

Modified Abelian gauge theories

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The topological properties of field configurations in gauge theory contain important data about the (generalized) global symmetries of the theory as well as potential inconsistencies in the form of gauge anomalies. In this work we modify the topological classes of Abelian p -form fields, generating new global variants of gauge theories. These modifications implement constraints directly on the classifying space of the gauge field and its cohomology classes via homotopy fiber construction. This general approach allows us to investigate the universal effects of the constraints on the conserved global charges encoded in gauge characteristic classes. We further demonstrate that this procedure generically leads to new topological sectors introducing additional global charges and anomalies in the modified gauge theories.

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I. INTRODUCTION

Nature's fundamental interactions are very effectively captured by quantum field theory. This involves the appearance of particles that transmit the forces which are the excitations of the underlying gauge field. These gauge theories are further subject to redundancies associated to gauge symmetries whose breaking would render the theory inconsistent. This remains to be true after the inclusion of quantum corrections and demands the absence of so-called anomalies that classify the properties of the partition function under gauge transformations. Moreover, the partition function splits into separate sectors which correspond to topologically distinct gauge field configurations. Since it is the topological features of the fields that specify the individual sectors, they are invariant under continuous deformations and can be thought of as defining conserved charges. The goal of this work is to modify the topological properties of gauge theories and study how this affects the topological charges as well as anomalies of the underlying theory.

While 1-form gauge fields can describe Abelian or non-Abelian gauge equivalence, their higher-form variants are

generically Abelian. These Abelian p -form fields frequently appear in higher-dimensional supergravity theories as well as string theory. Even in the Abelian case the theories are far from trivial and present many interesting phenomena. Many of these are described by the analysis of their generalized (categorical) symmetries [1] (see [2–4] for reviews) which have advanced our understanding tremendously within the last years.

One class of generalized symmetries, typically called magnetic (higher-form) symmetries, originates from the fact that the field strength is closed. In case of a p -form gauge field the field strength is described by a $(p+1)$ -form $F^{(p+1)}$ and defines a conserved current $j \sim *F^{(p+1)}$

$$d * j \sim d * (*F^{(p+1)}) = \pm dF^{(p+1)} = 0. \quad (1.1)$$

Since the same is true for a power (wedge product) of the field strength this leads to a whole set of (composite) conserved currents, e.g., for p odd one can define the k -fold product current,

$$d * (*F^{(p+1)} \wedge \dots \wedge F^{(p+1)}) = 0. \quad (1.2)$$

Each of these currents is associated to a generalized global symmetry, sometimes referred to as Chern-Weil symmetry, [5] (see also [6]). The corresponding conserved charges are given by the integral of $*j$ over a submanifold of appropriate dimension, which in the case above can be written as

$$Q \sim \int_{\Sigma_{kp+k}} (*F^{(p+1)} \wedge \dots \wedge F^{(p+1)}). \quad (1.3)$$

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This formula suggests a close connection between these generalized higher-form symmetries, their conserved charges, and the characteristic classes of the underlying gauge theory. This follows simply from the fact, that the characteristic classes on a spacetime manifold X are given by elements of $H^*(X; \mathbb{Z})$ pulled back from the associated classifying space (see, e.g., [7,8]).¹ Since these classes are closed by definition they describe conserved quantities which include the U(1) Chern-Weil symmetries above, but as cohomology classes with integer coefficients they can also include discrete $\mathbb{Z}_k$ torsion charges. Thus, the characteristic classes of gauge theories are an important piece of information for a full understanding of the gauge theory and its (generalized) symmetries. This is further corroborated by the fact that typically the values of these characteristic classes split the gauge theory in topologically distinct sectors which need to be summed over in the full partition function.

This raises the question of whether there can be theories that differ in these symmetry sectors. Such modified Abelian gauge theories and their construction are the subject of this work. In particular we construct theories where some characteristic classes, i.e., elements of $H^*(X; \mathbb{Z})$, are restricted to satisfy certain constraints. This procedure can be understood as a modification of the underlying global higher-form symmetry and leads to (global) variants of Abelian p -form theories.

The technique that we use in order to implement such constraints is insensitive to the choice of spacetime manifold X because it directly modifies the classifying space. This is because characteristic classes on X , come from (the pullback of) cohomology classes of the classifying space itself. The concrete implementation uses what is known as a homotopy fiber construction, see, e.g., [8], which can be understood as gluing additional cells into the classifying space trivializing the wanted (co)homology classes. However, these constructions tend to also introduce additional characteristic classes, which come from the implementation of the constraint. Thus, we find that while we can get rid of certain conserved global charges this often comes at the price of introducing new ones. Interestingly, the homotopy fiber construction also provides a physical interpretation for this since the modified classifying space can be understood as the total space of a fibration, with the original classifying space appearing as the base space. While the fibration can trivialize cohomology classes of the base, the fiber in general introduces new ones, which are precisely the additional classes we encounter.

While this type of modification might seem exotic, we know several examples where similar constraints are implemented. Most prominently, this appears in the coupling of non-Abelian gauge dynamics to gravity and is also manifested in the Green-Schwarz anomaly cancellation

mechanism [9], where the presence and transformation properties of a 2-form field trivializes otherwise conserved currents by making them exact, and hence trivial in cohomology. An explicit example would be

$$dH^{(3)} \sim F^{(2)} \wedge F^{(2)}, \quad (1.4)$$

where $H^{(3)}$ is the (gauge invariant) field strength of a 2-form field. Such an equation is also called the Bianchi identity. Since, here, we are working with cohomology classes with integer coefficients, we are also sensitive to the proportionality factor in (1.4) as well as torsion classes. In string theory and supergravity examples one further can include characteristic classes of the tangent bundle accounting for a gravitational contribution as is the case for the original Green-Schwarz mechanism in heterotic string theory. Here, we focus on the field theory side without the appearance of the curvature 2-form in the constraints, and we analyze how a modification of the gauge theory affects its characteristic classes and anomalies.

Our main topic is the modification of U(1) p -form gauge theories, whose classifying space $B^p\text{U}(1)$ is given by the Eilenberg-MacLane space $K(\mathbb{Z}, p+1)$, which is defined to have a single nontrivial homotopy group $\pi_m(K(A, m)) = A$ in positive degree. An analogous approach could be used for discrete $\mathbb{Z}_n$ p -form gauge theories with classifying space $K(\mathbb{Z}_n, p)$. Implementing a constraint on a characteristic class in $H^k(X; \mathbb{Z})$ describes the modified classifying space as a fibration of $K(A, k-1)$ over $B^p\text{U}(1)$ (where A is an Abelian group). This suggests that the constraints are enforced by the mixing with higher-form fields with gauge group A , very similar to what happens in the Bianchi identity (1.4) above and leads to a so-called higher-group symmetry (see, e.g., [10–12]). In this sense the homotopy fiber approach is very physical, but also general in that it does not need the specification of the number of spacetime dimensions, spacetime manifold X , or the explicit formulation at the level of a Lagrangian. The resulting theory will have a different set of conserved charges and therefore a different set of (generalized) symmetries.

These modifications have further consequences for the possible gauge anomalies. For perturbative anomalies this can be seen rather straightforwardly from the fact that the anomaly polynomial, which encodes the nontrivial phase of the partition function under small gauge transformations, is built out of characteristic classes of the gauge theory. Thus, once one restricts the characteristic classes, also the gauge anomalies are modified. Including diffeomorphisms, one can further include the characteristic classes of the tangent bundle and find additional constraints capturing mixed gravitational anomalies which might also be modified. However, the newly introduced characteristic classes might lead to additional anomalies, which were absent before and demand extra consistency conditions. For nonperturbative anomalies (including diffeomorphisms)

¹We work in Euclidean signature.

the analysis becomes more tricky and potential anomalies are classified by nontrivial bordism classes, [13–19], for which the gauge theory characteristic classes are important input data. That a modification of a gauge theory should have an influence on the anomalies of the underlying theory should not come as a surprise since this is exactly the purpose of the Bianchi identity in the Green-Schwarz mechanism, whose more global versions have been studied in for example [20–31]. The application of the homotopy fiber construction for fermion anomaly cancellation was further discussed in [32], from which we adopt many of the relevant techniques.

Since the mathematics involved can be quite heavy (see [18,33–35]), we try to illustrate the general principles in various explicit examples. In particular, we will be interested in D -dimensional Maxwell theory, i.e., with $U(1)$ 1-form field, whose first nontrivial characteristic class, the first Chern class, $c_1 \in H^2(X; \mathbb{Z})$, remains unmodified, but whose composite class c_1^2 is subject to some constraints. For the constraints we primarily focus on a “mod n ” constraint with $c_1^2 \bmod n \in H^4(X; \mathbb{Z}_n)$ being trivial; and a “times n ” constraint with $nc_1^2 \in H^4(X; \mathbb{Z})$ being trivial. The effect on the local dynamics of these theories depends on the way they are implemented at the level of differential cohomology and goes beyond the topological analysis of the homotopy fiber in the main part of this work. We comment on this issue in the Appendix A. Nevertheless, even on the topological level we find interesting new characteristic classes and modifications of the gauge anomalies of these global modifications of Maxwell theory.

The rest of this work is organized as follows: In Sec. II we discuss the topological sectors of gauge theories with focus on $U(1)$ p -form fields, and their relation to characteristic classes. We will also introduce the two restrictions; “times n ” and “mod n ,” which we will implement throughout this work. In Sec. III we explain the realization of the constraints using the homotopy fiber construction and demonstrate the evaluation of the characteristic classes of the new theory $H^*(Q; \mathbb{Z})$ in various interesting examples. We further explore what happens in case one wants to implement more than one constraint. In Sec. IV we further include gravitational information for some of the discussed examples and explore the oriented and spin bordism groups of the modified gauge theories which classify the potential anomalies of the modified theory. We point out various applications and generalizations of the techniques described here in Sec. V. In Appendix A, we connect Bianchi identities and the homotopy fiber construction and further discuss different ways to implement the constraints beyond the topological level. We provide the cohomology classes of Eilenberg-MacLane space of the form $K(\mathbb{Z}, m)$ and $K(\mathbb{Z}_2, m)$ in the Appendix B and determine some bordism groups we use in the main text in the Appendix C.

II. TOPOLOGICAL SECTORS OF ABELIAN GAUGE THEORIES

In this section we describe the topological sectors of Abelian p -form gauge fields. In particular, we describe how the different gauge configurations are distinguished by their realization of characteristic classes. These characteristic classes can be understood, by pulling back the nontrivial cohomology classes from the classifying space BG to the spacetime manifold X . We then describe two possible modifications of the topological sectors, which can be understood as a modification of the magnetic, sometimes also called solitonic, (higher-form) symmetries. While we focus on Abelian, especially $U(1)$ gauge fields, an analogous approach also works in other cases.

A. Abelian gauge fields and differential cohomology

A convenient way to describe $U(1)$ p -form fields uses differential cohomology, see, e.g., [36,37] and [38,39] for reviews. A $U(1)$ p -form field is fully described by a differential character, i.e., by an element of $\check{H}^{p+1}(X)$. This differential character contains information about both the topological features of the configurations as well as the continuous information encoded for example in the connection or field strength.

For our purpose it will be useful to consider one of the short exact sequences describing $\check{H}^{p+1}(X)$, given by

$$0 \rightarrow \frac{\Omega^p(X)}{\Omega_{\mathbb{Z}}^p(X)} \rightarrow \check{H}^{p+1}(X) \rightarrow H^{p+1}(X; \mathbb{Z}) \rightarrow 0. \quad (2.1)$$

Here, $\Omega^p(X)$ [$\Omega_{\mathbb{Z}}^p(X)$] denote differential p -forms (with integer periods). The first entry, can be understood as the topologically trivial fields, where the $U(1)$ gauge field is defined as a global differential form $\Omega^p(X)$, up to (large) gauge transformations $\Omega_{\mathbb{Z}}^p(X)$. The last entry, $H^{p+1}(X; \mathbb{Z})$, captures the topological class $[N^{(p+1)}]$ of the gauge configuration in integer cohomology, whose representative cocycle is given by $N^{(p+1)}$. For a $U(1)$ p -form field, this decomposition of a differential character $\check{A} \in \check{H}^{(p+1)}(X)$ also motivates a split of the path integral into a discrete sum over topological sectors, labeled by elements in $H^{p+1}(X; \mathbb{Z})$, and a continuous integral over topologically trivial differential characters $\check{a} \in \check{H}^{p+1}(X)$ which we write as

$$\int D\check{A} \rightarrow \sum_i \int D\check{a}. \quad (2.2)$$

In the following we will be interested in modifying the discrete sum by restricting the allowed topological classes of the gauge theory. However, we will not only modify the topological class in $H^{p+1}(X; \mathbb{Z})$, but also more general

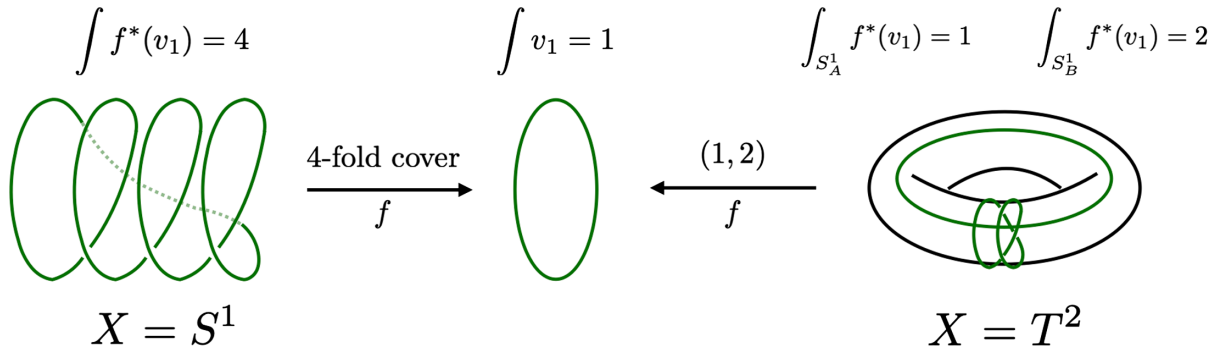


FIG. 1. Illustration of classifying maps f from $X = S^1$ and $X = T^2$ to the classifying space of a periodic scalar field, $K(\mathbb{Z}, 1) \simeq S^1$.

characteristic classes in higher degree, e.g., powers of the topological class $(N^{(p+1)})^k$. In the case of a U(1) 1-form field described by an element in $\check{H}^2(X)$ we will also use the common notation

$$[N^{(2)}] = c_1 \in H^2(X; \mathbb{Z}), \quad (2.3)$$

in terms of the first Chern class of the bundle.

The fate of the continuous integral over $\check{\alpha}$ above, goes beyond the topological data of the classifying space and depends on the particular implementation of the modification, as we argue in the Appendix A. In the following we concentrate on the topological features that modify the discrete sum.

B. Characteristic classes and classifying spaces

Every topological class of a gauge configuration with gauge group G on the spacetime manifold X can be identified with a homotopy class of maps from X into the classifying space BG (see, e.g., [7,40]),

$$f: X \rightarrow BG. \quad (2.4)$$

For that reason f is called the classifying map. Note however, that there is more physical data in the gauge configuration, e.g., the choice of connection, which requires information beyond the topological class. Denoting the homotopy class of a map from X to BG , by $[X, BG]$ each separate class defines a topologically distinct gauge configuration. Since the topological sectors of the gauge theory are distinguished by characteristic classes, these must be part of the information of the classifying space BG .

Indeed, the possible characteristic classes of the gauge theory can be identified with nontrivial elements of the cohomology of the classifying space, sometimes also referred to as group cohomology, $H^*(BG; \mathbb{Z})$. The characteristic classes of a particular gauge background on X , specified by the homotopy class of its classifying map f , are then obtained by pulling back, f^* , the cohomology classes $H^*(BG; \mathbb{Z})$ to spacetime. Since these are

integer-valued classes, they remain constant under continuous deformations of the map f , exactly as we would expect since only the homotopy class $[X, BG]$ enters. Let us discuss this in some simple examples.

One can understand a periodic scalar field $\varphi \sim \varphi + 2\pi$ as a 0-form U(1) gauge field (the “large gauge transformations” make the scalar periodic). The associated classifying space is given by S^1 , the circle, which is also an Eilenberg-MacLane space, $K(\mathbb{Z}, 1)$ (see [8]). The topological classes of this periodic scalar field are then classified by homotopy classes of maps from X to S^1 . Since in positive degree the only nontrivial cohomology group of the circle is given by $H^1(S^1; \mathbb{Z}) = \mathbb{Z}$, the topological classes of periodic scalars are fully classified by specifying the pullback of this class under the classifying map f , i.e., specifying the corresponding element in $H^1(X; \mathbb{Z})$. In the case that spacetime X is also given by a circle, we know that homotopy classes of maps from X to $BG \simeq S^1$ are fully specified by an integer n , which describes how often X winds around BG , see Fig. 1. This defines the value of $\int_X N_\varphi^{(1)}$, i.e., the winding of the periodic scalar around the spacetime circle.² Denoting the generator of the cohomology group by $v_1 \in H^1(S^1; \mathbb{Z})$, one has

$$\int_X f^*(v_1) = \int_{f_*X} v_1 = \int_{nS^1} v_1 = n \int_{S^1} v_1 = n. \quad (2.5)$$

Similar, considerations can be done for higher-dimensional X , where one can determine the winding number of φ around the nontrivial 1-cycles of X in the same way. For the torus, T^2 , for instance, the homotopy classes of maps $T^2 \rightarrow S^1$ are specified by two integers, which characterize the winding numbers of the periodic scalar around a basis of 1-cycles (with representatives S_A^1 and S_B^1), Fig. 1.

Next, let us move to U(1) 1-form gauge fields, i.e., Maxwell theory. In this case the classifying space is infinite-dimensional and given by the infinite-dimensional complex projective space

²We avoid the use of $d\varphi$ since φ is not globally well defined due to large gauge transformations.

$$BU(1) \simeq \mathbb{CP}^\infty \simeq K(\mathbb{Z}, 2), \quad (2.6)$$

which again can be identified with an Eilenberg-MacLane space. While this seems to present a significant complication when compared to the periodic scalar field the logic proceeds analogously. The different topological sectors are described by homotopy classes $[X, BG]$, and the characteristic classes on X can be obtained via pullback. For example, the first Chern class $c_1 \in H^2(X; \mathbb{Z})$ is obtained by pulling back the generator of $H^2(\mathbb{CP}^\infty; \mathbb{Z}) = \mathbb{Z}$ which we denote by v_2 . For $X = S^2 \simeq \mathbb{CP}^1$, there is again the analog of a “winding number” which here characterizes the magnetic flux through S^2 , since homotopy classes of maps $[S^2; \mathbb{CP}^\infty]$ are described by $H^2(S^2; \mathbb{Z}) = \mathbb{Z}$, which can be identified as

$$\int_{S^2} c_1 = \int_{S^2} f^*(v_2). \quad (2.7)$$

However, as opposed to the circle above, $\mathbb{CP}^\infty$ has more nontrivial cohomology classes, which correspond to powers of the generator v_2 , since $H^*(\mathbb{CP}^\infty; \mathbb{Z}) = \mathbb{Z}[v_2]$. For a four-dimensional spacetime manifold X , and using naturality of the cup product, one can define

$$f^*(v_2 \cup v_2) = f^*(v_2) \cup f^*(v_2) = c_1 \cup c_1 \in H^4(X; \mathbb{Z}), \quad (2.8)$$

a characteristic class of higher degree.

The classifying space of a $U(1)$ p -form field is given by

$$B^p U(1) \simeq K(\mathbb{Z}, p+1), \quad (2.9)$$

which again can be identified with an Eilenberg-MacLane space. We see that the topological classes of the higher-form gauge field can be identified with homotopy classes $[X, K(\mathbb{Z}, p+1)]$ from spacetime into the classifying space, which are isomorphic to $H^{p+1}(X; \mathbb{Z})$. This is exactly as expected from our discussion of differential cohomology in Sec. II A, where the topological class was determined by $[N^{(p+1)}] \in H^{p+1}(X; \mathbb{Z})$. More general characteristic classes can then be obtained by pulling back cohomology classes of the Eilenberg-MacLane spaces $K(\mathbb{Z}, p+1)$ of higher degree to X .

Our goal is to restrict the allowed values of characteristic classes on X , for which we need to modify the classifying spaces BG .

C. Modifications of topological sectors

In this work we explore how to describe a modification of the sum over topological sectors of Abelian $U(1)$ p -form gauge theories. For $U(1)$ global symmetries associated to characteristic classes valued in $\mathbb{Z}$, we consider the following two modifications:

- (i) The “times n ” constraint: $n[N^{(p+1)}]^k = 0 \in H^{kp+k}(X; \mathbb{Z})$: The case $n = 1$ enforces the characteristic

class to be topologically trivial. It can be understood as gauging the full associated $U(1)$ symmetry. For $n \neq 1$, this still presents a gauging of the full associated $U(1)$ global symmetry,³ but there is a discrete remnant in terms of potential torsion classes in $H^{kp+k}(X; \mathbb{Z})$. These can be understood as the topological classes of a $(kp + k - 1)$ -form $\mathbb{Z}_n$ gauge field, since these correspond to the image of $H^{kp+k-1}(X; \mathbb{Z}_n)$ in $H^{kp+k}(X; \mathbb{Z})$, via the Bockstein map $\beta_{\mathbb{Z}}$ explained below.

- (ii) The “mod n ” constraint: $\mathbf{m}_n([N^{(p+1)}]^k) = 0 \in H^{kp+k}(X; \mathbb{Z}_n)$: This constraint implements that the mod n reduction $\mathbf{m}_n$ of a certain characteristic class vanishes and can be regarded as the gauging of a $\mathbb{Z}_n$ subgroup of the associated $U(1)$ global symmetry.

That indeed we are dealing with a nontrivial gauging becomes clear in later sections, using spectral sequences techniques, where the trivialized characteristic classes become the target of a differentials making them exact. For continuous $U(1)$ symmetries this precisely corresponds to the definition of gauging in [5], in which the associated charge density becomes exact and does not lead to a conserved charge when integrated over closed submanifolds.

The two constraints above are closely related as can be seen by considering the long exact sequence in cohomology associated to the short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \xrightarrow{\mathbf{m}_n} \mathbb{Z}_n \longrightarrow 0, \quad (2.10)$$

given by

$$\begin{aligned} \dots &\longrightarrow H^{r-1}(X; \mathbb{Z}_n) \xrightarrow{\beta_{\mathbb{Z}}} H^r(X; \mathbb{Z}) \xrightarrow{\cdot n} H^r(X; \mathbb{Z}) \\ &\xrightarrow{\mathbf{m}_n} H^r(X; \mathbb{Z}_n) \longrightarrow H^{r+1}(X; \mathbb{Z}) \longrightarrow \dots \end{aligned} \quad (2.11)$$

Here, $\beta_{\mathbb{Z}}$ denotes the Bockstein homomorphism associated to (2.10) (see [8]).

The times n constraint demands that the modified characteristic class is in the kernel of $\cdot n$, which by exactness means that it is in the image of $\beta_{\mathbb{Z}}$, i.e., there is a class $x_n \in H^{kp+k-1}(X; \mathbb{Z}_n)$ such that

$$\beta_{\mathbb{Z}}(x_n) = [N^{(p+1)}]^k. \quad (2.12)$$

The mod n constraint on the other hand means that the modified characteristic class is in the kernel of $\mathbf{m}_n$ and hence by exactness in the image of $\cdot n$, i.e., there exists an $x \in H^{kp+k}(X; \mathbb{Z})$ such that

³This can also be understood at the level of symmetry generator (see for example [1]) $\exp(\frac{i\alpha}{2\pi} \int [N^{(p+1)}]^k) = \exp(\frac{i\alpha}{2\pi n} \int n[N^{(p+1)}]^k)$ which is trivial for $U(1)$ parameter α .

$$nx = [N^{(p+1)}]^k. \quad (2.13)$$

Note that these two properties are not mutually exclusive, as we will also see in Sec. III E.

Let us now describe how we can implement these constraints very efficiently in a model-independent way.

III. THE HOMOTOPY FIBER CONSTRUCTION

In this section we implement the modification of the Abelian gauge theories discussed above, using a homotopy fiber construction, see [8] [Sec. IV.3] as well as its applications in [32]. This procedure does not require the explicit implementation at the level of the Lagrangian, whose connection to Bianchi identities we discuss in the Appendix A, but rather proceeds via a modification of the classifying space of the underlying gauge theory.

A. The homotopy fiber

Since our goal is to restrict the “allowed” characteristic classes on X , we can achieve that by modifying the classifying space of the underlying gauge theory. In particular, we want a new space Q , which has a map p into the original BG , but has restricted homotopy classes of maps $[X, Q]$ as compared to $[X, BG]$. To be more precise, we want that a map $[X, BG]$ only lifts to a map $[X, Q]$, if the restriction on the characteristic classes is satisfied. Such a space can be constructed as the fiber in a fibration of the form

$$Q \xrightarrow{p} BG \xrightarrow{\alpha} Y, \quad (3.1)$$

where the space Y and the map α are chosen to implement the constraint. This construction guarantees that a map f from X to BG lifts to a map f_Q from X to Q if and only if

$$\alpha \circ f = 0, \quad (3.2)$$

meaning that the map is in the trivial class $[X, Y]$. The homotopy fiber Q constructed in such a way is unique (up to homotopy).

Next, let us consider how to choose Y and α for the type of constraints defined above. For that we note that we aim for a restriction of characteristic classes on X , i.e., elements of $H^m(X; A)$. Thus, to implement such a constraint we ideally would like Y to be sensitive to that, but only that information. Indeed, these spaces are known and given precisely by the Eilenberg-MacLane spaces $K(A, m)$.⁴ To be more specific, the homotopy classes of maps from X to

$K(A, m)$ are in one-to-one correspondence with cohomology classes $H^m(X; A)$

$$[X; K(A, m)] \simeq H^m(X; A). \quad (3.3)$$

In some sense Eilenberg-MacLane spaces can be understood as very simply spectra, that define cohomology theories in exactly the way described above. They can also be understood as the minimal choice, since the homotopy classes of maps exclusively depend on this particular cohomology element.

Having identified the spaces $K(A, m)$ as natural choice for Y , we now turn to the maps α . Due to the properties just discussed, we know that homotopy classes of α are fully determined by

$$[BG, K(A, m)] \simeq H^m(BG; A). \quad (3.4)$$

Pulling back these elements to X , we choose α in such a way that it corresponds to the characteristic class that should be trivialized. With BG , Y , and α fixed, the space Q naturally appears as fiber in the sequence (3.1) above. Let us again try to illustrate this in several simple examples.

Let us once more consider the periodic scalar field φ with associated classifying space S^1 . Since the only cohomology class in positive dimension is given by $H^1(S^1; \mathbb{Z}) = \mathbb{Z}$, the only spaces Y we want to consider are of the form $K(A, 1)$. And to be more precise we choose A to be a free $\mathbb{Z}$ or cyclic $\mathbb{Z}_n$ group, leading to

$$K(\mathbb{Z}, 1) \simeq S^1, \quad K(\mathbb{Z}_n, 1) \simeq L_n^\infty \simeq B\mathbb{Z}_n. \quad (3.5)$$

The first choice is appropriate to implement the times n constraint, for which we choose α in the class

$$nv_1 \in H^1(S^1; \mathbb{Z}) \simeq [S^1, S^1]. \quad (3.6)$$

Thus, α is an n -fold covering. The fiber is thus given by $\mathbb{Z}_n$, which is 0-dimensional. From that we learn that, for any X , implementing the constraint $nf^*(v_1) = 0 \in H^1(X; \mathbb{Z})$ trivializes all topological sectors, see Fig. 2.

The second choice of Y allows us to implement the mod n constraint, in particular we have that α is described by a class in

$$[S^1, B\mathbb{Z}_n] \simeq H^1(S^1; \mathbb{Z}_n) \simeq \mathbb{Z}_n. \quad (3.7)$$

Choosing it to be given by $e^{2\pi i/n} \in \mathbb{Z}_n$, will successfully restrict $f^*(v_1) \in H^1(X; \mathbb{Z})$ to vanish in $H^1(X; \mathbb{Z}_n)$.⁵ Here, the homotopy fiber Q can be interpreted as an n -fold covering of S^1 , since those map to the trivial class in $[S^1, B\mathbb{Z}_n]$. From this construction we find that

⁴One can alternatively choose the base space to be $Y = E_m$ where E_m is a classifying space for the generalized cohomology $E^m(X)$. This would correspond to a modification of the so called solitonic symmetries corresponding to the generalized cohomology theories, see [41].

⁵One could also choose other elements $e^{2\pi i k/n}$ with k coprime to n .

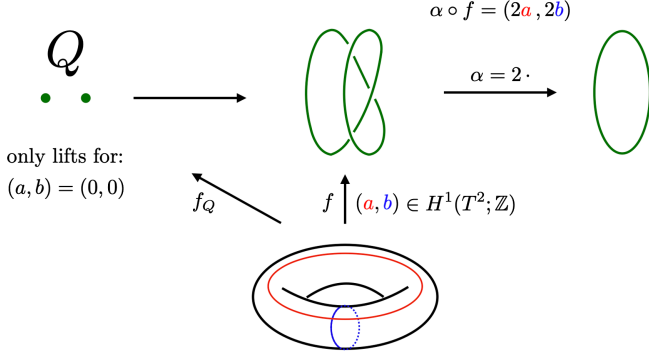


FIG. 2. Illustration of the homotopy fiber construction of the times n constraint for a periodic scalar field (for $X = T^2$ and α a double cover). Here, $f^*(v_1) = (a, b) \in H^1(T^2; \mathbb{Z})$ denotes the pullback of the fundamental class of the circle, v_1 , with respect to f .

$f^*(v_1) \in H^1(X; \mathbb{Z})$ needs to vanish mod n , in order to lift to Q , exactly as intended, see Fig. 3.

Below we will study more interesting examples where we modify the characteristic classes of $U(1)$ p -form fields, but before that let us describe Q in a way that will be more convenient for us. The sequence (3.1) can be extended to the left, leading to a description of Q itself given as the total space of a fibration

$$Z \longrightarrow Q \xrightarrow{p} BG. \quad (3.8)$$

The fiber Z is homotopically equivalent to ΩY the loop space of Y . Luckily, the Eilenberg-MacLane spaces have a nice behavior under Ω and one has

$$\Omega K(A, m) \simeq K(A, m-1). \quad (3.9)$$

With Q given as a total space of a fibration and the knowledge of the cohomology of the fiber and base, we can use spectral sequences techniques in order to extract the cohomology classes

$$H^*(Q; \mathbb{Z}), \quad (3.10)$$

which correspond to the new set of characteristic classes of the modified gauge theory (after pullback to X via f_Q^*).⁶ In the following we will find that while the restrictions are successfully implemented, there can be new characteristic classes, i.e., new conserved global charges, originating from the $K(A, m-1)$ fiber.

⁶An explicit way to construct Q is to consider a path fibration of $K(A, m)$ i.e. $K(A, m-1) \rightarrow \mathcal{P}K(A, m) \rightarrow K(A, m)$ where $\mathcal{P}K(A, m)$ is the space of paths in $K(A, m)$ starting at the based point and which is homotopically trivial. Q can be constructed as a pullback of this fibration with the map α , i.e., $Q \simeq \alpha^* \mathcal{P}K(A, m)$ and one automatically obtains the fibration structure [42].

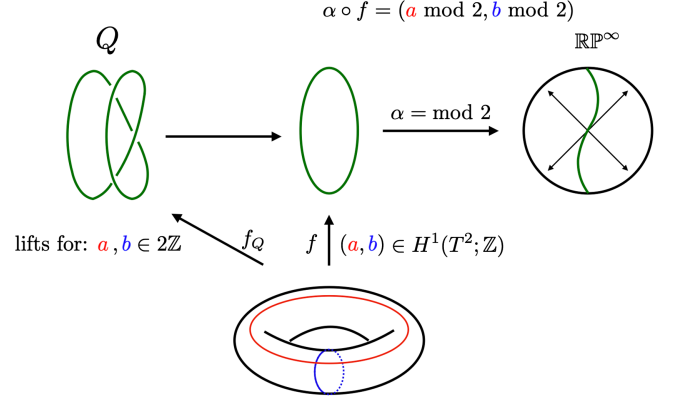


FIG. 3. Illustration of the homotopy fiber construction of the mod n constraint for a periodic scalar field (for $X = T^2$ and α the nontrivial element in $H^1(S^1; \mathbb{Z}_2)$, i.e., $n = 2$).

Even though the low degrees make the examples above somewhat special, let us try to apply this strategy. One has

$$\Omega K(\mathbb{Z}, 1) = K(\mathbb{Z}, 0) \simeq \mathbb{Z}, \quad (3.11)$$

This means that we obtain $\mathbb{Z}_n$ as a fibration of $\mathbb{Z}$ over S^1 . This can be made more explicit by noting that $\text{pt} \simeq \mathbb{R}$ and the fact that fibering $\mathbb{Z}$ over S^1 can turn it into a line. The fibration is defined such that going around the base circle relates points in $\mathbb{Z}$ which are n steps apart. One therefore obtains $\mathbb{R}^{\oplus n} \simeq \text{pt}^{\oplus n} \simeq \mathbb{Z}_n$ as the total space, see Fig. 4. Similarly, for the mod n constraint one has

$$\Omega K(\mathbb{Z}_n, 1) = K(\mathbb{Z}_n, 0) \simeq \mathbb{Z}_n, \quad (3.12)$$

and the fibration of Q , being the n -fold cover of the base circle has indeed a fiber given by $\mathbb{Z}_n$, reproducing the fibration structure depicted in Fig. 2. This relation of the two different fibration structures once more suggests the relation between the times n and mod n constraint.

Let us now move to the discussion of $U(1)$ 1-form gauge fields.

B. Warm-up example: Modification of c_1

As a warm-up example we consider the modification of the first Chern class in a $U(1)$ gauge theory as described in Sec. II C.

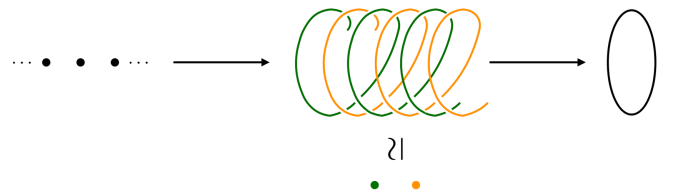


FIG. 4. Illustration of Q in times n constraint as $\mathbb{Z}$ fibered over S^1 .

For the times n constraint, we construct the homotopy fiber similar to the construction for the periodic scalar above⁷

$$Q \longrightarrow BU(1) \xrightarrow{\alpha} K(\mathbb{Z}, 2), \quad (3.13)$$

where the map α is multiplication by n in the following sense. From the discussion in Sec. II B, we know that the homotopy class of α is determined by

$$[BU(1), K(\mathbb{Z}, 2)] \simeq H^2(BU(1); \mathbb{Z}) \simeq \mathbb{Z}. \quad (3.14)$$

Let us denote the generator of $H^2(BU(1); \mathbb{Z})$ by v_2 , then we can identify the different maps α with an element nv_2 , which is obtained via pullback

$$\alpha^*(v_2) = nv_2 \in H^2(BU(1); \mathbb{Z}). \quad (3.15)$$

Thus, to lift to a map into Q , one needs

$$f^*(nv_2) = nf^*(v_2) = nc_1 = 0 \in H^2(X; \mathbb{Z}), \quad (3.16)$$

to be trivial in cohomology, as desired.

Let us start by considering $n = 1$, in which case we can choose α to be the identity map. However, this means that Q itself is a point, whose cohomology classes $H^*(\text{pt}; \mathbb{Z})$ in positive degree vanish identically. Indeed, this means that the new gauge theory with classifying space Q has no nontrivial characteristic classes. One nevertheless has a nontrivial theory since one needs to integrate over topologically trivial gauge fields, which in particular do not need to be flat ($F = 0$).⁸ Taking loop spaces one can rephrase Q as

$$K(\mathbb{Z}, 1) \simeq S^1 \rightarrow Q \rightarrow BU(1), \quad (3.17)$$

i.e., as a circle fibration over $BU(1)$ and another way to interpret Q is given by the universal bundle $EU(1)$ which is contractible and therefore can be identified with a point. Since further $S^1 \simeq B\mathbb{Z}$ and $Q \simeq B\mathbb{R}$ one can think of this fibration as a fibration of classifying spaces corresponding to the exact sequence of groups

$$0 \rightarrow \mathbb{Z} \hookrightarrow \mathbb{R} \rightarrow U(1) \rightarrow 0, \quad (3.18)$$

with an intriguing relation to gauge fields with gauge group $\mathbb{R}$, which recently featured in the description of symmetry topological field theories for continuous symmetries, see, e.g., [43,44].

⁷Even though $BU(1) \simeq K(\mathbb{Z}, 2)$, we use $BU(1)$ to distinguish between the base and the total space.

⁸In fact, in Euclidean space $\mathbb{R}^d$ one could not distinguish between the two versions of $U(1)$ gauge theory, since the cohomology of spacetime is trivial.

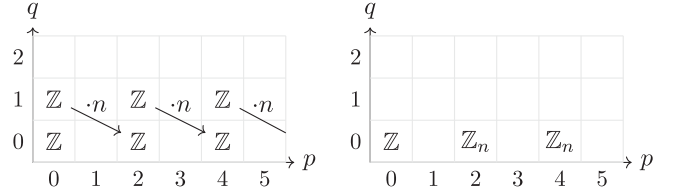


FIG. 5. $E_2^{p,q}$ and $E_\infty^{p,q}$ page of Q implementing $nc_1 = 0$.

Now consider $n > 1$, extending to the left we again find

$$K(\mathbb{Z}, 1) \simeq S^1 \rightarrow Q \rightarrow BU(1), \quad (3.19)$$

but with different fibration structure as compared to (3.17). With the cohomology of S^1 and $BU(1)$ known, we can apply the Leray-Serre spectral sequence (LSSS), [33,34] and [32,45] for recent applications, whose second page is given by

$$E_2^{p,q} = H^p(BU(1); H^q(S^1; \mathbb{Z})) \Rightarrow H^{p+q}(Q; \mathbb{Z}), \quad (3.20)$$

and summarized in Fig. 5. We also draw the differentials which indicate the nontrivial fibration and trivialize nc_1 . Since there can be no other differentials the $E_\infty^{p,q}$ page is fully determined and also depicted in Fig. 5. There are no extension problems and one finds the integer cohomology of Q to be given by

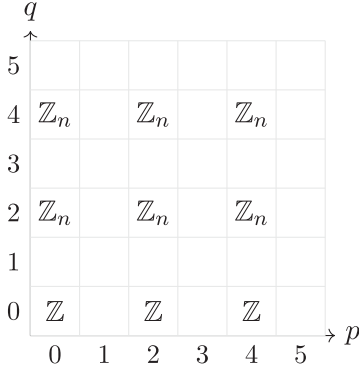
$$H^k(Q; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & k = 0, \\ \mathbb{Z}_n, & k > 0, \text{ even}, \\ 0, & \text{otherwise}. \end{cases} \quad (3.21)$$

This precisely matches our expectations that requiring nc_1 to be trivial also restricts the higher topological classes c_1^k to respect

$$nc_1^k = (nc_1)c_1^{k-1} = 0 \in H^{2k}(X; \mathbb{Z}). \quad (3.22)$$

In fact, we see that the cohomology groups of Q coincide with that of $K(\mathbb{Z}_n, 1) \simeq B\mathbb{Z}_n$ and the two theories have the same classes of topological sectors.⁹

⁹The fact that the cohomology groups agree is not sufficient to show that the two spaces are homotopy equivalent, but is necessary if they are. One way to see that these spaces agree follows from the fact that $B\mathbb{Z}_n$ is part of the fibration $B\mathbb{Z}_n \rightarrow BU(1) \rightarrow BU(1)$ induced by the exact sequence of groups $0 \rightarrow \mathbb{Z}_n \rightarrow U(1) \rightarrow U(1) \rightarrow 0$, so it can be realized as a total space of the same fibration as in (3.19). Since this fibration is fully classified by the Postnikov class $\alpha \in H^2(BU(1); \mathbb{Z}) \cong \mathbb{Z}$, i.e. by a choice of an integer $m \in \mathbb{Z}$, we see from the computation above that obtaining the desired cohomology one needs to take $m = n$ meaning that $B\mathbb{Z}_n$ is the total space of the same fibration as our Q so they must be equal up to homotopy.

FIG. 6. $E_2^{p,q} = E_\infty^{p,q}$ page of Q implementing $\mathbf{m}_n(c_1) = 0$.

The intuition behind this is as follows; the constraint $nc_1 = 0$, requires the image of c_1 in the free part of $H^2(X; \mathbb{Z})$ to be trivial. While this does not mean that the gauge configurations are flat, it means that each gauge configuration is continuously connected to a flat gauge field, which one can take as the representative of the topological sector. Thus, the topological data is fully specified by an element in $H^1(X; U(1))$. After specifying n the relevant data reduces to the image of $H^1(X; \mathbb{Z}_n)$ in $H^2(X; \mathbb{Z})$ under the Bockstein map $\beta_{\mathbb{Z}}$, see (2.11), precisely the topological class encoded by a $\mathbb{Z}_n$ gauge field.

Let us move to the mod n constraint, i.e., the trivialization of $\mathbf{m}_n(c_1) \in H^2(X; \mathbb{Z}_n)$. Here, we use a homotopy fiber constructed as

$$Q \rightarrow BU(1) \rightarrow K(\mathbb{Z}_n, 2). \quad (3.23)$$

The second map is defined such that it pulls back the generator $u_2 \in H^2(K(\mathbb{Z}_n, 2); \mathbb{Z}) \simeq \mathbb{Z}_n$ to the generator $\mathbf{m}_n(v_2)$ in $H^2(BU(1); \mathbb{Z}_n) \simeq \mathbb{Z}_n$. Extending to the left we find

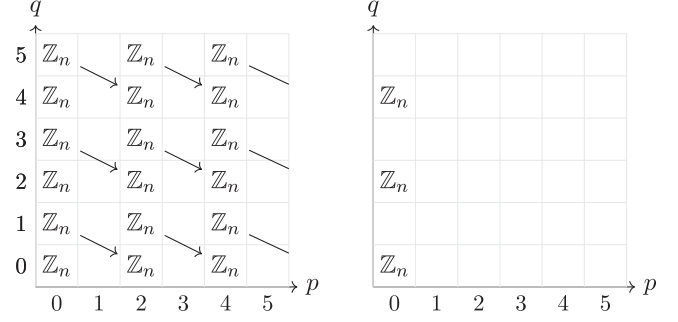
$$K(\mathbb{Z}_n, 1) \simeq B\mathbb{Z}_n \rightarrow Q \rightarrow BU(1), \quad (3.24)$$

which suggests that the constraint involves the topological classes of a $\mathbb{Z}_n$ 1-form field, whose classifying space appears as the fiber. Again, we know the cohomology groups of both base and fiber and can apply the LSSS, whose second page coincides with the ∞ -page and is provided in Fig. 6.

There can be no differentials and the only thing one needs to resolve are extension problems, since for example one has

$$0 \rightarrow \mathbb{Z} \rightarrow H^2(Q; \mathbb{Z}) \rightarrow \mathbb{Z}_n \rightarrow 0, \quad (3.25)$$

and there are several options, including $\mathbb{Z}$ and $\mathbb{Z} \oplus \mathbb{Z}_d$, where d divides n . To resolve this, we can run the same spectral sequence, but this time for cohomology with $\mathbb{Z}_n$ coefficients. For that, consider the LSSS with the second page:

FIG. 7. $E_2^{p,q}$ and $E_\infty^{p,q}$ of Q implementing $\mathbf{m}_n(c_1) = 0$ with $\mathbb{Z}_n$ coefficients.

$$E_2^{p,q} = H^p(BU(1); H^q(K(\mathbb{Z}_n, 1); \mathbb{Z}_n)) \Rightarrow H^{p+q}(Q; \mathbb{Z}_n), \quad (3.26)$$

given by Fig. 7. Because we want $H^2(BU(1); \mathbb{Z}_n) = E_2^{2,0} \rightarrow E_\infty^{2,0} \subset H^2(Q; \mathbb{Z}_n)$ to be zero we require the differential $d_2: E_2^{0,1} \rightarrow E_2^{2,0}$ to be an isomorphism. To determine other differentials we can use the ring structure of $H^*(K(\mathbb{Z}_n, 1); \mathbb{Z}_n)$ given by (B12) or (B13) and the fact that differentials respect the ring structure. To be more specific, with x and y denoting the generators of $H^*(B\mathbb{Z}_n; \mathbb{Z}_n)$ in degree 1 and 2, respectively, we require

$$d_2x = \mathbf{m}_n(v_2), \quad (3.27)$$

from which we see that $d_2(xv_2) = \mathbf{m}_n(v_2^2)$, which means that $d_2y = 0$. Thus, we obtained the full action of d_2 , which enables us to determine the third page E_3 , which in this case is the ∞ -page, see Fig. 7. From there we read off the cohomology groups

$$H^k(Q; \mathbb{Z}_n) = \begin{cases} \mathbb{Z}_n, & k \geq 0, \text{ even,} \\ 0, & \text{otherwise.} \end{cases} \quad (3.28)$$

Now using the universal coefficient theorem for cohomology, [46],

$$H^{2k-1}(Q; \mathbb{Z}_n) \simeq H^{2k-1}(Q; \mathbb{Z}) \otimes \mathbb{Z}_n \oplus \text{Tor}(H^{2k}(Q; \mathbb{Z}); \mathbb{Z}_n) \simeq 0 \quad (3.29)$$

we can solve the extension problem in Fig. 6 by realizing that if there were any finite part in $H^{2k}(Q; \mathbb{Z})$ the torsion functor in the universal coefficient theorem would be nontrivial, leading to a contradiction with the computation above. The conclusion is that one has the maximal non-trivial extension and the correct cohomology groups are

$$H^k(Q; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & k \geq 0, \text{ even,} \\ 0, & \text{otherwise.} \end{cases} \quad (3.30)$$

Now, these seem to be identical to $H^*(BU(1); \mathbb{Z})$ but the extensions lead to a different normalization. While pulled back from $BU(1)$ the topological class c_1^k takes arbitrary integer values they are divisible by n^k when lifting through Q . Thus, the $\mathbb{Z}$ -valued topological classes for Q are schematically given by $\frac{1}{n^k} c_1^k$ making the nontrivial extensions apparent. The fibration (3.24) can be interpreted as corresponding to the nontrivial extension of gauge groups:

$$0 \rightarrow \mathbb{Z}_n \hookrightarrow U(1) \xrightarrow{n} U(1) \rightarrow 0. \quad (3.31)$$

This extended $U(1)$ allows for smaller electric charges of the form $q \in \frac{1}{n} \mathbb{Z}$. Dirac quantization, then requires the magnetic fluxes to be divisible by n , precisely as imposed by the constraint $\mathbf{m}_n(c_1) = 0$.

Now that we have some more familiarity with the necessary techniques we can go to more interesting examples, where we modify powers of c_1 .

C. Modification of c_1^2

In this next set of examples we still remain within the realm of $U(1)$ gauge theories, but instead of restricting c_1 , we restrict c_1^2 , see also [47,48] for similar constructions. In analogy to the treatment above, we focus on the two situations

$$\begin{aligned} \text{times } n: nc_1^2 &= 0 \in H^4(X; \mathbb{Z}), \\ \text{mod } (n=2): \mathbf{m}_2(c_1^2) &= 0 \in H^4(X; \mathbb{Z}_2). \end{aligned} \quad (3.32)$$

Here, we restrict the discussion of the mod n restriction to the case $n=2$, since more is known about the associated Eilenberg-MacLane spaces, but a similar treatment would also be possible for different n .

In the times n case we consider a homotopy fiber of the form

$$Q \rightarrow BU(1) \xrightarrow{\alpha} K(\mathbb{Z}, 4). \quad (3.33)$$

From the discussion above we know that the homotopy classes of maps α are determined by an element in $H^4(BU(1); \mathbb{Z})$ which in the case above is given by nv_2^2 which is trivialized in Q . In particular, this means that pulling back the generator $v_4 \in H^4(K(\mathbb{Z}, 4); \mathbb{Z})$ to $BU(1)$ one has $\alpha^*(v_4) = nv_2^2$. Extending to the left we find

$$K(\mathbb{Z}, 3) \rightarrow Q \rightarrow BU(1), \quad (3.34)$$

which suggests the involvement of a $U(1)$ 2-form field, with classifying space $K(\mathbb{Z}, 3)$ appearing as the fiber. The integer cohomology of $K(\mathbb{Z}, 3)$ is known, see, e.g., [49], and summarized in the Appendix B. From this we can find the second page of the LSSS, see Fig. 8.

The nontriviality of the fibration is indicated by a differential d_4 , which trivializes nv_2^2 and higher order topological classes. We also know that the $\mathbb{Z}_2$ generator

in $E_2^{0,9}$ is given by the product of the element in $E_2^{0,6}$ and $E_2^{0,3}$, see [32,49], and with that we can argue that there is another differential acting as indicated. Note also that this action is only nontrivial for n odd [we denote the greatest common divisor of r and s , by (r, s)]. With this we can draw the relevant portion of $E_\infty^{p,q}$ to determine the cohomology groups of $H^*(Q; \mathbb{Z})$, which we also show in Fig. 8.

The only remaining questions are extension problems for the cohomology groups $H^6(Q; \mathbb{Z})$, $H^8(Q; \mathbb{Z})$, and $H^{10}(Q; \mathbb{Z})$. For that we can rerun the same spectral sequence but for cohomology with $\mathbb{Z}_2$ and $\mathbb{Z}_3$ coefficients. The first advantage is that there are no extension problems. The second is that it still can distinguish between the various possibilities for $H^*(Q; \mathbb{Z})$. The third is that by knowing the differentials on the lower degrees, using the ring and Steenrod algebra structure one can obtain all the differentials more easily.

For example, for n even, the two options¹⁰ for $H^6(Q; \mathbb{Z})$ are $\mathbb{Z}_2 \oplus \mathbb{Z}_n$ or $\mathbb{Z}_{2n}$; in the first case one would produce $H^6(Q; \mathbb{Z}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$, but in the second case one would find a single $\mathbb{Z}_2$ factor. For $\mathbb{Z}_2$ coefficients we find the second page of the LSSS sequence in Fig. 8. There is a nontrivial differential $d_4: E_4^{0,3} = E_2^{0,3} \rightarrow E_4^{0,4} = E_2^{0,4}$ only in the case of n odd. From this one can argue that for n even all relevant differentials are trivial. Denoting the generator of $H^3(K(\mathbb{Z}, 3); \mathbb{Z}_2) \simeq \mathbb{Z}_2$ by $\tilde{v}_3$, in the odd case, the new differential appears, since the element in $E_2^{0,8}$, $\tilde{v}_3 Sq^2 \tilde{v}_3$, is the product of the element in $E_2^{0,5} (Sq^2 \tilde{v}_3)$ and $E_2^{0,3} (\tilde{v}_3)$. We see that there is a difference depending whether n is even or odd and we have, for example

$$H^6(Q; \mathbb{Z}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_{(n,2)}, \quad (3.35)$$

which shows that

$$H^6(Q; \mathbb{Z}) = \mathbb{Z}_n \oplus \mathbb{Z}_2, \quad (3.36)$$

and solves the extension problem in degree 6. Similarly, we have

$$H^8(Q; \mathbb{Z}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_{(n,2)} \oplus \mathbb{Z}_{(n,2)}. \quad (3.37)$$

Tracking the factors, through various applications of the universal coefficient theorem demonstrates that $\mathbb{Z}_2$ appears as a separate factor in $H^8(Q; \mathbb{Z})$ and the remaining extension problem is the $\mathbb{Z}_3$ in $E_\infty^{0,8}$, which is only relevant if n is divisible by 3. To resolve the last piece of the extension problem we can use the spectral sequence for cohomology with $\mathbb{Z}_3$ coefficients, see Fig. 8. Analogous to the strategy with $\mathbb{Z}_2$ coefficients one can use the extra structure on the cohomology groups $H^*(BU(1); \mathbb{Z}_3)$ and $H^*(K(\mathbb{Z}, 3); \mathbb{Z}_3)$, namely of $\mathcal{A}_3$ -modules, where $\mathcal{A}_3$ is a

¹⁰For n odd these options are identical, since $\mathbb{Z}_n \oplus \mathbb{Z}_2 \cong \mathbb{Z}_{2n}$ (n odd).

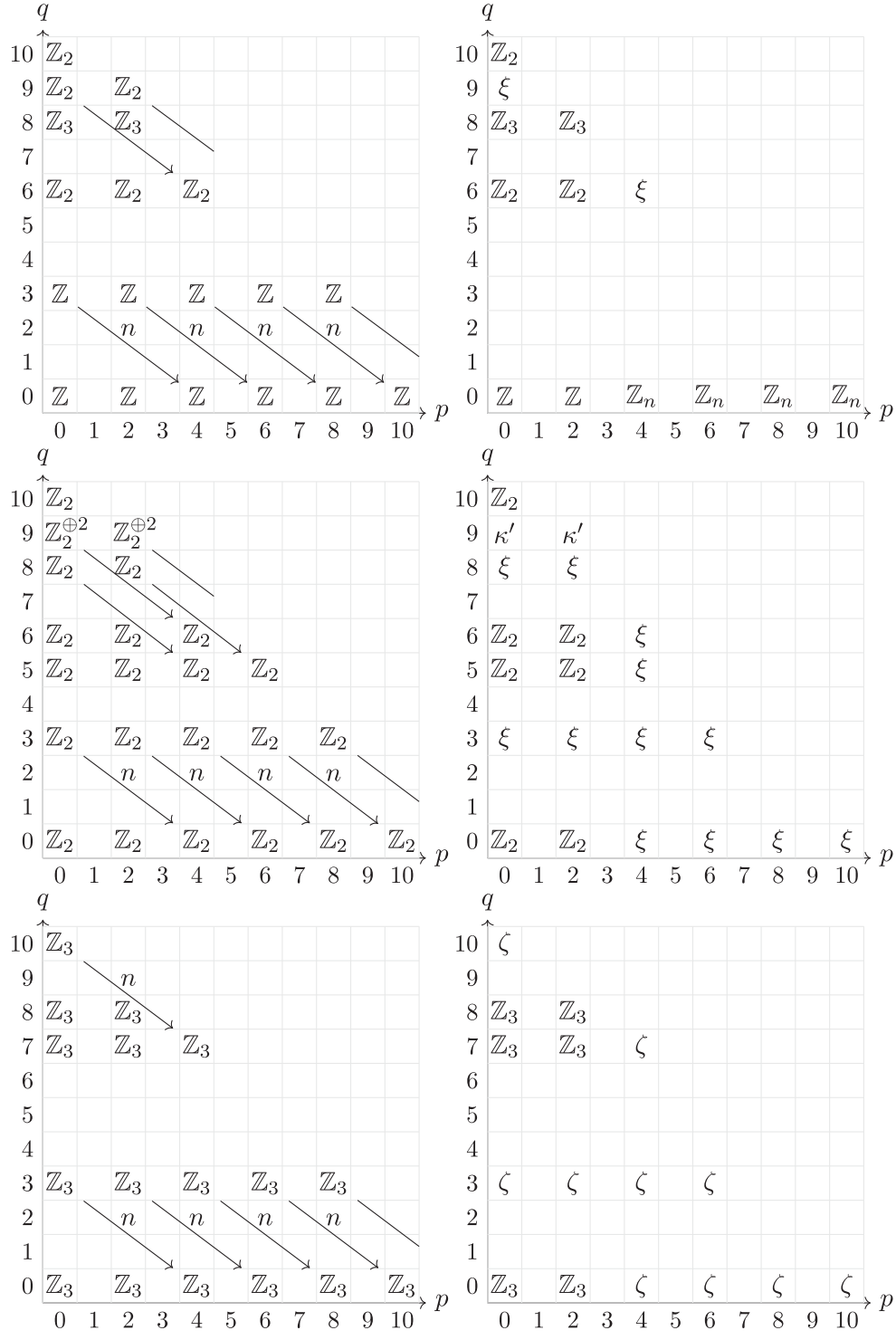


FIG. 8. Relevant entries for $E_2^{p,q}$ and $E_\infty^{p,q}$ of Q implementing $nc_1^2 = 0$ (only entries with $p + q \leq 10$ are indicated). Upper diagrams are for $\mathbb{Z}$ coefficients, middle ones for $\mathbb{Z}_2$ coefficients (where we denote $\mathbb{Z}_{(n,2)}$ by ξ and $\mathbb{Z}_{(n,2)} \oplus \mathbb{Z}$ by κ') and lower ones for $\mathbb{Z}_3$ coefficients (with $\mathbb{Z}_{(n,3)}$ labeled by ζ).

mod-3 Steenrod algebra.¹¹ As for $\mathbb{Z}_2$ coefficients, cohomology groups of Eilenberg-MacLane spaces with $\mathbb{Z}_3$ coefficients are freely generated graded-commutative algebras generated by the fundamental class and the admissible sequence of Steenrod operations acting on it [33]. In our case,

$$\begin{aligned} H^*(BU(1); \mathbb{Z}_3) &= \mathbb{Z}_3[v'_2], \\ H^*(K(\mathbb{Z}, 3); \mathbb{Z}_3) &= \mathbb{Z}_3[v'_3, P^1 v'_3, \beta P^1 v'_3, \dots], \end{aligned} \quad (3.38)$$

where v'_k denotes the mod 3 reduction of the generator $v_k \in H^k(K(\mathbb{Z}, k); \mathbb{Z})$. Since $d_4(v'_3) = n(v'_2)^2$ we have a nontrivial differential only in the case of n not being divisible by 3 and as above, one can argue that in this case all differentials are trivial. Otherwise there is an extra relevant

nontrivial differential $d_4(v'_3 P^1 v'_3) = n(v'_2)^2 P^1 v'_3$. One can just read off the $H^*(Q; \mathbb{Z}_3)$ from the infinity page. Most importantly one obtains

$$H^7(Q, \mathbb{Z}_3) = \mathbb{Z}_3 \oplus \mathbb{Z}_{(3,n)}. \quad (3.39)$$

Since from the universal coefficient theorem for cohomology

$$H^7(Q; \mathbb{Z}_3) = H^7(Q; \mathbb{Z}) \otimes \mathbb{Z}_3 \oplus \text{Tor}(H^8(Q; \mathbb{Z}); \mathbb{Z}_3) \quad (3.40)$$

one sees that we get a trivial extension of $\mathbb{Z}_3$ with $\mathbb{Z}_n$ in the cohomology of degree 8, and similarly in degree 10. Finally, we obtain the cohomology groups up to degree 10:

k	0	1	2	3	4	5	6	7	8	9	10
$H^k(Q; \mathbb{Z})$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}_n$	0	$\mathbb{Z}_n \oplus \mathbb{Z}_2$	0	$\mathbb{Z}_n \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_2$	$\mathbb{Z}_{(n,2)}$	$\mathbb{Z}_n \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{(n,2)}$

(3.41)

Also, since all the extensions were trivial we have that under the pullback map

$$k \geq 2: \quad \mathbb{Z}[v'_2] \cong H^{2k}(BU(1); \mathbb{Z}) \rightarrow H^{2k}(Q; \mathbb{Z}) \cong \mathbb{Z}_n \oplus \text{extra}, \quad (3.42)$$

the generator v'_2 , with $k \geq 2$, is sent to the generator of $\mathbb{Z}_n \subset H^{2k}(Q; \mathbb{Z})$. We can also understand the extra contributions, in particular the factors of $\mathbb{Z}_2$ and $\mathbb{Z}_3$ above are simply inherited from the fiber $K(\mathbb{Z}, 3)$ and thus describe topologically nontrivial configurations of a U(1) 2-form field. The factors $\mathbb{Z}_{(n,2)}$ are also inherited from the fiber, but only survive the differentials in case n is even. One therefore sees, that in these low degrees the characteristic classes can be associated to the modified U(1) classes and that of a U(1) 2-form gauge field. Further note that even in the case $n = 1$ the presence of the fiber generates new characteristic classes.

Now let us move to the mod n constraint for the particular case $n = 2$, for which the homotopy fiber reads

$$Q \rightarrow BU(1) \rightarrow K(\mathbb{Z}_2, 4), \quad (3.43)$$

where the generator $u_4 \in H^4(K(\mathbb{Z}_2, 4); \mathbb{Z})$ pulls back to $\mathbf{m}_2(v'_2) \in H^4(BU(1); \mathbb{Z}_2)$. Again, we can extend to the left to find a description of Q as fibration

$$K(\mathbb{Z}_2, 3) \rightarrow Q \rightarrow BU(1). \quad (3.44)$$

¹¹The mod-3 Steenrod algebra is generated by a Bockstein map $\beta: H^k(X; \mathbb{Z}_3) \rightarrow H^{k+1}(X; \mathbb{Z}_3)$ which correspond to a short exact sequence of groups $0 \rightarrow \mathbb{Z}_3 \rightarrow \mathbb{Z}_9 \rightarrow \mathbb{Z}_3 \rightarrow 0$ and Steenrod powers $P^i: H^k(X; \mathbb{Z}_3) \rightarrow H^{k+4i}(X; \mathbb{Z}_3)$, which satisfy the analogous properties to Steenrod squares (for more details see [8]).

Here, we find the classifying space for a $\mathbb{Z}_2$ 3-form field as a fiber. With the cohomology groups in [50] and summarized in the Appendix B, we can write down the second page of the LSSS, see Fig. 9. We can extract useful information about differentials and extensions from the spectral sequence with $\mathbb{Z}_2$ coefficients, which we draw in Fig. 10. We will not be able to fully pin down all the relevant differentials, but we will nevertheless manage to determine all the cohomology groups of Q in degree 10 and below as well as their relations to the characteristic classes of $BU(1)$. We use Serre's description of the cohomology $H^*(K(\mathbb{Z}_2, 3); \mathbb{Z}_2)$ as a polynomial algebra given in terms of Steenrod operations [33,34]. To be more precise, in terms of the generator $\tilde{u}_3 \in H^3(K(\mathbb{Z}_2, 3); \mathbb{Z}_2)$ one has

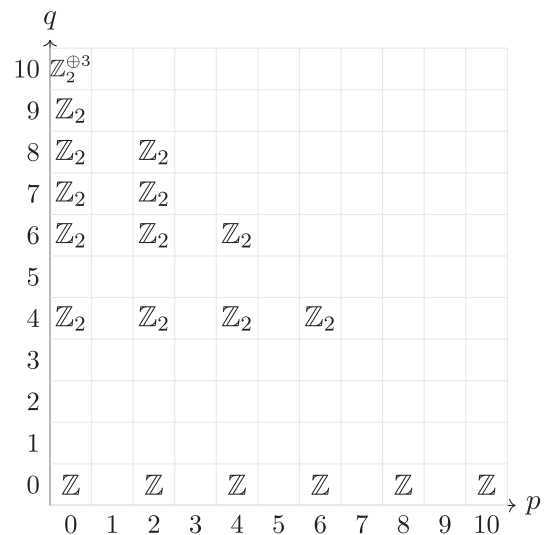


FIG. 9. $E_2^{p,q}$ page of Q implementing $\mathbf{m}_2(c_1^2) = 0$, for $p + q \leq 10$.

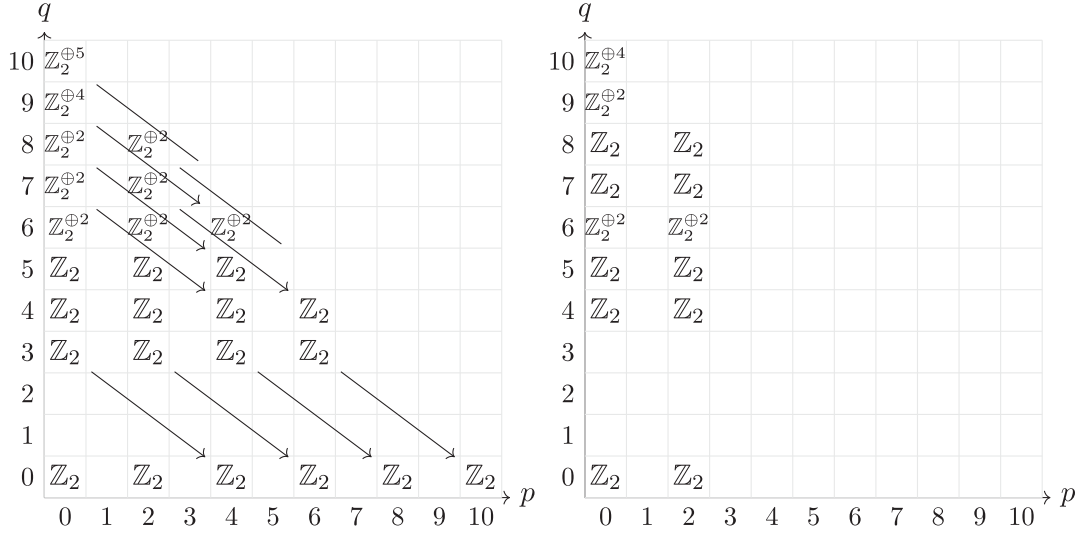


FIG. 10. $E_2^{p,q}$ and $E_\infty^{p,q}$ of Q implementing $\mathfrak{m}_2(c_1^2) = 0$ with $\mathbb{Z}_2$ coefficients and $p + q \leq 10$.

$$H^*(K(\mathbb{Z}_2, 3); \mathbb{Z}_2) = \mathbb{Z}_2[\tilde{u}_3, Sq^1\tilde{u}_3, Sq^2\tilde{u}_3, Sq^2Sq^1\tilde{u}_3, Sq^3Sq^1\tilde{u}_3, Sq^4Sq^2\tilde{u}_3, Sq^4Sq^2Sq^1\tilde{u}_3, \dots]. \quad (3.45)$$

By design, there is a nontrivial differential $d_4(\tilde{u}_3) = \mathfrak{m}_n(v_2^2)$. Using that d_4 is a differential with respect to the ring structure determines all the nontrivial differentials¹² in the Fig. 10. From the ∞ -page we read out the cohomology groups with $\mathbb{Z}_2$ coefficients:

k	0	1	2	3	4	5	6	7	8	9	10
$H^k(Q; \mathbb{Z}_2)$	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{\oplus 3}$	$\mathbb{Z}_2^{\oplus 2}$	$\mathbb{Z}_2^{\oplus 3}$	$\mathbb{Z}_2^{\oplus 3}$	$\mathbb{Z}_2^{\oplus 5}$

(3.46)

Comparing this result with the LSSS for $\mathbb{Z}$ coefficients in combination with (3.29), we can resolve some of the extension problems, similar to our strategy above. We find that the groups with integer coefficients have to be of the form

$$\begin{aligned} H^4(Q; \mathbb{Z}) &= \mathbb{Z}, & H^6(Q; \mathbb{Z}) &= \mathbb{Z} \oplus \mathbb{Z}_{2^{\ell_1}}, & H^8(Q; \mathbb{Z}) &= \mathbb{Z} \oplus \mathbb{Z}_{2^{\ell_2}}, \\ H^9(Q; \mathbb{Z}) &= \mathbb{Z}_{2^{\ell_3}}, & H^{10}(Q; \mathbb{Z}) &= \mathbb{Z} \oplus \mathbb{Z}_{2^{\ell_4}} \oplus \mathbb{Z}_{2^{\ell_5}}, \end{aligned} \quad (3.47)$$

and all that is left to determine are the explicit extensions encoded in the ℓ_i .

Since $H^4(Q; \mathbb{Z}) = \mathbb{Z}$ there is a nontrivial extension. Denoting the generator of the free factor in $H^{2k}(Q; \mathbb{Z})$ by α_{2k} , this means

$$p^*(v_2^2) = 2\alpha_4, \quad (3.48)$$

¹²For that, note that Sq^i commutes with the transgression map $\tau = d_r: E_r^{0,r-1} \rightarrow E_r^{r,0}$, which is the name for all differentials that take a class coming from the fiber and sends it to a class of the base [33,34]. From that one finds $\tau(Sq^i\tilde{u}_3) = Sq^i(\tau(\tilde{u}_3)) = Sq^i(\mathfrak{m}_2(v_2^2))$ which vanishes for Sq^1 and Sq^2 , which in our case is enough to show the triviality of the differential on the remaining elements in $H(K(\mathbb{Z}_2, 3); \mathbb{Z}_2)$, for $k \leq 10$.

where p is given by projection onto the base in (3.44). In degree 6, there are two possibilities, given by $\mathbb{Z} \oplus \mathbb{Z}_2$ and $\mathbb{Z} \oplus \mathbb{Z}_4$, the second case, however is inconsistent with the ring structure, since it would imply $p^*(v_2^3) = \alpha_6$ but one has

$$p^*(v_2^3) = p^*(v_2) \cup p^*(v_2^2) = 2\alpha_2\alpha_4, \quad (3.49)$$

which is clearly divisible by 2. Thus, there is a nontrivial extension, which is inherited from degree 4 and implements $p^*(v_2^3) = 2\alpha_6 = 2\alpha_2\alpha_4$. Using similar arguments one can show that, again, there are nontrivial extensions in degree 8, which can be inferred from

$$p^*(v_2^4) = p^*(v_2^2) \cup p^*(v_2^2) = 4\alpha_4^2, \quad (3.50)$$

which needs extensions with two $\mathbb{Z}_2$ factors and singles out

$$H^8(Q; \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}_2, \quad (3.51)$$

as the correct cohomology group.

In degree 9 we encounter yet another phenomenon; the appearance of a nontrivial differential in the LSSS with integer coefficients, which is not determined by the ring structure above. This differential takes care of one of the $\mathbb{Z}_2$ entries in the second page and one finds $H^9(Q; \mathbb{Z}) = \mathbb{Z}_2$. The other possible option would have been that one has a nontrivial enhancement to $\mathbb{Z}_4$. In both cases the universal coefficient theorem (3.29) leads to contributions for $H^8(Q; \mathbb{Z}_2)$ and $H^9(Q; \mathbb{Z}_2)$. However, since Sq^1 can be understood as the Bockstein homomorphism associated to the short exact sequence

$$0 \rightarrow \mathbb{Z}_2 \rightarrow \mathbb{Z}_4 \rightarrow \mathbb{Z}_2 \rightarrow 0, \quad (3.52)$$

its action is different, since one has

$$\dots \rightarrow H^8(Q; \mathbb{Z}_2) \rightarrow H^9(Q; \mathbb{Z}_2) \rightarrow H^9(Q; \mathbb{Z}_4) \rightarrow \dots \quad (3.53)$$

Denoting the relevant element in $H^8(Q; \mathbb{Z}_2)$ by b , one sees that for $H^9(Q; \mathbb{Z}) = \mathbb{Z}_4$, $Sq^1(b)$ needs to be trivial by exactness. For $H^9(Q; \mathbb{Z}) = \mathbb{Z}_2$ on the other hand, multiplication by 2 acts trivial and $Sq^1(b)$ is nontrivial. The problem is that, while we have the Steenrod algebra for $BU(1)$ and $K(\mathbb{Z}_2, 3)$, we do need the action of the Steenrod operations on the cohomology elements of Q . For our particular problem, here, we can use the spectral Steenrod operations¹³ developed by Araki [51]. While this only fully

determines the Steenrod operations on the associated graded groups induced by the filtration of $E_\infty^{p,q}$, it is enough for us. Let us denote the generator of $H^4(K(\mathbb{Z}_2, 3); \mathbb{Z}_2)$ by $\tilde{u}_3$, then one has

$$Sq^1: E_2^{2,6} \rightarrow E_2^{2,7} \text{ with } (Sq^2 Sq^1 \tilde{u}_3) \tilde{v}_2 \mapsto (Sq^3 Sq^1 \tilde{u}_3) \tilde{v}_2, \quad (3.54)$$

inducing an isomorphism. The spectral Steenrod operation make sure that this remains nontrivial on the ∞ -page, i.e., there is a nontrivial map

$$Sq^1: E_\infty^{2,6} \rightarrow E_\infty^{2,7}. \quad (3.55)$$

Since these factors are the building blocks of $H^*(Q; \mathbb{Z}_2)$ there is a nontrivial action also here. However, due to nontrivial extensions encoded in the fibration structure the action

$$Sq^1: H^8(Q; \mathbb{Z}_2) \rightarrow H^9(Q; \mathbb{Z}_2), \quad (3.56)$$

might also have contributions from other $E_\infty^{9-i,i}$, which here is only $E_\infty^{0,9}$. What remains, is that the nontrivial Sq^1 action demands that

$$H^9(Q; \mathbb{Z}) = \mathbb{Z}_2, \quad (3.57)$$

due to an additional (hidden) differential. This will also trivialize a $\mathbb{Z}_2$ entry in of $E_\infty^{10-i,i}$ and hence modify $H^{10}(Q; \mathbb{Z})$ accordingly. One can determine the cohomology in degree ten in the same way. At the level of spectral sequence there is a relevant nontrivial action of the Steenrod operation

$$Sq^1: E_2^{0,9} \rightarrow E_2^{0,10} \text{ with } Sq^4 Sq^2 \tilde{u}_3 \rightarrow Sq^5 Sq^2 \tilde{u}_3, \\ Sq^1 \tilde{u}_3 Sq^2 \tilde{u}_3 \rightarrow \tilde{u}_3 Sq^1 \tilde{u}_3, \quad (3.58)$$

which survives to the infinity page and implies the nontrivial action of the Steenrod operation

$$Sq^1: H^9(Q; \mathbb{Z}_2) \rightarrow H^{10}(Q; \mathbb{Z}_2), \quad (3.59)$$

on two $\mathbb{Z}_2$ summands inside of $H^9(Q; \mathbb{Z}_2)$. With the same argument as for degree 9, there are two $\mathbb{Z}_2$ summands in $H^{10}(Q; \mathbb{Z})$ and when compared with (3.47) one fixes the cohomology to

$$H^{10}(Q; \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \quad (3.60)$$

Thus, one has

¹³Spectral sequence Steenrod operations are operations defined on every page of a spectral sequence related to the fibration $F \rightarrow E \rightarrow B$ such that $Sq^k: E_r^{p,q} \rightarrow E_r^{p,q+k}$ if $k \leq q$ and $Sq^k: E_r^{p,q} \rightarrow E_t^{p+k-q,2q}$ if $q \leq k$ and t is an integer typically larger than r making Sq^k well defined, however in the case of $r = 2$ one can take $t = 2$. These operations commute with the differentials, meaning that by defining them on the second page the action on the later pages is directly induced. On the second page they are defined as $(1 \otimes Sq^k): H^p(B; \mathbb{Z}_2) \otimes H^q(F; \mathbb{Z}_2) \rightarrow H^p(B; \mathbb{Z}_2) \otimes H^{q+k}(F; \mathbb{Z}_2)$ if $k \leq q$ and $Sq^{k-q} \otimes Sq^q: H^p(B; \mathbb{Z}_2) \otimes H^q(F; \mathbb{Z}_2) \rightarrow H^{p+k-q}(B; \mathbb{Z}_2) \otimes H^{2q}(F; \mathbb{Z}_2)$, where $Sq^i \otimes Sq^j$ are standard Steenrod operations acting on the cohomology of the base and the fiber, respectively. These spectral Steenrod operations do not fully determine the action of the Steenrod algebra on the cohomology of the total space, but converges to it in the sense that the spectral Steenrod operations on the infinity page are equal to those induced from the Steenrod operations on the total space, e.g., $Sq^k: E_\infty^{p,q} \cong \frac{F^p H^{p+q}(E; \mathbb{Z}_2)}{F^{p+1} H^{p+q}(E; \mathbb{Z}_2)} \rightarrow E_\infty^{p,q+r} \cong \frac{F^p H^{p+q+r}(E; \mathbb{Z}_2)}{F^{p+1} H^{p+q+r}(E; \mathbb{Z}_2)}$. For more details see [51].

$$H^k(Q; \mathbb{Z}) \mid \begin{array}{cccccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ \mathbb{Z} & 0 & \mathbb{Z} & 0 & \mathbb{Z} & 0 & \mathbb{Z} \oplus \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z} \oplus \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \end{array}, \quad (3.61)$$

describing the cohomology groups of Q up to degree 10.

As for $BU(1)$ we find factors of $\mathbb{Z}$ in even degrees, however, now these are normalized differently as is encoded in the nontrivial extensions. For example consider the factor of $\mathbb{Z}$ in degree 8, which is extended by a factor of 4, (3.50), and the generator in terms of the original first Chern class of the $U(1)$ bundle is given by

$$\frac{1}{4}c_1^4 \in H^8(Q; \mathbb{Z}). \quad (3.62)$$

We would expect the same factor 4 for the generator of $\mathbb{Z}$ in degree 10, which however depends on the existence of a nontrivial differential from elements in $E_j^{11-r,r}$ which go beyond our discussion here.

Beyond this change of normalization as compared to the characteristic classes of $BU(1)$ we also find additional factors of $\mathbb{Z}_2$. From the second page of the LSSS, Fig. 9, these can be traced back to characteristic classes of the fiber $K(\mathbb{Z}_2, 3)$, as well as combinations of classes from fiber and base. Once more we find that the modification of the gauge theory leads to desired redefinition of characteristic classes in combination with the introduction of new ones.

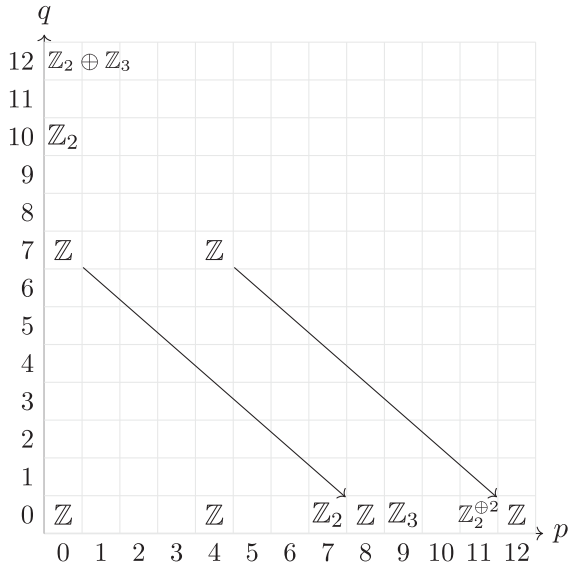


FIG. 11. $E_2^{p,q}$ page of Q implementing $n[N^{(4)}]^2 = 0$ in the range $p + q \leq 12$.

D. Modification of classes of higher-form fields

Since in the discussion above we have focused on the modification of $U(1)$ 1-form fields, we want to illustrate the applicability of the homotopy fiber construction also for higher-form fields, we do so for the modification of a $U(1)$ 3-form field with $[N^{(4)}]$ as its topological class, in particular we consider the constraints

$$\begin{aligned} \text{times } n: n[N^{(4)}]^2 &= 0 \in H^8(X; \mathbb{Z}), \\ \text{mod } (n=2): \mathbf{m}_2([N^{(4)}]^2) &= 0 \in H^8(X; \mathbb{Z}_2), \end{aligned} \quad (3.63)$$

in analogy to the cases above.

First, let us discuss the times n constraint, with homotopy fiber Q described by

$$K(\mathbb{Z}, 7) \rightarrow Q \rightarrow K(\mathbb{Z}, 4). \quad (3.64)$$

We provide the cohomology groups of the fiber and base in the Appendix B, which leads to the second page of the LSSS indicated in Fig. 11. The only low degree differentials act by multiplication by n and we can immediately read off the integer cohomology groups of Q up to degree 9. For degrees 10, 11, and 12 we need to determine whether there are nontrivial differentials $d_{11}: E_{11}^{0,10} \rightarrow E_{11}^{11,0}$, or $d_{13}: E_{13}^{0,12} \rightarrow E_{13}^{13,0}$. Beyond that, there is a nontrivial extension problem in degree 12, where one has

$$0 \rightarrow \mathbb{Z}_n \rightarrow H^{12}(Q; \mathbb{Z}) \rightarrow \mathbb{Z}_2 \oplus \mathbb{Z}_3 \rightarrow 0. \quad (3.65)$$

As above, this can be nontrivial only in cases when n is divisible by 2 or 3. To resolve these issues we once more can use the LSSS with $\mathbb{Z}_2$ and $\mathbb{Z}_3$ coefficients, see Fig. 12 and Fig. 13.

In Fig. 12 we show the second and the ∞ -page. The $\mathbb{Z}_2$ version of the differential implementing the constraint is given by

$$d\tilde{v}_7 = \mathbf{m}_2(nv_4^2) = n\tilde{v}_4^2, \quad (3.66)$$

where $\tilde{v}_i$ generates $H^i(K(\mathbb{Z}, i); \mathbb{Z}_2) \simeq \mathbb{Z}_2$ and v_i generates $H^i(K(\mathbb{Z}, i); \mathbb{Z}) \simeq \mathbb{Z}$. All other differentials follow from the Steenrod algebra structure. In particular, one has with transgression map τ ,

$$\tau(Sq^4\tilde{v}_7) = Sq^4(\tau(\tilde{v}_7)) = nSq^4(\tilde{v}_4^2) = n(Sq^2\tilde{v}_4)^2, \quad (3.67)$$

which is nontrivial for n odd. This leads to the nontrivial differential indicated by the dashed arrow in Fig. 12. This allows us to obtain

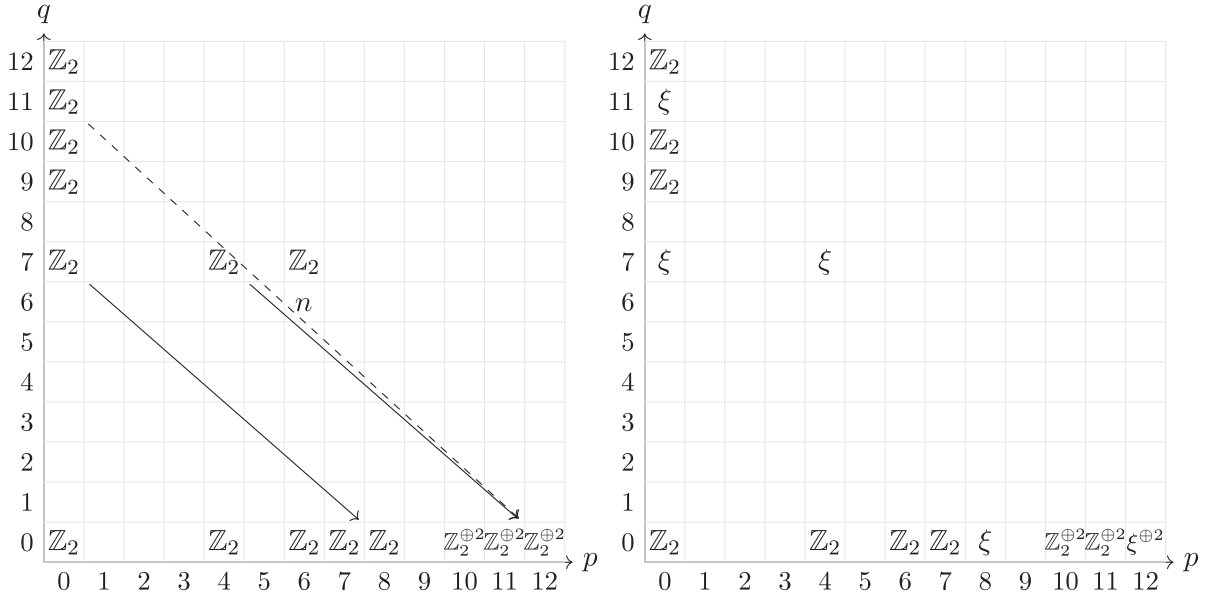


FIG. 12. $E_2^{p,q}$ and $E_\infty^{p,q}$ of Q implementing $n[N^{(4)}]^2 = 0$ with $\mathbb{Z}_2$ coefficients and $p + q \leq 12$ ($\xi = \mathbb{Z}_{(n,2)}$).

k	4	5	6	7	8	9	10	11	12
$H^k(Q; \mathbb{Z}_2)$	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_{(n,2)}$	$\mathbb{Z}_{(n,2)}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{\oplus 3}$	$\mathbb{Z}_2^{\oplus 2} \oplus \mathbb{Z}_{(n,2)}^{\oplus 2}$	$\mathbb{Z}_2 \oplus \mathbb{Z}_{(n,2)}^{\oplus 2}$

(3.68)

from the ∞ -page with $\mathbb{Z}_2$ coefficients.

Finally, to resolve the extensions for the prime $p = 3$ parts, we use the spectral sequence with $\mathbb{Z}_3$ coefficients, Fig. 13, together with the mod 3 Steenrod algebra of the Eilenberg-MacLane space (remember that P^1 increases the degree by 4), where we denote the generators of $H^i(K(\mathbb{Z}, i); \mathbb{Z}_3)$ by v'_i ,

$$\begin{aligned} H^*(K(\mathbb{Z}, 4); \mathbb{Z}_3) &= \mathbb{Z}_3[v'_4, P^1 v'_4, \beta P^1 v'_4, \dots], \\ H^*(K(\mathbb{Z}, 7); \mathbb{Z}_3) &= \mathbb{Z}_3[v'_7, P^1 v'_7, \beta P^1 v'_7, \dots]. \end{aligned} \quad (3.69)$$

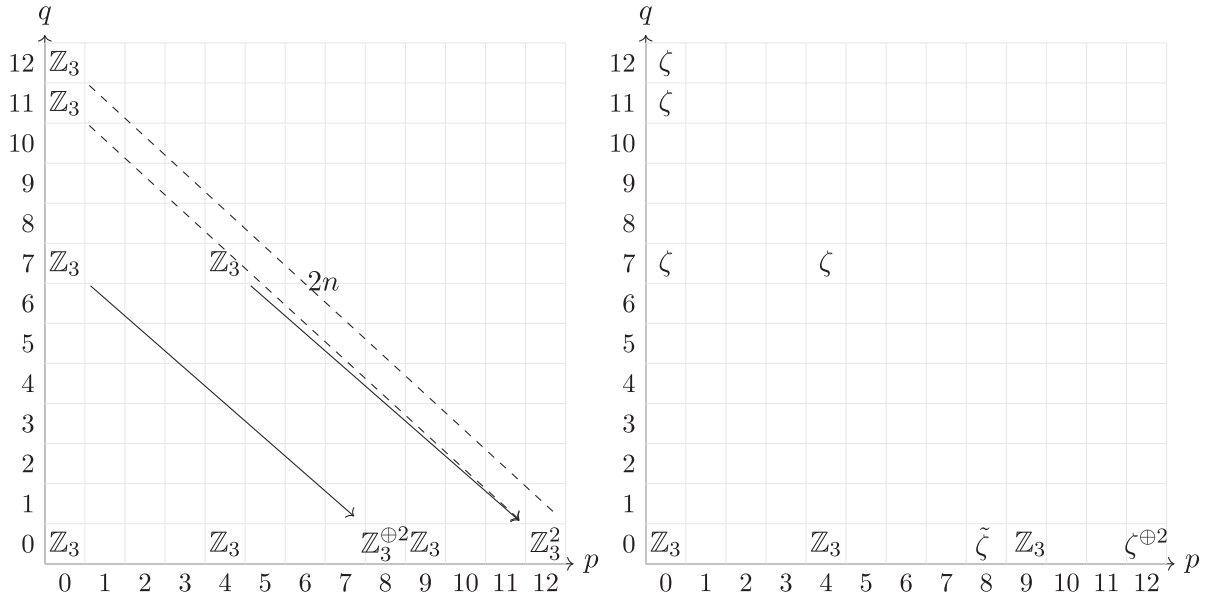


FIG. 13. $E_2^{p,q}$ and $E_\infty^{p,q}$ of Q implementing $n[N^{(4)}]^2 = 0$ with $\mathbb{Z}_3$ coefficients and $p + q \leq 12$ ($\zeta = \mathbb{Z}_{(n,3)}$ and $\tilde{\zeta} = \mathbb{Z}_3 \oplus \mathbb{Z}_{(n,3)}$).

Further using $d_8 v'_7 = \tau(v'_7) = \mathfrak{m}_3(n v_4^2) = n(v'_4)^2$ we find that all differentials are trivial for n divisible by 3 and otherwise use that Steenrod operations commute with transgression. So for example one has

$$\begin{aligned} d_{11}(P^1 v'_7) &= \tau(P^1 v'_7) = P^1(\tau(v'_7)) = nP^1(v'_4)^2 = 2nv'_4 P^1 v'_4, \\ d_{12}(\beta P^1 v'_7) &= 2nv'_4 \beta P^1 v'_4, \end{aligned} \quad (3.70)$$

which are both nontrivial for n not divisible by 3, and are indicated by the dashed lines in Fig. 13. With this we find

$$\begin{array}{c|cccccccccc} k & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ \hline H^k(Q; \mathbb{Z}_3) & \mathbb{Z}_3 & 0 & 0 & \mathbb{Z}_{(n,3)} & \mathbb{Z}_3 \oplus \mathbb{Z}_{(n,3)} & \mathbb{Z}_3 & 0 & \mathbb{Z}_{(n,3)}^{\oplus 2} & \mathbb{Z}_{(n,3)}^{\oplus 3} \end{array} \quad (3.71)$$

from the ∞ -page.

With this knowledge we can return to cohomology with integer coefficients, fixing extension problems with the application of the universal coefficient theorem. We obtain

$$\begin{array}{c|cccccccccc} k & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ \hline H^k(Q; \mathbb{Z}) & \mathbb{Z} & 0 & 0 & \mathbb{Z}_2 & \mathbb{Z}_n & \mathbb{Z}_3 & \mathbb{Z}_2 & \mathbb{Z}_2^{\oplus 2} & \mathbb{Z}_n \oplus \mathbb{Z}_{(n,2)} \oplus \mathbb{Z}_{(n,3)} \end{array} \quad (3.72)$$

as the final result.

We see that the cohomology groups are sensitive to the prime factors of n , since some differentials become trivial in case n is divisible by 2 or 3. However, this only happens in degree 12, and below we find a straightforward identification of characteristic classes of Q and in terms of $K(\mathbb{Z}, 4)$ and $K(\mathbb{Z}, 7)$. Especially, we note that for $k \geq 2$, v_4^k pulls back to the generator of the $\mathbb{Z}_n$ factor of $H^{4k}(Q; \mathbb{Z})$ under the projection, which is very similar what we saw for the modification of c_1^2 above.

Now we consider the mod 2 constraint of the 3-form gauge field.¹⁴ We have the homotopy fiber Q described by

$$Q \rightarrow K(\mathbb{Z}, 4) \rightarrow K(\mathbb{Z}_2, 8), \quad (3.73)$$

where u_8 , the generator of $H^8(K(\mathbb{Z}_2, 8); \mathbb{Z})$, pulls back to $\mathfrak{m}_2(v_4^2) \in H^8(K(\mathbb{Z}, 4); \mathbb{Z}_2)$. Extending to the left leads to the more useful fibration

$$K(\mathbb{Z}_2, 7) \rightarrow Q \rightarrow K(\mathbb{Z}, 4). \quad (3.74)$$

With the cohomology groups summarized in the Appendix B, this fibration leads to second page of the LSSS indicated in Fig. 14. To determine differentials and extensions we can use the same approach as described for the other cases above, in particular, the switch to $\mathbb{Z}_2$ coefficients, see Fig. 15. From this we can read off the cohomology groups with $\mathbb{Z}_2$ coefficients:

$$\begin{array}{c|cccccccccc} k & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ \hline H^k(Q; \mathbb{Z}_2) & \mathbb{Z}_2 & 0 & \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2^{\oplus 4} & \mathbb{Z}_2^{\oplus 3} & \mathbb{Z}_2^{\oplus 3} \end{array} \quad (3.75)$$

This helps to determine the extensions and one finds

$$\begin{array}{c|cccccccccc} k & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ \hline H^k(Q; \mathbb{Z}) & \mathbb{Z} & 0 & 0 & \mathbb{Z}_2 & \mathbb{Z} & \mathbb{Z}_3 & \mathbb{Z}_2 & \mathbb{Z}_2^{\oplus 3} & \mathbb{Z} \end{array} \quad (3.76)$$

a rather rich structure for the cohomology groups with

¹⁴The case of n odd prime can be treated analogously using the mod- p Steenrod algebra.

integer coefficients, which indicates a nontrivial extension for the $\mathbb{Z}$ factor in degree 8 and 12.

This nontrivial extension also helps for the physical interpretation of the results, by pulling back classes from the base $K(\mathbb{Z}; 4)$ to the homotopy fiber Q , using the associated projection map p . For example, we have

$$p^*(v_4) = \alpha_4, \quad p^*(v_4^2) = 2\alpha_8, \quad (3.77)$$

where α_{4i} are generators of the $\mathbb{Z}$ factor of $H^{4i}(Q; \mathbb{Z})$. The specific realization of $p^*(v_4^3)$ requires a full knowledge of the differentials from elements in $E_j^{13-r,r}$, which goes

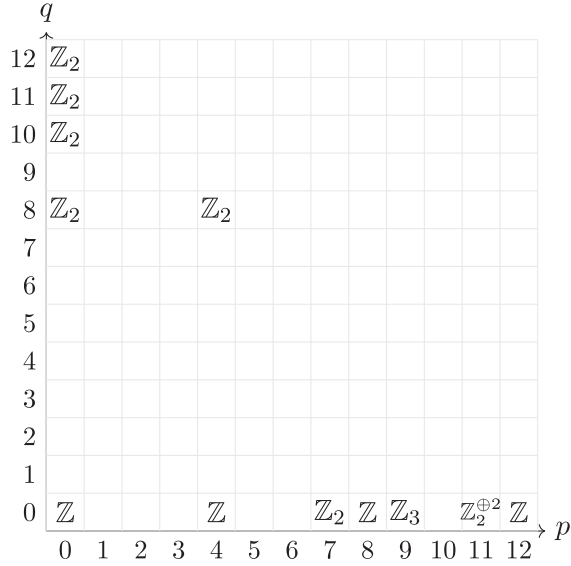


FIG. 14. $E_2^{p,q}$ for Q implementing $m_2([N^{(4)}]^2) = 0$, for $p + q \leq 12$.

beyond the our scope here. From the physics interpretation we would expect $p^*(v_4^3) = 2\alpha_{12}$, inherited from $p^*(v_4^2)$. If this is indeed correct it does require the existence of a nontrivial differential.

E. Multiple constraints

In this section we want to explore what happens if we implement more than one constraint. This can be done by consecutive constructions of homotopy fibers in the following sense. Let us start with BG on which we want to restrict two specific topological classes of the gauge bundle

via the homotopy fiber construction. Then we implement the first constraint via the fibration

$$Q_1 \rightarrow BG \rightarrow Y_1, \quad (3.78)$$

after which we can implement the second constraint on Q_1

$$Q \rightarrow Q_1 \rightarrow Y_2. \quad (3.79)$$

Equivalently, we can switch the order of implementation with the two fibers given by

$$\begin{aligned} Q_2 &\rightarrow BG \rightarrow Y_2, \\ \tilde{Q} &\rightarrow Q_2 \rightarrow Y_1. \end{aligned} \quad (3.80)$$

That these two procedures lead to the same result, i.e., that Q is homotopy equivalent to $\tilde{Q}$, can be seen by combining this two-step process into a single fibration. Suppose we aim to trivialize the topological classes $\alpha \in H^n(BG; A)$ and $\beta \in H^m(BG; B)$, represented by maps labeled by the same name

$$\begin{aligned} \alpha: BG &\rightarrow K(A, n) = Y_1, \\ \beta: BG &\rightarrow K(B, m) = Y_2. \end{aligned} \quad (3.81)$$

In the first step we construct a fibration $Q_1 \xrightarrow{p_1} BG$ for trivializing the class α and then, in order to further trivialize the class β in BG , we implement the homotopy fiber for the class represented by $\pi = \beta \circ p_1$ in $H^m(Q_1; B)$ that leads to the fibration $Q \xrightarrow{p} Q_1$ as follows:

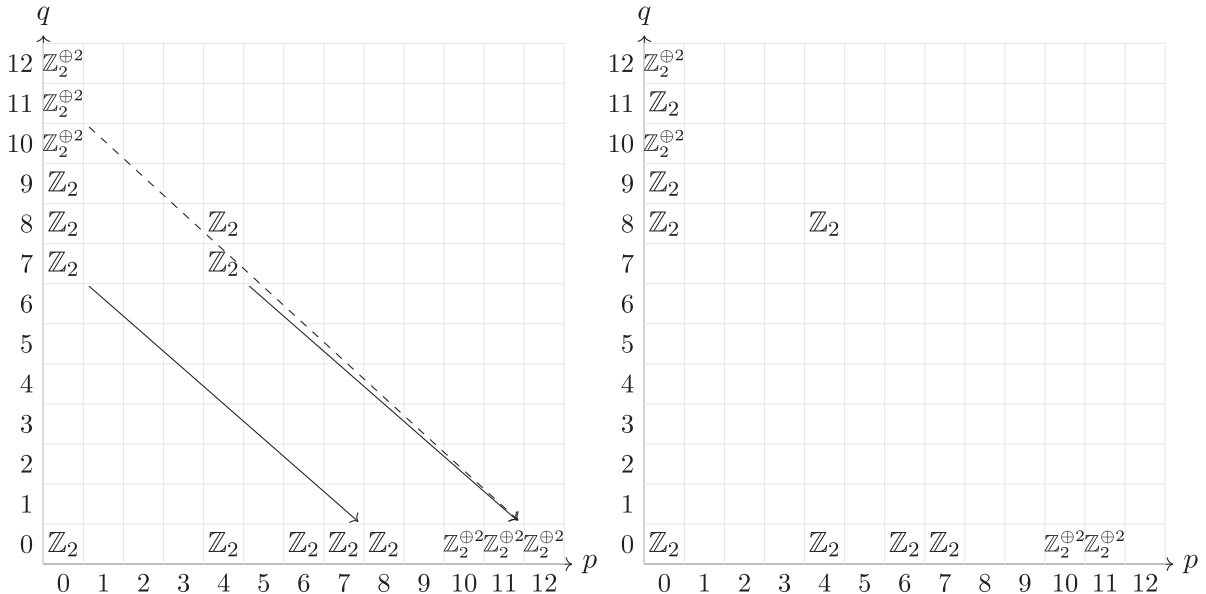


FIG. 15. $E_2^{p,q}$ and $E_\infty^{p,q}$ of Q implementing $m_2([N^{(4)}]^2) = 0$ with $\mathbb{Z}_2$ coefficients and $p + q \leq 12$.

$$\begin{array}{ccc}
K(B, m-1) & \longrightarrow & Q \\
& & \downarrow p \\
K(A, n-1) & \xrightarrow{i} & Q_1 \xrightarrow{\pi} K(B, m) \\
& & \downarrow p_1 \quad \nearrow \beta \\
& & BG \xrightarrow{\alpha} K(A, n).
\end{array} \quad (3.82)$$

We can combine this sequence of fibrations into a single fibration $Q \xrightarrow{p_1 \circ p} BG$ with fiber Z given by a fibration of $K(B, m-1)$ over $K(A, n-1)$ twisted by a class $i^*\pi \in H^m(K(A, n-1); B)$

$$\begin{array}{ccc}
K(B, m-1) & \longrightarrow & Z \\
& & \downarrow \\
K(A, n-1) & \xrightarrow{i^*\pi} & K(B, m).
\end{array} \quad (3.83)$$

Since from the diagram (3.82) one sees that $\pi \circ i \simeq \beta \circ p_1 \circ i$ is homotopy trivial since $p_1 \circ i$ is homotopy trivial as maps in a fibration, the class that measures the twisting in the fibration above $i^*\pi = 0$ is trivial. This means that Z is homotopy equivalent to a direct product leading to

$$\begin{array}{ccc}
K(A, n-1) \times K(B, m-1) & \longrightarrow & Q \\
& & \downarrow \\
& & BG \xrightarrow{\alpha \times \beta} K(A, n) \times K(B, m).
\end{array} \quad (3.84)$$

Analogously, by doing the procedure in the opposite order, one would arrive at the same fibration leading to the conclusion that the order does not matter and we can even do it simultaneously, with the fiber given by the product $\Omega Y_1 \times \Omega Y_2$.

Note that the product structure of the fiber is due to the fact that we impose both constraints in the same space BG . Alternatively, one could impose one constraint on BG and another, independent one, on the homotopy fiber Q_1 . This new constraint can now involve classes that did not exist in BG , which for example originate from the fiber of Q_1 . If this is the case, the result would not be a product space and there might be nontrivial twists. In the following, we will focus on the situation where we impose constraints on the same space BG .

Let us illustrate this in a simple example, where we impose two different constraints in various ways. For that we consider gauge theory with a $U(1)$ 1-form and impose the two constraints

$$\begin{aligned}
mc_1 &= 0 \in H^2(X; \mathbb{Z}), \\
m_n(c_1) &= 0 \in H^2(X; \mathbb{Z}_n),
\end{aligned} \quad (3.85)$$

i.e., we simultaneously impose a times m and mod n constraint.

If we first construct the homotopy fiber Q_1 to trivialize $m_n(v_2)$ [as usual v_2 is the generator of $H^2(BU(1); \mathbb{Z})$] we end up with the space discussed in (3.23), for which we already computed the cohomology groups $H^*(Q_1; \mathbb{Z}) \cong \mathbb{Z}[x]$, with $x \in H^2(Q_1; \mathbb{Z})$ a generator in degree two. Under the pullback with respect to $p_1: Q_1 \rightarrow BU(1)$ one has

$$p_1^*(v_2) = nx. \quad (3.86)$$

Then, imposing the constraint $mv_2 = 0 \in H^2(BU(1); \mathbb{Z})$ translates to the constraint

$$p_1^*(mv_2) = mp_1^*(v_2) = mnx = 0 \in H^2(Q_1; \mathbb{Z}), \quad (3.87)$$

on cohomology classes on Q_1 , defining the second homotopy fiber to be given by

$$Q \rightarrow Q_1 \xrightarrow{\cdot nm} K(\mathbb{Z}, 2), \quad (3.88)$$

where $\cdot nm$ is understood in the same way as in (3.13). Using the result of the analysis there, we find

$$H^k(Q; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & k = 0, \\ \mathbb{Z}_{mn}, & k > 0, \text{ even}, \\ 0, & \text{otherwise.} \end{cases} \quad (3.89)$$

From the full construction we can follow the fate of the generator $v_2 \in H^2(BU(1); \mathbb{Z})$ and its powers, through the pullbacks using the projection maps

$$p^*(p_1^*(v_2^k)) = p^*(p_1^*(v_2)^k) = p^*(n^k x^k) = n^k y^k \quad (3.90)$$

where the generator $y = p^*(x) \in H^2(Q; \mathbb{Z})$ satisfies $nmy = 0$, and equivalently for its powers. Note that even in the case $m = n$ one does not fully trivialize c_1 since one has

$$(p_1 \circ p)^* v_2 = ny, \quad (3.91)$$

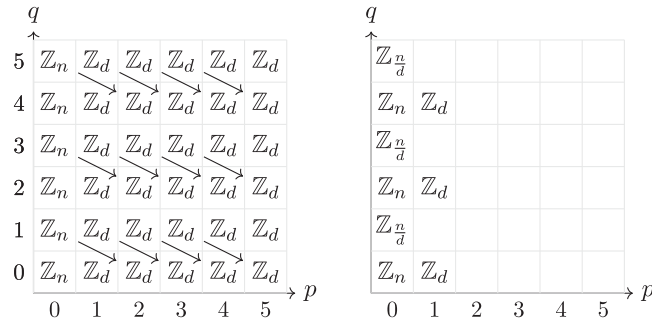


FIG. 16. $E_2^{p,q}$ and $E_\infty^{p,q}$ of $\tilde{Q}$ implementing $\mathfrak{m}_n(c_1) = 0$ after $mc_1 = 0$ with $\mathbb{Z}_n$ coefficients [$d = (n, m)$].

which generically is nonzero since the triviality condition for $n = m$ reads $n^2y = 0$.¹⁵

Now let us implement the two constraints in different order. First, we construct the fiber Q_2 imposing $mc_1 = 0$, which we have already considered in (3.13), and leading to

$$H^k(Q_2; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & k = 0, \\ \mathbb{Z}_m, & k > 0, \text{ even}, \\ 0 & \text{otherwise.} \end{cases} \quad (3.92)$$

We denote the generator of $H^2(Q_2; \mathbb{Z}) \simeq \mathbb{Z}_m$ by x_2 , whose powers also generate $H^{2i}(Q; \mathbb{Z})$ and satisfies

$$p_2^*(v_2) = x_2. \quad (3.93)$$

The second constraint $\mathfrak{m}_n(v_2) = 0$ is lifted to

$$\mathfrak{m}_n(x_2) = 0 \in H^2(Q_2; \mathbb{Z}), \quad (3.94)$$

in terms of cohomology classes of Q_2 . As usual we implement that by the homotopy fiber construction obtaining

$$K(\mathbb{Z}_n, 1) \rightarrow \tilde{Q} \rightarrow Q_2, \quad (3.95)$$

where $\tilde{Q}$ implements both constraints. Let us determine the LSSS with $\mathbb{Z}_n$ coefficients, which leads to the second page

$$E_2^{p,q} = H^p(Q_2; H^q(K(\mathbb{Z}_n, 1); \mathbb{Z}_n)) \Rightarrow H^{p+q}(\tilde{Q}; \mathbb{Z}_n), \quad (3.96)$$

in Fig. 16. Combining this with the spectral sequence with integer coefficients, Fig. 17, one deduces that

$$H^k(\tilde{Q}; \mathbb{Z}_n) = \mathbb{Z}_n, \quad k \geq 0, \quad (3.97)$$

¹⁵To get an intuition for that, consider a manifold with $H^2(X; \mathbb{Z}) = \mathbb{Z}_{n^2}$, then there are elements that are zero when multiplied by n but are also n times an element. In particular these are given by the $\mathbb{Z}_n \hookrightarrow \mathbb{Z}_{n^2}$ subgroup.

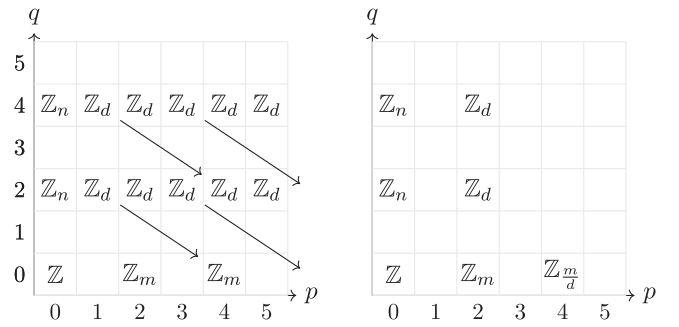


FIG. 17. $E_2^{p,q}$ (left) and $E_\infty^{p,q}$ (right) page of Q implementing $\mathfrak{m}_n(c_1) = 0$ after $mc_1 = 0$ with $\mathbb{Z}$ coefficients [$d = (n, m)$].

due to nontrivial extensions. This in turn determines the cohomology with integer coefficients

$$H^k(\tilde{Q}; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & k = 0, \\ \mathbb{Z}_{mn}, & k > 0, \text{ even}, \\ 0, & \text{otherwise,} \end{cases} \quad (3.98)$$

as necessary for $Q \simeq \tilde{Q}$.

Finally, we implement both constraints simultaneously using

$$Q \rightarrow BU(1) \xrightarrow{((mv_2, \mathfrak{m}_n(v_2)))} K(\mathbb{Z}, 2) \times K(\mathbb{Z}_n, 2), \quad (3.99)$$

leading to a fiber given by a product of Eilenberg-MacLane spaces

$$K(\mathbb{Z}, 1) \times K(\mathbb{Z}_n, 1) \rightarrow Q \xrightarrow{p} BU(1). \quad (3.100)$$

For the sake of simplicity we focus on the case $m = n$ and use Künneth theorem to obtain the cohomology groups of the fiber as well as its ring structure, from the two individual factors (e.g., [52])

$$\begin{aligned} H^*(K(\mathbb{Z}, 1) \times K(\mathbb{Z}_n, 1); A) \\ = H^*(K(\mathbb{Z}, 1); A) \otimes H^*(K(\mathbb{Z}_n, 1); A), \end{aligned} \quad (3.101)$$

for a field A , e.g., one can choose $A = \mathbb{Z}_p$ with p prime. Here, because cohomology of $K(\mathbb{Z}, 1) \simeq S^1$ is free, we can even use this for $\mathbb{Z}$ and $\mathbb{Z}_n$, since there are no extra terms.¹⁶ Starting with $\mathbb{Z}_n$ coefficients we find Fig. 18, where the differentials are given in terms of the (2×2) -matrices

¹⁶In general one has, [8],

$$\begin{aligned} 0 \rightarrow \oplus_i (H_i(X; R) \otimes_R H_{k-i}(Y; R)) \rightarrow H_k(X \times Y; R) \\ \rightarrow \oplus_i \text{Tor}_R(H_i(X; R); H_{k-i-1}(Y; R)) \rightarrow 0, \end{aligned} \quad (3.102)$$

for a principle ideal domain R , which splits, and similar for cohomology.

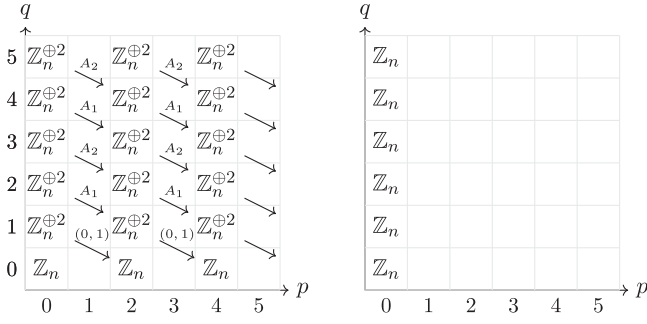


FIG. 18. $E_2^{p,q}$ and $E_\infty^{p,q}$ of Q implementing $\mathfrak{m}_n(c_1) = 0$ and $nc_1 = 0$ simultaneously, with $\mathbb{Z}_n$ coefficients.

$A_i \in M_2(\mathbb{Z}_n)$. Choosing the basis

$$\{\tilde{v}_1 \tilde{u}_j, \tilde{u}_{j+1}\} \in H^{j+1}(K(\mathbb{Z}, 1) \times K(\mathbb{Z}_n, 1); \mathbb{Z}_n), \quad (3.103)$$

where $\tilde{v}_1$ generates $H^1(K(\mathbb{Z}, 1); \mathbb{Z}_n) \simeq \mathbb{Z}_n$ and $\tilde{u}_j$ generates $H^j(K(\mathbb{Z}_n, 1); \mathbb{Z}_n)$, see also (B12) and (B13), these are given by¹⁷

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad (3.104)$$

encoding the differential $d_2 \tilde{u}_1 = \mathfrak{m}_n(v_2)$. From that it is apparent that

$$H^k(Q; \mathbb{Z}) = \mathbb{Z}_n, \quad k \geq 0. \quad (3.105)$$

Combining this with the LSSS with integer coefficients in Fig. 19, one finds that there are nontrivial differentials d_2 as well as d_4 . This leads to the final result

$$H^k(Q; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & k = 0, \\ \mathbb{Z}_{n^2}, & k > 0, \text{ even}, \\ 0, & \text{otherwise}, \end{cases} \quad (3.106)$$

precisely as expected in the case $n = m$. Moreover, pulling back under the projection map for the product fiber P , we can track the fate of the of the characteristic classes of $BU(1)$

$$\begin{aligned} P^*(v_2) &= nx \in H^2(Q; \mathbb{Z}) \simeq \mathbb{Z}_{n^2}, \\ P^*(v_2^k) &= n^k x^k = 0 \in H^{2k}(Q; \mathbb{Z}) \simeq \mathbb{Z}_{n^2} \end{aligned} \quad (3.107)$$

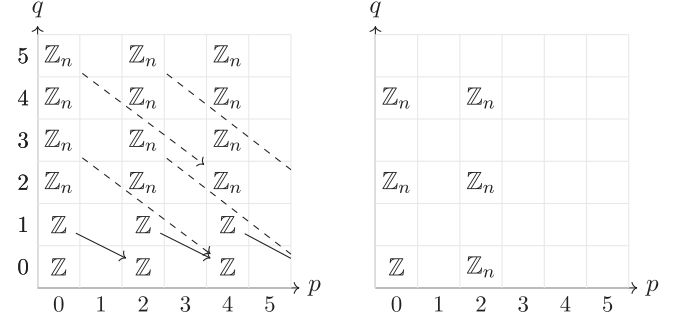


FIG. 19. $E_2^{p,q}$ and $E_\infty^{p,q}$ of Q implementing $\mathfrak{m}_n(c_1) = 0$ and $nc_1 = 0$ simultaneously with $\mathbb{Z}$ coefficients.

with x generating $H^2(Q; \mathbb{Z})$. Again, this matches our expectations from the stepwise implementation for the special case $m = n$.

We close this section with another illustrative and important example for the implementation of two constraints, now in different degrees. Assume, we want to modify a 1-form $U(1)$ gauge theory by

$$nc_1 = 0 \quad \text{and} \quad nc_1^2 = 0. \quad (3.108)$$

Since the first condition implies the second, one would naively expect to obtain the same modified classifying space Q_1 as in (3.13), i.e., that only of the constraint $nc_1 = 0$. However, since we still introduce the fiber for the second constraint

$$K(\mathbb{Z}, 3) \rightarrow Q \rightarrow Q_1, \quad (3.109)$$

which is now trivially fibered, one finds the product space

$$Q \simeq Q_1 \times K(\mathbb{Z}, 3). \quad (3.110)$$

Thus, the characteristic classes are enhanced to include those of an additional $U(1)$ 2-form gauge field, that does not appear in the Bianchi identities of the modified gauge theory. Using Künneth theorem, now with potential torsion [8], we find

k	0	1	2	3	4	5	6	7
$H^k(Q; \mathbb{Z})$	$\mathbb{Z}$	0	$\mathbb{Z}_n$	$\mathbb{Z}$	$\mathbb{Z}_n$	$\mathbb{Z}_n$	$\mathbb{Z}_n \oplus \mathbb{Z}_2$	$\mathbb{Z}_n \oplus \mathbb{Z}_{(n,2)}$

(3.111)

as the cohomology groups with integer coefficient.

¹⁷The elements for $p > 0$ are given by multiplication with the appropriate power of $\mathfrak{m}_n(v_2^k)$.

IV. BORDISM GROUPS AND ANOMALIES

The modification of the gauge theory can also have interesting and important consequences for the possible anomalies of the theory. In fact, excluding gravitational anomalies, the group cohomology of Q , that we determined above, is precisely what captures such an anomaly for bosonic theories, see, e.g., [53]. However, once we include diffeomorphisms as a symmetry¹⁸ of the system anomalies are described by the Anderson dual of the associated bordism group, [54,55]. The Anderson dual, $(I_{\mathbb{Z}}\Omega^{\xi})^{D+2}(Q)$, is given by the short exact sequence [24,37]

$$\begin{aligned} 0 \rightarrow \text{Ext}(\Omega_{D+1}^{\xi}(Q); \mathbb{Z}) \rightarrow (I_{\mathbb{Z}}\Omega^{\xi})^{D+2}(Q) \\ \rightarrow \text{Hom}(\Omega_{D+2}^{\xi}(Q); \mathbb{Z}) \rightarrow 0, \end{aligned} \quad (4.1)$$

where D denotes the spacetime dimension of the theory of interest and ξ its tangential structure, i.e., a restriction of the allowed spacetime manifolds. For a bosonic theory for example one can choose to work on oriented spacetime manifolds, $\xi = \text{SO}$, in the presence of fermions one needs to include a spin structure, $\xi = \text{spin}$, or generalizations thereof. In the following we will focus on these two possibilities. We see that the relevant bordism groups are $\Omega_{D+1}^{\text{SO/spin}}(Q)$ and $\Omega_{D+2}^{\text{SO/spin}}(Q)$, which roughly capture the potential nonperturbative and perturbative anomalies, respectively.

Since the bordism groups, as deformation classes of smooth manifolds with classifying maps into Q , depend on the modification of the gauge theory, also anomalies generically change. This becomes obvious when considering the Atiyah-Hirzebruch spectral sequence (AHSS) for oriented and spin bordism, whose second page is given by

$$E_{p,q}^2 = H_p(Q; \Omega_q^{\text{SO/spin}}(\text{pt})) \Rightarrow \Omega_{p+q}^{\text{SO/spin}}(Q). \quad (4.2)$$

Since Q has different (co)homology groups when compared to the original classifying space BG , also the bordism groups and hence the associated potential anomalies can be different. From the spectral sequence we further see that if $H_*(Q; A)$ changes only in degree k and above, the bordism groups can change earliest in degree $(k-1)$.

In the oriented case, there is a significant simplification due to the statement that all differentials in the spectral sequence for oriented bordism groups have an image given by odd torsion groups and also extensions only affect these odd torsion pieces [56,57], implying that

$$\Omega_k^{\text{SO}}(Q) \cong \oplus_p H_p(Q; \Omega_{k-p}^{\text{SO}}(\text{pt})) \quad \text{modulo odd torsion}, \quad (4.3)$$

¹⁸This is not yet necessarily gravity, for which we would also have to make the spacetime manifold dynamical and sum over different configurations in the path integral.

as a sum of the homology group of Q . This once more underlines the importance of the (co)homology groups in the evaluation of anomalies.

Before we perform some explicit calculations of the oriented and spin bordism groups of some of the theories above, let us provide a very simple example: Consider $U(1)$ gauge theory with fermions in $D = 2$, which has a pure gauge anomaly captured by

$$\frac{1}{2} c_1^2 \in \text{Hom}(\Omega_4^{\text{spin}}(BU(1)); \mathbb{Z}). \quad (4.4)$$

In fact this presents a generator, in that the minimal value (e.g., for $S^2 \times S^2$ with single unit of flux on each sphere) is given by 1. Now let us implement the mod n constraint on c_1 with new classifying space Q . While it is still true that $\frac{1}{2} c_1^2$ is an element of $\text{Hom}(\Omega_4^{\text{spin}}(Q); \mathbb{Z})$ it is not a generator any more since we can divide it by n^2 and still obtain an integer, changing the anomaly.¹⁹ This easy example also suggests that it is interesting to compare the anomalies of the modified theory with the original one in order to be sensitive to such relative factors.

Now let us move to some more interesting examples.

A. Bordism groups for modified c_1^2

Let us focus on the bordism groups for Maxwell theory with a modification of c_1^2 .

As above we start the discussion with the times n constraint, $nc_1^2 = 0$. With our knowledge of the integer homology groups of Q , and the homology groups with $\mathbb{Z}_2$ coefficients after application of the universal coefficient theorem (e.g., [52]), given by

k	$H_k(Q; \mathbb{Z})$	$H_k(Q; \mathbb{Z}_2)$	
0	$\mathbb{Z}$	$\mathbb{Z}_2$	
1	0	0	
2	$\mathbb{Z}$	$\mathbb{Z}_2$	
3	$\mathbb{Z}_n$	$\mathbb{Z}_{(n,2)}$	
4	0	$\mathbb{Z}_{(n,2)}$	
5	$\mathbb{Z}_n \oplus \mathbb{Z}_2$	$\mathbb{Z}_{(n,2)} \oplus \mathbb{Z}_2$	
6	0	$\mathbb{Z}_{(n,2)} \oplus \mathbb{Z}_2$	
7	$\mathbb{Z}_n \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_2$	$\mathbb{Z}_{(n,2)} \oplus \mathbb{Z}_2$	
8	$\mathbb{Z}_{(n,2)}$	$\mathbb{Z}_{(n,2)}^{\oplus 2} \oplus \mathbb{Z}_2$	
9	$\mathbb{Z}_n \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{(n,2)}$	$\mathbb{Z}_{(n,2)}^{\oplus 3} \oplus \mathbb{Z}_2$	(4.5)

¹⁹A fermionic spectrum with anomaly for $BU(1)$ would still be anomalous for Q , but due to the additional n^2 prefactor might be easier to cancel.

we can apply the Atiyah-Hirzebruch spectral sequence (see [18] for an introduction) in order to obtain information about $\Omega_k^{SO/spin}(Q)$. To determine the second page we also need to provide the groups $\Omega_k^{SO/spin}(pt)$, which up to degree 8 are given by

k	0	1	2	3	4	5	6	7	8
$\Omega_k^{SO}(pt)$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	0	0	$\mathbb{Z} \oplus \mathbb{Z}$
$\Omega_k^{spin}(pt)$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z} \oplus \mathbb{Z}$

(4.6)

With the knowledge of the bordism groups of a point we can easily find the second page of the AHSS for $\Omega_k^{SO}(Q)$, see Fig. 20. Below degree 7 we do not need to worry about differentials or extensions and find²⁰

k	0	1	2	3	4	5	6
$\Omega_k^{SO}(Q)$	$\mathbb{Z}$	0	$\mathbb{Z}$	$\mathbb{Z}_n$	$\mathbb{Z}$	$\mathbb{Z}_n \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}$
$\Omega_k^{SO}(BU(1))$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z} \oplus \mathbb{Z}$

(4.7)

which clearly differs from $\Omega_k^{SO}(BU(1))$, which we also indicate. The topological classes that detect the $\mathbb{Z}_n$ factors in degree 3 and 5 are $m_n(c_1^2)$ and $m_n(c_1^3)$, the mod n reduction of powers of the Chern class in cohomology.²¹ The two factors of $\mathbb{Z}_2$ in degree 5 are detected by $w_2 \cup N^{(3)}$ and $w_2 \cup w_3$, where w_i denotes the Stiefel-Whitney classes of the tangent bundle of X and $N^{(3)}$ the pullback of the fiber class $v_3 \in H^3(K(\mathbb{Z}, 3); \mathbb{Z})$ of Q to X . The $\mathbb{Z}$ factors in degree 4 and 6 are inherited from the unmodified U(1) theory.

For example, one generator of $\Omega_6^{SO}(BU(1))$ in dimension 6 given by $\mathbb{CP}^3$, with c_1 given by the generator of $H^2(\mathbb{CP}^3; \mathbb{Z})$, does not survive the modification, since now one needs to satisfy

$$nc_1^3 = c_1(nc_1^2) = 0, \quad (4.8)$$

which would not be satisfied for $\mathbb{CP}^3$ with the indicated flux. Instead one only finds a remnant $\mathbb{Z}_n$ factor of the pure gauge anomaly that now shifts into $\Omega_5^{SO}(Q)$, indicating that it becomes a nonperturbative anomaly. From its origin in the spectral sequence we see that it is associated to the element $H_5(Q; \mathbb{Z})$, which, since it is torsion, appears as part of $H^6(Q; \mathbb{Z})$, which can be identified with $m_n(c_1^3)$. From this logic we find, that a new generator can be chosen to be L_n^5 , a five-dimensional lens space with a flat U(1) gauge bundle classified by $c_1 \in H^2(L_n^5; \mathbb{Z}) \simeq \mathbb{Z}_n$. This version of an anomaly interplay, [58,59], resembles that of a $\mathbb{Z}_n$ gauge anomaly²² with the difference that the mixed gauge

gravitational piece is still $\mathbb{Z}$ -valued and perturbative. Thus it seems that, at least when it comes to the description of topological sectors, the resulting theory has features of both a U(1) and $\mathbb{Z}_n$ gauge theory.

Another way to understand this modification is as an incomplete anomaly cancellation mechanism, in which not the full U(1) pure gauge anomaly is canceled, but it is reduced to a remnant $\mathbb{Z}_n$. The mixed gauge gravitational anomaly on the other hand is unmodified, since it would require the homotopy fiber to be sensitive to gravitational characteristic classes, e.g., $p_1 \in H^4(X; \mathbb{Z})$, and would rather be described by a twisted string structure (see, e.g., [29,63–65] and references therein).

Now let us move to the evaluation of $\Omega_k^{spin}(Q)$ for which the second page of the AHSS is depicted in Fig. 21. Again, this looks very different from that of Maxwell theory see, e.g., [18]. Let us focus on the anomalies for theories in dimension $D \leq 4$, for which we are interested in $\Omega_k^{spin}(Q)$ with $k \leq 6$.

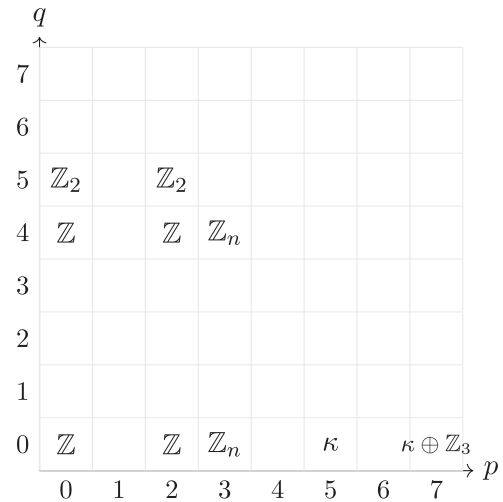


FIG. 20. $E_2^{p,q}$ page for oriented bordism of Q implementing $nc_1^3 = 0$ up to order $p + q \leq 7$ (κ denotes $\mathbb{Z}_n \oplus \mathbb{Z}_2$).

²⁰One can argue that differentials in the higher degrees are also trivial since the image has to be odd, see (4.3), but there could possibly be nontrivial extensions. For instance, in degree 7 in case n is odd there is an extension problem of the form $0 \rightarrow \mathbb{Z}_n \rightarrow E \subset \Omega_7^{SO}(Q) \rightarrow \mathbb{Z}_n \oplus \mathbb{Z}_3 \rightarrow 0$.

²¹Note the usual shift in degree between torsion in homology and cohomology.

²²See [18,22,28,31,60–62] for some of the works on $\mathbb{Z}_n$ anomalies and their cancellation.

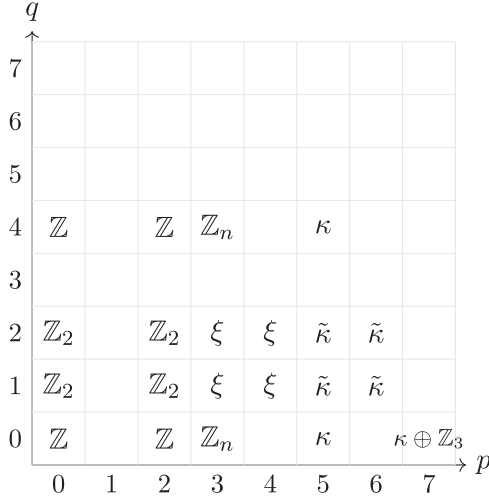


FIG. 21. $E_2^{p,q}$ for AHSS for spin bordism of Q implementing $nc_1^2 = 0$ up to order $p + q \leq 7$ (ξ denotes $\mathbb{Z}_{(n,2)}$, κ denotes $\mathbb{Z}_n \oplus \mathbb{Z}_2$, and $\tilde{\kappa}$ denotes $\mathbb{Z}_{(n,2)} \oplus \mathbb{Z}_2$).

Since $H_*(Q; A)$, for Abelian A , is modified earliest in degree three and there are no nontrivial differentials from the diagonal $p + q = 3$ the bordism groups with $k \in \{0, 1, 2\}$ are unchanged and isomorphic to that of $BU(1)$.

The first change happens for $k = 3$, where still the differentials are trivial but one encounters the extension problem for even n

$$0 \rightarrow \mathbb{Z}_2 \rightarrow \Omega_3^{\text{spin}}(Q) \rightarrow \mathbb{Z}_n \rightarrow 0. \quad (4.9)$$

In higher degrees differentials can also be nontrivial and we encounter the typical problem of deducing their action. Luckily, since Q is the total space of a fibration, we can employ another approach, see also [32].

The alternative version of the AHSS is given by, see [32],

$$E_{p,q}^2 = H_p(BU(1); \Omega_q^{\text{spin}}(K(\mathbb{Z}, 3))) \Rightarrow \Omega_{p+q}^{\text{spin}}(Q). \quad (4.10)$$

We calculate the spin bordism of $K(\mathbb{Z}, 3)$ in the Appendix C, with result up to degree 7;

k	0	1	2	3	4	5	6	7
$\Omega_k^{\text{spin}}(K(\mathbb{Z}, 3))$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}$	0	0	$\mathbb{Z}$

(4.11)

With that we can determine the second page, which we summarize in Fig. 22.

Since the cohomology of $K(\mathbb{Z}, 3)$ is concentrated in degree 3 and above, one has the isomorphism

$$\Omega_k^{\text{spin}}(\text{pt}) \simeq \Omega_k^{\text{spin}}(K(\mathbb{Z}, 3)), \quad \text{for } k \in \{0, 1, 2\}. \quad (4.12)$$

This also means that the differentials in the lowest two rows are given by the dual of the Steenrod square operation, e.g., [66, 67], which we know how to evaluate. The fibration of $K(\mathbb{Z}, 3)$ over $BU(1)$, on the other hand is, as inherited from the LSSS above, encoded in a differential on the fourth page $E_{p,q}^4$. More explicitly, since $\Omega_3^{\text{spin}}(K(\mathbb{Z}, 3)) \simeq H_3(K(\mathbb{Z}, 3))$, we can use the homological version of the LSSS to show that the differential there is essentially multiplication by n

$$H_4(BU(1); \mathbb{Z}) \simeq \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \simeq H_3(K(\mathbb{Z}, 3)), \quad (4.13)$$

and analogously for $H_6(BU(1); \mathbb{Z})$. This induces the action of the differential on $E_{4,0}^4 = \ker(d_2)$, given by the even elements in $H_4(BU(1); \mathbb{Z})$ and $E_{6,0}^4 \simeq H_6(BU(1); \mathbb{Z})$. The result is shown in Fig. 22, from which we can read off the spin bordism groups in degree 5 and below

k	0	1	2	3	4	5
$\Omega_k^{\text{spin}}(Q)$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z} \oplus \mathbb{Z}_2$	$\mathbb{Z}_{2n}$	$\mathbb{Z}$	$\mathbb{Z}_n$

(4.14)

While we find the same $\mathbb{Z}_n$ as for oriented bordism above, the $\mathbb{Z}_n$ in degree is extended to $\mathbb{Z}_{2n}$. This is a remnant of the fact that c_1^2 is divisible by 2 on spin 4-manifolds (the same does not hold for c_1^3). Similarly, the $\mathbb{Z}$ factor in degree 6 now is detected by $\frac{1}{48} p_1$ as opposed to the oriented case. For the two $\mathbb{Z}_2$ factors, coming from $\Omega^{\text{spin}}(\text{pt})$, in degree 1 and 2 one further needs to include fermionic classes, like η -invariants.

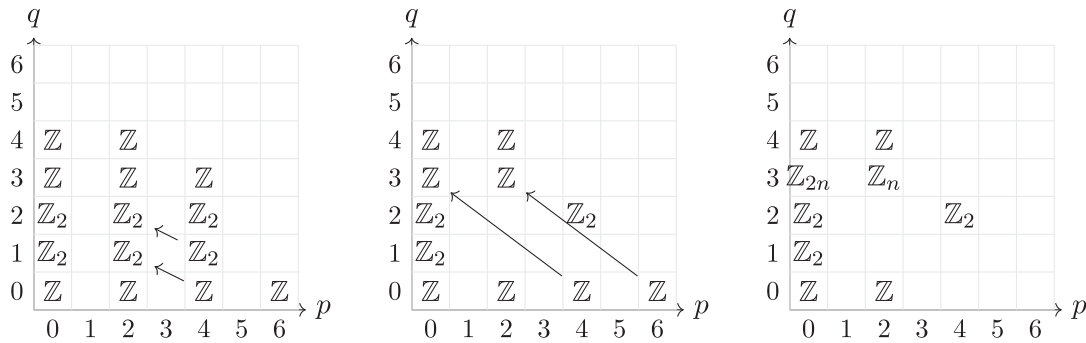


FIG. 22. $E_2^{p,q}$ of the AHSS for the spin bordism of Q implementing $nc_1^2 = 0$, where we utilize the fibration structure of Q (left), as well as $E_{p,q}^4$ (middle) and $E_{4,0}^4$ (right) of the same construction, for $p + q \leq 6$.

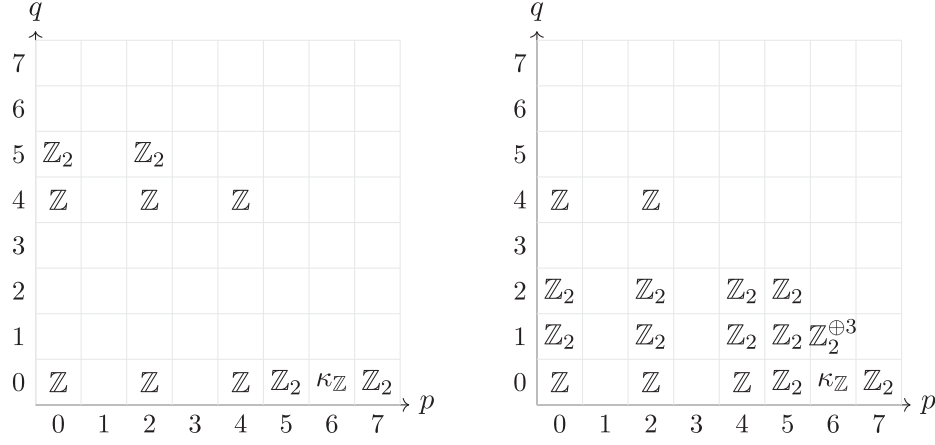


FIG. 23. $E_{p,q}^2$ for the AHSS for oriented (left) and spin (right) bordism of Q implementing $\mathbf{m}_2(c_1^2) = 0$ up to order $p + q \leq 7$. ($\kappa_{\mathbb{Z}}$ denotes $\mathbb{Z} \oplus \mathbb{Z}_2$).

This leaves us with an extension problem in degree 6, given by

$$0 \rightarrow \mathbb{Z} \rightarrow \Omega_6^{\text{spin}}(Q) \rightarrow \mathbb{Z}_2 \rightarrow 0. \quad (4.15)$$

While this does not change the fact, that there is a perturbative anomaly associated to a single $\mathbb{Z}$ factor it might change its normalization, i.e., the normalization of $c_1 p_1$ in the associated anomaly polynomial.

Now let us assume we are in a four dimensional theory, with $U(1)$ charged chiral fermions. From the structure of the spectral sequences we see that while an anomaly free theory still requires a cancellation of a perturbative mixed anomaly, the absence of pure gauge anomalies is no longer $\mathbb{Z}$ -valued. This means that the typical requirement

$$\sum_q n_q q^3 = 0 \in \mathbb{Z}, \quad (4.16)$$

is replaced by some discrete remnant in the modified Abelian gauge theory. Indeed this can be understood as a form of incomplete anomaly cancellation [32], as described above.

Next, let us move to the mod n constraint (again for $n = 2$) for which the relevant homology groups with $\mathbb{Z}$ and $\mathbb{Z}_2$ coefficients can be determined from (3.61), and are given by

k	$H_k(Q; \mathbb{Z})$	$H_k(Q; \mathbb{Z}_2)$
0	$\mathbb{Z}$	$\mathbb{Z}_2$
1	0	0
2	$\mathbb{Z}$	$\mathbb{Z}_2$
3	0	0
4	$\mathbb{Z}$	$\mathbb{Z}_2$
5	$\mathbb{Z}_2$	$\mathbb{Z}_2$
6	$\mathbb{Z} \oplus \mathbb{Z}_2$	$\mathbb{Z}_2^{\oplus 3}$
7	$\mathbb{Z}_2$	$\mathbb{Z}_2^{\oplus 2}$
8	$\mathbb{Z} \oplus \mathbb{Z}_2$	$\mathbb{Z}_2^{\oplus 3}$

(4.17)

With that it is straightforward to obtain the second page of the AHSS for oriented and spin bordism which we summarize in Fig. 23. Since in this case neither $H_*(Q; \mathbb{Z})$ nor $\Omega_*^{\text{SO}}(\text{pt})$ contain odd torsion factors the AHSS collapses on the second page (4.3) and we can simply read off the oriented bordism groups:

k	0	1	2	3	4	5	6
$\Omega_k^{\text{SO}}(Q)$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2$

(4.18)

Again this differs, from the groups of $BU(1)$ in several ways. For instance, it contains extra $\mathbb{Z}_2$ factors, which capture anomalous contributions of the fiber [e.g., one of the $\mathbb{Z}_2$ in degree 5 is of this kind and can be detected for example by $w_2 \cup f_Q^*(\tilde{u}_3)$, with $\tilde{u}_3 \in H^3(K(\mathbb{Z}_2; 3); \mathbb{Z}_2)$]. But also the $\mathbb{Z}$ factors differ in their normalization, as discussed already at the beginning of this Sec. IV. For example the constraint $\mathbf{m}_2(c_1^2) = 0$ implies, that one can divide c_1^2 by 2 in the modified gauge theory.²³

For spin bordism the story becomes more complicated, since now the lower rows in the AHSS are nontrivial as can be seen in Fig. 23. Once more one way to proceed is to consider a different AHSS for the fibration (3.44) with the second page

$$E_{p,q}^2 = H_p(BU(1); \Omega_q^{\text{spin}}(K(\mathbb{Z}_2, 3))), \quad (4.19)$$

where we use the bordism groups computed in the Appendix C to obtain the second page depicted in Fig. 24. The differentials in the two lowest rows are one more inherited from the spin bordism of $BU(1)$. The action of a differential on the fourth page, analogously to the

²³This is typically not the case for oriented manifolds; take, for example, $\mathbb{CP}^2$, with c_1 given by the generator of $H^2(\mathbb{CP}^2; \mathbb{Z}) \simeq \mathbb{Z}$.

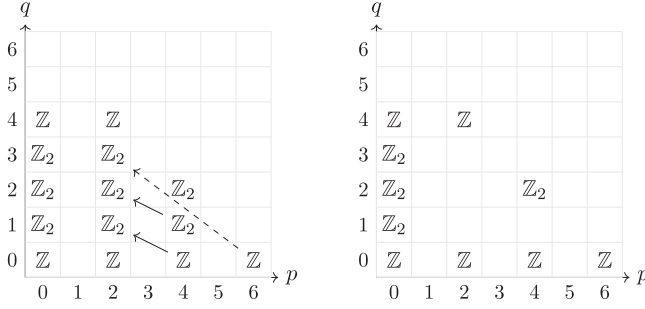


FIG. 24. $E_{p,q}^2$ and $E_{p,q}^\infty$ of the AHSS for spin bordism of Q implementing $\mathbf{m}_2(c_1^2) = 0$ using its fibration structure, with $p + q \leq 6$.

discussion above, is fixed by LSSS for cohomology. In the range of interest in the example this gives the action

$$d_4: E_{6,0}^4 \simeq H_6(BU(1); \mathbb{Z}) \simeq \mathbb{Z} \xrightarrow{\mathbf{m}_2} \mathbb{Z}_2 \simeq E_{2,3}^4. \quad (4.20)$$

This allows us to determine the ∞ -page for spin bordism, see Fig. 24, up to $p + q \leq 6$ and leads to the spin bordism groups

k	0	1	2	3	4	5	(4.21)
$\Omega_k^{\text{spin}}(Q)$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z} \oplus \mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z} \oplus \mathbb{Z}$	0	

The most interesting entry is the factor $\mathbb{Z}_2$ in degree 3. This element does not exist for the unmodified $U(1)$ gauge theory, because of a nontrivial differential dual to the Steenrod square Sq^2 from $H^2(BU(1); \mathbb{Z}_2)$ to $H^4(BU(1); \mathbb{Z}_2)$. However, for Q this differential is trivial and the associated class survives. From Fig. 24 we also see that it is associated to the nontrivial $\Omega_3^{\text{spin}}(K(\mathbb{Z}_2, 3)) \simeq \mathbb{Z}_2$ related to the mod 2 class $u_3 \in H^3(K(\mathbb{Z}_2, 3); \mathbb{Z})$, see Appendix C. It is therefore suggestive that this nontrivial class is detected by a nontrivial $\mathbb{Z}_2$ -valued integral of the pullback of this fiber class to spacetime X .

The first extension problem appears in degree 6, in which the $\mathbb{Z}_2$ factor could be absorbed by the $\mathbb{Z} \simeq E_{2,4}^\infty$, precisely as above. Again, this would only change the normalization, but not the fact that there is perturbative mixed gravitational anomaly in four dimensions. As opposed to the times n constraint now we also have a perturbative pure gauge anomaly, which however has a different quantization, since c_1^3 can be divided by 2. We also find additional bordism groups due to the fiber, for example $\Omega_3^{\text{spin}}(Q) \cong \mathbb{Z}_2$.

We see that even in these rather minimal examples the modification of the topological sectors of Abelian gauge theories does have a profound impact on both the characteristic classes, as well as the anomalies classified by bordism groups. While it indeed can be used to reduce the conserved charges and anomalies, the homotopy fiber construction often introduces new topological sectors, which also can contribute to anomalies and hence needs to be considered.

V. CONCLUSIONS AND OUTLOOK

We have demonstrated that the modification of the characteristic classes of an Abelian gauge theory at the level of the classifying space has several important consequences. These include the proliferation of constraints to higher-dimensional characteristic classes as well as the appearance of additional topological gauge field configurations coming from the modified classifying space as a fibration. These modifications at the level of (co)homology will also impact the deformation classes of the associated modified Abelian gauge theories described by bordism groups. Since these classify the potential anomalies of the theory the implementation of the constraints further affects the anomalies of the modified theory. With the general techniques of the homotopy fiber construction and the corresponding phenomena, we now want to point out several avenues for a continuation of this work.

We already mentioned the close connection between the homotopy fiber construction and higher groups. However, at least when the higher group contains a 0-form symmetry there is one additional piece of data that specifies its higher-group structure and is given roughly by the action of the 0-form symmetry on the higher-form pieces [10,12,68]. To implement such an action one would likely have to pass to cohomology with local coefficients, whose local nature encode the nontrivial 0-form symmetry action as an automorphism on the higher-symmetry pieces. This will also change the realization of characteristic classes as well as anomalies (see, e.g., [12,69–71]) and might provide further applications in the context of anomaly cancellation. Similarly, one should include the characteristic classes of the tangent bundle and its mixing with the gauge symmetries to allow for a treatment of twisted tangential structures, which often appear in type II string theories and their compactification [72–76], and are essential for the heterotic string (e.g., [64,65,77], for a mathematical description), see also [78].

Similar techniques as discussed here, can also be used for non-Abelian 0-form symmetries, e.g., for non-Abelian gauge theories described by a Lie group. Beyond the implications for anomalies and their cancellation, see [32], this can have clear phenomenological applications. In particular, a restriction on the sum of topological sectors or a coupling to a topological field theory sector, see, e.g., [79–87], can have important consequences for fields that couple to the topological charge density, such as axions, e.g., [88–92] and [93] for a review. Since the group cohomology classes $H^D(Q; U(1))$ or bordism groups $\Omega_D^E(Q)$ classify such topological couplings they will generically be affected by the modification. It would further be interesting to connect the modification of a characteristic class described by a top-form when pulled back to spacetime with the decomposition of the theory in several individual universes, see, e.g., [94]. Moreover, restrictions of the type discussed in this work can be implemented on

the lattice (e.g., [95,96] for modifications of c_1) which should allow for a treatment of challenging strong coupling dynamics.

The transition from group (co)homology to bordism groups is in general very interesting, since it allows one to understand the effect of gravity in the modification of (generalized) symmetries and the connected nonconservation of some of the charges [97]. This is closely related to the Swampland cobordism conjecture [98]. In particular, the homotopy fiber analysis encodes a gauging of these global symmetries and verifies the expectation that such a gauging can introduce new conserved charges which need to be violated in the full theory of quantum gravity.

Finally, in this work we mainly focused on the topological classes, which for U(1) gauge theories are concentrated in even degrees and thus most apparent for theories in even dimensions. These modifications would however also modify the Chern-Simons terms in odd dimensions and can allow for fractional levels, even in the absence of a spin structure, see [99] and references therein for the importance of spin structure. The generality of the homotopy fiber approach which is viable also for higher-form gauge fields might therefore be a unifying description of these and related phenomena, such as the anyonic statistics of extended objects described in [100,101].

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: HOMOTOPY FIBERS AND BIANCHI IDENTITIES

We use this Appendix to make the connection between the homotopy fiber construction and Bianchi identities more explicit and point out some subtleties associated to the realization of the higher-form fields. We will concentrate on the times n constraint for c_1^2 , which involves the appearance of a U(1) higher-form field.

If we write out the action of the differential in the top left of Fig. 8, we obtain the equation

$$dv_3 = nv_2^2, \quad (\text{A1})$$

where v_3 is the generator of $H^3(K(\mathbb{Z}, 3); \mathbb{Z})$ and v_2 the generator of $H^2(BU(1); \mathbb{Z})$. Writing this in a more physics-oriented way, by pulling back to spacetime X and using Chern-Weil representatives this becomes

$$dH^{(3)} = \frac{n}{4\pi^2} F^{(2)} \wedge F^{(2)}. \quad (\text{A2})$$

This precisely corresponds to a usual Bianchi identity. However, there are two important subtleties:

- (i) Equation (A2) does not include all the information of (A1), because it is not sensitive to torsion classes.
- (ii) Equation (A2) contains way too much information, since it is a local identity of differential forms, i.e., valid at each point in X , whereas (A1) only equates the cohomology classes, independent of their representatives.

Instead of imposing the Bianchi identity on top of the equations of motion, we can also include it as enforced by an equation of motion of an additional Lagrange multiplier field. In order to do so we however need to fix the number of spacetime dimensions, D , and use a specific Lagrangian description. To do this consistently we move to the description in terms of differential characters, [36–39], see [65,102] for a mathematical description of Bianchi identities on the level of differential cohomology. In this framework, the Eq. (A1) is contained in the topological class of the composite differential character $\check{A} * \check{A}$ and can be written as

$$n[N_{A*A}^{(4)}] = n[N_A^{(2)}] \cup [N_A^{(2)}] = 0 \in H^4(X; \mathbb{Z}). \quad (\text{A3})$$

This triviality of the topological class is enforced in the Bianchi identity (A2) by providing an explicit trivialization. This choice of trivialization in (A2) is precisely what contains the additional local information and beyond restriction of the topological class, also determines the field strength $F_{A*A}^{(4)} = F^{(2)} \wedge F^{(2)}$.

Let us now describe, how this can be encoded in the properties of a Lagrange multiplier $\check{\lambda}$ and we include a term

$$\Delta_{\check{\lambda}} S = 2\pi i \int_X (\check{\lambda}, \check{q}(\check{A})) \quad (\text{A4})$$

in the Euclidean action. Here, $\check{q}(\check{A}) \in \check{H}^4(X)$ denotes a differential character that depends on $\check{A}$, and $(\cdot, \cdot)$ is the differential cohomology pairing (see also [103]).

1. Flat gauging

The first realization is also known as “flat gauging” in the literature [44] and is used to implement the topological constraint (A1). For that we want $\check{\lambda} \in \check{H}^{D-3}(X)$ to be a flat differential character, i.e., using the short exact sequence

$$0 \rightarrow H^{k-1}(X; U(1)) \rightarrow \check{H}^k(X) \rightarrow \Omega_{\mathbb{Z}}^k(X) \rightarrow 0, \quad (\text{A5})$$

we see that $\check{\lambda}$ is fully characterized as an element in $H^{D-4}(X; U(1))$. Further choosing $\check{q}(\check{A}) = n\check{A} * \check{A}$, (A4) becomes

$$\Delta_{\check{\lambda}} S = 2\pi i \int_X \lambda \cup [N_A^{(2)} \cup N_A^{(2)}] \quad (\text{A6})$$

which realizes the perfect pairing

$$H^{D-k}(X; U(1)) \times H^k(X; \mathbb{Z}) \rightarrow U(1) \quad (\text{A7})$$

described in [104]. The path integral over $\lambda \in H^{D-4}(X; U(1))$ then implements (A1) without additional local constraints.

But then one might ask, whether the flatness condition on $\check{\lambda}$ should also be implemented at the level of the equations of motion. This can be done using the other perfect pairing described in [104], given by

$$\Omega_{\mathbb{Z}}^{D-k}(X) \times \frac{\Omega^k(X)}{\Omega_{\mathbb{Z}}^k(X)} \rightarrow U(1). \quad (\text{A8})$$

For that we introduce another Lagrange multiplier $\check{\alpha} \in \check{H}^4(X)$ which is a topologically trivial differential character described by a globally defined differential form $\alpha^{(3)} \in \frac{\Omega^3(X)}{\Omega_{\mathbb{Z}}^3(X)}$ defined up to large gauge transformations.²⁴ Now we modify the action (A4) by adding

$$\Delta_{\check{\alpha}} S = 2\pi i \int_X (\check{\lambda}, \check{\alpha}) = 2\pi i \int_X F_{\lambda}^{(D-3)} \wedge \alpha^{(3)}, \quad (\text{A9})$$

where the second equality follows from the simplification of the differential cohomology pairing in case one of the differential characters is topologically trivial. Integrating out $\check{\alpha}$ in (A9) imposes the flatness condition $F_{\lambda} = 0$ on the field $\check{\lambda}$ reducing (A4) to (A6).

2. Full gauging

If instead we take the Lagrange multiplier $\check{\lambda} \in \check{H}^{D-3}(X)$ to be an unconstrained differential character, Eq. (A4) implements a stronger constraint. In particular, the path integral over $\check{\lambda}$ now includes the integration over topologically trivial characters described by elements in $\Omega_{\mathbb{Z}}^{D-4}(X)/\Omega_{\mathbb{Z}}^{D-4}(X)$, which due to the pairing (A8), enforces

²⁴One is inclined to call a topologically trivial gauge field an $\mathbb{R}$ gauge field, but there is a subtlety that $\mathbb{R}$ gauge field transformations are only “small” in the sense that gauge transformations are always exact forms, and for the purpose of using (A8), it is important to include “large” gauge transformations.

$$F_{q(A)}^{(4)} = 0 \in \Omega_{\mathbb{Z}}^4(X). \quad (\text{A10})$$

For the same choice as above, i.e., $\check{q} = n(\check{A} * \check{A})$, this yields

$$\frac{n}{4\pi^2} (F^{(2)} \wedge F^{(2)}) = 0 \in \Omega_{\mathbb{Z}}^4(X), \quad (\text{A11})$$

which is the Bianchi identity (A2) for trivial $H^{(3)}$ (or the “homogeneous solution”). To modify this to obtain (A1) and (A2) one needs to introduce a new field $\check{H} \in \check{C}^3(X)$ which is a differential cochain, defined for example in [37], i.e., it is not necessarily closed. Including this field we consider the action

$$\Delta S = 2\pi i \int_X (\check{\lambda}, \check{q}(\check{A}) - d\check{H}), \quad (\text{A12})$$

where d is a differential defined on differential cochains.²⁵ Since $d^2 = 0$ we see that $d\check{H}$ is closed, i.e., the Lagrangian contains a well-defined differential character. Integrating out the Lagrange multiplier field $\check{\lambda}$ in (A12) one obtains the constraint, see also [22],

$$d\check{H} = \check{q}(\check{A}), \quad (\text{A13})$$

which written in components, $\check{H} = (h, h_H, H^{(3)})$, for the case $\check{q}(\check{A}) = n(\check{A} * \check{A})$ contains

$$dH^{(3)} = \frac{n}{4\pi^2} F^{(2)} \wedge F^{(2)}, \quad (\text{A14})$$

identical to (A2), as desired.

APPENDIX B: (CO)HOMOLOGY OF EILENBERG-MACLANE SPACES

In this Appendix we summarize the (co)homology of Eilenberg-MacLane spaces of the form $K(\mathbb{Z}, m)$ and $K(\mathbb{Z}_n, m)$ with a focus on $\mathbb{Z}_n = \mathbb{Z}_2$. These spaces are defined by their homotopy groups, see, e.g., [8, 34],

$$\pi_k(K(A, m)) = \begin{cases} A, & k = m, \\ 0, & k > 0, \text{ otherwise,} \end{cases} \quad (\text{B1})$$

where in our case A is Abelian.

²⁵If a differential cochain $\check{c} \in \check{C}^p(X)$ is given by the triplet $(c, h_c, F_c) \in C^p(X; \mathbb{Z}) \times C^{p-1}(X; \mathbb{R}) \times \Omega^p(X)$, the differential $d: \check{C}^p(X) \rightarrow \check{C}^{p+1}(X)$ is given by $d(c, h_c, F_c) = (dc, F_c - dh_c - c, dF_c)$ where the differentials on the individual entries represent differentials for singular cochains and differential forms, respectively.

1. $K(\mathbb{Z}, m)$

These Eilenberg-MacLane spaces can be interpreted as the classifying spaces of $U(1)$ $(m-1)$ -from fields. In low degree they are given by

$$K(\mathbb{Z}, 1) = U(1) = S^1, \quad K(\mathbb{Z}, 2) = BU(1) = \mathbb{CP}^\infty. \quad (\text{B2})$$

From that we can easily extract their cohomology groups with integer coefficients

$$H^k(K(\mathbb{Z}, 1); \mathbb{Z}) = \begin{cases} \mathbb{Z}, & k \in \{0, 1\}, \\ 0, & \text{otherwise,} \end{cases} \quad H^k(K(\mathbb{Z}, 2); \mathbb{Z}) = \begin{cases} \mathbb{Z}, & k \geq 0, k \text{ even,} \\ 0, & \text{otherwise.} \end{cases} \quad (\text{B3})$$

Moreover, the cohomology of $K(\mathbb{Z}, 2)$ is built from a single generator v_2 in degree 2, defining the (polynomial) ring structure

$$H^*(K(\mathbb{Z}, 2); \mathbb{Z}) = \mathbb{Z}[v_2]. \quad (\text{B4})$$

In higher degrees $m \geq 3$, the cohomology groups become more complicated quickly and can be extracted from the homology groups with integer coefficients in [49], give by

k	3	4	5	6	7	8	9	10	11	12	
$H_k(K(\mathbb{Z}, 3); \mathbb{Z})$	$\mathbb{Z}$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_3$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_2 \oplus \mathbb{Z}_5$	$\mathbb{Z}_2$	
$H_k(K(\mathbb{Z}, 4); \mathbb{Z})$	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	0	$\mathbb{Z} \oplus \mathbb{Z}_3$	0	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	0	$\mathbb{Z} \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_4$	
$H_k(K(\mathbb{Z}, 7); \mathbb{Z})$	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2 \oplus \mathbb{Z}_3$	0	

(B5)

which translates to

k	3	4	5	6	7	8	9	10	11	12	
$H^k(K(\mathbb{Z}, 3); \mathbb{Z})$	$\mathbb{Z}$	0	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_3$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_2 \oplus \mathbb{Z}_5$	
$H^k(K(\mathbb{Z}, 4); \mathbb{Z})$	0	$\mathbb{Z}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}_3$	0	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}$	
$H^k(K(\mathbb{Z}, 7); \mathbb{Z})$	0	0	0	0	$\mathbb{Z}$	0	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2 \oplus \mathbb{Z}_3$	

(B6)

where we always have that $H^0(K(\mathbb{Z}, m); \mathbb{Z}) = \mathbb{Z}$ since these spaces are connected as well as $H^k(K(\mathbb{Z}, m); \mathbb{Z}) = 0$ for $0 < k < m$. One further has the generic generator of $H^m(K(\mathbb{Z}, m); \mathbb{Z}) = \mathbb{Z}$ in degree m , which we will denote by v_m .

Using the universal coefficient theorem we can also extract the cohomology with $\mathbb{Z}_2$ coefficients, that will be useful for the Adams spectral sequence below. In particular, one has

k	3	4	5	6	7	8	9	10	11	12	
$H^k(K(\mathbb{Z}, 3); \mathbb{Z}_2)$	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2^{\oplus 2}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{\oplus 2}$	
$H^k(K(\mathbb{Z}, 4); \mathbb{Z}_2)$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}_2^{\oplus 2}$	$\mathbb{Z}_2^{\oplus 2}$	$\mathbb{Z}_2^{\oplus 2}$	
$H^k(K(\mathbb{Z}, 7); \mathbb{Z}_2)$	0	0	0	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	

(B7)

The nontrivial generators can be identified with the action of all admissible actions of the Steenrod operations Sq^n , without the appearance of Sq^1 , see, e.g., [34], on the generator $\tilde{v}_m \in H^m(K(\mathbb{Z}, m); \mathbb{Z}_2) \simeq \mathbb{Z}_2$. For example for $K(\mathbb{Z}, 7)$ one has

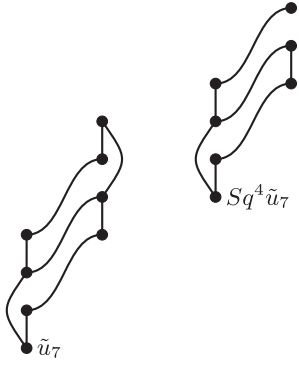
$$H^*(K(\mathbb{Z}, 7); \mathbb{Z}_2) = \mathbb{Z}_2[\tilde{v}_7, Sq^2\tilde{v}_7, Sq^3\tilde{v}_7, Sq^4\tilde{v}_7, Sq^5\tilde{v}_7, \dots], \quad (\text{B8})$$

where we indicated all elements in degree ≤ 12 . Its $\mathcal{A}(1)$ structure is captured by Fig. 26. Indeed just considering the different elements we find a match with (B7), derived from the universal coefficient theorem as needed.

2. $K(\mathbb{Z}_n, m)$

As above, the low degree case is known to be determined by

$$K(\mathbb{Z}_n, 1) = B\mathbb{Z}_n = L_n^\infty, \quad (\text{B9})$$

FIG. 25. The $\mathcal{A}(1)$ structure of $K(\mathbb{Z}_2, 7)$.

leading to the cohomology groups

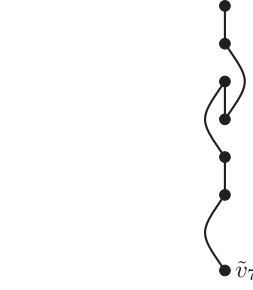
$$H^k(K(\mathbb{Z}_n, 1); \mathbb{Z}) = \begin{cases} \mathbb{Z}, & k = 0, \\ \mathbb{Z}_n, & k > 0, k \text{ even}, \\ 0, & \text{otherwise.} \end{cases} \quad (\text{B10})$$

and with $\mathbb{Z}_n$ coefficients which will also be useful for us

$$H^k(K(\mathbb{Z}_n, 1); \mathbb{Z}_n) = \mathbb{Z}_n, \quad k \geq 0. \quad (\text{B11})$$

Again, there is a ring structure which in the case of n odd is

$$H^*(B\mathbb{Z}_n; \mathbb{Z}_n) = \frac{\mathbb{Z}_n[x, y]}{(x^2)}, \quad (\text{B12})$$

FIG. 26. The $\mathcal{A}(1)$ structure of $K(\mathbb{Z}, 7)$.

where x and y are of degree 1 and 2, respectively. For the case of n even

$$H^*(B\mathbb{Z}_n; \mathbb{Z}_n) = \frac{\mathbb{Z}_n[x, y]}{(x^2 - \frac{n}{2}y)}, \quad (\text{B13})$$

with the action of a Bockstein map relating the two, see [8].

In higher degree the cohomology groups for general n are calculated and we instead will focus on $n = 2$, for which some results can be found in [50]. In particular, the cohomology of $K(\mathbb{Z}_2, m)$ with $\mathbb{Z}_2$ coefficients can be extracted from the Steenrod action on a generator $\tilde{u}_m$ of $H^m(K(\mathbb{Z}_2, m); \mathbb{Z}_2)$, [34,42], in a way similar to the case $K(\mathbb{Z}, m)$. However, this time one also allows for the appearance of Sq^1 in the enumeration of all admissible actions, [34]. In the example of $K(\mathbb{Z}_2, 7)$ one has

$$H^*(K(\mathbb{Z}_2, 7); \mathbb{Z}_2) = \mathbb{Z}_2[\tilde{u}_7, Sq^1 \tilde{u}_7, Sq^2 \tilde{u}_7, Sq^3 \tilde{u}_7, Sq^2 Sq^1 \tilde{u}_7, Sq^4 \tilde{u}_7, Sq^3 Sq^1 \tilde{u}_7, Sq^5 \tilde{u}_7, Sq^4 Sq^1 \tilde{u}_7, \dots]. \quad (\text{B14})$$

From this we can simply read off the cohomology groups

k	0	1	2	3	4	5	6	7	8	9	10	11	12
$H^*(K(\mathbb{Z}_2, 7); \mathbb{Z}_2)$	$\mathbb{Z}_2$	0	0	0	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{\oplus 2}$	$\mathbb{Z}_2^{\oplus 2}$	$\mathbb{Z}_2^{\oplus 2}$

(B15)

The $\mathcal{A}(1)$ structure in the relevant degree is given in Fig. 25. Forgetting about the Sq^2 data, we can also use this Adams spectral sequence for $H_*(K(\mathbb{Z}_2, 7); \mathbb{Z}_2)^\vee$, the 2-completion of its integer cohomology. Moreover, since we are interested in $K(\mathbb{Z}_2, 7)$ and not $K(\mathbb{Z}, 7)$ the cohomology localizes in prime 2, and the approach captures everything. The extensions now are taken with respect to $\mathcal{A}(0)$, generated only by Sq^1 . With this we find the degrees relevant to us and convert to cohomology with the universal coefficient theorem,²⁶ leading to

k	0	1	2	3	4	5	6	7	8	9	10	11	12
$H_*(K(\mathbb{Z}_2, 7); \mathbb{Z})$	$\mathbb{Z}$	0	0	0	0	0	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
$H^*(K(\mathbb{Z}_2, 7); \mathbb{Z})$	$\mathbb{Z}$	0	0	0	0	0	0	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$

(B16)

²⁶This can also be seen from an argument in [34], saying that if there are two elements $x \in H^k(X; \mathbb{Z}_2)$ and $y \in H^{k+1}(X; \mathbb{Z}_2)$ related via $y = Sq^1 x$, then there is a $\mathbb{Z}_2$ factor in $H_k(X; \mathbb{Z})$ where they both originate from.

With similar techniques one can derive the cohomology groups of $K(\mathbb{Z}_2, 3)$ and $K(\mathbb{Z}_2, 2)$ with $\mathbb{Z}_2$ coefficients

k	2	3	4	5	6	7	8	9	10	11	12
$H^k(K(\mathbb{Z}_2, 2); \mathbb{Z}_2)$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{\oplus 2}$	$\mathbb{Z}_2^{\oplus 2}$	$\mathbb{Z}_2^{\oplus 2}$	$\mathbb{Z}_2^{\oplus 3}$	$\mathbb{Z}_2^{\oplus 4}$	$\mathbb{Z}_2^{\oplus 4}$	$\mathbb{Z}_2^{\oplus 5}$	$\mathbb{Z}_2^{\oplus 6}$
$H^k(K(\mathbb{Z}_2, 3); \mathbb{Z}_2)$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{\oplus 2}$	$\mathbb{Z}_2^{\oplus 2}$	$\mathbb{Z}_2^{\oplus 2}$	$\mathbb{Z}_2^{\oplus 4}$	$\mathbb{Z}_2^{\oplus 5}$	$\mathbb{Z}_2^{\oplus 5}$	$\mathbb{Z}_2^{\oplus 7}$
$H^k(K(\mathbb{Z}_2, 7); \mathbb{Z}_2)$	0	0	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{\oplus 2}$	$\mathbb{Z}_2^{\oplus 2}$	$\mathbb{Z}_2^{\oplus 2}$

(B17)

which can also be verified by comparing with the integer cohomology groups calculated in [50]

k	3	4	5	6	7	8	9	10	11	12
$H^k(K(\mathbb{Z}_2, 2); \mathbb{Z})$	$\mathbb{Z}_2$	0	$\mathbb{Z}_4$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_8$	$\mathbb{Z}_2^{\oplus 2}$	$\mathbb{Z}_2^{\oplus 2}$	$\mathbb{Z}_2^{\oplus 3}$
$H^k(K(\mathbb{Z}_2, 3); \mathbb{Z})$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{\oplus 3}$	$\mathbb{Z}_2^{\oplus 2}$	$\mathbb{Z}_2^{\oplus 3}$

(B18)

Using the Adams spectral sequence for $K(\mathbb{Z}, 7)$, one obtains the 2-completion

k	0	1	2	3	4	5	6	7	8	9	10	11
$H^k(K(\mathbb{Z}, 7); \mathbb{Z})_2^\vee$	$\mathbb{Z}$	0	0	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$

(B19)

but one needs to be careful regarding to the appearance of torsional elements at other primes, as we know appear in degree 12, see (B6).

APPENDIX C: BORDISM GROUPS OF EILENBERG-MACLANE SPACES

Beyond the cohomology of the Eilenberg-MacLane spaces, we are also interested in their oriented and spin bordism groups of the form

$$\Omega_k^{\text{SO/spin}}(K(A, m)). \quad (\text{C1})$$

These can be constrained using the Atiyah-Hirzebruch and the Adams spectral sequence, see for example [27, 105].

The second page of the AHSS needs the integer homology groups $H_*(K(G, m); \mathbb{Z})$ as input. Let us illustrate this approach with $K(\mathbb{Z}, 3)$ and $K(\mathbb{Z}_2, 3)$, which we also use in Sec. IV. The second page of the AHSS is given by

$$E_{p,q}^2 = H_p(K(A, m); \Omega_q^{\text{SO/spin}}(\text{pt})), \quad (\text{C2})$$

where the oriented and spin bordism groups of a point are given in (4.6). We provide the second page of the AHSS for oriented bordism of $K(\mathbb{Z}_2, 3)$ and $K(\mathbb{Z}, 3)$ in Fig. 27. From there we see that in the range $p + q \leq 7$ there are no nontrivial differentials. The first extension problems only occur in degree 7 and above, and we find

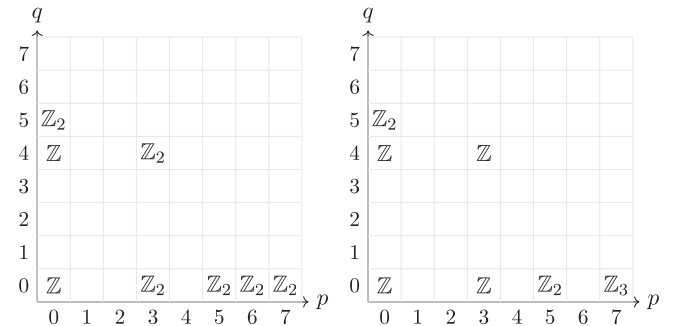
k	0	1	2	3	4	5	6
$\Omega_k^{\text{SO}}(K(\mathbb{Z}_2, 3))$	$\mathbb{Z}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2$
$\Omega_k^{\text{SO}}(K(\mathbb{Z}, 3))$	$\mathbb{Z}$	0	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	0

(C3)

For the spin bordism group the second page of the AHSS looks a little more involved and is depicted in Fig. 28. One way to make progress is to move to the Adams spectral sequence that will have information about the 2-completion and is sufficient to determine the spin bordism groups in dimensions 6 and below. Its second page is given by

$$E_2^{s,t} = \text{Ext}_{\mathcal{A}(1)}^{s,t}((\mathbb{Z}_2 \oplus \Sigma^8 \mathbb{Z}_2 \oplus \Sigma^{10} J \oplus \dots) \otimes_{\mathbb{Z}_2} \tilde{H}^*(Y; \mathbb{Z}_2); \mathbb{Z}_2) \quad (\text{C4})$$

and converges to the 2-completion of the reduced spin bordism group $\tilde{\Omega}_*^{\text{spin}}(Y)_2^\vee$. Moreover, since we are only



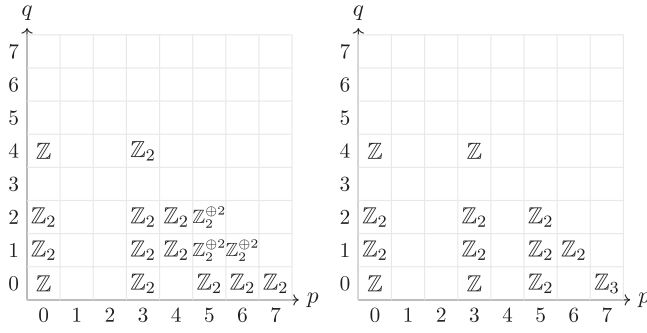


FIG. 28. The second page of the AHSS for the spin bordism groups of $K(\mathbb{Z}_2, 3)$ (left) and $K(\mathbb{Z}, 3)$ (right); for $p + q \leq 7$.

interested in degree 6 and below this simplifies to

$$E_2^{s,t} = \text{Ext}_{\mathcal{A}(1)}^{s,t}(\tilde{H}^*(Y; \mathbb{Z}_2); \mathbb{Z}_2), \quad (\text{C5})$$

and we need the $\mathcal{A}(1)$ module structure of $Y = K(\mathbb{Z}_2, 3)$ and $K(\mathbb{Z}, 3)$. This is depicted in Fig. 29 and Fig. 30, respectively. This can be converted to the Adams chart, see, e.g., [35], from which one can read off the bordism group. In dimension 6 and below the Adams chart for $K(\mathbb{Z}_2, 3)$ has a single $\mathbb{Z}_2$ in degree 3. For $K(\mathbb{Z}, 3)$, the first $\mathcal{A}(1)$ module, the “upside-down question mark,” can be found in [35] From this we find²⁷

k	0	1	2	3	4	5	6
$\tilde{\Omega}_k^{\text{spin}}(K(\mathbb{Z}_2, 3))_2^\vee$	0	0	0	$\mathbb{Z}_2$	0	0	0
$\tilde{\Omega}_k^{\text{spin}}(K(\mathbb{Z}, 3))_2^\vee$	0	0	0	$\mathbb{Z}$	0	0	0

(C6)

which combined with the results of the AHSS and $\Omega^{\text{spin}}(\text{pt})$ yields

²⁷Using the exact sequence of $\mathcal{A}(1)$ modules we can further resolve the extension question in [106], confirming that $\Omega_7^{\text{spin}}(K(\mathbb{Z}_2, 3)) = \mathbb{Z}_4$.

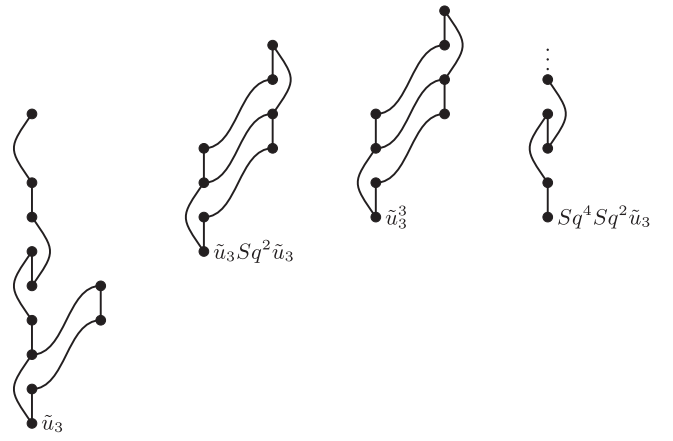


FIG. 29. The $\mathcal{A}(1)$ structure of $K(\mathbb{Z}_2, 3)$; $\tilde{u}_3$ is the generator of $H^3(K(\mathbb{Z}_2, 3); \mathbb{Z}_2)$.

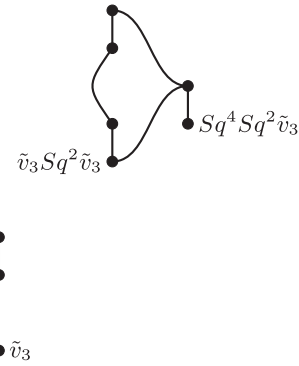


FIG. 30. The $\mathcal{A}(1)$ structure of $K(\mathbb{Z}, 3)$; $\tilde{v}_3$ is the generator of $H^3(K(\mathbb{Z}, 3); \mathbb{Z}_2)$.

k	0	1	2	3	4	5	6
$\Omega_k^{\text{spin}}(K(\mathbb{Z}_2, 3))$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
$\Omega_k^{\text{spin}}(K(\mathbb{Z}, 3))$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}$	0	0

(C7)

which agrees with [106] and we use as input for the AHSS

$$E_{p,q}^2 = H_p(BU(1); \Omega_q^{\text{spin}}(K(\mathbb{Z}_2, 3))), \quad (\text{C8})$$

in Sec. IV.

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