frac{i\sigma}{2} (U_r^\dagger M_r - M_r^\dagger U_r)_P \simeq -\sigma M \phi_r \equiv -\chi m_\pi^2 \phi_r. \quad (83)$$

Here, $X_P \equiv 2\text{tr}(t^a X) t^a$ denotes the projection of a matrix X to the group space, and the mass of the NG boson, $m_\pi^2 = \sigma M / \chi$. The linearized dispersion relation for each ϕ_r^a is again identical to the U(1) case (49).

We next discuss possible terms in the second order of M_r . Following the same steps in Sec. III B 2 for the U(1) case, we consider a KMS-invariant term $\text{tr}[F(KF)]$, with the most general form of F ,

$$F = c_1 \mathcal{M}_a + c_2 \mathcal{M}_a^T + c_3 \mathcal{M}_a^* + c_4 \mathcal{M}_a^\dagger, \quad (84)$$

where $\mathcal{M}_a$ as well as $\mathcal{M}_a^T$, $\mathcal{M}_a^*$, $\mathcal{M}_a^\dagger$, are all gauge-invariant matrices.⁸ We have the KMS transformation for each operator:

$$\begin{aligned} K \mathcal{M}_a &= -\mathcal{M}_a^* - \beta \partial_t (M_r^* U_r^T), \\ K \mathcal{M}_a^T &= -\mathcal{M}_a^\dagger - \beta \partial_t (U_r M_r^\dagger), \end{aligned} \quad (85a)$$

⁸One can also consider a KMS-invariant term of the form, $\text{tr}(F) \text{tr}(KF)$. However, due to $\text{tr}(t_a) = 0$ for $\text{SU}(N)$, this structure does not produce the interesting terms responsible for a linearized charge relaxation.

$$\begin{aligned} K\mathcal{M}_a^* &= -\mathcal{M}_a + \beta\partial_t(M_r U_r^\dagger), \\ K\mathcal{M}_a^\dagger &= -\mathcal{M}_a^\dagger + \beta\partial_t(U_r^* M_r^\dagger). \end{aligned} \quad (85b)$$

It can be shown that $\text{tr}[F(KF)]$ is a linear combination of the following six independent KMS-invariant terms consistent with unitarity and stability,

$$L_1 = -\frac{i}{2}\text{tr}[\mathcal{M}_a(K\mathcal{M}_a^\dagger) + \mathcal{M}_a^\dagger(K\mathcal{M}_a)], \quad (86a)$$

$$L_2 = -\frac{i}{2}\text{tr}[\mathcal{M}_a(K\mathcal{M}_a^*) + \mathcal{M}_a^\dagger(K\mathcal{M}_a^\dagger)], \quad (86b)$$

$$L_3 = -\beta^{-1}\text{tr}[\mathcal{M}_a(K\mathcal{M}_a^\dagger) - \mathcal{M}_a^\dagger(K\mathcal{M}_a^*)], \quad (86c)$$

$$L_4 = -\frac{i}{2}\text{tr}[\mathcal{M}_a(K\mathcal{M}_a) + \mathcal{M}_a^\dagger(K\mathcal{M}_a^\dagger)], \quad (86d)$$

$$L_5 = -\frac{i}{2}\text{tr}[\mathcal{M}_a(K\mathcal{M}_a^\dagger) + \mathcal{M}_a^\dagger(K\mathcal{M}_a)], \quad (86e)$$

$$L_6 = -\beta^{-1}\text{tr}[\mathcal{M}_a(K\mathcal{M}_a) - \mathcal{M}_a^\dagger(K\mathcal{M}_a^\dagger)], \quad (86f)$$

which means that a general KMS-invariant term satisfying the unitarity and stability conditions is a linear combination of L_n 's with positive real coefficients. The L_3 and L_6 are purely real and nondissipative,

$$L_3 = \text{tr}[\mathcal{M}_a\partial_t(U_r M_r^\dagger) + \mathcal{M}_a^\dagger\partial_t(M_r U_r^\dagger)], \quad (87a)$$

$$L_6 = \text{tr}[\mathcal{M}_a\partial_t(M_r^* U_r^\dagger) + \mathcal{M}_a^\dagger\partial_t(U_r^* M_r^\dagger)], \quad (87b)$$

which are higher orders in M and ∂_t relative to $\mathcal{L}_M$, so that we will not discuss about them. Among the remaining dissipative terms, only L_1 is invariant under the full shift symmetry, which transforms $U_r \rightarrow \Lambda(\mathbf{x})U_r$ and $\mathcal{M}_a \rightarrow \mathcal{M}_a\Lambda^\dagger(\mathbf{x})$,

$$L_1 = \text{tr}\left\{i\mathcal{M}_a\mathcal{M}_a^\dagger + \frac{i\beta}{2}[\mathcal{M}_a\partial_t(U_r M_r^\dagger) - \mathcal{M}_a^\dagger\partial_t(M_r U_r^\dagger)]\right\}. \quad (88)$$

The next interesting term is L_2 ,

$$\begin{aligned} L_2 = \text{tr}\left\{\frac{i}{2}[\mathcal{M}_a^2 + (\mathcal{M}_a^\dagger)^2] \right. \\ \left. - \frac{i\beta}{2}[\mathcal{M}_a\partial_t(M_r U_r^\dagger) - \mathcal{M}_a^\dagger\partial_t(U_r M_r^\dagger)]\right\}, \end{aligned} \quad (89)$$

which is not invariant under the shift symmetry, but its effect is independent of the generator t_a , i.e., all ϕ_r^a receive the same effect from this term. On the other hand, L_4 and L_5 , which contain terms that are proportional to $\text{tr}[\mathcal{M}_a(\mathcal{M}_a^*)]$ and $\text{tr}[\mathcal{M}_a(\mathcal{M}_a^\dagger)]$, respectively, give in general different contributions for different group directions since $t_a^* = t_a^\dagger$ is not universally related to t_a , unless the representation is real, e.g., $\text{SU}(2)$ where $t_a = \sigma_a/2$ and $t_a^* = -\sigma_2 t_a \sigma_2$. These terms may be possible if the pattern of the SSB is such as to allow a nonuniversality among different group directions, but we will consider only L_1 and L_2 here for simplicity.

In the symmetry restored phase, we have only L_1 in the action, and using the condition $D_r M_r = 0$ for physical value, we find

$$2\gamma_1\beta^{-1}L_1 = -\gamma_1\text{tr}(\phi_a\{M_r^\dagger M_r, \mu\}) + 2i\gamma_1\beta^{-1}\text{tr}(\phi_a^2 M_r^\dagger M_r), \quad (90)$$

where $\{A, B\} = AB + BA$. The EOM from this is

$$D_\mu J_r^\mu = -\frac{\gamma_1}{2}\{M_r^\dagger M_r, \mu\}_{\mathcal{P}} + \xi, \quad (91)$$

which describes stochastic relaxation dynamics for the dynamical degree of freedom μ . Here, the noise $\xi = \xi^a t_a$ has the correlation

$$\langle \xi^a(x)\xi^b(x') \rangle = 2\gamma_1 T \text{tr}(\{t_a, t_b\} M_r^\dagger M_r) \delta^{(4)}(x - x'). \quad (92)$$

In the SSB phase, the L_2 term is also present,

$$\begin{aligned} 2\gamma_2\beta^{-1}L_2 = & -\gamma_2\text{tr}[\phi_a(U_r^\dagger M_r \mu U_r^\dagger M_r + M_r^\dagger U_r \mu M_r^\dagger U_r)] \\ & + i\gamma_2\beta^{-1}\text{tr}(\phi_a U_r^\dagger M_r \phi_a U_r^\dagger M_r \\ & + \phi_a M_r^\dagger U_r \phi_a M_r^\dagger U_r). \end{aligned} \quad (93)$$

The stability condition gives a constraint $\gamma_1 > |\gamma_2|$, and the EOM is

$$\begin{aligned} D_\mu J_r^\mu = & -\frac{i\sigma}{2}(M_r U_r^\dagger - U_r M_r^\dagger)_{\mathcal{P}} - \frac{\gamma_1}{2}\{M_r^\dagger M_r, \mu\}_{\mathcal{P}} \\ & - \frac{\gamma_2}{2}(U_r^\dagger M_r \mu U_r^\dagger M_r + M_r^\dagger U_r \mu M_r^\dagger U_r)_{\mathcal{P}} + \xi, \end{aligned} \quad (94)$$

with the noise $\xi = \xi^a t_a$, where

$$\langle \xi^a(x)\xi^b(x') \rangle = 2T[\gamma_1\text{tr}(\{t_a, t_b\} M_r^\dagger M_r) + \gamma_2\text{tr}(t_a U_r^\dagger M_r t_b U_r^\dagger M_r + t_a M_r^\dagger U_r t_b M_r^\dagger U_r)] \delta^{(4)}(x - x'). \quad (95)$$

If we assume $M_r = M\mathbf{1}$, the linearized dispersion relation for ϕ_r^a is independent of t_a and has precisely the same form as in the U(1) case, (76).

V. THE CASE OF CHIRAL SYMMETRY $SU(N)_L \times SU(N)_R$

We now discuss our main application of the developed formalism: the chiral symmetry of QCD, $SU(N)_L \times SU(N)_R$. We introduce the two group fields, U_L and U_R , which transform under the chiral symmetry as

$$U_L \rightarrow U_L V_L^\dagger, \quad U_R \rightarrow U_R V_R^\dagger, \quad (96)$$

where $V_{L/R} \in SU(N)_{L/R}$. In the classical limit, we have the r- and a-type fields, (U_{Lr}, U_{Rr}) and (ϕ_{La}, ϕ_{Ra}) , respectively for each $SU(N)_L$ and $SU(N)_R$. These are simply two copies of the $SU(N)$ case in the previous section. We also have the external gauge fields $A_{L\mu}$ and $A_{R\mu}$, and the gauge-invariant B fields as before,

$$\begin{aligned} B_{Lr\mu} &= iU_{Lr}(D_\mu U_{Lr}^\dagger) = -i(D_\mu U_{Lr})U_{Lr}^\dagger, \\ B_{La\mu} &= U_{Lr}(D_\mu \phi_{La})U_{Lr}^\dagger, \end{aligned} \quad (97)$$

and similarly for $B_{Rr\mu}$ and $B_{Ra\mu}$. Recall that the gauge transformations for the group fields are

$$M_r \rightarrow V_{Lr} M_r V_{Rr}^\dagger, \quad m_{La} \rightarrow V_{Lr}(m_{La} - v_{La})V_{Lr}^\dagger, \quad m_{Ra} \rightarrow V_{Rr}(m_{Ra} - v_{Ra})V_{Rr}^\dagger. \quad (101)$$

The KMS transformation for the M fields is $M_r(t) \rightarrow M_r^*(-t)$ and

$$m_{La}(t) \rightarrow -m_{La}^*(-t) - \frac{\beta}{2}(\partial_t M_r^*)(M_r^*)^{-1}(-t), \quad m_{Ra}(t) \rightarrow -m_{Ra}^*(-t) + \frac{\beta}{2}(M_r^*)^{-1}(\partial_t M_r^*)(-t), \quad (102)$$

while the KMS transformation for $U_{L/R}$ fields is the same as in the previous section,

$$U_{Lr}(t) \rightarrow U_{Lr}^*(-t), \quad \phi_{La}(t) \rightarrow -\phi_{La}^*(-t) + \beta U_{Lr}^T(\partial_t U_{Lr}^*)(-t), \quad (103)$$

and similarly for the right-handed fields.

We first discuss the symmetric fluid at the zeroth order in M_r . If the chiral symmetry is not spontaneously broken, we impose the full $SU(N)_L \times SU(N)_R$ chiral shift symmetry,

$$U_{Lr} \rightarrow \Lambda_L U_{Lr}, \quad U_{Rr} \rightarrow \Lambda_R U_{Rr}, \quad (104)$$

and the action at quadratic order in derivatives should be a simple sum of two independent $SU(N)$ actions in the previous section, and the dynamics of left and right chiral symmetries decouple. Since the QCD interactions are identical for the left- and right-handed quarks, we additionally impose a $\mathbb{Z}_2$ symmetry on the action between $U_L \leftrightarrow U_R$. Then, the parameters in the action for the

$$U_{Lr} \rightarrow U_{Lr} V_{Lr}^\dagger, \quad \phi_{La} \rightarrow V_{Lr}(\phi_{La} - v_{La})V_{Lr}^\dagger, \quad (98)$$

and similarly for U_{Rr} and ϕ_{Ra} .

The explicit symmetry-breaking parameter we consider is the quark mass matrix M , which transforms under the chiral symmetry as

$$M \rightarrow V_L M V_R^\dagger, \quad (99)$$

and in the SK contour in the classical limit, we have the r field and a field, (M_r, m_{La}, m_{Ra}) , defined by

$$\begin{aligned} M_1 &= \left(1 - \frac{i\hbar}{2} m_{La}\right) M_r \left(1 + \frac{i\hbar}{2} m_{Ra}\right) + \dots, \\ M_2 &= \left(1 + \frac{i\hbar}{2} m_{La}\right) M_r \left(1 - \frac{i\hbar}{2} m_{Ra}\right) + \dots. \end{aligned} \quad (100)$$

The (m_{La}, m_{Ra}) are not independent, and only the combination $(m_{La} M_r - M_r m_{Ra})$ is meaningful, and the action we construct later will contain only this combination. This way of splitting will be useful to see the gauge-invariant structure of the action terms later, as the gauge transformation can be chosen to be

left- and right-handed chiral symmetries should be identical to each other.

The more interesting case is the phase with the SSB to the vector part of the chiral symmetry. To figure out the corresponding partial shift symmetry that we need to impose on the action, it is helpful to think of the condensate responsible for the symmetry breaking as being proportional to the group field $\Sigma \equiv U_L^\dagger U_R$, which transforms under the chiral symmetry as the usual group field in the chiral effective theory

$$\Sigma \rightarrow V_L \Sigma V_R^\dagger. \quad (105)$$

We want our effective theory to contain similar terms that appear in the chiral effective theory constructed in terms of

Σ , which means that Σ should be invariant under the partial shift symmetry we impose on the fields. This fixes the partial shift symmetry to be the vector-like one with $\Lambda_L = \Lambda_R = \Lambda(\mathbf{x})$, i.e., $U_{L/R} \rightarrow \Lambda(\mathbf{x})U_{L/R}$ with the same $\Lambda(\mathbf{x})$, under which Σ is indeed invariant. The gauge-invariant B fields transform under the shift symmetry as

$$(B_{Lr}, B_{Rr}, B_{La\mu}, B_{Ra\mu}) \rightarrow \Lambda(\mathbf{x})(B_{Lr}, B_{Rr}, B_{La\mu}, B_{Ra\mu})\Lambda^\dagger(\mathbf{x}), \quad (106)$$

$$(B_{Lri}, B_{Rri}) \rightarrow \Lambda(\mathbf{x})(B_{Lri}, B_{Rri})\Lambda^\dagger(\mathbf{x}) + i\Lambda(\mathbf{x})\partial_i\Lambda^\dagger(\mathbf{x}). \quad (107)$$

Note that the axial B field, $B_{Ari} \equiv B_{Lri} - B_{Rri} = iU_{Lr}D_i\Sigma U_{Rr}^\dagger$, transforms covariantly under the shift symmetry, and the most general action respecting all symmetries is

$$\mathcal{L}_0 = \frac{1}{2}\text{tr} \left[\sum_{\alpha=V,A} \chi_\alpha B_{\alpha a} B_{\alpha r} - f_\pi^2 B_{Aa} B_{Ari} - \sum_{\alpha=V,A} \chi_\alpha^2 \kappa_\alpha B_{\alpha a} (\partial_t B_{\alpha r} - i\beta^{-1} B_{\alpha a}) - \sum_{\alpha=V,A} \lambda_\alpha B_{\alpha a} (\partial_t B_{\alpha r} - i\beta^{-1} B_{\alpha a}) \right], \quad (108)$$

where we also defined $B_{Aa\mu} \equiv B_{La\mu} - B_{Ra\mu}$, and the vector-like B fields, $B_{Vr\mu} \equiv B_{Lr\mu} + B_{Rr\mu}$, $B_{Va\mu} \equiv B_{La\mu} + B_{Ra\mu}$, and $\chi_{V/A}$ are vector/axial susceptibilities, respectively. The vector-like symmetry sector remains as a simple charge diffusion system, while the axial sector, where f_π^2 -term is nonzero, is described by the propagating NG pions, $U_{Lr}^\dagger = U_{Rr} = e^{i\phi_r}$, with a velocity v_s^2 . The linearized dispersion relation for $\phi_r = \phi_r^a t_a$ is identical to the U(1) and SU(N) case in the previous sections. The axial part of the action can also be related to the more conventional chiral field $\Sigma = U_L^\dagger U_R$. Expanding $\Sigma_{1,2} = \Sigma_r \pm i\hbar/2(\Sigma_r \phi_{Ra} - \phi_{La} \Sigma_r) \equiv \Sigma_r \pm \hbar/2\Sigma_a$ with $\Sigma_r = U_{Lr}^\dagger U_{Rr}$, it can be shown that the classical limit of the conventional chiral effective theory reduces to the leading derivative term

of the axial part in the above action ($v_s = 1$),

$$\lim_{\hbar \rightarrow 0} \frac{1}{\hbar} [\text{tr}(D_\mu \Sigma_1^\dagger D_\mu \Sigma_1 - D_\mu \Sigma_2^\dagger D_\mu \Sigma_2)] = \text{tr}(D_\mu \Sigma_r^\dagger D_\mu \Sigma_a + D_\mu \Sigma_a^\dagger D_\mu \Sigma_r) = \text{tr}(B_{Aa\mu} B_{Ar\mu}). \quad (109)$$

We define the left-handed chemical potentials as in the SU(N) case,

$$\mu_L = U_{Lr}^\dagger B_{Lr} U_{Lr} = iD_t U_{Lr}^\dagger U_{Lr}, \quad (110)$$

similarly for μ_R . Note that $\Sigma_r^\dagger D_t \Sigma_r = -i(\Sigma_r^\dagger \mu_L \Sigma_r - \mu_R)$ and $D_t \Sigma_r \Sigma_r^\dagger = -i(\mu_L - \Sigma_r \mu_R \Sigma_r^\dagger)$. The chiral currents obtained from variations with respect to external gauge fields are

$$J_{Lr}^i = \frac{1}{4} [\chi_V(\mu_L + \Sigma_r \mu_R \Sigma_r^\dagger) + \chi_A(\mu_L - \Sigma_r \mu_R \Sigma_r^\dagger) - \chi_V^2 \kappa_V (D_t \mu_L + \Sigma_r D_t \mu_R \Sigma_r^\dagger) - \chi_A^2 \kappa_A (D_t \mu_L - \Sigma_r D_t \mu_R \Sigma_r^\dagger)] + \frac{1}{2} U_{Lr}^\dagger (\xi_{\kappa V} + \xi_{\kappa A}) U_{Lr}, \quad (111a)$$

$$J_{Rr}^i = \frac{1}{4} [\chi_V(\mu_R + \Sigma_r^\dagger \mu_L \Sigma_r) + \chi_A(\mu_R - \Sigma_r^\dagger \mu_L \Sigma_r) - \chi_V^2 \kappa_V (D_t \mu_R + \Sigma_r^\dagger D_t \mu_L \Sigma_r) - \chi_A^2 \kappa_A (D_t \mu_R - \Sigma_r^\dagger D_t \mu_L \Sigma_r)] + \frac{1}{2} U_{Rr}^\dagger (\xi_{\kappa V} - \xi_{\kappa A}) U_{Rr}, \quad (111b)$$

$$J_{Lr}^i = \frac{1}{4} [if_\pi^2 \Sigma_r D_i \Sigma_r^\dagger - \lambda_V [D_i \mu_L - E_{Li} + \Sigma_r (D_i \mu_R - E_{Ri}) \Sigma_r^\dagger] - \lambda_A [D_i \mu_L - E_{Li} - \Sigma_r (D_i \mu_R - E_{Ri}) \Sigma_r^\dagger]] + \frac{1}{2} U_{Lr}^\dagger (\xi_V^i + \xi_A^i) U_{Lr}, \quad (111c)$$

$$J_{Rr}^i = \frac{1}{4} \{if_\pi^2 \Sigma_r^\dagger D_i \Sigma_r - \lambda_V [D_i \mu_R - E_{Ri} + \Sigma_r^\dagger (D_i \mu_L - E_{Li}) \Sigma_r] - \lambda_A [D_i \mu_R - E_{Ri} - \Sigma_r^\dagger (D_i \mu_L - E_{Li}) \Sigma_r]\} + \frac{1}{2} U_{Rr}^\dagger (\xi_V^i - \xi_A^i) U_{Rr}, \quad (111d)$$

where the thermal noises, $\xi_{\kappa\alpha} = \xi_{\kappa\alpha}^a t_a$ and $\xi_\alpha^i = \xi_\alpha^{ia} t_a$ ($\alpha = V, A$), have the correlation functions,

$$\langle \xi_{\kappa\alpha}^a(x) \xi_{\kappa\alpha}^b(x') \rangle = 2T \chi_\alpha^2 \kappa_\alpha \delta^{ab} \delta^{(4)}(x - x'), \quad (112a)$$

$$\langle \xi_{\alpha}^{ia}(x) \xi_{\alpha}^{jb}(x') \rangle = 2T \lambda_\alpha \delta^{ab} \delta^{ij} \delta^{(4)}(x - x'), \quad (112b)$$

with $(\alpha = V, A)$. The EOM following from the variations of the effective action with respect to ϕ_{La} and ϕ_{Ra} are given by the conservation laws,

$$D_\mu J_{Lr}^\mu = 0, \quad (113)$$

and likewise for the right current J_R .

We next discuss the linear term in M_r , which exists only in the SSB phase. It is the classical limit of the usual mass term in the chiral effective theory, $\text{tr}(U_L M U_R^\dagger) + \text{h.c.} = \text{tr}(M \Sigma^\dagger) + \text{h.c.}$, which is put in the SK contour,

$$\begin{aligned} L_M &= \frac{1}{\hbar} \sigma \text{tr}(M_1 \Sigma_1^\dagger - M_2 \Sigma_2^\dagger) + \text{h.c.} \\ &= -i\sigma \text{tr}\{[M_r(\phi_{Ra} - m_{Ra}) - (\phi_{La} - m_{La})M_r] \Sigma_r^\dagger\} \\ &\quad + \text{h.c.} \end{aligned} \quad (114)$$

Again, we emphasize that this term is excluded by the chiral shift symmetry in the symmetry-restored phase. It is easily shown to be KMS invariant up to the total derivative term

$$-i\sigma\beta\partial_t\{\text{tr}[M_r^\dagger(-t)\Sigma_r(-t)]\} - \text{h.c.} \quad (115)$$

The EOM is modified as

$$\begin{aligned} D_\mu J_{Lr}^\mu &= \frac{i\sigma}{2} (M_r \Sigma_r^\dagger - \Sigma_r M_r^\dagger)_P, \\ D_\mu J_{Rr}^\mu &= \frac{i\sigma}{2} (M_r^\dagger \Sigma_r - \Sigma_r^\dagger M_r)_P. \end{aligned} \quad (116)$$

The linearized dispersion relation for the pions is again

$$\begin{aligned} 2\gamma_1 \beta^{-1} L_1 &= 2\gamma_1 \beta^{-1} \text{tr} \left\{ i\mathcal{M}_a \mathcal{M}_a^\dagger + \frac{i\beta}{2} [\mathcal{M}_a \partial_t (U_{Rr} M_r^\dagger U_{Lr}^\dagger) - \mathcal{M}_a^\dagger \partial_t (U_{Lr} M_r U_{Rr}^\dagger)] \right\} \\ &= -\gamma_1 \text{tr}[\phi_{La} \{M_r M_r^\dagger, \mu_L\} + \phi_{Ra} \{M_r^\dagger M_r, \mu_R\} - 2(\phi_{La} M_r \mu_R M_r^\dagger + \phi_{Ra} M_r^\dagger \mu_L M_r)] \\ &\quad + 2i\gamma_1 \beta^{-1} \text{tr}(\phi_{La}^2 M_r M_r^\dagger + \phi_{Ra}^2 M_r^\dagger M_r - 2\phi_{La} M_r \phi_{Ra} M_r^\dagger). \end{aligned} \quad (121)$$

Therefore, in the phase where chiral symmetry is not spontaneously broken, only the L_1 is present, and it is the leading mechanism of axial charge relaxation. The EOM takes the form

$$D_\mu J_{Lr}^\mu = -\frac{\gamma_1}{2} (\{M_r M_r^\dagger, \mu_L\} - 2M_r \mu_R M_r^\dagger)_P + \xi_L, \quad (122a)$$

$$D_\mu J_{Rr}^\mu = -\frac{\gamma_1}{2} (\{M_r^\dagger M_r, \mu_R\} - 2M_r^\dagger \mu_L M_r)_P + \xi_R, \quad (122b)$$

identical to the $SU(N)$ case with the pion mass $m_\pi^2 = 4\sigma M/\chi_A$, if $Mr = M1$.

Finally, we construct the terms to the second order in M_r . We find that the procedure and the outcome go parallel with those in the $SU(N)$ case. The unique gauge-invariant matrix $\mathcal{M}_a$ that has the necessary properties we discussed in the previous section is

$$\mathcal{M}_a = U_{Lr}[M_r(\phi_{Ra} - m_{Ra}) - (\phi_{La} - m_{La})M_r]U_{Rr}^\dagger, \quad (117)$$

and we consider a most general KMS-invariant structure in the action, $\text{tr}(F(KF))$, where F is a general gauge-invariant matrix given by (84). The KMS transformation for $(\mathcal{M}_a, \mathcal{M}_a^T, \mathcal{M}_a^*, \mathcal{M}_a^\dagger)$ has a very similar structure as in the $SU(N)$ case (85),

$$\begin{aligned} K\mathcal{M}_a &= -\mathcal{M}_a^* - \beta\partial_t(U_{Lr}^* M_r^* U_{Rr}^T), \\ K\mathcal{M}_a^T &= -\mathcal{M}_a^\dagger - \beta\partial_t(U_{Rr} M_r^\dagger U_{Lr}^\dagger), \end{aligned} \quad (118)$$

$$\begin{aligned} K\mathcal{M}_a^* &= -\mathcal{M}_a + \beta\partial_t(U_{Lr} M_r U_{Rr}^\dagger), \\ KF^\dagger &= -\mathcal{M}_a^T + \beta\partial_t(U_{Rr}^* M_r^T U_{Lr}^T), \end{aligned} \quad (119)$$

and the most general KMS-invariant term, which is consistent with the unitarity and stability conditions, is given by a linear combination of the six independent structures, L_i , ($i = 1, \dots, 6$), whose expression in terms of $(\mathcal{M}_a, \mathcal{M}_a^T, \mathcal{M}_a^*, \mathcal{M}_a^\dagger)$ takes the precisely same form as in the $SU(N)$ case (86). Note that $\mathcal{M}_a$ transforms under the full chiral shift symmetry as

$$\mathcal{M}_a \rightarrow \Lambda_L \mathcal{M}_a \Lambda_R^\dagger, \quad (120)$$

and only L_1 is invariant under this, neglecting the non-dissipative L_3 and L_6 ,

with the thermal noise correlations

$$\langle \xi_L^a(x) \xi_L^b(x') \rangle = 2\gamma_1 T \text{tr}(\{t_a, t_b\} M_r M_r^\dagger) \delta^{(4)}(x - x'), \quad (123a)$$

$$\langle \xi_R^a(x) \xi_R^b(x') \rangle = 2\gamma_1 T \text{tr}(\{t_a, t_b\} M_r^\dagger M_r) \delta^{(4)}(x - x'), \quad (123b)$$

and

$$\langle \xi_L^a(x) \xi_R^b(x') \rangle = -4\gamma_1 T \text{tr}(t_a M_r t_b M_r^\dagger) \delta^{(4)}(x - x'). \quad (124)$$

Recall that imposing the full chiral shift symmetry in this phase also dictates that the leading derivative terms in terms of B fields decouple between the left- and right-handed sectors, or equivalently we have $f_\pi^2 = 0$ and $(\chi_V, \kappa_V, \lambda_V) = (\chi_A, \kappa_A, \lambda_A)$. From these, the chiral currents are given only in terms of the chemical potentials, without mixing between the left- and right-handed sectors. Since

the right-hand side of the EOM in the above also involves only the chemical potentials without Σ_r field, the chiral hydrodynamics in this phase is described completely in terms of the chemical potentials $\mu_{L/R}$.

On the other hand, in the SSB phase, we impose only the vector-like shift symmetry, $\mathcal{M}_a \rightarrow \Lambda \mathcal{M}_a \Lambda^\dagger$, we find that L_2 is additionally allowed to appear in the action, while L_4 and L_5 are still not consistent with the shift symmetry,

$$\begin{aligned} 2\gamma_2 \beta^{-1} L_2 &= 2\gamma_2 \beta^{-1} \left(\frac{-i}{2} \right) \text{tr} \{ -\mathcal{M}_a^2 - (\mathcal{M}_a^\dagger)^2 + \beta [\mathcal{M}_a \partial_t (U_{Lr} M_r U_{Rr}^\dagger) - \mathcal{M}_a^\dagger \partial_t (U_{Rr} M_r^\dagger U_{Lr}^\dagger)] \} \\ &= -\gamma_2 \text{tr} [(\phi_{La} M_r - M_r \phi_{Ra}) \Sigma_r^\dagger (\mu_L M_r - M_r \mu_R) \Sigma_r^\dagger + \text{h.c.}] \\ &\quad + i\gamma_2 \beta^{-1} \text{tr} [(M_r^\dagger \phi_{La} - \phi_{Ra} M_r^\dagger) \Sigma_r (M_r^\dagger \phi_{La} - \phi_{Ra} M_r^\dagger) \Sigma_r + \text{h.c.}]. \end{aligned} \quad (125)$$

The full EOM in this phase is

$$\begin{aligned} D_\mu J_{Lr}^\mu &= \frac{i\sigma}{2} (M_r \Sigma_r^\dagger - \Sigma_r M_r^\dagger)_P - \frac{\gamma_1}{2} (\{M_r M_r^\dagger, \mu_L\} - 2M_r \mu_R M_r^\dagger)_P \\ &\quad - \frac{\gamma_2}{2} [M_r \Sigma_r^\dagger (\mu_L M_r - M_r \mu_R) \Sigma_r^\dagger + \Sigma_r (M_r^\dagger \mu_L - \mu_R M_r^\dagger) \Sigma_r M_r^\dagger]_P + \xi_L, \end{aligned} \quad (126a)$$

$$\begin{aligned} D_\mu J_{Rr}^\mu &= \frac{i\sigma}{2} (M_r^\dagger \Sigma_r - \Sigma_r^\dagger M_r)_P - \frac{\gamma_1}{2} (\{M_r^\dagger M_r, \mu_R\} - 2M_r^\dagger \mu_L M_r)_P \\ &\quad - \frac{\gamma_2}{2} [M_r^\dagger \Sigma_r (\mu_R M_r^\dagger - M_r^\dagger \mu_L) \Sigma_r + \Sigma_r^\dagger (M_r \mu_R - \mu_L M_r) \Sigma_r^\dagger M_r]_P + \xi_R, \end{aligned} \quad (126b)$$

and the noise correlation functions are

$$\langle \xi_L^a(x) \xi_L^b(x') \rangle = 2T [\gamma_1 \text{tr}(\{t_a, t_b\} M_r M_r^\dagger) + \gamma_2 \text{tr}(M_r^\dagger t_a \Sigma_r M_r^\dagger t_b \Sigma_r + \Sigma_r^\dagger t_a M_r \Sigma_r^\dagger t_b M_r)] \delta^{(4)}(x - x'), \quad (127a)$$

$$\langle \xi_R^a(x) \xi_R^b(x') \rangle = 2T [\gamma_1 \text{tr}(\{t_a, t_b\} M_r^\dagger M_r) + \gamma_2 \text{tr}(t_a M_r^\dagger \Sigma_r t_b M_r^\dagger \Sigma_r + \Sigma_r^\dagger M_r t_a \Sigma_r^\dagger M_r t_b)] \delta^{(4)}(x - x'), \quad (127b)$$

$$\langle \xi_L^a(x) \xi_R^b(x') \rangle = -2T [\gamma_1 \text{tr}(t_a M_r t_b M_r^\dagger) + \gamma_2 \text{tr}(M_r^\dagger t_a \Sigma_r t_b M_r^\dagger \Sigma_r + \Sigma_r^\dagger M_r t_b \Sigma_r^\dagger t_a M_r)] \delta^{(4)}(x - x'). \quad (127c)$$

Again, the stability condition dictates that $\gamma_1 > |\gamma_2|$.

VI. LINEARIZED DYNAMICS AND THE KUBO FORMULA

In this section, we study the linearized hydrodynamics of the SSB phase of QCD, and derive the Kubo formula for the dissipative kinetic coefficients $\gamma_{1,2}$ that we introduced in the previous section. The linearized dynamics of axial and

vector charges decouple, and the vector charge dynamics is a simple hydrodynamics diffusion of a conserved charge, so we will focus only on the axial charge dynamics which features a relaxation dynamics due to the explicit symmetry breaking by the quark mass matrix that we assume to be $M_r = M \mathbf{1}$ with the positive constant $M > 0$.

For our purpose, we expand the fields and keep only the linearized fluctuations. In the absence of external gauge fields, we obtain

$$\begin{aligned} U_{Lr} &= e^{-i\phi_r} \simeq 1 - i\phi_r, & U_{Rr} &= e^{i\phi_r} \simeq 1 + i\phi_r, & \Sigma_r &= U_{Lr}^\dagger U_{Rr} \simeq 1 + 2i\phi_r, \\ \phi_{La} &= -\phi_{Ra} \equiv -\phi_a, & B_{A\mu} &\simeq -2\partial_\mu \phi_r, & B_{Aa\mu} &\simeq -2\partial_\mu \phi_a, & \mu_A &\equiv (\mu_L - \mu_R)/2 \simeq -\partial_t \phi_r. \end{aligned} \quad (128)$$

It is sufficient to consider the quadratic action in (ϕ_r, ϕ_a) for linearized dynamics. The total effective Lagrangian is $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_M + \mathcal{L}_\gamma$, where

$$\mathcal{L}_0 = 2\text{tr}[\chi_A(\partial_t\phi_a)(\partial_t\phi_r) - f_\pi^2(\partial_t\phi_a)(\partial_t\phi_r) - \chi_A^2\kappa_A(\partial_t\phi_a)(\partial_t^2\phi_r - i\beta^{-1}\partial_t\phi_a) - \lambda_A(\partial_t\phi_a)(\partial_t\partial_t\phi_r - i\beta^{-1}\partial_t\phi_a)], \quad (129a)$$

$$\mathcal{L}_M = -8\sigma M\text{tr}(\phi_a\phi_r), \quad (129b)$$

$$\mathcal{L}_\gamma = -8\gamma M^2\text{tr}(\phi_a\partial_t\phi_r) + 8i\gamma M^2\beta^{-1}\text{tr}(\phi_a^2). \quad (129c)$$

In the chiral symmetry restored phase, we have $f_\pi^2 = \sigma = \gamma_2 = 0$, and the action can be written only in terms of ϕ_a and $\mu_A = -\partial_t\phi_r$. On the other hand, in the SSB phase, all the coefficients are nonvanishing in general, and the pseudo-NG bosons (pions), ϕ_r , become propagating dynamical modes. Note that γ_1 and γ_2 in this phase appear in the combination of γ in the linearized dynamics, and they are distinguishable only in nonlinear dynamics.

Introducing the random noise fields, $\xi_{\kappa A}$, $\xi_A = \xi_A^i$, and $\xi_{\gamma A}$, the $\mathcal{L}_0$ and $\mathcal{L}_\gamma$ can also be written as

$$\begin{aligned} \mathcal{L}_0 = 2\text{tr} \left[\chi_A(\partial_t\phi_a)(\partial_t\phi_r) - f_\pi^2(\partial_t\phi_a)(\partial_t\phi_r) - \chi_A^2\kappa_A(\partial_t\phi_a)(\partial_t^2\phi_r) - \lambda_A(\partial_t\phi_a)(\partial_t\partial_t\phi_r) \right. \\ \left. - \xi_{\kappa A}(\partial_t\phi_a) - \xi_A^i(\partial_t\phi_a) + \frac{i}{2\chi_A^2\kappa_A T}\xi_{\kappa A}^2 + \frac{i}{2\lambda_A T}\xi_A^i\xi_A^i \right], \end{aligned} \quad (130a)$$

$$\mathcal{L}_\gamma = -8\gamma M^2\text{tr}(\phi_a\partial_t\phi_r) - 2\text{tr}(\xi_{\gamma A}\phi_a) + \frac{i}{8\gamma M^2 T}\text{tr}(\xi_{\gamma A}^2). \quad (130b)$$

The path integral of the noise fields reproduces the original action, showing the equivalence. The EOM for ϕ_r field in this formalism takes the form of a stochastic hydrodynamic equation with random thermal noises. Another advantage is that the action becomes linear in the a-type external gauge fields A_a , and therefore there are no contact terms in the correlation functions of the “r”-type charge currents J_r which is conjugate to A_a ; a feature that is shared by the microscopic QCD action.

The EOM can be written as (the violation of) the conservation, or the Ward-Takahashi identity for the (approximate) axial symmetry,

$$\partial_\mu J_A^\mu = 4\sigma M\phi_r - 4\gamma M^2\mu_A + \xi_{\gamma A} \equiv S_A, \quad (131)$$

where the axial current is

$$J_A^i = \chi_A\mu_A - \chi_A^2\kappa_A\partial_t\mu_A + \xi_{\kappa A}, \quad J_A^i = f_\pi^2\partial_i\phi_r - \lambda_A\partial_t\mu_A + \xi_A^i, \quad (132)$$

and we defined the source operator, S_A , for the axial charge violation. In underlying QCD, the source operator would correspond to $S_A^a = iM\bar{q}\gamma^5 t_a q$, where q is the quark field (Dirac spinor) with the group index a . The noise correlations are given by

$$\langle \xi_{\kappa A}^a(x) \xi_{\kappa A}^b(x') \rangle = 2T\chi_A^2\kappa_A\delta^{ab}\delta^{(4)}(x-x'), \quad (133a)$$

$$\langle \xi_A^{ia}(x) \xi_A^{jb}(x') \rangle = 2T\lambda_A\delta^{ij}\delta^{ab}\delta^{(4)}(x-x'), \quad (133b)$$

$$\langle \xi_{\gamma A}^a(x) \xi_{\gamma A}^b(x') \rangle = 8T\gamma M^2\delta^{ab}\delta^{(4)}(x-x'). \quad (133c)$$

In the phase where there is no SSB ($\sigma = 0$ and $\gamma = \gamma_1$), the source-source correlation function at zero spatial momentum is computed to be⁹

$$\begin{aligned} G_{\Pi}^{S_A}(\omega, \mathbf{k} = 0) &\equiv \langle S_A(\omega) S_A(-\omega) \rangle \\ &= 16\gamma_1 M^2 \langle \mu_A \mu_A \rangle + \langle \xi_{\gamma A} \xi_{\gamma A} \rangle \\ &= \frac{8T\gamma_1 M^2 \omega^2 (\chi_A^2 + \chi_A^4 \kappa_A^2 \omega^2 + 4\chi_A^2 \kappa_A \gamma_1 M^2)}{\chi_A^2 \omega^2 + (\chi_A^2 \kappa_A \omega^2 + 4\gamma_1 M^2)^2}. \end{aligned} \quad (134)$$

Here, to get the last line, we have used (133c) and

$$\langle \mu_A(\omega) \mu_A(-\omega) \rangle = \frac{8T\gamma_1 M^2 + 2\omega^2 T\chi_A^2 \kappa_A}{\chi_A^2 \omega^2 + (\chi_A^2 \kappa_A \omega^2 + 4\gamma_1 M^2)^2}, \quad (135)$$

which can be derived using the EOM (131) in the symmetry restored phase with (132) and (133c). Using the final expression of $G_{\Pi}^{S_A}$, (134), we obtain the Kubo formula for γ_1 ,

$$\gamma_1 = \frac{1}{8T} \lim_{\omega \rightarrow 0} \lim_{M \rightarrow 0} \lim_{\mathbf{k} \rightarrow 0} \frac{1}{M^2} G_{\Pi}^{S_A}(\omega, \mathbf{k}), \quad (136)$$

⁹We omit group index for simplicity. The left side is $\langle S_A^a S_A^b \rangle$ and the right side has δ^{ab} factor.

where the order of the two limits, $\omega \rightarrow 0$ and $M \rightarrow 0$, is important. The axial current correlation functions are

$$\begin{aligned}\langle J_A^t(\omega, \mathbf{k}) J_A^t(-\omega, -\mathbf{k}) \rangle &= \frac{2T(\lambda_A k^2 + 4\gamma_1 M^2)[\chi_A^2 + \chi_A^4 \kappa_A^2 \omega^2 + \chi_A^2 \kappa_A(\lambda_A k^2 + 4\gamma_1 M^2)]}{\chi_A^2 \omega^2 + (\chi_A^2 \kappa_A \omega^2 + \lambda_A k^2 + 4\gamma_1 M^2)^2}, \\ \langle J_A^i(\omega, \mathbf{k}) J_A^j(-\omega, -\mathbf{k}) \rangle &= 2T\lambda_A \left[\delta^{ij} - \hat{k}^i \hat{k}^j \frac{\lambda_A k^2 (\chi_A^2 \kappa_A \omega^2 + \lambda_A k^2 + 4\gamma_1 M^2)}{\chi_A^2 \omega^2 + (\chi_A^2 \kappa_A \omega^2 + \lambda_A k^2 + 4\gamma_1 M^2)^2} \right].\end{aligned}\quad (137)$$

We obtain the Kubo formula for the conductivity as usual,

$$\lambda_A = \frac{1}{2T} \lim_{\omega, \mathbf{k}, M \rightarrow 0} \frac{\omega^2}{k^2} G_{\pi}''(\omega, \mathbf{k}), \quad (138)$$

while the Kubo formula for κ_A in this phase requires a higher order term in momentum expansion, which is understandable since κ_A in this phase is a higher order effect in gradient expansion.

In the SSB phase, the pions are propagating modes satisfying the EOM,

$$(\chi_A \partial_t^2 - \chi_A v_s^2 \nabla^2 + \chi_A m_\pi^2 - \chi_A^2 \kappa_A \partial_t^3 - \lambda_A \nabla^2 \partial_t + 4\gamma M^2 \partial_t) \phi_r = \partial_t \xi_{\kappa_A} + \partial_t \xi_A^i - \xi_{\gamma_A}, \quad (139)$$

where $\gamma = \gamma_1 + \gamma_2$, $m_\pi^2 = 4\sigma M/\chi_A$ is the pion mass square, and v_s^2 is the pion velocity square given by (55) with $\chi = \chi_A$. The current correlation functions are computed to be

$$G_{\pi}''(\omega, \mathbf{k}) = \frac{2T\chi_A^2 \{(\lambda_A k^2 + 4\gamma M^2)[1 + \chi_A^2 \kappa_A^2 \omega^2 + \kappa_A(\lambda_A k^2 + 4\gamma M^2)]\omega^2 + \chi_A^2 \kappa_A \omega_k^4\}}{\chi_A^2 (\omega^2 - \omega_k^2)^2 + (\chi_A^2 \kappa_A \omega^2 + \lambda_A k^2 + 4\gamma M^2)^2 \omega^2}, \quad (140)$$

$$G_{\pi}^{ij}(\omega, \mathbf{k}) = 2T \left[\lambda_A \delta^{ij} + k^i k^j \frac{(f_\pi^4 - \lambda_A^2 \omega^2)(\chi_A^2 \kappa_A \omega^2 + \lambda_A k^2 + 4\gamma M^2) + 2\lambda_A \chi_A f_\pi^2 (\omega^2 - \omega_k^2)}{\chi_A^2 (\omega^2 - \omega_k^2)^2 + (\chi_A^2 \kappa_A \omega^2 + \lambda_A k^2 + 4\gamma M^2)^2 \omega^2} \right], \quad (141)$$

where $\omega_k^2 \equiv v_s^2 k^2 + m_\pi^2$. From this, we obtain a simple Kubo formula for κ_A as

$$\kappa_A = \frac{1}{2T\chi_A^2} \lim_{\omega_k \rightarrow 0} \lim_{\omega \rightarrow 0} G_{\pi}''(\omega, \mathbf{k}). \quad (142)$$

Note that $\omega \rightarrow 0$ limit must be taken in advance of $k \rightarrow 0$ or $M \rightarrow 0$ (i.e., $\omega_k \rightarrow 0$), where the order of these last two limits does not matter. Another Kubo formula can also be obtained using the source fluctuation correlation function. We have $G_{\pi}^{S_A}(\omega, \mathbf{k} = 0) = \omega^2 G_{\pi}''(\omega, \mathbf{k} = 0)$ by the Ward-Takahashi identity, and we have the Kubo formula for κ_A in this phase as

$$\kappa_A = \frac{1}{2T\chi_A^2} \lim_{\omega \rightarrow 0} \lim_{M \rightarrow 0} \lim_{k \rightarrow 0} \frac{\omega^2}{m_\pi^4} G_{\pi}^{S_A}(\omega, \mathbf{k}) = \frac{1}{32T\sigma^2} \lim_{\omega \rightarrow 0} \lim_{M \rightarrow 0} \lim_{k \rightarrow 0} \frac{\omega^2}{M^2} G_{\pi}^{S_A}(\omega, \mathbf{k}). \quad (143)$$

The Kubo formula for γ in this phase requires a higher order term in momentum expansion since the effect of γ in this phase is subleading compared to that of κ_A .

VII. CONCLUSION

In this paper, we constructed the low-energy effective theory on the SK contour for systems with $U(1)$, $SU(N)$, and the chiral $SU(N)_L \times SU(N)_R$ symmetries, incorporating their small explicit symmetry-breaking parameters in a fully nonlinear and/or non-Abelian manner. For each symmetry, we investigated relaxation dynamics both in the symmetry-broken and symmetry-restored phases. We carefully analyzed the spurious symmetry transformations

associated with explicit symmetry breaking and identified all possible terms that maintain the gauge invariance when the explicit breaking parameters also transform under spurious symmetry transformations. We found that the relaxation term in the approximate conservation law should be second order in the explicit symmetry-breaking parameter, such as the quark mass in QCD for the chiral symmetry.

There are several future directions possibly stemming from our work. While our EFT approach to the hydrodynamics of approximate symmetries is a powerful method, there are complementary theoretical approaches capturing dynamics in the quasihydrodynamic regime. It would be interesting to compare the present results with

those from other approaches, e.g., the thermodynamic approach based on the second law of thermodynamics [49,50], the statistical mechanical approaches such as the projection operator method [51–53] or the nonequilibrium statistical operator method [54–56], and microscopic approaches based on, e.g., the kinetic theory (see, e.g., Ref. [57] for a recent review) or the holography [58–61]. In these formalisms, the dissipative terms to second order in the quark mass have not been systematically studied. We hope to address the thermodynamic approach in future work. It would also be interesting to take into account the energy-momentum sector that we have neglected in this work.

While we have shown that the relaxation term is forced to appear only at higher order in the small explicit symmetry-breaking parameter—where the spurious method is theoretically well justified (e.g., for the light-quark mass in the QCD case)—it would also be interesting to investigate situations in which the explicit breaking is not small and the relaxation term could play a more prominent role. Incorporating this relaxation term into

fluctuating hydrodynamic simulations (see recent developments, e.g., Refs. [62,63]) and testing its phenomenological impact would be a natural direction for future work.

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DATA AVAILABILITY

No data were created or analyzed in this study.

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