

Enhancement of the photon emission rate near the QCD critical point

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We compute photon emission rate enhancement near the QCD critical point using an effective theory of dynamic critical phenomena and derive a universal photon spectrum. The emission rate scales similar to conductivity, increasing with the correlation length (ξ), diverging at the critical point. The spectrum exhibits $\omega dN_\gamma/d^3k \propto \omega^{-1/2}$ in the scaling regime, with the transition occurring at a frequency comparable to shear damping rate $\omega \sim \gamma_\eta/\xi^2$, reflecting the nonequilibrium properties of the near-critical liquid.

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I. INTRODUCTION

In thermodynamics, a critical point represents the end point of a first-order phase transition and plays a crucial role in characterizing the phase structure of physical systems. The properties of these points adhere to universal laws, determined by only a few microscopic theory characteristics such as symmetries and conservation laws. In the study of matter governed by strong interactions, as described by Quantum Chromodynamics (QCD), several critical points have been proposed [1]. Among these, the point predicted in the region of temperatures and densities of order the typical strong interaction scale $\mathcal{O}(100)$ MeV draws the most attention [2,3] due to the possibility of its discovery in heavy-ion collisions [4,5]. The search for this QCD critical point is actively pursued through the Beam Energy Scan programs at the Relativistic Heavy-Ion Collider [6] and is planned for future experimental facilities [7].

Promising signatures of the critical point in relativistic heavy-ion collisions based on the nonmonotonic behavior of fluctuations were proposed [4,8]. These signatures come with challenges and limitations [9]. Among those, the most

obvious one is due to the hydrodynamic expansion inherent in heavy-ion collisions, which hinders the system from achieving full criticality [10]. One of the strategies to overcome this challenge is to analyze higher-order (third- and fourth-order) cumulants of the baryon number, which are expected to be more sensitive to the critical point effects [11]. It is clear that a dynamic modeling of heavy-ion collisions, including the effects of criticality on fluctuations, is needed. Various research groups have been developing corresponding analytical and numerical approaches [12–29]. One might anticipate the emergence of a universal nonequilibrium scaling regime in the slow expansion limit offering a potential framework for studying dynamic critical phenomena [30–33].

Complementary information could be gained through the electromagnetic probes, which experience practically no interaction with strongly interacting matter and thus retain information from the production point. Recent model calculations have indicated a potential divergence in the production rate of dileptons with low energy and momentum. It is attributed to the critical behavior of electric conductivity [34,35], which, in these models, is different from that of Model H expected to describe universal critical dynamics of the QCD critical point [36]. The difference comes from the lack of corresponding effective infrared dynamics in terms of soft modes, obscuring the physical interpretation of such a divergence.

In this paper, we predict the photon emission rate from matter near the QCD critical point without relying on any microscopic models (Fig. 1). Instead, we utilize the universality of the critical fluid, as described by a framework known as Model H in the Hohenberg-Halperin classification [36,37], to calculate the spectrum. We find

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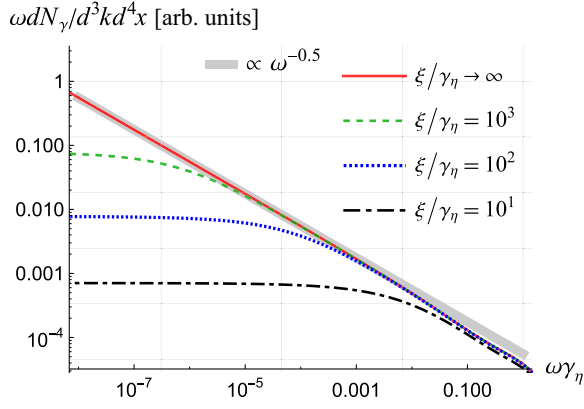


FIG. 1. Photon spectra from critical fluctuations near the critical point for various correlation length ξ/γ_η . The absolute magnitude is proportional to an unknown coefficient; see the later discussion around Eq. (16). We assume that bare transport coefficients satisfy $\bar{\lambda} \equiv K\lambda/\gamma_\eta^3 = 1$.

that it diverges as $\omega dN_\gamma/d^3k \propto \omega^{-1/2}$ at low energies $\omega = k$. Slightly away from the critical point this divergence is regularized to $\omega dN_\gamma/d^3k \propto \xi$ at $\omega \sim \gamma_\eta/\xi^2$. Here $\gamma_\eta = \bar{\eta}/w$ represents the diffusion coefficient of the transverse momentum ($\bar{\eta}$ is the shear viscosity and $w = e + p$ is the enthalpy density) and ξ denotes the correlation length, which diverges at the critical point. This transition occurs when the photon frequency ω matches the shear damping rate of the critical fluid γ_η/ξ^2 , indicating a universal expression in terms of $\nu \equiv \omega\xi^2/\gamma_\eta$ for $\omega \ll T$,

$$\omega \frac{dN_\gamma}{d^3kd^4x} \propto \frac{T^2\xi}{\gamma_\eta} \cdot \frac{1 - (1-\nu)\sqrt{\nu/2}}{1 + \nu^2} = \frac{T^2\xi}{\gamma_\eta} \Phi(\nu). \quad (1)$$

Here, the universal scaling function $\Phi(\nu)$ incorporates frequency and transport coefficients, reflecting the non-equilibrium properties of the critical fluid probed by the on-shell photon, $\omega = k$. This stands in stark contrast to the critical damping rate observed in quasielastic photon scattering experiments [38], which probe the near-static regime $\omega \simeq 0$ with fixed $\mathbf{k}$ and the longitudinal channel instead of the transverse channel of velocity fluctuations. These experiments are beautifully described by the universal function $\Omega_K(y) = \frac{3}{4}y^{-2}[1 + y^2 + (y^3 - y^{-1})\arctan y]$ with $y = k\xi$, known as the Kawasaki function [39]. For comparison with our analysis, a derivation of the Kawasaki function is presented in Appendix A.

This paper is organized as follows. In Sec. II, the Model H, which describes the critical dynamics of the QCD critical point, is reviewed. In Sec. III, the photon emission rate is computed at one-loop level of thermal fluctuations. Since the photon rate is defined not at a static momentum $k^\mu = (0, \mathbf{k})$ but at an on-shell momentum $([\mathbf{k}], \mathbf{k})$, the effect of sound mode is discussed, which is neglected in Model H due to its fast oscillation. In Sec. IV, we summarize the paper.

II. CRITICAL DYNAMICS

Hydrodynamics near the QCD critical point is governed by conserved densities and the order parameter. Because of the mixing of the conserved densities and order parameter fields, it suffices to consider the conserved densities in the description at long time and length scales [36]. Specifically, these are the density fluctuations of baryon number, energy, and momentum. Among these five modes, the slowest mode is the specific entropy $\hat{s} \equiv s/n$ —a diffusive scalar mode whose fluctuations can be represented by a linear combination of the energy (δe) and baryon (δn) density fluctuations $\delta\hat{s} \equiv \delta(s/n) = \frac{1}{nT}\delta e - \frac{e+p}{n^2T}\delta n$. The evolution of $\delta\hat{s}$ is delayed by critical slowing down. The other modes, such as sound and shear modes, evolve much faster than the diffusive mode at long wavelengths. At the nonlinear level, the relevant interaction for the diffusive mode $\delta\hat{s}$, denoted as ψ in Model H, arises from the mode coupling with the shear mode $\mathbf{v}_T$, which is the transverse projection of $\mathbf{v}$. This leads us to critical hydrodynamics, also known as Model H [36].

The Langevin equation of Model H is written as [37]

$$\dot{\psi} = -\frac{1}{w}\nabla \cdot \left(\psi \frac{\delta F}{\delta \mathbf{v}_T} \right) + \lambda \nabla^2 \frac{\delta F}{\delta \psi} + \nabla \cdot \boldsymbol{\zeta}_\psi, \quad (2a)$$

$$w\dot{\mathbf{v}}_T = P_T \left(-\psi \nabla \frac{\delta F}{\delta \psi} + \frac{\bar{\eta}}{w} \nabla^2 \frac{\delta F}{\delta \mathbf{v}_T} + \boldsymbol{\zeta}_v \right), \quad (2b)$$

$$F = \int d^3x K \left[\frac{(\nabla \psi)^2}{2} + \frac{a\psi^2}{2} + \frac{b_3\psi^3}{3} + \frac{b_4\psi^4}{4} \right] + \frac{wv_T^2}{2}. \quad (2c)$$

Here, K, a, b_3, b_4 are parameters of the effective potential, $\lambda = w^2\sigma_B/(n^4T^2)$ is proportional to baryon conductivity σ_B , and $P_T(\mathbf{V})$ denotes the transverse projection of a vector $\mathbf{V}$, i.e., $P_T(\mathbf{V}) \equiv \mathbf{V} - (\mathbf{V} \cdot \mathbf{k})\mathbf{k}/k^2$ in $\mathbf{k}$ space. The property of noises $\boldsymbol{\zeta}_\psi$ and $\boldsymbol{\zeta}_v$ is determined by requiring the fluctuation-dissipation relation:

$$\langle \zeta_{\psi k}(x) \zeta_{\psi l}(x') \rangle = 2T\lambda \delta_{kl} \delta^4(x - x'), \quad (3a)$$

$$\langle \zeta_{vk}(x) \zeta_{vl}(x') \rangle = -2T\bar{\eta} \delta_{kl} \nabla^2 \delta^4(x - x'). \quad (3b)$$

The parameters satisfy $K = n^2T/(c_p a)$ and $a = 1/\xi^2$, where c_p is isobaric specific heat [40]. If we neglect the self-interaction of ψ , $c_p \propto \xi^2$ and thus K approaches a constant at the critical point. The self-interaction modifies the relation to $c_p \propto \xi^{2-\eta} = a^{-1+\eta/2}$, where the scaling exponent is $\eta = \mathcal{O}(\epsilon^2)$ in the ϵ -expansion scheme ($\epsilon = 4 - d$ and d is the spatial dimension). A combination $K\lambda a$ is the (heat) diffusion coefficient, expressed using baryon conductivity by $D = K\lambda a = w^2\sigma_B/(Tn^2c_p)$.

III. PHOTON EMISSION RATE

Formula for photon emission rate from medium with temperature T is [41]

$$k \frac{dN_\gamma}{d^3k d^4x} = \frac{\alpha}{2\pi^2} \frac{\text{Im} \Pi_R^{Tii}(k, \mathbf{k})}{e^{k/T} - 1} \simeq \frac{\alpha}{2\pi^2} \Pi_S^{Tii}(k, \mathbf{k}), \quad (4)$$

where $\Pi_{R/S}^{Tii}(\omega, \mathbf{k}) = (\delta_{ij} - k_i k_j / k^2) \Pi_{R/S}^{ij}(\omega, \mathbf{k})$ is the trace of transverse projection of the response/correlation function $\Pi_{R/S}^{ij}(\omega, \mathbf{k})$. When $\omega \ll T$, the fluctuation-dissipation theorem dictates $\Pi_S^{Tij}(\omega, \mathbf{k}) \simeq \frac{T}{\omega} \text{Im} \Pi_R^{Tij}(\omega, \mathbf{k})$. The correlation function is defined by the electric current $\mathbf{J}_{\text{em}}$ (in units of the elementary charge $e = \sqrt{4\pi\alpha}$),

$$\Pi_S^{ij}(\omega, \mathbf{k}) \equiv \int d^4x e^{i\omega t - i\mathbf{k} \cdot \mathbf{x}} \langle J_{\text{em}}^i(x) J_{\text{em}}^j(0) \rangle_{T, \mu}. \quad (5)$$

Corresponding to the transverse polarization of photons, only the transverse components contribute to the emission rate. Here, our focus lies on the soft momentum region, which is predominantly influenced by critical fluctuations. The electric current in $N_f = 2$ is decomposed as $J_{\text{em}}^\mu = \frac{1}{2} J_B^\mu + J_I^\mu$, where J_B^μ represents the baryon current and J_I^μ denotes the isospin current. Furthermore, near the critical point, the isospin fluctuation is shown to be irrelevant [36], leading to the approximation $J_{\text{em}}^\mu \simeq \frac{1}{2} J_B^\mu$. Hence, it becomes necessary to define the baryon current J_B^μ within the framework of Model H.

In hydrodynamics, baryon current is expanded by fluctuations and their derivatives

$$J_B^0 = n + \left(\frac{\partial n}{\partial \hat{s}} \right)_p \delta \hat{s} + \left(\frac{\partial n}{\partial p} \right)_s \delta p + \frac{nv^2}{2} + \dots, \quad (6a)$$

$$\mathbf{J}_B = \left[n + \left(\frac{\partial n}{\partial \hat{s}} \right)_p \delta \hat{s} + \left(\frac{\partial n}{\partial p} \right)_s \delta p \right] \mathbf{v} - \sigma_B T \nabla \left(\frac{\mu}{T} \right) + \dots, \quad (6b)$$

where the thermodynamic derivatives can be expressed by $(\partial n / \partial \hat{s})_p = -n^2(1 - \gamma^{-1})/b$ and $(\partial n / \partial p)_s = n/(c_s^2 w)$. Here $b \equiv (\partial p / \partial T)_n$ is finite and the adiabatic index $\gamma \equiv c_p / c_v$ is divergent at the critical point. The diffusive mode is mostly composed of $\delta \hat{s}$, while the sound mode mostly consists of the fluctuations of pressure δp and longitudinal momentum $\mathbf{v}_L$. In Model H for critical dynamics, the latter decouples due to its fast oscillation. Furthermore, the squared amplitudes of the transverse momentum v_T^2 is negligible compared to critical fluctuation $\delta \hat{s}$. Thus, we can write

$$J_B^0 \simeq n + \mathcal{A} \psi, \quad (7a)$$

$$\mathbf{J}_B \simeq n \mathbf{v}_T + \mathcal{A} [\psi \mathbf{v}_T + K \lambda \nabla [(\nabla^2 - a) \psi] - \boldsymbol{\zeta}_\psi], \quad (7b)$$

with $\mathcal{A} \equiv (\partial n / \partial \hat{s})_p$.

A. One-loop calculation of Model H

Using the Langevin equation of Model H, we calculate the correlation function of the transverse electric current $\Pi_S^{Tij}(\omega, \mathbf{k})$ at the one-loop level. Since the transverse electric current is given by

$$\mathbf{J}_{\text{em}}^T \equiv P_T(\mathbf{J}_{\text{em}}) \simeq \frac{\mathcal{A}}{2} P_T(\psi \mathbf{v}_T - \boldsymbol{\zeta}_\psi) + \frac{n}{2} \mathbf{v}_T, \quad (8)$$

the two-point function consists of various combinations of terms. At tree level, $\langle \zeta_{\psi k}(x) \zeta_{\psi l}(x') \rangle$ and $\langle v_{Tk}(x) v_{Tl}(x') \rangle$ contribute and we obtain for lightlike momenta k^μ as $\Pi_S^{Tii}(k, \mathbf{k})|_{\text{tree}} \simeq \mathcal{A}^2 T \lambda + \frac{n^2}{w} T \gamma_\eta$ in the limit $\gamma_\eta k \ll 1$. At the one-loop level, we need to calculate $\langle \psi v_{Tk}(x) \cdot \psi v_{Tl}(x') \rangle$, one-loop correction to $\langle v_{Tk}(x) v_{Tl}(x') \rangle$, and $\langle \psi v_{Tk}(x) \cdot \zeta_{\psi l}(x') \rangle$, while the other terms contribute only at the two-loop level and beyond. For lightlike momenta k^μ , we obtain for $\langle \psi v_{Tk} \cdot \psi v_{Tl} \rangle$,

$$\begin{aligned} \Pi_S^{Tii}(k, \mathbf{k})|_{1\text{-loop}} &\simeq \text{diagram} \\ &= \frac{\mathcal{A}^2}{4} \left(\delta_{ij} - \frac{k_i k_j}{k^2} \right) \int \frac{d^4 p}{(2\pi)^4} G_{S0}^{\psi\psi}(p) G_{S0}^{Tij}(k-p), \end{aligned} \quad (9)$$

where $G_{S0}^{\psi\psi}$ and G_{S0}^{Tij} are tree-level correlation functions given by

$$G_{S0}^{\psi\psi}(p) = \text{diagram} = \frac{2T \lambda \mathbf{p}^2}{(p^0)^2 + (K \lambda)^2 \mathbf{p}^4 (a + \mathbf{p}^2)^2}, \quad (10a)$$

$$G_{S0}^{Tij}(p) = \text{diagram} = \frac{2T w^{-1} \gamma_\eta (\delta_{ij} \mathbf{p}^2 - p_i p_j)}{(p^0)^2 + \gamma_\eta^2 \mathbf{p}^4}. \quad (10b)$$

As shown below, Eq. (9) behaves as $\propto \omega^{-1/2}$ in the low photon energy $\omega = k \rightarrow 0$ at the critical point $a = 0$, contributing to the renormalization of λ . The next singular contribution comes from one-loop correction to $\langle v_{Tk}(x) v_{Tl}(x') \rangle$, which behaves as $\propto \omega^{-1/4}$ and contributes to the renormalization of γ_η , which will be reported in a separate paper. Finally, $\langle \psi v_{Tk}(x) \cdot \zeta_{\psi l}(x') \rangle$ does not yield any singular contribution at low energy.

When both $a = 1/\xi^2$ and ω approach 0 near the critical point, a natural scaling variable is the ratio between the photon frequency ω and the typical damping rate γ_η / ξ^2 of $\mathbf{v}_T$, yielding $\nu = \omega \xi^2 / \gamma_\eta = \omega / (\gamma_\eta a)$. Note that the damping rate of $\mathbf{v}_T$ with the wavelength ξ is relevant because of the mode coupling between ψ and $\mathbf{v}_T$. The one-loop integral near the critical point can be approximated as

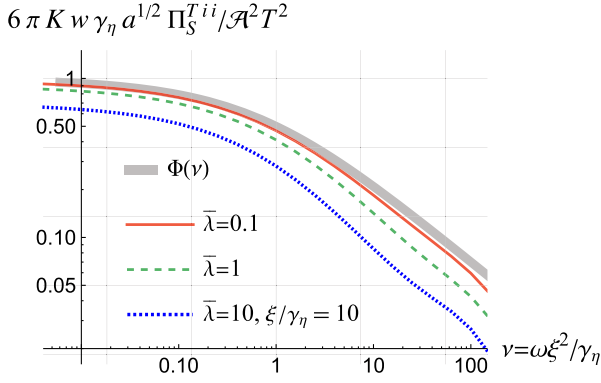


FIG. 2. $\Phi(\nu)$ is compared with Eq. (9) calculated using Eq. (10) with $\bar{a} \equiv a\gamma_\eta^2 = 0.01$ ($\xi/\gamma_\eta = 10$) and $\bar{\lambda} \equiv K\lambda/\gamma_\eta^3 = 0.1, 1, 10$. This corresponds to comparing $\Phi(\nu)$ and $\frac{3\bar{a}^{1/2}}{4\pi} F(\nu, \bar{a}, \bar{\lambda})$.

$$\Pi_S^{Tii}(k, k)|_{1\text{-loop}} \simeq \frac{\mathcal{A}^2 T^2}{6\pi K w \gamma_\eta a^{1/2}} \Phi(\nu). \quad (11)$$

The left-hand side can be expressed using a dimensionless function $F(\nu, \bar{a}, \bar{\lambda})$ with three scaled variables ν , $\bar{a} = a\gamma_\eta^2$, and $\bar{\lambda} = K\lambda/\gamma_\eta^3$,

$$\Pi_S^{Tii}(k, k)|_{1\text{-loop}} = \frac{\mathcal{A}^2}{4} \frac{T^2}{2\pi^2 K w} F(\nu, \bar{a}, \bar{\lambda}). \quad (12)$$

Thus, the critical scaling behavior (11) is equivalent to

$$\lim_{\bar{a} \rightarrow 0} \frac{3\bar{a}^{1/2}}{4\pi} F(\nu, \bar{a}, \bar{\lambda}) = \Phi(\nu). \quad (13)$$

Near the critical point, $F(\nu, \bar{a}, \bar{\lambda})$ is independent of $\bar{\lambda}$ because the soft mode ψ is effectively frozen due to the critical slowing down.

In Fig. 2, $\Phi(\nu)$ is compared with the numerical integration of the left-hand side of (11), with a proper multiplicative constant. A weak dependence on $\bar{\lambda}$ is found for a fixed $\bar{a} = 0.01$. By using the asymptotic forms of $\Phi(0) = 1$ and $\Phi(\nu \gg 1) \simeq 1/\sqrt{2\nu}$, we obtain a formula for the critical enhancement of the photon emission rate for $\omega = k \ll T$ (see Appendix B for details),

$$\omega \frac{dN_\gamma}{d^3 k d^4 x} \simeq \begin{cases} \frac{\alpha \mathcal{A}^2}{12\pi^3 K w} \frac{T^2 \xi}{\gamma_\eta} & (\omega \ll \frac{\gamma_\eta}{\xi^2}), \\ \frac{\alpha \mathcal{A}^2}{12\pi^3 K w} \frac{T^2}{\sqrt{2\gamma_\eta \omega}} & (\frac{\gamma_\eta}{\xi^2} \ll \omega \ll \frac{c_s^2}{\gamma_\eta}). \end{cases} \quad (14)$$

Around $\omega \sim c_s^2/\gamma_\eta$, we need to consider the sound mode contributions to the photon production as will be discussed below.

The emission rate for soft photons increases as $\omega^{-1/2}$ in the infrared until it reaches ξ/γ_η at $\omega \sim \gamma_\eta/\xi^2$. In the presence of photon background, the system is driven out of equilibrium for larger frequencies, while it remains in

the Ohmic regime for smaller frequencies. The transition occurs when the frequency balances the typical damping rate of $\mathbf{v}_T$, i.e., $\omega = \gamma_\eta/\xi^2$. Thus, the photon spectra reflect the nonequilibrium dynamical properties of the critical liquid. In the hydrodynamic regime, as $\omega \rightarrow 0$, the correlation function behaves as $\Pi_S^{Tii} \simeq \mathcal{A}^2 T [\lambda + T\xi/(6\pi K \bar{\eta})] = \mathcal{A}^2 T \lambda_{\text{hydro}}$, leading to $\lambda_{\text{hydro}} \propto \xi$. Thus, the diffusion coefficient $D = K\lambda/\xi^2$ is renormalized as $D_{\text{hydro}} = D + T/(6\pi \bar{\eta} \xi) \propto 1/\xi$. This relationship is well known and leads to the determination of the dynamic critical exponent of Model H as approximately $z \simeq 3$ [42,43].

B. Sound mode contribution

Here we evaluate the contribution of sound mode to the photon production rate. In Model H, the sound mode decouples from the slow diffusive dynamics because of its fast oscillatory phase with frequency $k^0 = c_s |\mathbf{k}| \gg \gamma_\eta k^2$. However, when we consider on-shell photon production, the photon involves an even larger frequency $k^0 = |\mathbf{k}|$ and thus the sound mode does not necessarily decouple.

The baryon current (6b) contains a nonlinear coupling $\propto \delta \hat{s} \mathbf{v}$ as well as a linear term $\propto \mathbf{v}$, which both include the sound mode $\mathbf{v}_L$ (longitudinal projection of $\mathbf{v}$). Although there is no tree contribution from the sound mode to the photon production rate due to the transverse projection, the sound mode contributes from the one-loop order. The nonlinear coupling between the sound mode $\mathbf{v}_L$ and the critical mode $\delta \hat{s}$ may enhance the photon production rate via the thermal loop effect.

The one-loop integral is similar to Eq. (9). One only needs to replace the transverse correlator $G_{S0}^{Tij}(p)$ with the longitudinal one,

$$G_{S0}^{Lij}(p) = \frac{2T w^{-1} \gamma_\xi(p^0)^2 p_i p_j}{[(p^0)^2 - c_s^2 \mathbf{p}^2]^2 + (\gamma_\xi p^0 \mathbf{p}^2)^2}. \quad (15)$$

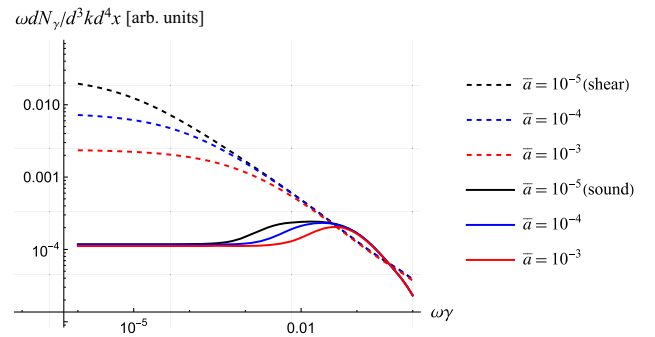


FIG. 3. Photon spectrum from sound mode loop is compared with that from Model H calculation near the critical point $\bar{a} \equiv a\gamma_\eta^2 = 10^{-5}, 10^{-4}, 10^{-3}$. The absolute magnitude is proportional to an unknown coefficient; see the later discussion around Eq. (16). We assume that bare transport coefficients satisfy $\bar{\lambda} \equiv K\lambda/\gamma_\eta^3 = 1$ and $\gamma_\eta = \gamma_\xi/2$ and that sound velocity is $c_s^2 = 1/3$.

For photon frequencies $\omega \lesssim c_s^2/\gamma_\eta$, the one-loop integral is found to be $\propto a^0$, showing no critical enhancement in the photon emission rate $\omega dN_\gamma/d^3k \propto 1/c_s$ (see Appendix C for details).

The photon emission rate from sound mode is numerically calculated in Fig. 3 and compared with the Model H results. The overall magnitude from the sound loop is $\propto a^0$, showing no enhancement with $a \rightarrow 0$. In contrast, the Model H calculation shows critical enhancement with $\omega dN_\gamma/d^3k \propto (\gamma_\eta \omega)^{-1/2}$ for $\omega \gg \gamma_\eta/\xi^2$ and dominates over the sound mode contribution for $\omega \gamma_\eta \ll c_s^2$. Therefore, the scaling behavior of the Model H calculation persists in a regime $\gamma_\eta/\xi^2 \ll \omega \ll c_s^2/\gamma_\eta$.

IV. DISCUSSION AND OUTLOOK

In this paper, we have derived a formula (14) for the enhanced emission rate of soft photons from a critical liquid. It is expressed using a universal function $\Phi(\nu)$ [see Eq. (1)], which characterizes the nonequilibrium property of the critical liquid probed by on-shell photons. Around $\omega \sim 0$, we estimate the magnitude of the critical photon spectra from a spacetime volume V_4 by

$$\omega \frac{dN_\gamma}{d^3k} \Big|_{\text{crit}} \simeq \frac{\alpha}{12\pi^3} \cdot \frac{T^2}{\bar{\eta}} \cdot \frac{\chi_T}{\xi^2} \cdot \xi V_4. \quad (16)$$

Here we use the relations near the critical point, $\mathcal{A}^2/K = n^2(1 - \gamma^{-1})^2 c_p a / (Tb^2) \simeq n^2 c_p / (Tb^2 \xi^2)$ and $T\chi_T = (1 - \gamma^{-1})n^2 c_p / b^2 \simeq n^2 c_p / b^2$, where $\chi_T \equiv (\partial n / \partial \mu)_T$ is the baryon number susceptibility. Note that χ_T/ξ^2 approaches a constant near the critical point when we neglect the self-interaction of ψ . If we include the full effects of critical fluctuations, the scaling would be estimated by $\xi^{1-\eta-x_\eta} = \xi^{x_\lambda}$. Here x_λ and x_η are scaling exponents of λ and $\bar{\eta}$ and are calculated as $x_\lambda = 18\epsilon/19 + \mathcal{O}(\epsilon^2)$ and $x_\eta = \epsilon/19 + \mathcal{O}(\epsilon^2)$ in the ϵ -expansion scheme. There is an uncertainty in the overall magnitude due to the unknown constant χ_T/ξ^2 (or $\chi_T/\xi^{2-\eta}$), which makes it difficult to predict the absolute magnitude. A crude estimate would be made by dimensional analysis. Since the ratio χ/ξ^2 has mass dimension four, the typical scale is given by the QCD scale such as the temperature T_c . Then $\omega dN_\gamma/d^3k \sim \alpha(T^2/\bar{\eta})T^4 \xi V_4 \sim \alpha T^3 \xi V_4$ for $\bar{\eta} \sim T^3$. As a reference, the photon from normal quark-gluon plasma can also be estimated as $\omega dN_\gamma/d^3k \sim \alpha T^2 V_4$. The critical enhancement is given by a factor $\xi T \sim 3$ for $\xi \sim 3$ fm and $T \sim 1$ fm $^{-1}$. As for the scaling $\omega dN_\gamma/d^3k \propto \omega^{-1/2}$, it is predicted robustly. For example, we can distinguish from the photon spectrum due to microscopic scattering in a weakly coupled quark-gluon plasma, which is predicted to be $\omega dN_\gamma/d^3k \propto \omega^{-3/2}$ for photons with low frequencies

ω , such that $eT, g_s^4 T \ln g_s^{-1} \ll \omega \ll T$ [44], where g_s is the QCD coupling constant.

Although observing the critical enhancement of low-energy photons may be challenging in heavy-ion collisions, our formula with the universal function $\Phi(\nu)$ holds significant theoretical value. Since photons probe lightlike kinematic regimes, the spectrum reflects the nonequilibrium properties of the critical fluid. This is indicated by the variable $\nu = \omega \xi^2/\gamma_\eta$ in the universal function $\Phi(\nu)$, which incorporates the photon frequency ω and the transport coefficient γ_η .

The result presented in this paper highlights the most singular behavior of the photon spectrum. However, there exists another singular contribution arising from the momentum density fluctuation $\langle v_{Tk}(x)v_{Tl}(x') \rangle$, which renormalizes γ_η (or $\bar{\eta}$) at one-loop level. Incorporating this contribution and applying a renormalization group treatment to the photon spectrum would enable us to derive small deviations from the exponent $\omega dN_\gamma/d^3k \propto \omega^{-1/2}$. With this analysis, we can complete the one-loop calculation of the current correlation function with lightlike momentum. Then, the next target after the photon spectrum would naturally be the dilepton spectrum.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

APPENDIX A: KAWASAKI FUNCTION

It is worth noting that the Kawasaki function $\Omega_K(y) = \frac{3}{4}y^{-2}[1 + y^2 + (y^3 - y^{-1}) \arctan y]$ [39] is derived from an integral akin to that in Eq. (B3) when computing the one-loop correction to the conductivity λ . The key distinctions lie in its evaluation at $k^0 = 0$ and the longitudinal projection $k^i k^j / k^2$, instead of the transverse projection. Then, the k -dependent one-loop correction $\Delta\lambda(\mathbf{k})$ reads

$$\begin{aligned}\Delta\lambda(\mathbf{k}) &= \frac{1}{2T} \frac{k_i k_j}{\mathbf{k}^2} \int \frac{d^4 p}{(2\pi)^4} G_{S0}^{\psi\psi}(p) G_{S0}^{Tij}(k-p) \Big|_{k^0=0} \\ &= \frac{T}{4\pi^2 K w} \int_0^\infty p^2 dp \int_{-1}^1 dz p^2 (p^2 - p^2 z^2) \frac{1}{p^2(a+p^2)} \frac{1}{k^2 - 2kpz + p^2} \frac{1}{K\lambda p^2(a+p^2) + \gamma_\eta(k^2 - 2kpz + p^2)}.\end{aligned}\quad (\text{A1})$$

Introducing dimensionless variables $x \equiv p/\sqrt{a}$ and $y \equiv k/\sqrt{a}$, it becomes

$$\Delta\lambda(\mathbf{k}) = \frac{T}{4\pi^2 K w \sqrt{a}} \int_0^\infty \frac{x^2 dx}{1+x^2} \int_{-1}^1 dz \frac{1}{y^2 - 2yxz + x^2} \frac{x^2 - x^2 z^2}{K\lambda a x^2(1+x^2) + \gamma_\eta(y^2 - 2yxz + x^2)}.\quad (\text{A2})$$

In the limit as $a \rightarrow 0$, while keeping $y \sim 1$, the integral is dominated by the region where $x \sim 1$, allowing us to approximate it as

$$\begin{aligned}\Delta\lambda(\mathbf{k}) &\simeq \frac{T}{4\pi^2 K w \gamma_\eta \sqrt{a}} \int_0^\infty \frac{x^4 dx}{1+x^2} \int_{-1}^1 dz \frac{1-z^2}{(y^2 - 2yxz + x^2)^2} \\ &= \frac{T}{8\pi K w \gamma_\eta \sqrt{a}} \frac{1}{y^2(1+y^2)} [1 + y^2 + (y^3 - y^{-1}) \arctan y] = \frac{T}{6\pi K w \gamma_\eta \sqrt{a}} \frac{\Omega_K(y)}{1+y^2}.\end{aligned}\quad (\text{A3})$$

The damping rate of ψ near the critical point is predominantly governed by the one-loop contribution, which scales as $a^{-1/2}$ and thus is expressed as

$$\Gamma_\psi(\mathbf{k}) = K(\lambda + \Delta\lambda(\mathbf{k}))k^2(k^2 + a) \simeq K\Delta\lambda(\mathbf{k})k^2(k^2 + a).\quad (\text{A4})$$

Using the scaling variable $y = k\xi = k/\sqrt{a}$, it is written as

$$\Gamma_\psi(\mathbf{k}) \simeq \frac{T}{6\pi\eta\xi^3} y^2 \Omega_K(y).\quad (\text{A5})$$

With the asymptotic behaviors $\Omega_K(0) = 1$ and $\Omega_K(y \gg 1) \simeq 3\pi y/8$, we can infer that $\lim_{a \rightarrow 0} \Gamma_\psi(\mathbf{k}) \propto k^3$ and $z = 3$.

APPENDIX B: DERIVATION OF EQ. (11)

Here, we provide detailed calculation of Eq. (11) in the main text. The propagators are defined by the two-point functions in the linearized approximation, denoted as $\langle \cdots \rangle_0$,

$$G_{S0}^{\psi\psi}(p) \equiv \int d^4 x e^{-ip \cdot x} \langle \psi(x) \psi(0) \rangle_0 = \frac{2T\lambda \mathbf{p}^2}{(p^0)^2 + (K\lambda)^2 \mathbf{p}^4 (a + \mathbf{p}^2)^2},\quad (\text{B1})$$

$$G_{S0}^{Tij}(p) \equiv \int d^4 x e^{-ip \cdot x} \langle v_T^i(x) v_T^j(0) \rangle_0 = \frac{2Tw^{-1}\gamma_\eta(\delta_{ij}\mathbf{p}^2 - p_i p_j)}{(p^0)^2 + \gamma_\eta^2 \mathbf{p}^4}.\quad (\text{B2})$$

Then, the one-loop integral is calculated as

$$\begin{aligned}&\left(\delta_{ij} - \frac{k_i k_j}{\mathbf{k}^2}\right) \left[\int \frac{d^4 p}{(2\pi)^4} G_{S0}^{\psi\psi}(p) G_{S0}^{Tij}(k-p) \right] \\ &= \left(\delta_{ij} - \frac{k_i k_j}{\mathbf{k}^2}\right) \int \frac{d^4 p}{(2\pi)^4} \frac{2T\lambda \mathbf{p}^2}{(p^0)^2 + (K\lambda)^2 \mathbf{p}^4 (a + \mathbf{p}^2)^2} \frac{2Tw^{-1}\gamma_\eta[\delta_{ij}(\mathbf{k} - \mathbf{p})^2 - (k_i - p_i)(k_j - p_j)]}{(k^0 - p^0)^2 + \gamma_\eta^2 (\mathbf{k} - \mathbf{p})^4} \\ &= \frac{4T^2 \lambda \gamma_\eta}{w} \int \frac{d^3 p}{(2\pi)^3} p^2 \left((\mathbf{k} - \mathbf{p})^2 + \frac{(\mathbf{k}^2 - \mathbf{k} \cdot \mathbf{p})^2}{\mathbf{k}^2} \right) \left[\frac{1}{2K\lambda \mathbf{p}^2 (a + \mathbf{p}^2)} + \frac{1}{2\gamma_\eta (\mathbf{k} - \mathbf{p})^2} \right] \frac{1}{(k^0)^2 + [K\lambda \mathbf{p}^2 (a + \mathbf{p}^2) + \gamma_\eta (\mathbf{k} - \mathbf{p})^2]^2} \\ &= \frac{T^2 \lambda \gamma_\eta}{2\pi^2 w} \int_0^\infty p^2 dp \int_{-1}^1 dz p^2 (k^2 - 2kpz + p^2 + (k - pz)^2) \\ &\quad \times \left[\frac{1}{K\lambda p^2 (a + p^2)} + \frac{1}{\gamma_\eta (k^2 - 2kpz + p^2)} \right] \frac{1}{(k^0)^2 + [K\lambda p^2 (a + p^2) + \gamma_\eta (k^2 - 2kpz + p^2)]^2}.\end{aligned}\quad (\text{B3})$$

From here, we use $\omega = k_0 = k$ for lightlike external momentum,

$$(B3) \Big|_{k_0=k=\omega} = \frac{T^2}{2\pi^2 K w} \int_0^\infty dp \int_{-1}^1 dz \left(\frac{\gamma_\eta p^2}{a+p^2} + \frac{K\lambda p^4}{\omega^2 - 2\omega pz + p^2} \right) \frac{2\omega^2 - 4\omega pz + p^2 + p^2 z^2}{\omega^2 + [K\lambda p^2(a+p^2) + \gamma_\eta(\omega^2 - 2\omega pz + p^2)]^2} \quad (B4)$$

$$= \frac{T^2}{2\pi^2 K w} F(\omega/(\gamma_\eta a), a\gamma_\eta^2, K\lambda/\gamma_\eta^3), \quad (B5)$$

where we introduce a dimensionless function $F(\nu, \bar{a}, \bar{\lambda})$ with variables $\nu \equiv \omega/(\gamma_\eta a)$, $\bar{a} \equiv a\gamma_\eta^2$ and $\bar{\lambda} \equiv K\lambda/\gamma_\eta^3$. Below, we show that the function $F(\nu, \bar{a}, \bar{\lambda})$ behaves near the critical point as

$$F(\nu, \bar{a}, \bar{\lambda}) \xrightarrow{\bar{a} \rightarrow 0} \frac{4\pi}{3\bar{a}^{1/2}} \Phi(\nu), \quad (B6)$$

independent of $\bar{\lambda}$ because of the critical slowing down of the soft mode ψ .

Using dimensionless quantities $x = p/a^{1/2} = p\xi$ and $\nu = \omega/(\gamma_\eta a) = \omega\xi^2/\gamma_\eta$, the first term in Eq. (B4) becomes

$$\frac{T^2 \gamma_\eta}{2\pi^2 K w} \int_0^\infty \frac{a^{3/2} x^2 dx}{a + ax^2} \int_{-1}^1 dz \frac{2(\gamma_\eta a \nu)^2 - 4(\gamma_\eta a \nu) a^{1/2} x z + ax^2 + ax^2 z^2}{(\gamma_\eta a \nu)^2 + [K\lambda ax^2(a + ax^2) + \gamma_\eta \{(\gamma_\eta a \nu)^2 - 2(\gamma_\eta a \nu) a^{1/2} x z + ax^2\}]^2}. \quad (B7)$$

Near the critical point $a \rightarrow 0$ while keeping $\nu \sim 1$, the integral is dominated by the $x \sim 1$ region so that it is approximated by

$$\frac{T^2 \gamma_\eta}{2\pi^2 K w a^{1/2}} \int_0^\infty \frac{x^2 dx}{1 + x^2} \int_{-1}^1 dz \frac{x^2 + x^2 z^2}{\gamma_\eta^2 \nu^2 + \gamma_\eta^2 x^4} = \frac{4T^2}{3\pi^2 K w \gamma_\eta a^{1/2}} \int_0^\infty \frac{x^4 dx}{1 + x^2} \frac{1}{\nu^2 + x^4} \propto a^{-1/2}. \quad (B8)$$

Similarly, the second term in Eq. (B4) is

$$\frac{T^2 \lambda}{2\pi^2 w} \int_0^\infty dx \int_{-1}^1 dz \frac{a^{5/2} x^4}{(\gamma_\eta a \nu)^2 - 2(\gamma_\eta a \nu) a^{1/2} x z + ax^2} \frac{2(\gamma_\eta a \nu)^2 - 4(\gamma_\eta a \nu) a^{1/2} x z + ax^2 + ax^2 z^2}{(\gamma_\eta a \nu)^2 + [K\lambda ax^2(a + ax^2) + \gamma_\eta \{(\gamma_\eta a \nu)^2 - 2(\gamma_\eta a \nu) a^{1/2} x z + ax^2\}]^2}. \quad (B9)$$

This integral is more complicated to analyze than the first one. One needs to understand that the dominant contribution is from $x \sim (\gamma_\eta/K\lambda)^{1/2} a^{-1/2}$ instead of $x \sim 1$. Then, by taking the limit $a \rightarrow 0$ while keeping $\nu \sim 1$, the integral is approximated by

$$\frac{T^2 \lambda a^{1/2}}{2\pi^2 w} \int_0^\infty x^2 dx \int_{-1}^1 dz \frac{x^2 + x^2 z^2}{(K\lambda ax^4 + \gamma_\eta x^2)^2} = \frac{4T^2 \lambda}{3\pi^2 w} \int_0^\infty \frac{a^{1/2} dx}{(K\lambda ax^2 + \gamma_\eta)^2} \propto a^0. \quad (B10)$$

Therefore, the first term ($\propto a^{-1/2}$) dominates the second term ($\propto a^0$) and we obtain

$$\left(\delta_{ij} - \frac{k_i k_j}{\mathbf{k}^2} \right) \int \frac{d^4 p}{(2\pi)^4} G_{S0}^{\psi\psi}(p) G_{S0}^{Tij}(k-p) \simeq \frac{4T^2}{3\pi^2 K \bar{\eta} a^{1/2}} \int_0^\infty \frac{x^4 dx}{1 + x^2} \frac{1}{\nu^2 + x^4}. \quad (B11)$$

The last integration is calculated as follows:

$$\begin{aligned} \int_0^\infty \frac{x^4 dx}{1 + x^2} \frac{1}{\nu^2 + x^4} &= \frac{1}{2} \int_{-\infty}^\infty \frac{x^4 dx}{(x^2 + 1)(x^2 - i\nu)(x^2 + i\nu)} \\ &= \frac{2\pi i}{2} \times \left[\frac{i^4}{2i(-1 - i\nu)(-1 + i\nu)} + \frac{(e^{i\pi/4} \sqrt{\nu})^4}{2e^{i\pi/4} \sqrt{\nu}(i\nu + 1)(2i\nu)} + \frac{(e^{3i\pi/4} \sqrt{\nu})^4}{2e^{3i\pi/4} \sqrt{\nu}(-i\nu + 1)(-2i\nu)} \right] \\ &= \frac{\pi}{2} \times \frac{1 - (1 - \nu)\sqrt{\nu/2}}{1 + \nu^2} = \frac{\pi}{2} \Phi(\nu), \end{aligned} \quad (B12)$$

with which we finally obtain

$$(9) = \frac{\mathcal{A}^2}{4K} \left(\delta_{ij} - \frac{k_i k_j}{k^2} \right) \int \frac{d^4 p}{(2\pi)^4} G_{S0}^{\psi\psi}(p) G_{S0}^{Tij}(k-p) \simeq \frac{\mathcal{A}^2 T^2}{6\pi K \bar{\eta} a^{1/2}} \Phi(\nu). \quad (\text{B13})$$

This derivation assumes a regime $\nu \sim 1$, i.e., $\omega \sim \gamma_\eta / \xi^2$. It can be extended to a regime $\omega \sim 1/\xi$. In this regime, the first term in the integral (B4) is dominated by $x \sim \gamma_\eta^{-1/2} a^{-1/4} \gg 1$ instead of $x \sim 1$, while the second term is by the same range $x \sim (\gamma_\eta / K\lambda)^{1/2} a^{-1/2}$. This difference only affects the approximate form of the integral (B8) by $x^2/(1+x^2) \simeq 1$ and thus the final formula (B13) is actually applicable to $\omega \sim 1/\xi$.

APPENDIX C: SOUND MODE CONTRIBUTIONS TO THE PHOTON PRODUCTION RATE

Expanding the baryon current up to the first nonlinear or dissipative corrections, we have as in Eq. (6b),

$$\mathbf{J}_B \simeq \left[n + \left(\frac{\partial n}{\partial \hat{s}} \right)_p \delta \hat{s} + \left(\frac{\partial n}{\partial p} \right)_s \delta p \right] \mathbf{v} - \sigma_B T \nabla \left(\frac{\mu}{T} \right) + \dots \quad (\text{C1})$$

Because of the transverse projection in the photon production rate, the sound mode contributes only from the one-loop order. The nonlinear coupling between the sound mode $\mathbf{v}_L$ (longitudinal projection of $\mathbf{v}$) and the critical mode $\delta \hat{s}$ may enhance the photon production rate via the thermal loop effect. Two-point function of momentum density is

$$G_{S0}^{ij}(p) = \frac{2Tw^{-1}\gamma_\eta(\mathbf{p}^2\delta_{ij} - p_i p_j)}{(p^0)^2 + \gamma_\eta^2 \mathbf{p}^4} + \frac{2Tw^{-1}\gamma_\zeta(p^0)^2 p_i p_j}{[(p^0)^2 - c_s^2 \mathbf{p}^2]^2 + (\gamma_\zeta p^0 \mathbf{p}^2)^2} = G_{S0}^{Tij}(p) + G_{S0}^{Lij}(p), \quad (\text{C2})$$

where G_{S0}^{Tij} is the transverse mode contribution and G_{S0}^{Lij} is the sound mode contribution. We focus on the regime $|\mathbf{p}| \ll c_s/\gamma_\zeta$, where sound poles are well-defined peaks such that the correlator G_{S0}^{Lij} can be split into

$$G_{S0}^{Lij}(p) \simeq \frac{Tw^{-1}\gamma_\zeta p_i p_j}{(p^0 - c_s |\mathbf{p}|)^2 + \gamma_\zeta^2 \mathbf{p}^4/4} + \frac{Tw^{-1}\gamma_\zeta p_i p_j}{(p^0 + c_s |\mathbf{p}|)^2 + \gamma_\zeta^2 \mathbf{p}^4/4}. \quad (\text{C3})$$

We calculate the one-loop integral

$$\begin{aligned} & \left(\delta_{ij} - \frac{k_i k_j}{k^2} \right) \int \frac{d^4 p}{(2\pi)^4} G_{S0}^{\psi\psi}(p) G_{S0}^{Lij}(k-p) \\ &= \left(\delta_{ij} - \frac{k_i k_j}{k^2} \right) \int \frac{d^4 p}{(2\pi)^4} \frac{2T\lambda \mathbf{p}^2}{(p^0)^2 + (K\lambda)^2 \mathbf{p}^4 (a + \mathbf{p}^2)^2} \frac{Tw^{-1}\gamma_\zeta (k_i - p_i)(k_j - p_j)}{(k^0 - p^0 - c_s |\mathbf{k} - \mathbf{p}|)^2 + \gamma_\zeta^2 (\mathbf{k} - \mathbf{p})^4/4} + (c_s \rightarrow -c_s) \\ &= \frac{2T^2 \lambda \gamma_\zeta}{w} \int \frac{d^3 p}{(2\pi)^3} \mathbf{p}^2 \left((\mathbf{k} - \mathbf{p})^2 - \frac{(\mathbf{k}^2 - \mathbf{k} \cdot \mathbf{p})^2}{k^2} \right) \\ & \quad \times \left[\frac{1}{2K\lambda \mathbf{p}^2 (a + \mathbf{p}^2)} + \frac{1}{\gamma_\zeta (\mathbf{k} - \mathbf{p})^2} \right] \frac{1}{(k^0 - c_s |\mathbf{k} - \mathbf{p}|)^2 + [K\lambda \mathbf{p}^2 (a + \mathbf{p}^2) + \gamma_\zeta (\mathbf{k} - \mathbf{p})^2/2]^2} + (c_s \rightarrow -c_s) \\ &= \frac{T^2 \lambda \gamma_\zeta}{4\pi^2 w} \int_0^\infty p^2 dp \int_{-1}^1 dz p^2 (k^2 - 2kpz + p^2 - (k - pz)^2) \left[\frac{1}{K\lambda p^2 (a + p^2)} + \frac{2}{\gamma_\zeta (k^2 - 2kpz + p^2)} \right] \\ & \quad \times \frac{1}{(k^0 - c_s \sqrt{k^2 - 2kpz + p^2})^2 + [K\lambda p^2 (a + p^2) + \gamma_\zeta (k^2 - 2kpz + p^2)/2]^2} + (c_s \rightarrow -c_s). \end{aligned} \quad (\text{C4})$$

From here, we use $\omega = k_0 = k$ for lightlike external momentum,

$$\begin{aligned} & \frac{T^2}{4\pi^2 K w} \int_0^\infty dp \int_{-1}^1 dz \left(\frac{\gamma_\zeta p^2}{a + p^2} + \frac{2K\lambda p^4}{\omega^2 - 2\omega pz + p^2} \right) \\ & \quad \times \frac{p^2 (1 - z^2)}{(\omega - c_s \sqrt{\omega^2 - 2\omega pz + p^2})^2 + [K\lambda p^2 (a + p^2) + \gamma_\zeta (\omega^2 - 2\omega pz + p^2)/2]^2} + (c_s \rightarrow -c_s). \end{aligned} \quad (\text{C5})$$

For $\omega \lesssim c_s^2/\gamma_\zeta$, the main contribution to this integral comes from $p \sim c_s/\gamma_\zeta \sim (\gamma_\zeta/K\lambda)^{1/2}$ and the result is $\propto a^0$ with no critical enhancement.

The integral (C5) does not diverge but has a physical cutoff scale at $p \sim c_s/\gamma_\zeta \sim (\gamma_\zeta/K\lambda)^{1/2}$ for hydrodynamic description. We can estimate the integral for $\omega = 0$ and $a = 0$ by

$$\begin{aligned} & \frac{T^2}{4\pi^2 K w} \int_0^{c_s/\gamma_\zeta} dp \int_{-1}^1 dz (\gamma_\zeta + 2K\lambda p^2) \frac{p^2(1-z^2)}{c_s^2 p^2 + (K\lambda p^4 + \gamma_\zeta p^2/2)^2} \times 2 \\ & \simeq \frac{T^2}{2\pi^2 K w} \int_0^{c_s/\gamma_\zeta} dp \int_{-1}^1 dz (\gamma_\zeta + 2K\lambda p^2) \frac{(1-z^2)}{c_s^2} = \frac{2T^2}{3\pi^2 K w} \left(\frac{1}{c_s} + \frac{2K\lambda c_s}{3\gamma_\zeta^3} \right) \simeq \frac{T^2}{K w} \times 0.12, \end{aligned} \quad (\text{C6})$$

where the final expression is obtained by substituting the values used in Fig. 3, i.e., $\bar{\lambda} \equiv K\lambda/\gamma_\eta^3 = 1$, $\gamma_\eta = \gamma_\zeta/2$, and $c_s^2 = 1/3$. The main contribution to the integral (C5) comes from around the cutoff scale, resulting in a non-negligible deviation from our estimate. To be specific, the estimate in (C6) corresponds to the photon yield $\sim 10^{-5}$ at $\omega = 0$, which is about 10% of the numerical integration ($\sim 10^{-4}$) in Fig. 3.

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