

Generalized Casas-Ibarra parametrization for Majorana neutrino masses

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We present a simple and broadly applicable extension of the Casas–Ibarra parametrization that captures the structure of all Majorana neutrino mass models. Building directly on the original formulation, our approach naturally accommodates additional degrees of freedom and provides a unified, minimal framework for parametrizing the Yukawa sector. It significantly simplifies both analytical treatments and numerical scans, and can be universally applied to any Majorana neutrino mass model, regardless of the underlying dynamics. The approach also offers a unified framework for classifying neutrino mass models according to the structure of the neutrino mass matrix, which naturally motivates the proposal of an extended version of the scotogenic model. This classification scheme yields tree-level (loop-level) representative models; the seesaw (scotogenic model), the linear seesaw (the generalized scotogenic model), and the linear plus inverse seesaw (the extended scotogenic model). We provide ready-to-use explicit expressions for several well-known scenarios, including the Zee model where one of the Yukawa matrices is antisymmetric.

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I. INTRODUCTION

The origin of the light neutrino masses remains one of the most interesting questions in particle physics. Several possible mechanisms have been proposed, one of the most compelling explanations being that neutrinos are Majorana particles, as hinted by the Weinberg operator [1]. One of the simplest high-energy completions are heavy right-handed neutrinos, such that light neutrino masses are generated by the (Type-I) seesaw mechanism [2–6]. Other possibilities, like radiative neutrino mass models, have also been extensively explored [7–9].

To study the phenomenology and do parameter scans of neutrino mass models, it is very useful to parametrize the Yukawa couplings that appear in the neutrino mass matrix

in terms of light neutrino parameters¹ (masses, mixings, and phases) and other free parameters of the models. For this goal, several parametrizations have been proposed in the literature. In particular, this approach has been very successfully applied to the Type-I seesaw model using the Casas-Ibarra (CI) parametrization [18], studied for special Yukawa textures in Ref. [19], simplified in the case of just two right-handed neutrinos in Ref. [20], and generalized to include radiative corrections [21] in [22] (see also Refs. [23–28] for other studies). In Refs. [29,30], other exact parametrizations for models with right-handed neutrinos, valid outside the seesaw limit, have been derived. For Type-II seesaw [31–34], obtaining the (symmetric) Yukawa matrix in terms of neutrino masses is trivial, while for the Type-I + II seesaw a parametrization has been proposed in Ref. [35]. Furthermore, parametrizations similar to the CI one for Type-III and inverse seesaw can be found in Refs. [36,37], a modified CI parametrization for

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¹It is worth noting the existence of models in which lepton mixing is connected to quark mixing [10–17], without the need for grand unification. In such frameworks, the PMNS matrix is expressed in terms of the quark mixing parameters, so that the Yukawa couplings depend on both neutrino and quark observables, combined in specific ways depending on the model.

the Krauss-Nasri-Trodden model [38] was given in Ref. [39], and a generalized CI parametrization for an extension of the scotogenic model [40,41] with several inert doublets was proposed in [42].

It is worth noting that, although in many works the seesaw parameter space is explored by adopting the standard CI parametrization, where several degrees of freedom of the model are embedded in an orthogonal complex matrix, this is not the only possible approach. Alternative formulations have been proposed in the literature, aiming to relate more directly the parameters of the high-energy Lagrangian to low-energy observables. In particular, Refs. [43–45] have shown that the parametrization can be reformulated in terms of the Hermitian combination $Y^\dagger Y$ of the Yukawa couplings, which provides a complementary perspective and can be particularly advantageous in the supersymmetric seesaw. Driven by the same motivation, another interesting parametrization for the inverse seesaw was presented in [46].

In addition, other formulations for the Yukawa structure in less minimal models have been proposed. In Ref. [47], a parametrization of the $n_L \times n_L$ antisymmetric Yukawa matrix of the singly-charged scalar present in the Zee-Babu model [48], which appears quadratically, has been obtained using the fact that the latter has a zero eigenvalue and it has been extensively used in phenomenological studies [49–53]. Further, Ref. [54] also using this fact, obtained some restrictions on the antisymmetric Yukawa matrix of models in which it appears linearly, like in the Zee model [55–57] (see also Ref. [58]). Finally, in Refs. [59,60], another general but more involved parametrization for Majorana neutrino mass models has been proposed.

In this work, we demonstrate that the CI parametrization can be generalized to encompass all Majorana neutrino mass models, providing a unified framework for Majorana neutrino masses. Although related ideas have appeared in the literature, a self-contained and comprehensive formulation has so far been lacking. To illustrate the straightforward applicability of this approach, we apply it to several prototypical examples, like the linear/inverse seesaw models and the generalized scotogenic model. We also include the case in which one Yukawa matrix entering neutrino masses is antisymmetric, like in the Zee model, where we provide a completely explicit parametrization in terms of some chosen free parameters. As a spin-off, our approach naturally leads us to propose a new version of the scotogenic model, which we term the extended scotogenic model.

The rest of the paper is organized as follows. In Sec. II we describe the general neutrino mass matrix and the generalized CI parametrization. In Sec. III we follow the form of the neutrino mass matrix and study typical model realizations of it, both at tree level and at one loop: (i) The seesaw and the scotogenic model; (ii) The linear seesaw

and the generalized scotogenic model; (iii) the linear + inverse seesaw and the extended scotogenic model. In Sec. IV we discuss models with extra symmetries and the application of the generalized CI parametrization to the Zee Model. In Sec. V we discuss an approach to parametrize the Yukawas using the complete neutrino mass matrix in models with right-handed neutrinos. In the Appendix we derive the standard CI in the seesaw limit using this approach. Our conclusions are summarized in Sec. VI.

II. GENERAL NEUTRINO MASS MATRIX AND PARAMETRIZATION

We are interested in the case where the $n_L = 3$ light neutrinos ν_L acquire Majorana masses,

$$-\mathcal{L}_\nu = \frac{1}{2} \bar{\nu}_L^c m_\nu \nu_L + \text{H.c.} \quad (1)$$

In general, the $n_L \times n_L$ light complex symmetric Majorana neutrino mass matrix may receive several contributions, either at tree level and/or from radiative corrections. If there are several contributions to light neutrino masses, the latter may be expressed in all generality as

$$m_\nu = \sum_{i,j=1}^N Y_i^T M_{ij} Y_j, \quad (2)$$

where the complex Yukawa matrix Y_i has dimensions $n_i \times n_L$ ($i = 1, \dots, N$) and the complex mass matrix $M_{ij} = M_{ji}^T$ has dimensions $n_i \times n_j$. For simplicity, let us consider in the following the case of $N = 2$, which in neutrino mass models corresponds to just two different sets of Yukawa couplings, Y_1 and Y_2 . Then, the light neutrino mass matrix becomes

$$m_\nu = Y_1^T M_{11} Y_1 + Y_1^T M_{12} Y_2 + Y_2^T M_{21} Y_1 + Y_2^T M_{22} Y_2. \quad (3)$$

The latter form may be mapped onto neutrino-mass-model parameters, which we exploit in Sec. III. The sum of these contributions can be rewritten as

$$m_\nu = \mathcal{Y}^T \mathcal{M} \mathcal{Y}, \quad (4)$$

where

$$\mathcal{Y} = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}, \quad \text{and} \quad \mathcal{M} = \begin{pmatrix} M_{11} & M_{12} \\ M_{12}^T & M_{22} \end{pmatrix}, \quad (5)$$

with $\mathcal{M} = \mathcal{M}^T$ being complex symmetric. Therefore, the neutrino mass matrix m_ν can always be expressed in the form of Eq. (4), regardless of the underlying physics or the order at which the different mass contributions arise. Note that this clearly also holds for $N > 2$, using

appropriate block matrices. This has already been pointed out in the Appendix F of Ref. [60].²

To study the phenomenology, it is useful to express the Yukawa couplings in terms of light neutrino parameters and other free parameters. For Type-I seesaw, this may be done via the CI parametrization [18], which can be used in the seesaw limit (see Sec. III A 1 below). However, in the generic case, before being able to apply the CI procedure [18], the extended mass matrix $\mathcal{M}$ has to be diagonalized via the Autonne-Takagi factorization

$$\mathcal{M} = V^T D_{\mathcal{M}} V, \quad (6)$$

where $D_{\mathcal{M}}$ is a real diagonal matrix with non-negative entries and V is a unitary matrix, $V^\dagger V = \mathbb{I}_{n_1+n_2}$. Then exactly the same derivation as in Ref. [18] can be applied to $V\mathcal{Y}$. Also by means of the Autonne-Takagi factorization, the $n_L \times n_L$ light neutrino mass matrix may be written as

$$D_m = \text{diag}(m_1, m_2, m_3) = U^T m_\nu U, \quad (7)$$

with U a $n_L \times n_L$ unitary matrix. Then we can write the generalized Casas-Ibarra (GCI) parametrization as

$$\mathcal{Y} = V^\dagger D_{\mathcal{M}}^{-1/2} \mathcal{R} D_{\sqrt{m}} U^\dagger, \quad (8)$$

where $\mathcal{R}$ is an $(n_1 + n_2) \times n_L$ (semi)orthogonal matrix, $\mathcal{R}^T \mathcal{R} = \mathbb{I}_{n_L}$, and we assume that there are no strictly vanishing eigenvalues in $D_{\mathcal{M}}$. Here the word *generalized* highlights that, by writing the Majorana neutrino mass matrix as in Eq. (4) and diagonalizing also $\mathcal{M}$, the structure of the CI parametrization of Ref. [18] actually holds for any Majorana neutrino mass model.³

Note that this expression is valid for any field basis. Indeed, one can work in a basis where the charged lepton mass matrix is not diagonal, as discussed, e.g., in Refs. [19,20]. In this case, the matrix U can be written as $U = U_{\text{PMNS}} U_L$, where U_{PMNS} is the standard Pontecorvo-Maki-Nakagawa-Sakata (PMNS) mixing matrix [61–64] and U_L accounts for the charged lepton mixing.⁴ It is also possible to operate in a basis where, e.g.,

²Equation (1) of Refs. [59,60] presents the general Majorana neutrino mass matrix in a form that, while algebraically equivalent to ours, is less transparent when several diagonal and off-diagonal terms are present. In such cases—and when y_1 and y_2 in that equation are subject to specific constraints—a direct use of their master parametrization does not follow automatically.

³A generalized CI parametrization was proposed in [42], but in the context of one specific model, namely a scotogenic model with several inert doublets (rather than one as in [40,41]), and using an orthogonal matrix in one of the diagonalization steps, instead of the unitary matrix appearing in the Autonne-Takagi factorization of Eq. (6).

⁴The choice of mass ordering (normal or inverted) becomes relevant only at this point, via the parametrization of the PMNS matrix, U_{PMNS} , and the order of the eigenvalues in D_m .

the Majorana mass matrix of the right-handed neutrinos in the Type-I seesaw is not diagonal, as in Ref. [20]. However, we emphasize that in the GCI parametrization proposed in Eq. (8), the matrix V is the one that diagonalizes the extended mass matrix $\mathcal{M}$ and it is not necessarily related to a change of basis, as in [20].

We summarize the important result discussed in this section. Any $n_L \times n_L$ Majorana neutrino mass matrix, which may have an arbitrary number of contributions,

$$m_\nu = \sum_{i,j=1}^N Y_i^T M_{ij} Y_j, \quad (9)$$

with Yukawas Y_i of dimensions $n_i \times n_L$, may be always written in the usual seesaw way,

$$m_\nu = \mathcal{Y}^T \mathcal{M} \mathcal{Y}, \quad (10)$$

with $\mathcal{Y}$ an $(n_1 + \dots + n_N) \times n_L$ Yukawa matrix, and $\mathcal{M}$ an $(n_1 + \dots + n_N) \times (n_1 + \dots + n_N)$ complex symmetric matrix. Therefore, the seesaw expression is the most general form for any Majorana (neutrino) mass matrix. A generalized Casas-Ibarra (GCI) parametrization may be used,

$$\mathcal{Y} = V^\dagger D_{\mathcal{M}}^{-1/2} \mathcal{R} D_{\sqrt{m}} U^\dagger \quad \text{GCI parametrization}, \quad (11)$$

where $D_{\mathcal{M}} = V^* \mathcal{M} V^\dagger$ and $\mathcal{R}$ is a complex orthogonal matrix, $\mathcal{R}^T \mathcal{R} = \mathbb{I}_{n_L}$. This is a simple and well-suited parametrization for numerical scans relevant to phenomenological analyses.

A word of caution is in order. Note that, if the Yukawa matrices are not general, extra conditions on the matrix $\mathcal{R}$ may need to be imposed. For instance, this is the case of antisymmetric Yukawa matrices, $f = -f^T$, which appear for instance in the Zee (linear in the antisymmetric Yukawa) [65] and Zee-Babu models (quadratic in the antisymmetric Yukawa coupling) [48], see also Ref. [56]. For the Zee-Babu case, another parametrization which uses the fact that an antisymmetric matrix has a zero eigenvalue is much more straightforward to use for phenomenological studies [47,49,50,52,54], while for the Zee model [65] (see Refs. [66–68] for detailed studies of the phenomenology), in Sec. IV we derive a parametrization of $\mathcal{R}$ that ensures the antisymmetric condition.

A. Specific form of the (semi)orthogonal Casas-Ibarra $\mathcal{R}$ matrix

The key point of the CI parametrization lies in embedding all the free parameters of the neutrino sector—except for the eigenvalues of the $\mathcal{M}$ matrix—into the $\mathcal{R}$ matrix, thereby making the phenomenological analysis simple and straightforward. The matrix $\mathcal{R}$ may, in general, take a nonsquare form depending on the dimensions of the

Yukawa coupling matrices. To build a semiorthogonal $(n_1 + n_2) \times n_L$ $\mathcal{R}$ matrix we can start from the special orthogonal group $SO(n)$. It is well-known that its dimension is $d = n(n-1)/2$, so that a matrix $\mathcal{R} \in SO(n)$ can be parametrized with d rotation angles as the product of plane rotations in d coordinates, as follows:

$$\mathcal{R} = \prod_{j=2}^n \prod_{i=1}^{j-1} R_{ij}, \quad (12)$$

with

$$[R_{ij}]_{ab} = \begin{cases} c_{ij} & a = b = i, \text{ or } a = b = j, \\ s_{ij} & a = i, \quad b = j, \\ -s_{ij} & a = j, \quad b = i, \\ 1 & a = b \notin \{i, j\}, \\ 0 & \text{otherwise,} \end{cases} \quad (13)$$

where $c_{ij} = \cos \theta_{ij}$ and $s_{ij} = \sin \theta_{ij}$. A procedure to obtain the semiorthogonal matrix $\mathcal{R}$ with dimensions $(n_1 + n_2) \times n_L$ is to start from a square matrix belonging to $SO(n_L)$ if $n_1 + n_2 < n_L$, or to $SO(n_1 + n_2)$ otherwise, and then reduce its dimension so as to reach the desired $(n_1 + n_2) \times n_L$ form. In general, a $\mathcal{R}_{n \times m}$ semiorthogonal matrix can be obtained as

$$\mathcal{R}_{n \times m} = \begin{cases} (\mathbb{I}_n | 0_{n \times (m-n)}) \mathcal{R}_{m \times m}, & n < m, \\ \mathcal{R}_{n \times n} \left(\frac{\mathbb{I}_m}{0_{(n-m) \times m}} \right), & n > m, \end{cases} \quad (14)$$

where 0_{pq} is a $p \times q$ matrix of zeros. As an example, let us consider the case $n_1 = n_2 = 1$ and substitute explicitly $n_L = 3$. The most general $\mathcal{R} \in SO(3)$ is

$$\mathcal{R} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13} & c_{12}c_{23} - s_{12}s_{23}s_{13} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13} & -c_{12}s_{23} - s_{12}c_{23}s_{13} & c_{23}c_{13} \end{pmatrix}, \quad (15)$$

and we can obtain the desired $\mathcal{R}_{2 \times 3}$ following Eq. (14):

$$\mathcal{R}_{2 \times 3} \equiv \left(\mathbb{I}_2 \mid 0_{2 \times 1} \right) \times \mathcal{R}_{3 \times 3}. \quad (16)$$

III. IMPLEMENTATIONS OF THE MASS MATRIX AND THE GENERALIZED CASAS-IBARRA PARAMETRIZATION

In the following, we study in turn tree-level and loop-level implementations of diagonal, off-diagonal, and general neutrino mass matrices. We consider specific models

that are widely discussed in the literature and illustrate how to implement for them the GCI parametrization. In all cases we start from a basis in which the M_{ij} matrices defined above are diagonal, and we use the notation $\bar{M}_i$ for the eigenvalues of $\mathcal{M}$. Also note that, when working in the basis in which the charged lepton mass matrix is diagonal, the matrix U appearing in all the expressions we give below coincides with the PMNS matrix, U_{PMNS} . For the latest values of the oscillation parameters, see Refs. [69,70].

A. Diagonal mass matrix

In some cases, there may only be one source of neutrino masses. In the following we study the case in which the diagonal element M_{11} is nonzero, $M_{11} \neq 0$, while the rest of elements are zero, $M_{12} = M_{21} = M_{22} = 0$.

1. Tree level: The seesaw case

The most famous example of $M_{11} \neq 0$ at tree level is the usual seesaw limit of models with extra right-handed neutrinos. If $n_R = n_1$ right-handed neutrinos N are added to the SM, the Yukawa and mass terms in the Lagrangian read [2–6],

$$-\mathcal{L}_{\text{SS}} = \bar{L}Y\tilde{H}N + \frac{1}{2}\bar{N}^c m_R N + \text{H.c.}, \quad (17)$$

where m_R is a $n_R \times n_R$ complex Majorana mass matrix which, without loss of generality, can be chosen to be diagonal with positive eigenvalues, and Y is a $n_L \times n_R$ complex Yukawa matrix. Here $L = (\nu_L, e_L)^T$ are the left-handed lepton doublets.⁵ We denote by H the SM Higgs doublet, which in the unitary gauge after electroweak symmetry breaking is given by

$$H \equiv (0, (h + v)/\sqrt{2})^T, \quad (18)$$

with $v = 246$ GeV the vacuum expectation value (VEV) and h the Higgs boson. In the basis (ν_L, N^c) , after electroweak spontaneous symmetry breaking (EW SSB), the complete $(n_L + n_R) \times (n_L + n_R)$ neutral fermion mass matrix reads

$$M_\nu = \begin{pmatrix} 0_{n_L \times n_L} & m_D \\ m_D^T & m_R \end{pmatrix}, \quad (19)$$

where $m_D = Yv/\sqrt{2}$ (so m_D is $n_L \times n_R$). In the seesaw limit, $m_D \ll m_R$, Eq. (19) may be diagonalized to yield the mass matrices,

⁵This flavor structure also applies, for instance, to Type-III seesaw and Model A_1 in Ref. [71].

$$\begin{aligned} m_{\text{light}} &\simeq m_D m_R^{-1} m_D^T \in \mathbb{C}^{n_L \times n_L}, \\ m_{\text{heavy}} &\simeq m_R \in \mathbb{C}^{n_R \times n_R}. \end{aligned} \quad (20)$$

The spectrum contains $(n_L - n_R)$ massless states at leading order. Therefore, the light neutrino mass matrix can be written as

$$m_\nu = Y_1^T M_{11} Y_1, \quad (21)$$

where $Y_1 = Y^T$ and $M_{11} = v^2/2m_R^{-1}$. Therefore, it has the form of Eq. (3) with $M_{12} = M_{21} = M_{22} = 0$, and the usual CI parametrization [18] can be used to express the Yukawa Y_1 in terms of light neutrino observables and other unknown free parameters. The most general neutrino Yukawa coupling compatible with low-energy data, in the basis where the charged leptons Yukawa couplings and the M_{11} are diagonal, reads

$$\begin{aligned} Y_1 &= D_{M_{11}}^{-1/2} R_\ell D_{\sqrt{m}} U^\dagger \\ &= \frac{\sqrt{2}}{v} m_R^{1/2} R_\ell D_{\sqrt{m}} U^\dagger \\ \text{Standard CI parametrization,} \end{aligned} \quad (22)$$

where $D_{M_{11}}^{-1/2}$ is the diagonal matrix of the square roots of the eigenvalues of M_{11} , $D_{\sqrt{m}}$ is the diagonal matrix of roots of the physical masses m_i of the light neutrinos, and R_ℓ is the $n_R \times n_L$ complex orthogonal matrix that includes all the free parameters describing the neutrino sector. In Sec. V, we discuss a rewriting of the general neutrino mass matrix in Eq. (19), which could potentially serve to obtain a GCI parametrization outside of the seesaw limit, or one that includes higher-order corrections. We also show in the Appendix how the standard CI may be derived in the seesaw limit using this approach.

Several other models exist in the literature where only the M_{11} element is different from zero at tree level, like Type-III seesaw [72], Model A₁ in Ref. [71], and other variants of the Type-I seesaw [73–75].

2. Loop level: The scotogenic model

The scotogenic model (ScM) [40,41] is based on a $\mathbb{Z}_2$ symmetry, see also Ref. [7]. The particle content with the associated charges can be found in Table I. The relevant terms in the Lagrangian of the model are

$$-\mathcal{L}_{\text{ScM}} = \frac{1}{2} \overline{\psi}_R m_R \psi_R + (\overline{\psi}_R \tilde{\Phi}^\dagger Y L + \text{H.c.}), \quad (23)$$

where $\tilde{\Phi} \equiv i\sigma_2 \Phi^*$. As in the case of the seesaw, at least two copies of the fermion singlets are needed. The scalar potential for Φ and the SM Higgs doublet H is given by

TABLE I. Particle content and charge assignments of the scotogenic model.

Field	$\text{SU}(3)_C$	$\text{SU}(2)_L$	$\text{U}(1)_Y$	$\mathbb{Z}_2$
$\Phi \equiv (\phi^+, \phi_0)^T$	1	2	1/2	1
ψ_R	1	1	0	1

$$\begin{aligned} \mathcal{V}_{\text{ScM}} &= -m_H^2 H^\dagger H + \lambda_H (H^\dagger H)^2 + m_\Phi^2 \Phi^\dagger \Phi + \lambda_\Phi (\Phi^\dagger \Phi)^2 \\ &\quad + \lambda_{H\Phi} (H^\dagger H) (\Phi^\dagger \Phi) + \lambda_{H\Phi,2} (H^\dagger \Phi) (\Phi^\dagger H) \\ &\quad + \frac{\lambda_{H\Phi,3}}{2} [(H^\dagger \Phi)^2 + \text{H.c.}]. \end{aligned} \quad (24)$$

The scalar mass spectrum, derived from the potential in Eq. (24), reads

$$\begin{aligned} m_{\phi_0^R} &= \sqrt{m_\Phi^2 + \frac{1}{2}(\lambda_{H\Phi} + \lambda_{H\Phi,2} + \lambda_{H\Phi,3})v^2}, \\ m_{\phi_0^I} &= \sqrt{m_\Phi^2 + \frac{1}{2}(\lambda_{H\Phi} + \lambda_{H\Phi,2} - \lambda_{H\Phi,3})v^2}, \\ m_{\phi^+} &= \sqrt{m_\Phi^2 + \frac{1}{2}\lambda_{H\Phi}v^2}. \end{aligned} \quad (25)$$

Here ϕ_0^R and ϕ_0^I denote the real and imaginary parts of the neutral field ϕ_0 in the additional scalar doublet, and ϕ^+ its charged component. The scalars ϕ_0^R and ϕ_0^I acquire a mass splitting proportional to the quartic coupling $\lambda_{H\Phi,3}$; taking $\lambda_{H\Phi,3}$ small is technically natural in the 't Hooft sense [76].

The exact discrete $\mathbb{Z}_2$ prevents the singlet fermion to couple with the Higgs doublet and neutrinos remain massless at tree level. The latter only obtain a mass at one loop from the diagram shown in Fig. 1. The Majorana neutrino mass matrix reads,

$$(m_\nu)_{\alpha\beta} = \frac{1}{32\pi^2} \sum_i (Y_{\alpha i} Y_{\beta i} m_{R_i}) \mathcal{F}(m_{\phi_0^R}, m_{\phi_0^I}, m_{R_i}), \quad (26)$$

where the loop function is

$$\mathcal{F}(x, y, z) = \frac{x^2}{x^2 - z^2} \log \frac{x^2}{z^2} - \frac{y^2}{y^2 - z^2} \log \frac{y^2}{z^2}. \quad (27)$$

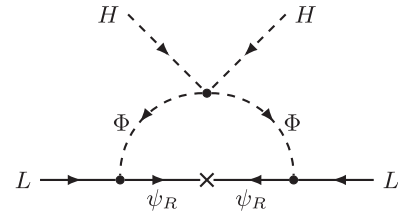


FIG. 1. Neutrino mass contributions at the one-loop level in the scotogenic model. The neutral scalars ϕ_0^R and ϕ_0^I run in the loop.

The neutrino mass matrix has the form of Eq. (21), $m_\nu = Y_1^T M_{11} Y_1$ with

$$Y_1 = Y^T, \quad M_{11} = \frac{m_{R_i}}{32\pi^2} \mathcal{F}(m_{\phi_0^R}, m_{\phi_0'}, m_{R_i}). \quad (28)$$

Therefore, the seesawlike flavor structure allows to use the standard CI parametrization [18], after trivially taking into account the different global factor. This result is in agreement with that found in Ref. [77], where the ScM was studied in the context of direct detection of Dark Matter (DM).

At one loop, another similar example is the scotosinglet model [78]; see Refs. [79–81] for other one-loop models and Ref. [8] for a review of radiative neutrino mass models.

B. Off-diagonal mass matrix

In the following we consider the scenario where only the crossed terms are nonvanishing, i.e., $M_{11} = M_{22} = 0$. We want to show how the CI parametrization can be implemented for such a broad type of scenario, without considering a particular model; this means that we do not consider if the neutrino mass matrix leads to a massless neutrino eigenvalue or to a specific mass ordering. We will go through some other examples in the following sections. The light neutrino mass matrix reads of Eq. (3) reads,

$$m_\nu = Y_1^T M_{12} Y_2 + Y_2^T M_{12}^T Y_1, \quad (29)$$

which can be rewritten in the form of Eq. (21) with

$$\mathcal{Y} = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \quad \text{and} \quad \mathcal{M} = \begin{pmatrix} 0 & M_{12} \\ M_{12}^T & 0 \end{pmatrix}. \quad (30)$$

The extended Yukawa matrix is $(n_1 + n_2) \times n_L$, and the extended mass matrix $\mathcal{M}$ is $(n_1 + n_2) \times (n_1 + n_2)$. The matrix $\mathcal{M}$ can be diagonalized by

$$V = \frac{1}{\sqrt{2}} \begin{pmatrix} i\mathbb{I}_{n_1 \times n_1} & -i\mathbb{I}_{n_1 \times n_2} \\ \mathbb{I}_{n_2 \times n_1} & \mathbb{I}_{n_2 \times n_2} \end{pmatrix}, \quad (31)$$

for arbitrary n_1 and n_2 , where the presence of the i ensures that all eigenvalues are positive, and with $D_{\mathcal{M}} = \text{diag}(M_{12}, M_{12})$. Therefore, we can proceed with the GCI parametrization provided in Eq. (11).

I. Casen₁ = n₂ = 1

In this case, the Yukawa couplings are three-dimensional vectors, so that the rank of the neutrino mass matrix is two. This implies that such a mass structure provides one massless neutrino. There is one parameter stemming from the $\mathcal{M}_{2 \times 2}$ matrix, together with the 12 real parameters needed to specify the Yukawa interactions. As is well-known, not all of them are physical quantities. Indeed, three of them can be absorbed by a rotation of the lepton doublets

and another one by the rescaling of the parameter in $\mathcal{M}_{2 \times 2}$. Therefore, there are only 10 effective parameters that determine the light-neutrino sector. Finally, we observe that the measurable parameters yield two masses, three mixing angles, and two phases, for a total of seven measurables, and we conclude that in order to describe the Majorana neutrino mass with off-diagonal mass matrix and $n_1 = n_2 = 1$, we need two real parameters. These parameters are included in the matrix $\mathcal{R}$, which can be obtained by following the procedure outlined in Eqs. (15) and (16), setting $\theta_{13} = \theta_{23} = 0$,⁶

$$\mathcal{R}_{2 \times 3} = \begin{pmatrix} \cos z & \mp \sin z & 0 \\ \sin z & \pm \cos z & 0 \end{pmatrix}, \quad (32)$$

where z is a complex angle, yielding the 2 real parameters discussed above. It must be noted that a $\pm$ in the second column has been included to account for possible reflections of the orthogonal $\mathcal{R}$ matrix. Let us now discuss how to diagonalize the $\mathcal{M}$ matrix in Eq. (30). Using Eq. (31), the matrix V of Eq. (6) can be directly obtained.

In standard seesaw models, the mixing of the charged leptons typically scales as $\Theta \sim \sqrt{m_\nu/M_N}$, with M_N the mass of the heavy leptons that leads to a suppression of the size of the Yukawa couplings, unless special mass textures are imposed in accordance with lepton number conservation [82–90]. This feature is shared by both the linear seesaw and the generalized scotogenic models, discussed below as examples of tree-level and loop-level neutrino mass generation, respectively. In the linear seesaw scenario, lepton number violation (LNV) arises only through a small perturbation in the mass matrix, while in the GScM nonzero tree level neutrino masses are protected by an additional $U(1)_{\text{DM}}$ symmetry. As a consequence, in both models the violation of lepton number is strongly suppressed, which naturally leads to a hierarchical Yukawa structure, i.e., $Y_2 \ll Y_1$. Therefore, within the CI parametrization, it is immediate to take the limit of large imaginary parts of the complex angle z ,⁷ corresponding to this strong hierarchy, which yields a rich phenomenology. For inverted ordering (IO), we obtain

$$\begin{aligned} Y_1 &= \frac{e^{-\text{Im } z}}{\sqrt{2M_1}} [u_1^* \sqrt{m_1} \mp i u_2^* \sqrt{m_2}], \\ Y_2 &= \frac{e^{\text{Im } z}}{\sqrt{2M_1}} [u_1^* \sqrt{m_1} \pm i u_2^* \sqrt{m_2}], \end{aligned} \quad (33)$$

⁶Since we need only one complex parameter two of the angles can be put to zero. The choice of which angles to set to zero depends on the structure of the diagonal neutrino mass matrix, i.e., on the position of its nonvanishing eigenvalues. By choosing the two nonzero eigenvalues to lie in the 2×2 upper-left sub-block of D_m , only θ_{12} has to appear in $\mathcal{R}$.

⁷In the large imaginary part limit of the angle z , the imaginary component is assumed to be positive.

while for the normal ordering (NO), we have

$$\begin{aligned} Y_1 &= \frac{e^{-\text{Im } z}}{\sqrt{2\bar{M}_1}} [u_2^* \sqrt{m_2} \mp i u_3^* \sqrt{m_3}], \\ Y_2 &= \frac{e^{\text{Im } z}}{\sqrt{2\bar{M}_1}} [u_2^* \sqrt{m_2} \pm i u_3^* \sqrt{m_3}], \end{aligned} \quad (34)$$

where u_i are the two columns of the PMNS mixing matrix. It is worth to notice that this result is the large mixing angle limit of the parametrization of Ref. [20], obtained for the seesaw scenario with two right-handed neutrinos. Also, a similar parametrization can be found in Ref. [91] in the context of low scale seesaw with two right-handed neutrinos.

2. Case $n_1 = n_2 = 2$

The matrix V which diagonalizes the matrix $\mathcal{M}_{4 \times 4}$ to the diagonal form,

$$D_{\mathcal{M}} = \text{diag}(\bar{M}_1, \bar{M}_2, \bar{M}_1, \bar{M}_2), \quad (35)$$

through the Autonne-Takagi decomposition in Eq. (6) is given by Eq. (31). Moreover, due to the increased dimensions of the Yukawa coupling matrices and of the matrix $\mathcal{M}_{4 \times 4}$, there are more free parameters to consider. With a similar parameter counting as the one performed for the previous case, we find that in this scenario there are 12 free real parameters. Using Eq. (14), we parametrize $\mathcal{R}_{4 \times 3}$ as

$$\mathcal{R}_{4 \times 3} = \mathcal{R}_{4 \times 4} \begin{pmatrix} \mathbb{I}_3 \\ 0_{1 \times 3} \end{pmatrix}, \quad (36)$$

which depends on six complex angles, contributing to the 12 real parameters required to describe the neutrino sector in this scenario.

3. Case $n_1 = n_2 = 3$

Using Eq. (31), the matrix V which diagonalizes the matrix $\mathcal{M}_{6 \times 6}$ to the diagonal form,

$$D_{\mathcal{M}} = \text{diag}(\bar{M}_1, \bar{M}_2, \bar{M}_3, \bar{M}_1, \bar{M}_2, \bar{M}_3),$$

through the Autonne-Takagi decomposition in Eq. (6) can be directly obtained. With a similar parameter counting performed as for the previous cases, we find that in this scenario there are 24 free real parameters. Using Eq. (14), we parametrize $\mathcal{R}_{6 \times 3}$ as

$$\mathcal{R}_{6 \times 3} = \mathcal{R}_{6 \times 6} \begin{pmatrix} \mathbb{I}_3 \\ 0_{3 \times 3} \end{pmatrix}, \quad (37)$$

which, in general, depends on 15 angles, for a total of 30 real parameters. Similarly to the case with $n_1 = n_2 = 1$, since there are more free parameters than needed to describe the neutrino sector, some of the angles can be

set to zero. In this scenario, three angles turn out to be redundant.

4. Tree level: The linear seesaw model

The linear seesaw (LSS) was first introduced in Refs. [92,93] for left-right symmetric models and also in grand unified theories [87,94]. Variations of the neutrino mass matrix have been proposed in Refs. [89,95]. More recently, models with new scalars in higher SU(2) representations have been proposed [71,96]. In general, the model consists of n_1 right-handed neutrinos N and n_2 singlet fermions S , so that the relevant terms in the Lagrangian are

$$-\mathcal{L}_{\text{LSS}} = \tilde{H}^\dagger \bar{N} Y_N L + \tilde{L} H Y_S S + \bar{N} M_R S + \text{H.c.}, \quad (38)$$

where $\tilde{L} \equiv i\sigma_2 C \bar{L}^T$, C the charge conjugation matrix, and M_R is a $n_1 \times n_2$ complex mass matrix and Y_N and Y_S are $3 \times n_1$ and $3 \times n_2$ complex Yukawa matrices, respectively. In the basis (ν_L, N^c, S) , after the EW SSB, the $(n_1 + n_2 + 3) \times (n_1 + n_2 + 3)$ neutral fermion mass matrix reads

$$M_\nu = \begin{pmatrix} 0 & \frac{v}{\sqrt{2}} Y_N^T & \frac{v}{\sqrt{2}} Y_S \\ \frac{v}{\sqrt{2}} Y_N & 0 & M_R \\ \frac{v}{\sqrt{2}} Y_S^T & M_R^T & 0 \end{pmatrix}. \quad (39)$$

Taking the limit $Y_N, Y_S \ll M_R/v$, the mass matrix of the light neutrinos is given by

$$m_\nu = \frac{v^2}{2} (Y_S M_R^{-1} Y_N + Y_N^T M_R^{T-1} Y_S^T), \quad (40)$$

which has the form of Eq. (29) with

$$Y_1 = Y_S^T, \quad Y_2 = Y_N, \quad M_{12} = \frac{v^2}{2} M_R^{-1}. \quad (41)$$

Note that the heavy fermions are degenerate in the aforementioned limit. The origin of the name *linear seesaw* becomes evident; the Yukawa couplings of the right-handed neutrinos enter linearly in the mass matrix, in contrast to the usual seesaw in which they enter quadratically. A schematic diagram of the neutrino mass generation is shown in Fig. 2.

One may directly use the GCI parametrization, Eq. (11), with V given by Eq. (31). For the $n_1 = n_2 = 1$ case, the simpler forms for the Yukawas provided in Eqs. (33) and (34) may be used for IO and NO, respectively. Note that Y_S violates lepton number in two units and therefore it is small in 't Hooft sense [76]. This correspond to the large imaginary part of the complex angle z . This flavor structure also applies to Models A_2 and B_i in Ref. [71]. The parametrization presented for the linear seesaw model can be mapped onto the parametrization used in

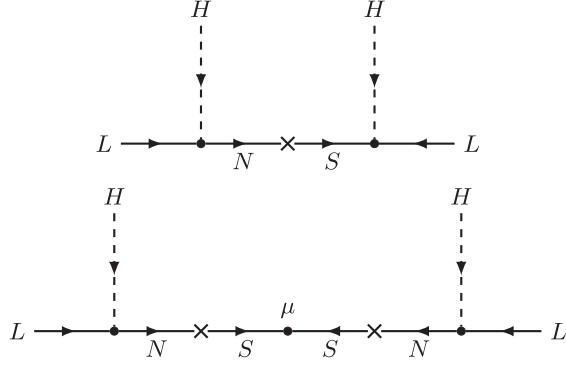


FIG. 2. Neutrino masses at tree level in the linear seesaw (left) and the inverse seesaw (right).

Ref. [90] in the case of degenerate heavy neutrinos by taking $\text{Re}[z] \rightarrow \theta/2$. Note that in that reference the Majorana phase is shifted by $\pi/2$ in order to provide all positive neutrino masses, whereas here the result is automatically obtained.

5. Loop level: Generalized scotogenic model

Here we discuss the generalized scotogenic model (GScM) [97], a generalization of the ScM Model [40,41] based on a global $U(1)_{\text{DM}}$ symmetry instead of the $\mathbb{Z}_2$ symmetry of ScM. We follow closely the notation of Ref. [97]. The SM is augmented by two additional scalar doublets and at least one vectorlike Dirac fermion, all charged under the $U(1)_{\text{DM}}$ symmetry. The particle content

TABLE II. Particle content and charge assignments of the generalized scotogenic model.

Field	$SU(3)_C$	$SU(2)_L$	$U(1)_Y$	$U(1)_{\text{DM}}$
$\Phi \equiv (\phi^+, \phi_0)^T$	1	2	1/2	1
$\Phi' \equiv (\phi'_0, \phi'^-)^T$	1	2	-1/2	1
Vectorlike fermion ψ	1	1	0	1

with the associated charges can be found in Table II. The relevant terms in the Lagrangian are

$$-\mathcal{L}_{\text{GScM}} = \bar{\psi} M_{\psi} \psi + (\bar{\psi} Y \tilde{\Phi}^\dagger L + \bar{\psi} Y'^* \tilde{\Phi}'^\dagger \tilde{L} + \text{H.c.}). \quad (42)$$

For the case of just one fermion, the neutrino Yukawa couplings Y, Y' are three-component row vectors. The scalar potential of Φ, Φ' , and H is given by

$$\begin{aligned} \mathcal{V}_{\text{GScM}} = & -m_H^2 H^\dagger H + \lambda_H (H^\dagger H)^2 + m_\Phi^2 \Phi^\dagger \Phi + \lambda_\Phi (\Phi^\dagger \Phi)^2 \\ & + m_{\Phi'}^2 \Phi'^\dagger \Phi' + \lambda_{\Phi'} (\Phi'^\dagger \Phi')^2 + \lambda_{H\Phi} (H^\dagger H) (\Phi^\dagger \Phi) \\ & + \lambda_{H\Phi'} (H^\dagger H) (\Phi'^\dagger \Phi') + \lambda_{\Phi\Phi'} (\Phi^\dagger \Phi) (\Phi'^\dagger \Phi') \\ & + \lambda_{H\Phi,2} (H^\dagger \Phi) (\Phi^\dagger H) + \lambda_{H\Phi',2} (H^\dagger \Phi') (\Phi'^\dagger H) \\ & + \lambda_{\Phi\Phi',2} (\Phi^\dagger \Phi') (\Phi'^\dagger \Phi) \\ & + \lambda_{H\Phi\Phi'} [(H^\dagger \Phi') (H^\dagger \Phi) + \text{H.c.}]. \end{aligned} \quad (43)$$

In the interaction basis (H_0, Φ_0, Φ'_0) , the scalar mass matrix is

$$M_S^2 = \begin{pmatrix} -m_H^2 + 2v^2\lambda_H & 0 & 0 \\ 0 & m_\Phi^2 + \frac{1}{2}v^2(\lambda_{H\Phi} + \lambda_{H\Phi,2}) & -\frac{1}{2}v^2\lambda_{H\Phi\Phi'} \\ 0 & -\frac{1}{2}v^2\lambda_{H\Phi\Phi'} & m_{\Phi'}^2 + \frac{1}{2}v^2(\lambda_{H\Phi'} + \lambda_{H\Phi',2}) \end{pmatrix}. \quad (44)$$

The neutral mass eigenstates are defined as

$$\begin{aligned} \eta_0 &= s_\theta \Phi_0 + c_\theta \Phi'_0, \\ \eta'_0 &= -c_\theta \Phi_0 + s_\theta \Phi'_0, \end{aligned} \quad (45)$$

with $c_\theta \equiv \cos \theta$ and $s_\theta \equiv \sin \theta$. The mixing angle is

$$\tan 2\theta = \frac{2c}{b-a}, \quad (46)$$

with

$$\begin{aligned} c &= -\frac{1}{2}\lambda_{H\Phi\Phi'}v^2, \\ a &= m_\Phi^2 + \frac{1}{2}v^2(\lambda_{H\Phi} + \lambda_{H\Phi,2}), \\ b &= m_{\Phi'}^2 + \frac{1}{2}v^2(\lambda_{H\Phi'} + \lambda_{H\Phi',2}). \end{aligned} \quad (47)$$

The mass eigenstates (h, η_0, η'_0) have masses,

$$\begin{aligned} m_h &= \sqrt{2\lambda_H}v, \\ m_{\eta_0} &= \sqrt{\frac{1}{2}\left(a+b+\sqrt{(a-b)^2-4c^2}\right)}, \\ m_{\eta'_0} &= \sqrt{\frac{1}{2}\left(a+b-\sqrt{(a-b)^2-4c^2}\right)}. \end{aligned} \quad (48)$$

Similarly to the scotogenic model, the exact global $U(1)_{\text{DM}}$ prevents the singlet fermion to couple with the

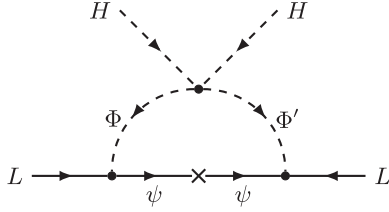


FIG. 3. Neutrino masses at the one-loop level in the generalized scotogenic model.

Higgs doublet and neutrinos remain massless at tree level. The latter only obtain a mass at one loop from the diagram shown in Fig. 3. The Majorana neutrino mass matrix reads,

$$(m_\nu)_{\alpha\beta} = \frac{\sin 2\theta}{32\pi^2} \sum_i (Y_{\alpha i} Y'_{\beta i} M_i + Y'_{\alpha i} Y_{\beta i} M_i) \times \mathcal{F}(m_{\eta_0}, m_{\eta'_0}, M_i), \quad (49)$$

where the loop function is defined in Eq. (27). The sum over the index i indicates the generalization to the case with multiple fermions, each with mass M_i . The neutrino mass matrix has the form of Eq. (29) with

$$Y_1 = Y, \quad Y_2 = Y', \quad M_{12} = \frac{\sin 2\theta}{32\pi^2} \text{diag}(M_i \mathcal{F}(M_i)). \quad (50)$$

For the $n_1 = n_2 = 1$ case, Eqs. (33) and (34) may be used for IO and NO, respectively. As a consistency check, we note that the parametrization derived from the GCI construction coincides with that employed in Refs. [97,98] by replacing $\zeta \rightarrow e^{-\text{Im} z}$. For suitable lepton number charge assignments of the additional scalar and fermion fields, lepton number violation originates from the Yukawa coupling Y , which can thus be regarded as naturally small in the sense of 't Hooft naturalness. This corresponds to the large imaginary part of the complex angle z .

Let us also mention that one of the interesting features of the GScM is that the lightest neutral particle of the dark sector (the dark fermion ψ or the lightest neutral dark scalar) may be a good DM candidate.

C. Diagonal plus off-diagonal mass matrix

This case corresponds to relaxing the assumption of vanishing diagonal entries in the matrix $\mathcal{M}$, thus obtaining the most general neutrino mass matrix m_ν , namely that of Eq. (9), for which the GCI parametrization applies, Eq. (11). As in the previous case with purely off-diagonal contributions, we will consider specific models in which neutrino masses arise either from tree-level diagrams (the linear plus inverse seesaw scenario) or from one-loop diagrams (the extended scotogenic model).

1. Tree level: Linear plus inverse seesaw model

In this section we consider the linear plus inverse seesaw model (LISS). It adds a small Majorana mass for S ,

$$-\mathcal{L}_{\text{ISS}} = \frac{1}{2} \overline{S^c} \mu S + \text{H.c.}, \quad (51)$$

to the Lagrangian of the LSS model, $\mathcal{L}_{\text{LSS}}$ given in Eq. (38),⁸ so that

$$\mathcal{L}_{\text{LISS}} = \mathcal{L}_{\text{LSS}} + \mathcal{L}_{\text{ISS}}. \quad (52)$$

Note that μ violates lepton number in two units and therefore is small in 't Hooft sense. The tree-level neutrino mass generation in the (LSS) ISS is shown on the left (right) panel of Fig. 2. The neutrino mass matrix reads,

$$M_\nu = \begin{pmatrix} 0 & \frac{v}{\sqrt{2}} Y_N^T & \frac{v}{\sqrt{2}} Y_S \\ \frac{v}{\sqrt{2}} Y_N & 0 & M_R \\ \frac{v}{\sqrt{2}} Y_S^T & M_R^T & \mu \end{pmatrix}. \quad (53)$$

Similarly to the usual seesaw limit, we can assume $Y_N v, Y_S v, \mu \ll M_R$, so that the mass matrix of the light neutrinos to the lowest order reads,

$$m_\nu = \frac{v^2}{2} (Y_S M_R^{-1} Y_N + Y_N^T M_R^{T-1} Y_S^T + Y_N^T M_R^{T-1} \mu M_R^{-1} Y_N), \quad (54)$$

where the first (second) contribution corresponds to the linear (inverse) seesaw. This has the form of Eq. (3) with

$$Y_1 = Y_S^T, \quad Y_2 = Y_N, \quad M_{22} = \frac{v^2}{2} M_R^{T-1} \mu M_R^{-1}, \quad M_{12} = \frac{v^2}{2} M_R^{-1}. \quad (55)$$

We now illustrate how the Yukawa parametrization is modified in the presence of a nonzero diagonal mass term for the case $n_1 = n_2 = 1$. In such a scenario, the $\mathcal{R}$ is the same as in Eq. (32) and V is obtained from Eq. (31). This scenario differs from the LSS one due to the small contribution to the $\mathcal{M}$ eigenvalues proportional to the LNV term μ . Therefore, now the heavy fermions form pseudo-Dirac pairs. The diagonalization of $\mathcal{M}$ gives

$$V^* \begin{pmatrix} 0 & M_{12} \\ M_{12} & M_{22} \end{pmatrix} V^\dagger \approx \begin{pmatrix} M_{12} + \frac{M_{22}}{2} & 0 \\ 0 & M_{12} - \frac{M_{22}}{2} \end{pmatrix}, \quad (56)$$

⁸One can also add another one in the M_ν^{11} element, but it will only contribute to neutrino masses at one loop. See Ref. [89,99,100] for other variants of the inverse seesaw.

TABLE III. Particle content and charge assignments of the extended scotogenic model.

Field	SU(3) _C	SU(2) _L	U(1) _Y	$\mathbb{Z}_2$
$\Phi \equiv (\phi^+, \phi_0)^T$	1	2	1/2	1
ψ_L	1	1	0	1
ψ_R	1	1	0	1

where $\approx$ indicates that this expression is valid in the limit $M_{22} \ll M_{12}$. For IO, we obtain,

$$\begin{aligned}
Y_1 &= \frac{e^{-\text{Im} z}}{\sqrt{2M_{12}}} [u_1^* \sqrt{m_1} \mp i u_2^* \sqrt{m_2}] \\
&\quad + \frac{e^{\text{Im} z}}{\sqrt{2M_{12}}} \frac{M_{22}}{4M_{12}} [u_1^* \sqrt{m_1} \pm i u_2^* \sqrt{m_2}], \\
Y_2 &= \frac{e^{\text{Im} z}}{\sqrt{2M_{12}}} [u_1^* \sqrt{m_1} \pm i u_2^* \sqrt{m_2}], \quad (57)
\end{aligned}$$

while for the NO, we have

$$\begin{aligned}
Y_1 &= \frac{e^{-\text{Im} z}}{\sqrt{2M_{12}}} [u_2^* \sqrt{m_2} \mp i u_3^* \sqrt{m_3}] \\
&\quad + \frac{e^{\text{Im} z}}{\sqrt{2M_{12}}} \frac{M_{22}}{4M_{12}} [u_2^* \sqrt{m_2} \pm i u_3^* \sqrt{m_3}], \\
Y_2 &= \frac{e^{\text{Im} z}}{\sqrt{2M_{12}}} [u_2^* \sqrt{m_2} \pm i u_3^* \sqrt{m_3}], \quad (58)
\end{aligned}$$

where u_i are the two columns of the PMNS mixing matrix. For completeness, we remark that also the Y_2 Yukawa receive contributions proportional to the LNV term; however, these are further suppressed by a factor $e^{-\text{Im} z}$ and are thus neglected. As a consistency check, we find that our results reproduce those of Ref. [90] upon identifying $M_{22} \rightarrow \Delta M$ and $M_{12} \rightarrow M$.

2. Loop level: Extended scotogenic model

Much like in the periodic table, where the presence of *gaps* has guided the search for missing elements, the absence of a radiative model incorporating both diagonal and off-diagonal mass contributions motivates the following proposal. The model is the generalization of the ScM [40,41], still based on a $\mathbb{Z}_2$ symmetry, but adding both chiralities of the fermion singlet. We term it the extended scotogenic model (EScM). The particle content with the associated charges can be found in Table III. The Lagrangian for the model is

$$\begin{aligned}
-\mathcal{L}_{\text{EScM}} &= \bar{\psi} m_D \psi + \frac{1}{2} \bar{\psi}_L^c m_L \psi_L + \frac{1}{2} \bar{\psi}_R^c m_R \psi_R \\
&\quad + (\bar{\psi}_R \tilde{\Phi}^\dagger Y L + \bar{\psi}_L \tilde{\Phi}^T Y'^* \tilde{L} + \text{H.c.}), \quad (59)
\end{aligned}$$

where for the case of just one copy of the fermion ψ , the neutrino Yukawa couplings Y, Y' are three-component row

vectors. The scalar potential is provided in Eq. (24). In the basis $(\nu_L, \psi_R^c, \psi_L)$, the complete neutral fermion mass matrix at tree-level reads,

$$M_\nu^{\text{tree}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & m_R & m_D \\ 0 & m_D^T & m_L \end{pmatrix}. \quad (60)$$

Therefore, active neutrinos are massless at tree level. One can diagonalize the heavy fermion mass matrix. For one family of fermions, so that m_R, m_D and m_L are numbers, one gets

$$M_{1,2} = \frac{1}{2} \left[m_R + m_L \pm \sqrt{(m_L - m_R)^2 + 4m_D^2} \right]. \quad (61)$$

The neutral mass eigenstates are defined as

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = V(\theta) \begin{pmatrix} \psi_R^c \\ \psi_L \end{pmatrix}, \quad V(\theta) = \begin{pmatrix} \sin \theta & \cos \theta \\ -\cos \theta & \sin \theta \end{pmatrix}, \quad (62)$$

with the mixing angle defined by

$$\tan 2\theta = \frac{2m_D}{m_L - m_R}. \quad (63)$$

We can distinguish several limiting cases:

(i) *Seesaw case*: $m_R \gg m_D \gg m_L$. We obtain:

$$M_1 \simeq m_L - \frac{m_D^2}{m_R}, \quad M_2 \simeq m_R. \quad (64)$$

Similarly for $m_L \gg m_D \gg m_R$, interchanging $m_R \leftrightarrow m_L$; and

(ii) *Pseudo-Dirac case*: $m_D \gg m_{L,R}$. We get

$$M_1 \simeq M_2 \simeq m_D. \quad (65)$$

This scenario corresponds to the off-diagonal case discussed above.

Light neutrino masses are generated at one loop, as shown in the diagrams of Fig. 4. The neutrino mass matrix can be written as in Eq. (4) with⁹

$$\begin{aligned}
\mathcal{Y} &= \begin{pmatrix} Y \\ Y' \end{pmatrix}, \\
\mathcal{M} &= \frac{1}{32\pi^2} \begin{pmatrix} F_1 M_1 s_\theta^2 + F_2 M_2 c_\theta^2 & (F_1 M_1 - F_2 M_2) s_\theta c_\theta \\ (F_1 M_1 - F_2 M_2) s_\theta c_\theta & F_1 M_1 c_\theta^2 + F_2 M_2 s_\theta^2 \end{pmatrix}, \quad (66)
\end{aligned}$$

⁹Note that the contributions to the neutrino mass matrix can be equivalently expressed in the form of Eq. (3).

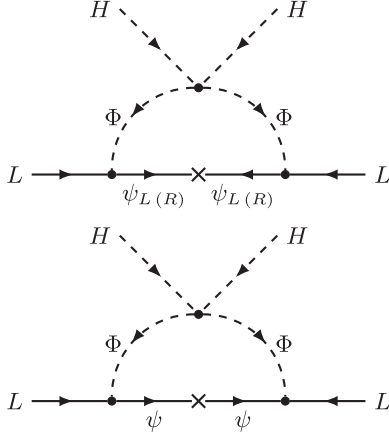


FIG. 4. Neutrino mass contributions at the one-loop level in the extended scotogenic model. *Left*: Contributions proportional to $m_{L(R)}$; *Right*: Contributions proportional to m_D .

where $s_\theta = \sin \theta$ and $c_\theta = \cos \theta$, with the mixing angle θ defined in Eq. (63). Here $F_k \equiv \mathcal{F}(m_{\phi_0^R}, m_{\phi_0^I}, M_k)$, where the loop function $\mathcal{F}$ is given in Eq. (27), and M_1 and M_2 are the mass eigenstates in Eq. (61). Importantly, even for the case of only one family of heavy fermions, two masses for light neutrinos are generated. It is therefore possible to use the GCI parametrization, see also Ref. [101]. First, let us consider the seesaw case for the heavy fermions, $m_R \gg m_D \gg m_L$, such that $s_\theta \simeq m_D/m_R$. The elements of $\mathcal{M}$ are given by

$$\begin{aligned} M_{11} &\simeq \frac{m_R}{32\pi^2} F_2, \\ M_{12} &\simeq -\frac{m_D}{32\pi^2} F_2, \\ M_{22} &\simeq \frac{1}{32\pi^2} \left[m_L F_1 + (F_2 - F_1) \frac{m_D^2}{m_R} \right]. \end{aligned} \quad (67)$$

In the limit $M_k \gg M_{\phi_0}$, the light neutrino mass is approximately given by

$$m_\nu \simeq \frac{1}{32\pi^2} Y'^T F_1 \left(m_L - \frac{m_D^2}{m_R} \right) Y'. \quad (68)$$

In this limit, the conventional seesaw scenario is reproduced, allowing to directly use the GCI parametrization. On the other hand, expanding $\mathcal{M}$ in Eq. (66) in the pseudo-Dirac limit, $m_D \gg m_{L,R}$, we have $s_\theta \simeq 1/\sqrt{2}$, and obtain

$$\begin{aligned} M_{11} &\simeq \frac{F_1 + F_2}{64\pi^2} \left[m_R + m_D \frac{F_1 - F_2}{F_1 + F_2} \right], \\ M_{12} &\simeq \frac{F_1 + F_2}{64\pi^2} \left[m_D + \frac{1}{2} (m_L + m_R) \frac{F_1 - F_2}{F_1 + F_2} \right], \\ M_{22} &\simeq \frac{F_1 + F_2}{64\pi^2} \left[m_L + m_D \frac{F_1 - F_2}{F_1 + F_2} \right]. \end{aligned} \quad (69)$$

Furthermore, neglecting the second term in the squared bracket, which is further suppressed by $(F_1 - F_2)/(F_1 + F_2) \ll 1$ as $M_1 \sim M_2 \simeq m_D$, we get the neat form,

$$\mathcal{M} \simeq \frac{F_1 + F_2}{64\pi^2} \begin{pmatrix} m_R & m_D \\ m_D & m_L \end{pmatrix}. \quad (70)$$

Note that the M_{11} contribution coincides with the scotogenic one, while the $M_{12} + M_{21}$ contribution is the same as in the generalized scotogenic model. Equivalently,

$$\begin{aligned} m_\nu &= \frac{1}{32\pi^2} (F_1 + F_2) [Y^T m_R Y + Y'^T m_L Y' \\ &\quad + Y^T m_D Y' + Y'^T m_D Y]. \end{aligned} \quad (71)$$

We thus recover the same neutrino mass structure discussed in the LSS + ISS case, with the difference being that in the present scenario $M_{11} \neq 0$. The presence of this term modifies the form of the eigenvalues of $\mathcal{M}$, which now receive an additional contribution proportional to it, leading to an enhanced splitting between the eigenvalues of $\mathcal{M}$ (as discussed in Ref. [90] for the LSS + ISS case). The parametrization of the Yukawa couplings, however, remains unchanged in its overall structure, and given by Eqs. (57) and (58) with M_{22} now replaced by $M_{11} + M_{22}$.

IV. IMPLEMENTATION OF THE GENERALIZED CASAS-IBARRA PARAMETRIZATION IN MODELS WITH EXTRA SYMMETRIES

The GCI parametrization is particularly useful in scenarios characterized by anarchic Yukawa couplings, i.e., when there is no specific underlying symmetry in them. Nonetheless, the GCI framework remains applicable and insightful also in situations where the Yukawas have special symmetries. This is the case of an antisymmetric matrix, $f = -f^T$, which for instance appears in the case of a singly-charged singlet, h^+ ,

$$\bar{L} f L h^+ + \text{H.c.} \quad (72)$$

As in this case neutrino masses cannot appear at tree level, we focus on a one-loop realization, the Zee model.

TABLE IV. Particle content and charge assignments of the Zee model.

Field	SU(3) _C	SU(2) _L	U(1) _Y
H_1	1	2	1/2
H_2	1	2	1/2
h^+	1	1	1

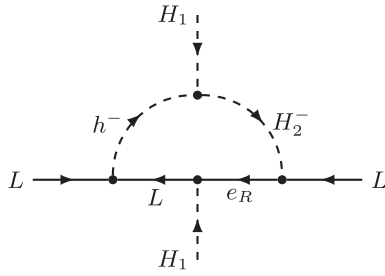


FIG. 5. Neutrino masses at the one-loop level in the Zee model. The mass eigenstates $h_{1,2}^+$ and the charged leptons run in the loop.

A. The Zee model

As an illustrative example, we consider the Zee model [55–57], see also Ref. [66] for a full analysis. The model provides an example of one-loop neutrino mass generation and consists of an extension of the Standard Model by an additional scalar doublet H_2 , together with a singly-charged scalar singlet h^+ . The particle content of the model is summarized in Table IV and the neutrino mass generation at one-loop is shown in Fig. 5. Without loss of generality, we work in the Higgs basis where only the scalar doublet H_1 takes a VEV $v = 246$ GeV. The Yukawa Lagrangian of the model reads,

$$-\mathcal{L}_{\text{Zee}} = \bar{L}(y_e^\dagger H_1 + \Gamma_e^\dagger H_2)e_R + \bar{\tilde{L}}fLh^+ + \text{H.c.}, \quad (73)$$

where e_R are the right-handed charged lepton singlets. Here y_e can be taken to be diagonal, i.e., we are in the charged lepton mass basis, with $M_e = y_e^\dagger v/\sqrt{2}$, whereas Γ_e is a general complex $n_L \times n_L$ matrix and f is a complex $n_L \times n_L$ antisymmetric matrix, $f^T = -f$. The scalar potential of the Zee model includes the LNV trilinear term,

$$V \subset \mu \tilde{H}_1^\dagger H_2 h^- + \text{H.c.}, \quad (74)$$

which, without loss of generality, can be chosen to be real and positive by a field redefinition of the singlet h^+ . After the EW SSB, it induces a mixing between the singly-charged scalars of the doublet H_2 and the singlet h^+ ,

$$\sin 2\varphi = \frac{\sqrt{2}v\mu}{m_{h_2^+}^2 - m_{h_1^+}^2}, \quad (75)$$

where h_1^+ and h_2^+ are the mass eigenstates, with $m_{h_2^+} > m_{h_1^+}$. The Majorana neutrino mass matrix reads,

$$m_\nu = \frac{\sin 2\varphi}{16\pi^2} (f M_e \Gamma_e + \Gamma_e^T M_e^T f^T) \log\left(\frac{m_{h_2^+}^2}{m_{h_1^+}^2}\right). \quad (76)$$

This mass matrix can be rewritten in the form of Eq. (10) with the identifications,

$$\mathcal{Y} = \begin{pmatrix} \hat{f} \\ \Gamma_e \end{pmatrix}, \quad \mathcal{M} = \begin{pmatrix} 0 & M_e \\ M_e^T & 0 \end{pmatrix}, \quad (77)$$

where M_e is the diagonal matrix of charged lepton masses and we have absorbed the global factors by defining

$$\hat{f} = \frac{\sin 2\varphi}{16\pi^2} \log\left(\frac{m_{h_2^+}^2}{m_{h_1^+}^2}\right) f^T. \quad (78)$$

The explicit form of the matrix V in $D_{\mathcal{M}} = V^* \mathcal{M} V^\dagger$ is given in Eq. (31). This allows for the use of the GCI parametrization, Eq. (11). However, in this scenario, the matrix $\mathcal{R}$ has less degrees of freedom than an arbitrary semiorthogonal complex matrix, because it must realize the antisymmetric structure of the Yukawa matrix associated with the singly-charged scalar h^+ , i.e., $f^T = -f$.

In the following, we show that it is possible to find an explicit parametrization of the $\mathcal{R}$ matrix that guarantees the antisymmetric condition of the Yukawa matrix f . For this, it is convenient to write $\mathcal{R}$ as

$$\mathcal{R} = \begin{pmatrix} R_1 \\ R_2 \end{pmatrix}, \quad (79)$$

where R_1 and R_2 are $n_L \times n_L$ complex matrices, with $n_L = 3$. In this way, the orthogonality condition $\mathcal{R}^T \mathcal{R} = \mathbb{I}_3$ reads

$$R_1^T R_1 + R_2^T R_2 = \mathbb{I}_3. \quad (80)$$

In addition, from the GCI parametrization, considering the upper block part, we get

$$\hat{f} = \frac{-i}{\sqrt{2}} M_e^{-1/2} (R_1 + iR_2) D_{\sqrt{m}} U^\dagger. \quad (81)$$

Therefore,

$$R_1 + iR_2 = A := i\sqrt{2} M_e^{1/2} \hat{f} U D_m^{-1/2}, \quad (82)$$

where we have introduced the matrix A and assumed that D_m is invertible. Transposing this equation, it is immediate to see that

$$R_1^T R_1 = A^T A - i(R_2^T A + A^T R_2) - R_2^T R_2. \quad (83)$$

Notably, from this equation and the orthogonality condition in Eq. (80), we arrive at a condition which is linear in R_2 , namely,

$$R_2^T A + A^T R_2 = i(\mathbb{I}_3 - A^T A). \quad (84)$$

Then it is possible to choose 6 free (complex) parameters and obtain analytical expressions for the remaining ones in

the following way. Two of the three Yukawa couplings in $\hat{f}$ can be freely chosen, while the third one is fixed by the condition that $\hat{f}$, being of rank 2, has a null eigenvector [54]. Specifically, writing $\hat{f}$ as

$$\hat{f} = \begin{pmatrix} 0 & \hat{f}_3 & -\hat{f}_2 \\ -\hat{f}_3 & 0 & \hat{f}_1 \\ \hat{f}_2 & -\hat{f}_1 & 0 \end{pmatrix}, \quad (85)$$

$\hat{v} = (\hat{f}_1, \hat{f}_2, \hat{f}_3)^T$ is a null eigenvector of $\hat{f}$, which satisfies

$$\hat{v}^T m_\nu \hat{v} = 0, \quad (86)$$

leading to a quadratic equation that fixes one of the elements of $\hat{f}$ in terms of the other two [54]. Moreover, once $\hat{f}$ is fixed, the matrix A is completely determined (for a given light neutrino mass matrix) and therefore Eq. (84) leads to a linear system of six equations for the nine elements of R_2 . Actually, given that $\hat{f}$ is not invertible, neither is A , and one can show that there are only five independent linear equations for the elements of R_2 . The linear system can always be solved analytically, yielding five elements of R_2 in terms of the other four that are chosen as free parameters. In turn, R_1 is determined from Eq. (82) (which can also be used to choose a different set of free parameters).

In order to write explicitly an independent set of linear equations for the elements of R_2 , note that from $\hat{f} \hat{v} = 0$ it follows that

$$A(D_{\sqrt{m}} U^\dagger \hat{v}) = 0. \quad (87)$$

From this condition it is possible to write the first column of A as a linear combination of the second and third columns, namely,

$$A_{i1} = \lambda_2 A_{i2} + \lambda_3 A_{i3}, \quad (88)$$

with $\lambda_{2,3}$ determined from $\hat{f}$ and m_ν ,

$$\begin{aligned} \lambda_2 &= -\frac{\sum_i \sqrt{m_2} U_{i2}^* \hat{f}_i}{\sum_i \sqrt{m_1} U_{i1}^* \hat{f}_i}, \\ \lambda_3 &= -\frac{\sum_i \sqrt{m_3} U_{i3}^* \hat{f}_i}{\sum_i \sqrt{m_1} U_{i1}^* \hat{f}_i}, \end{aligned} \quad (89)$$

for $\sum_i \sqrt{m_1} U_{i1}^* \hat{f}_i \neq 0$. Then, a proper system of five linear independent equations for the elements of $R_2 := [r_{ij}]$, writing also $A := [a_{ij}]$ and $i(\mathbb{I}_3 - A^T A) := [b_{ij}] = [b_{ji}]$, is

$$\begin{aligned} 2 \sum_j (\lambda_2 a_{j2} + \lambda_3 a_{j3}) r_{j1} &= b_{11} \\ \sum_j a_{j3} r_{j1} + (\lambda_2 a_{j2} + \lambda_3 a_{j3}) r_{j3} &= b_{13} \\ 2 \sum_j a_{j2} r_{j2} &= b_{22} \\ \sum_j a_{j3} r_{j2} + a_{j2} r_{j3} &= b_{23} \\ 2 \sum_j a_{j3} r_{j3} &= b_{33}, \end{aligned} \quad (90)$$

where, if not specified, the summations go from $i(j) = 1$ to $i(j) = 3$.¹⁰ Here it is possible to choose different sets of four free parameters and express the remaining ones in terms of them, typically by solving smaller subsystems at a time. For example, choosing r_{11} , r_{21} , r_{12} , and r_{13} as free parameters, the solution is

$$r_{31} = \frac{1}{\lambda_2 a_{32} + \lambda_3 a_{33}} \left(\frac{b_{11}}{2} - \sum_{j=1}^2 (\lambda_2 a_{j2} + \lambda_3 a_{j3}) r_{j1} \right), \quad (91)$$

$$\begin{aligned} \begin{pmatrix} r_{23} \\ r_{33} \end{pmatrix} &= \frac{1}{\lambda_2 d} \begin{pmatrix} a_{33} & -(\lambda_2 a_{32} + \lambda_3 a_{33}) \\ -a_{23} & \lambda_2 a_{22} + \lambda_3 a_{23} \end{pmatrix} \\ &\times \begin{pmatrix} b_{13} - \sum_{j=1}^3 a_{j3} r_{j1} - (\lambda_2 a_{12} + \lambda_3 a_{13}) r_{13} \\ b_{33}/2 - a_{13} r_{13} \end{pmatrix}, \end{aligned} \quad (92)$$

$$\begin{aligned} \begin{pmatrix} r_{22} \\ r_{32} \end{pmatrix} &= \frac{1}{d} \begin{pmatrix} a_{33} & -a_{32} \\ -a_{23} & a_{22} \end{pmatrix} \\ &\times \begin{pmatrix} b_{22}/2 - a_{12} r_{12} \\ b_{23} - a_{13} r_{12} - \sum_{j=1}^3 a_{j2} r_{j3} \end{pmatrix}, \end{aligned} \quad (93)$$

with $d = a_{22} a_{33} - a_{23} a_{32}$. Finally, R_1 can be obtained from R_2 using Eq. (82) and we arrive at an explicit parametrization for both $\hat{f}$ [see Eq. (81)] and Γ_e ,

$$\Gamma_e = \frac{1}{\sqrt{2}} M_e^{-1/2} (iR_1 + R_2) D_{\sqrt{m}} U^\dagger, \quad (94)$$

which results from considering the lower block part of the GCI parametrization.

This procedure gives a parametrization of the Zee model and other models with an antisymmetric Yukawa matrix, with analytical expressions for all the parameters in terms of a chosen set of free parameters. Note that once the Yukawa couplings in f are fixed, instead of following the GCI procedure outlined above, it would also be possible to

¹⁰The equation $\sum_j a_{j2} r_{j1} + (\lambda_2 a_{j2} + \lambda_3 a_{j3}) r_{j2} = b_{12}$ is a linear combination of the other ones and therefore it has been omitted.

set and solve analytically a linear system of equations directly for the Yukawa couplings in Γ_e . Also notice that a different parametrization was proposed in Ref. [102]¹¹ and it was applied to phenomenological studies in Refs. [67,103]. Further insight into the relationship among the different parameters of the Zee model and observational constraints, particularly on charged lepton flavor violation, was given in Ref. [58]. Since different parametrizations may have different virtues, comparing and applying these approaches could be an interesting direction for future work.

V. THE COMPLETE MASS MATRIX WITH RIGHT-HANDED NEUTRINOS: DISCUSSION AND OUTLOOK

It is suggestive to rewrite the complete neutrino mass matrix including right-handed neutrinos, M_ν in Eq. (19), in the form,

$$M_\nu = \tilde{Y}^T \tilde{\mathcal{M}} \tilde{Y}, \quad (95)$$

with

$$\tilde{Y} = \begin{pmatrix} Y^T & 0_{n_R \times n_R} \\ 0_{n_R \times n_L} & \mathbb{I}_{n_R} \end{pmatrix} \in \mathbb{C}^{2n_R \times (n_L + n_R)} \quad (96)$$

and

$$\tilde{\mathcal{M}} = \begin{pmatrix} 0_{n_R} & v/\sqrt{2}\mathbb{I}_{n_R} \\ v/\sqrt{2}\mathbb{I}_{n_R} & m_R \end{pmatrix} \in \mathbb{C}^{2n_R \times 2n_R}, \quad (97)$$

where we have written quantities with a tilde to point out that we are working with the complete mass matrix. Note that $\tilde{Y}$ is rectangular (unless $n_L = n_R$) and $\tilde{\mathcal{M}}$ is symmetric. In principle, it is possible to formally write a GCI structure in this case, similar to Eq. (11),

$$\tilde{Y} = V^\dagger D_{\tilde{\mathcal{M}}}^{-1/2} \tilde{\mathcal{R}} D_M^{1/2} W^\dagger, \quad (98)$$

where we used a Autonne-Takagi decomposition for the $(n_L + n_R) \times (n_L + n_R)$ complete neutrino mass matrix,

$$D_M = \text{diag}(m_1, \dots, m_{n_L + n_R}) = W^T M_\nu W. \quad (99)$$

Here V and W are $2n_R \times 2n_R$ and $(n_L + n_R) \times (n_L + n_R)$ unitary matrices, respectively, and $\tilde{\mathcal{R}}$ is a $2n_R \times (n_L + n_R)$ complex orthogonal matrix, $\tilde{\mathcal{R}}^T \tilde{\mathcal{R}} = \mathbb{I}_{n_L + n_R}$. Note that $\tilde{Y}$ is not general, as it has many zeroes and the identity matrix,

¹¹Note that in the Zee-Wolfenstein model [57] only one scalar doublet couples to the SM leptons (imposed via a discrete symmetry), and therefore $\text{Tr}(m_\nu) = 0$. However, this possibility is now excluded by data. Therefore, in general, both scalar doublets couple to the leptons and $\text{Tr}(m_\nu) \neq 0$.

therefore $\tilde{\mathcal{R}}$ is not arbitrary either. A natural question emerges;

Can one use $\tilde{Y}$ to obtain a parametrization for Y valid outside of the seesaw limit, or one that uses some heavy-light mixings as input?

Although deriving a generic parametrization valid outside of the seesaw limit using this approach would definitely be very interesting, it is beyond the scope of this work. For other parametrizations in models with right-handed neutrinos, which are exact and valid outside of the seesaw limit, see Refs. [29,30]. In the next section and in the Appendix we discuss the seesaw limit, which of course should give rise to the standard CI at leading order.

A. The standard Casas-Ibarra parametrization in the seesaw limit

In the seesaw limit, $m_D \ll m_R$ (small light-heavy mixing), the complete neutral fermion mass matrix M_ν can be approximately block-diagonalized to $D_M = \text{diag}(D_\nu, D_R)$ by the full unitary matrix W . It is convenient to factor the later as

$$W = \Omega(U \oplus \mathbb{I}_{n_R}), \quad (100)$$

where U is the PMNS matrix and Ω encodes the light-heavy mixing. The corresponding complete unitary mixing matrix that diagonalizes M_ν is

$$W \simeq \begin{pmatrix} (\mathbb{I}_{n_L} - \frac{1}{2}\Theta\Theta^\dagger)U & \Theta \\ -\Theta^\dagger U & \mathbb{I}_{n_R} - \frac{1}{2}\Theta^\dagger\Theta \end{pmatrix} + \mathcal{O}(\Theta^3),$$

$$\Theta = m_D m_R^{-1} \in \mathbb{C}^{n_L \times n_R}, \quad (101)$$

This gives rise to mass matrices in the diagonal of the form,

$$m_{\text{light}} \simeq m_D m_R^{-1} m_D^T \in \mathbb{C}^{n_L \times n_L},$$

$$m_{\text{heavy}} \simeq m_R \in \mathbb{C}^{n_R \times n_R}. \quad (102)$$

In the Appendix we explicitly show how the standard CI parametrization,

$$m_D = U^* D_\nu^{1/2} R_\ell D_R^{1/2}, \quad (103)$$

valid up to $\mathcal{O}(\Theta^2)$, may be derived in the seesaw limit, as expected, from Eq. (98). Here R_ℓ encodes the remaining freedom,

$$R_\ell \in \mathbb{C}^{n_L \times n_R}, \quad R_\ell R_\ell^T = \mathbb{I}_{n_L} \quad \text{if } n_R \geq n_L, \quad (104)$$

while for $n_R < n_L$ one has $R_\ell^T R_\ell = \mathbb{I}_{n_R}$.

VI. CONCLUSIONS

We have shown that a simple generalization of the CI parametrization can be applied to all Majorana neutrino mass models. In particular, it can be straightforwardly applied to scenarios where there are several contributions to the Majorana neutrino mass matrix, with both diagonal and off-diagonal terms in Yukawa space. This is highly relevant to perform phenomenological studies, and, in particular, it allows to control certain somewhat fine-tuned regions of the parameter space when doing a numerical scan, like those with large Yukawa hierarchies, which are governed by the imaginary parts of the complex angles of the (semi) orthogonal $\mathcal{R}$ matrix.

The approach, in turn, provides a unified framework for classifying neutrino mass models according to the structure of the neutrino mass matrix. This strategy has highlighted the lack of certain types of models, naturally leading to the proposal of an extended version of the scotogenic model whose phenomenology may be explored elsewhere. In particular, we have discussed different tree-level (one loop-level) examples, including the seesaw (the scotogenic model), the linear seesaw (the generalized scotogenic model), and the linear plus inverse seesaw (the newly proposed extended scotogenic model).

We have also provided ready-to-use expressions for the Yukawa matrices in several well-known models. In particular, for the Zee model we have derived the constraints on the $\mathcal{R}$ matrix and provided an explicit parametrization. With our procedure, using Eq. (98), it may also be possible to apply the parametrization outside of the seesaw limit, which is left for future work.

Note added. While this work was being finalized, Ref. [104] appeared on the *arXiv*, proposing a generalization of the Casas-Ibarra parametrization and demonstrating its equivalence to the master parametrization of Refs. [59,60]. We emphasize that Ref. [104] pursues a distinct aim; it focuses on conditions that reduce the number of free parameters, whereas our emphasis is on generality and on applications to widely studied models. The new model we propose is, in fact, motivated by this broader perspective.

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX: DERIVATION OF THE STANDARD CASAS-IBARRA FROM THE GENERAL ONE

To make contact with the usual CI parametrization, it is instructive to analyze the seesaw limit of the general relation given in Eq. (98),

$$\tilde{\mathcal{Y}} = V^\dagger D_{\mathcal{M}}^{-1/2} \tilde{\mathcal{R}} D_M^{1/2} W^\dagger, \quad (\text{A1})$$

where $D_M = \text{diag}(D_\nu, D_R)$. It is convenient to factor the full diagonalization matrix as

$$W = \Omega(U \oplus \mathbb{I}_{n_R}), \quad (\text{A2})$$

where U is the PMNS matrix and Ω encodes the light-heavy mixing, so that

$$\tilde{\mathcal{Y}} = X \tilde{\mathcal{R}} D_M^{1/2} (U^\dagger \oplus \mathbb{I}_{n_R}) \Omega^\dagger, \quad X \equiv V^\dagger D_{\mathcal{M}}^{-1/2}. \quad (\text{A3})$$

We partition the matrices as

$$X = \begin{pmatrix} X_{aa} & X_{ab} \\ X_{ba} & X_{bb} \end{pmatrix}, \quad \tilde{\mathcal{R}} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad \Omega = \begin{pmatrix} K & R \\ S & T \end{pmatrix}. \quad (\text{A4})$$

Then,

$$\begin{aligned}\tilde{Y} &= \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \begin{pmatrix} K^\dagger & S^\dagger \\ R^\dagger & T^\dagger \end{pmatrix} \\ &= \begin{pmatrix} A_1 K^\dagger + B_1 R^\dagger & A_1 S^\dagger + B_1 T^\dagger \\ C_1 K^\dagger + D_1 R^\dagger & C_1 S^\dagger + D_1 T^\dagger \end{pmatrix},\end{aligned}\quad (\text{A5})$$

where we defined the intermediate blocks,

$$\begin{aligned}A_1 &\equiv (X_{aa}A + X_{ab}C)D_\nu^{1/2}U^\dagger, \\ B_1 &\equiv (X_{aa}B + X_{ab}D)D_R^{1/2},\end{aligned}\quad (\text{A6})$$

$$\begin{aligned}C_1 &\equiv (X_{ba}A + X_{bb}C)D_\nu^{1/2}U^\dagger, \\ D_1 &\equiv (X_{ba}B + X_{bb}D)D_R^{1/2}.\end{aligned}\quad (\text{A7})$$

Equating with $\tilde{Y} = \text{diag}(Y^T, \mathbb{I}_{n_R})$ yields,

$$Y^T = A_1 K^\dagger + B_1 R^\dagger, \quad (\text{A8})$$

$$0 = A_1 S^\dagger + B_1 T^\dagger, \quad (\text{A9})$$

$$0 = C_1 K^\dagger + D_1 R^\dagger, \quad (\text{A10})$$

$$\mathbb{I} = C_1 S^\dagger + D_1 T^\dagger. \quad (\text{A11})$$

In the seesaw regime $\|m_D\| \ll \|m_R\|$, we have

$$\begin{aligned}\Omega &\simeq \begin{pmatrix} \mathbb{I}_{n_L} - \frac{1}{2}\Theta\Theta^\dagger & \Theta \\ -\Theta^\dagger & \mathbb{I}_{n_R} - \frac{1}{2}\Theta^\dagger\Theta \end{pmatrix}, \\ \Theta &\equiv m_D D_R^{-1} \in \mathbb{C}^{n_L \times n_R},\end{aligned}\quad (\text{A12})$$

so that, to leading order,

$$\begin{aligned}K^\dagger &= \mathbb{I} + \mathcal{O}(\Theta^2), & T^\dagger &= \mathbb{I} + \mathcal{O}(\Theta^2), \\ R^\dagger &= \Theta^\dagger + \mathcal{O}(\Theta^3), & S^\dagger &= -\Theta + \mathcal{O}(\Theta^3).\end{aligned}\quad (\text{A13})$$

Equations (A10)–(A11) imply

$$C_1 = \mathcal{O}(\Theta), \quad D_1 = \mathbb{I} + \mathcal{O}(\Theta^2), \quad (\text{A14})$$

and $B_1 = 0$ at zeroth order from Eq. (A9). Solving the B_1 and D_1 equations, we obtain at Θ^0 ,

$$C = -X_{bb}^{-1}X_{ba}A + \mathcal{O}(\Theta), \quad DD_R^{1/2} = S_b^{-1} + \mathcal{O}(\Theta^2), \quad (\text{A15})$$

where the Schur complements of X are (we assume that X_{aa}, X_{bb} are invertible)

$$S_a \equiv X_{aa} - X_{ab}X_{bb}^{-1}X_{ba}, \quad S_b \equiv X_{bb} - X_{ba}X_{aa}^{-1}X_{ab}. \quad (\text{A16})$$

With these, we obtain

$$\begin{aligned}A_1 &= S_a A D_\nu^{1/2} U^\dagger + \mathcal{O}(\Theta), \\ B_1 &= X_{aa} B D_R^{1/2} + X_{ab} S_b^{-1} + \mathcal{O}(\Theta).\end{aligned}\quad (\text{A17})$$

Using Eq. (A8), we get

$$Y^T = S_a A D_\nu^{1/2} U^\dagger + (X_{aa} B D_R^{1/2} + X_{ab} S_b^{-1}) \Theta^\dagger, \quad (\text{A18})$$

where the second term is $\mathcal{O}(\Theta^2)$. From the Autonne-Takagi decomposition of $\tilde{\mathcal{M}}$ given in Eq. (97),

$$\begin{aligned}D_{\tilde{\mathcal{M}}} &\simeq \text{diag}\left(\frac{v^2}{2}D_R^{-1}, D_R\right), \\ V &\simeq \begin{pmatrix} \mathbb{I}_{n_R} - \frac{1}{2}\Phi\Phi^\dagger & \Phi \\ -\Phi^T & \mathbb{I}_{n_R} - \frac{1}{2}\Phi^T\Phi^* \end{pmatrix}, \quad \Phi \equiv \frac{v}{\sqrt{2}}D_R^{-1},\end{aligned}\quad (\text{A19})$$

so neglecting $\mathcal{O}(\Phi^2)$ terms,

$$\begin{aligned}X &\equiv V^\dagger D_{\tilde{\mathcal{M}}}^{-1/2} \simeq \begin{pmatrix} \alpha\mathbb{I} & -\beta\Phi \\ \alpha\Phi & \beta\mathbb{I} \end{pmatrix}, \quad \alpha \equiv \frac{\sqrt{2}D_R^{1/2}}{v}, \\ \beta &\equiv D_R^{-1/2}.\end{aligned}\quad (\text{A20})$$

Then, as X_{aa}, X_{bb} are invertible because α, β are ($D_R > 0$),

$$\begin{aligned}S_a &= X_{aa} - X_{ab}X_{bb}^{-1}X_{ba} = \alpha(\mathbb{I} + \Phi^2) \\ &= \frac{\sqrt{2}D_R^{1/2}}{v} \left[\mathbb{I} + \mathcal{O}\left(\frac{v^2}{2}D_R^{-2}\right) \right].\end{aligned}\quad (\text{A21})$$

Dropping the $\mathcal{O}(\Theta^2)$ correction in Eq. (A18) and choosing

$$A \equiv R_\ell^T, \quad R_\ell \text{ complex orthogonal}, \quad (\text{A22})$$

with $R_\ell R_\ell^T = \mathbb{I}_{n_L}$ for $n_R \geq n_L$ and $R_\ell^T R_\ell = \mathbb{I}_{n_R}$ for $n_R < n_L$, we obtain

$$Y^T U \simeq \frac{\sqrt{2}}{v} D_R^{1/2} R_\ell^T D_\nu^{1/2} \Rightarrow m_D \simeq U^* D_\nu^{1/2} R_\ell D_R^{1/2}, \quad (\text{A23})$$

which is the familiar CI parametrization, valid up to $\mathcal{O}(\Theta^2)$.

- [1] S. Weinberg, Baryon and lepton nonconserving processes, *Phys. Rev. Lett.* **43**, 1566 (1979).
- [2] P. Minkowski, $\mu \rightarrow e\gamma$ at a rate of one out of 10^9 muon decays?, *Phys. Lett.* **67B**, 421 (1977).
- [3] T. Yanagida, Horizontal gauge symmetry and masses of neutrinos, *Conf. Proc. C* **7902131**, 95 (1979).
- [4] T. Yanagida, Horizontal symmetry and mass of the top quark, *Phys. Rev. D* **20**, 2986 (1979).
- [5] S. L. Glashow, The future of elementary particle physics, *NATO Sci. Ser. B* **61**, 687 (1980).
- [6] M. Gell-Mann, P. Ramond, and R. Slansky, Complex spinors and unified theories, *Conf. Proc. C* **790927**, 315 (1979).
- [7] E. Ma, Pathways to naturally small neutrino masses, *Phys. Rev. Lett.* **81**, 1171 (1998).
- [8] Y. Cai, J. Herrero-García, M. A. Schmidt, A. Vicente, and R. R. Volkas, From the trees to the forest: A review of radiative neutrino mass models, *Front. Phys.* **5**, 63 (2017).
- [9] J. Gargalionis and R. R. Volkas, Exploding operators for Majorana neutrino masses and beyond, *J. High Energy Phys.* **01** (2021) 074.
- [10] H. Minakata and A. Y. Smirnov, Neutrino mixing and quark-lepton complementarity, *Phys. Rev. D* **70**, 073009 (2004).
- [11] A. Giarnetti, S. Marciano, and D. Meloni, On quark-lepton mixing and the leptonic CP violation, *Universe* **10**, 345 (2024).
- [12] J. Harada, Neutrino mixing and CP violation from Dirac-Majorana bimaximal mixture and quark-lepton unification, *Europhys. Lett.* **75**, 248 (2006).
- [13] M. Picariello, B. C. Chauhan, J. Pulido, and E. Torrente-Lujan, Predictions from non-trivial quark-lepton complementarity, *Int. J. Mod. Phys. A* **22**, 5860 (2007).
- [14] B. Chauhan, M. Picariello, J. Pulido, and E. Torrente-Lujan, Quark-lepton complementarity with lepton and quark mixing data predict $\theta_{13}^{\text{PMNS}} = (9_{-2}^{+1})^\circ$, *Eur. Phys. J. C* **50**, 573 (2007).
- [15] J. Barranco, F. Gonzalez Canales, and A. Mondragon, Universal mass texture, CP violation and quark-lepton complementarity, *Phys. Rev. D* **82**, 073010 (2010).
- [16] Y. Zhang, X. Zhang, and B.-Q. Ma, Quark-lepton complementarity and self-complementarity in different schemes, *Phys. Rev. D* **86**, 093019 (2012).
- [17] K. Zhukovsky and A. A. Davydova, CP violation and quark-lepton complementarity of the neutrino mixing matrix, *Eur. Phys. J. C* **79**, 385 (2019).
- [18] J. A. Casas and A. Ibarra, Oscillating neutrinos and $\mu \rightarrow e, \gamma$, *Nucl. Phys.* **B618**, 171 (2001).
- [19] A. Ibarra and G. G. Ross, Neutrino properties from Yukawa structure, *Phys. Lett. B* **575**, 279 (2003).
- [20] A. Ibarra and G. G. Ross, Neutrino phenomenology: The case of two right-handed neutrinos, *Phys. Lett. B* **591**, 285 (2004).
- [21] A. Pilaftsis, Radiatively induced neutrino masses and large Higgs neutrino couplings in the standard model with Majorana fields, *Z. Phys. C* **55**, 275 (1992).
- [22] J. Lopez-Pavon, E. Molinaro, and S. T. Petcov, Radiative corrections to light neutrino masses in low scale type I seesaw scenarios and neutrinoless double beta decay, *J. High Energy Phys.* **11** (2015) 030.
- [23] A. Broncano, M. B. Gavela, and E. E. Jenkins, Neutrino physics in the seesaw model, *Nucl. Phys.* **672B**, 163 (2003).
- [24] W.-l. Guo, Z.-z. Xing, and S. Zhou, Neutrino masses, lepton flavor mixing and leptogenesis in the minimal seesaw model, *Int. J. Mod. Phys. E* **16**, 1 (2007).
- [25] Z.-z. Xing, Correlation between the charged current interactions of light and heavy Majorana neutrinos, *Phys. Lett. B* **660**, 515 (2008).
- [26] Z.-z. Xing, A full parametrization of the 6×6 flavor mixing matrix in the presence of three light or heavy sterile neutrinos, *Phys. Rev. D* **85**, 013008 (2012).
- [27] J. Heeck, Seesaw parametrization for n right-handed neutrinos, *Phys. Rev. D* **86**, 093023 (2012).
- [28] Z.-z. Xing, On the orthogonal matrix in the Casas-Ibarra parametrization for the Yukawa interactions of Majorana neutrinos, *Nucl. Phys.* **B1008**, 116723 (2024).
- [29] M. Blennow and E. Fernandez-Martinez, Parametrization of seesaw models and light sterile neutrinos, *Phys. Lett. B* **704**, 223 (2011).
- [30] A. Donini, P. Hernandez, J. Lopez-Pavon, M. Maltoni, and T. Schwetz, The minimal $3 + 2$ neutrino model versus oscillation anomalies, *J. High Energy Phys.* **07** (2012) 161.
- [31] M. Magg and C. Wetterich, Neutrino mass problem and gauge hierarchy, *Phys. Lett. B* **94**, 61 (1980).
- [32] R. N. Mohapatra and G. Senjanovic, Neutrino masses and mixings in gauge models with spontaneous parity violation, *Phys. Rev. D* **23**, 165 (1981).
- [33] G. Lazarides, Q. Shafi, and C. Wetterich, Proton lifetime and fermion masses in an $SO(10)$ model, *Nucl. Phys.* **B181**, 287 (1981).
- [34] J. Schechter and J. W. F. Valle, Neutrino masses in $SU(2) \times U(1)$ theories, *Phys. Rev. D* **22**, 2227 (1980).
- [35] E. K. Akhmedov and W. Rodejohann, A Yukawa coupling parameterization for type I + II seesaw formula and applications to lepton flavor violation and leptogenesis, *J. High Energy Phys.* **06** (2008) 106.
- [36] A. Das and N. Okada, Inverse seesaw neutrino signatures at the LHC and ILC, *Phys. Rev. D* **88**, 113001 (2013).
- [37] A. Das and S. Mandal, Bounds on the triplet fermions in type-III seesaw and implications for collider searches, *Nucl. Phys.* **B966**, 115374 (2021).
- [38] L. M. Krauss, S. Nasri, and M. Trodden, A model for neutrino masses and dark matter, *Phys. Rev. D* **67**, 085002 (2003).
- [39] R. Cepedello, M. Hirsch, P. Rocha-Morán, and A. Vicente, Minimal 3-loop neutrino mass models and charged lepton flavor violation, *J. High Energy Phys.* **08** (2020) 067.
- [40] Z. Tao, Radiative seesaw mechanism at weak scale, *Phys. Rev. D* **54**, 5693 (1996).
- [41] E. Ma, Verifiable radiative seesaw mechanism of neutrino mass and dark matter, *Phys. Rev. D* **73**, 077301 (2006).
- [42] A. Ahriché, A scotogenic model with two inert doublets, *J. High Energy Phys.* **02** (2023) 028.
- [43] S. Davidson and A. Ibarra, Determining seesaw parameters from weak scale measurements?, *J. High Energy Phys.* **09** (2001) 013.
- [44] J. R. Ellis, J. Hisano, M. Raidal, and Y. Shimizu, A new parametrization of the seesaw mechanism and applications

- in supersymmetric models, *Phys. Rev. D* **66**, 115013 (2002).
- [45] A. Ibarra, Reconstructing the two right-handed neutrino model, *J. High Energy Phys.* **01** (2006) 064.
- [46] J. Kriewald and A. M. Teixeira, Taming flavour violation in the inverse seesaw, *J. High Energy Phys.* **04** (2025) 027.
- [47] K. S. Babu and C. Macesanu, Two loop neutrino mass generation and its experimental consequences, *Phys. Rev. D* **67**, 073010 (2003).
- [48] K. S. Babu, Model of “calculable” Majorana neutrino masses, *Phys. Lett. B* **203**, 132 (1988).
- [49] D. Aristizabal Sierra and M. Hirsch, Experimental tests for the Babu-Zee two-loop model of Majorana neutrino masses, *J. High Energy Phys.* **12** (2006) 052.
- [50] M. Nebot, J. F. Oliver, D. Palao, and A. Santamaria, Prospects for the Zee-Babu model at the CERN LHC and low energy experiments, *Phys. Rev. D* **77**, 093013 (2008).
- [51] T. Ohlsson, T. Schwetz, and H. Zhang, Non-standard neutrino interactions in the Zee-Babu model, *Phys. Lett. B* **681**, 269 (2009).
- [52] J. Herrero-Garcia, M. Nebot, N. Rius, and A. Santamaria, The Zee-Babu model revisited in the light of new data, *Nucl. Phys. B* **885**, 542 (2014).
- [53] D. Schmidt, T. Schwetz, and H. Zhang, Status of the Zee-Babu model for neutrino mass and possible tests at a like-sign linear collider, *Nucl. Phys. B* **885**, 524 (2014).
- [54] T. Felkl, J. Herrero-Garcia, and M. A. Schmidt, The singly-charged scalar singlet as the origin of neutrino masses, *J. High Energy Phys.* **05** (2021) 122; **05** (2022) 073(E).
- [55] A. Zee, A theory of lepton number violation, neutrino Majorana mass, and oscillation, *Phys. Lett.* **93B**, 389 (1980); **95B**, 461(E) (1980).
- [56] T. P. Cheng and L.-F. Li, Neutrino masses, mixings and oscillations in $SU(2) \times U(1)$ models of electroweak interactions, *Phys. Rev. D* **22**, 2860 (1980).
- [57] L. Wolfenstein, A theoretical pattern for neutrino oscillations, *Nucl. Phys. B* **175**, 93 (1980).
- [58] C.-H. Chen, Structure of leptonic Yukawa couplings in the Zee model, [arXiv:2508.18757](https://arxiv.org/abs/2508.18757).
- [59] I. Cordero-Carrión, M. Hirsch, and A. Vicente, Master Majorana neutrino mass parametrization, *Phys. Rev. D* **99**, 075019 (2019).
- [60] I. Cordero-Carrión, M. Hirsch, and A. Vicente, General parametrization of Majorana neutrino mass models, *Phys. Rev. D* **101**, 075032 (2020).
- [61] B. Pontecorvo, Mesonium and antimesonium, *Sov. Phys. JETP* **6**, 429 (1958).
- [62] B. Pontecorvo, Inverse beta processes and nonconservation of lepton charge, *Sov. Phys. JETP* **7**, 172 (1958).
- [63] Z. Maki, M. Nakagawa, and S. Sakata, Remarks on the unified model of elementary particles, *Prog. Theor. Phys.* **28**, 870 (1962).
- [64] B. Pontecorvo, Neutrino experiments and the problem of conservation of leptonic charge, *Zh. Eksp. Teor. Fiz.* **53**, 1717 (1967).
- [65] A. Zee, Quantum numbers of Majorana neutrino masses, *Nucl. Phys. B* **264**, 99 (1986).
- [66] J. Herrero-García, T. Ohlsson, S. Riad, and J. Wirén, Full parameter scan of the Zee model: Exploring Higgs lepton flavor violation, *J. High Energy Phys.* **04** (2017) 130.
- [67] J. Heeck and A. Thapa, Zee-model predictions for lepton flavor violation, *Phys. Lett. B* **841**, 137910 (2023).
- [68] K. S. Babu, P. S. B. Dev, S. Jana, and A. Thapa, Non-standard interactions in radiative neutrino mass models, *J. High Energy Phys.* **03** (2020) 006.
- [69] I. Esteban, M. C. Gonzalez-Garcia, M. Maltoni, I. Martinez-Soler, J. P. Pinheiro, and T. Schwetz, NuFit-6.0: Updated global analysis of three-flavor neutrino oscillations, *J. High Energy Phys.* **12** (2024) 216.
- [70] NuFIT Collaboration, Nufit 6.0, <https://www.nu-fit.org>, 2024.
- [71] A. Giarnetti, J. Herrero-García, S. Marciano, D. Meloni, and D. Vatsyayan, Neutrino masses from new seesaw models: low-scale variants and phenomenological implications, *Eur. Phys. J. C* **84**, 803 (2024).
- [72] R. Foot, H. Lew, X. G. He, and G. C. Joshi, Seesaw neutrino masses induced by a triplet of leptons, *Z. Phys. C* **44**, 441 (1989).
- [73] G. C. Branco, J. T. Penedo, P. M. F. Pereira, M. N. Rebelo, and J. I. Silva-Marcos, Type-I seesaw with eV-scale neutrinos, *J. High Energy Phys.* **07** (2020) 164.
- [74] S. Centelles Chuliá, A. Herrero-Brocal, and A. Vicente, The type-I seesaw family, *J. High Energy Phys.* **07** (2024) 060.
- [75] W. Grimus and L. Lavoura, Softly broken lepton numbers and maximal neutrino mixing, *J. High Energy Phys.* **07** (2001) 045.
- [76] G. 't Hooft, Naturalness, chiral symmetry, and spontaneous chiral symmetry breaking, *NATO Sci. Ser. B* **59**, 135 (1980).
- [77] A. Ibarra, C. E. Yaguna, and O. Zapata, Direct detection of fermion dark matter in the radiative seesaw model, *Phys. Rev. D* **93**, 035012 (2016).
- [78] A. Beniwal, J. Herrero-García, N. Leerdam, M. White, and A. G. Williams, The scoto singlet model: A scalar singlet extension of the scotogenic model, *J. High Energy Phys.* **21** (2020) 136.
- [79] G. Cacciapaglia and M. Rosenlyst, Loop-generated neutrino masses in composite Higgs models, *J. High Energy Phys.* **09** (2021) 167.
- [80] F. Bonnet, M. Hirsch, T. Ota, and W. Winter, Systematic study of the $d = 5$ Weinberg operator at one-loop order, *J. High Energy Phys.* **07** (2012) 153.
- [81] D. Restrepo, O. Zapata, and C. E. Yaguna, Models with radiative neutrino masses and viable dark matter candidates, *J. High Energy Phys.* **11** (2013) 011.
- [82] D. Wyler and L. Wolfenstein, Massless neutrinos in left-hand symmetric models, *Nucl. Phys. B* **218**, 205 (1983).
- [83] R. N. Mohapatra, Mechanism for understanding small neutrino mass in superstring theories, *Phys. Rev. Lett.* **56**, 561 (1986).
- [84] R. N. Mohapatra and J. W. F. Valle, Neutrino mass and baryon number nonconservation in superstring models, *Phys. Rev. D* **34**, 1642 (1986).
- [85] J. Bernabéu, A. Santamaria, J. Vidal, A. Mendez, and J. Valle, Lepton flavour non-conservation at high energies in a superstring inspired standard model, *Phys. Lett. B* **187**, 303 (1987).

- [86] G. C. Branco, W. Grimus, and L. Lavoura, The seesaw mechanism in the presence of a conserved lepton number, *Nucl. Phys.* **B312**, 492 (1989).
- [87] S. M. Barr, A Different seesaw formula for neutrino masses, *Phys. Rev. Lett.* **92**, 101601 (2004).
- [88] J. Kersten and A. Y. Smirnov, Right-Handed neutrinos at CERN LHC and the mechanism of neutrino mass generation, *Phys. Rev. D* **76**, 073005 (2007).
- [89] M. B. Gavela, T. Hambye, D. Hernandez, and P. Hernandez, Minimal flavour seesaw models, *J. High Energy Phys.* **09** (2009) 038.
- [90] P. Hernandez, J. Lopez-Pavon, N. Rius, and S. Sandner, Bounds on right-handed neutrino parameters from observable leptogenesis, *J. High Energy Phys.* **12** (2022) 012.
- [91] A. Ibarra, E. Molinaro, and S. T. Petcov, Low energy signatures of the TeV scale seesaw mechanism, *Phys. Rev. D* **84**, 013005 (2011).
- [92] E. K. Akhmedov, M. Lindner, E. Schnapka, and J. W. F. Valle, Left-right symmetry breaking in NJL approach, *Phys. Lett. B* **368**, 270 (1996).
- [93] E. K. Akhmedov, M. Lindner, E. Schnapka, and J. W. F. Valle, Dynamical left-right symmetry breaking, *Phys. Rev. D* **53**, 2752 (1996).
- [94] M. Malinsky, J. C. Romao, and J. W. F. Valle, Novel supersymmetric SO(10) seesaw mechanism, *Phys. Rev. Lett.* **95**, 161801 (2005).
- [95] A. Abada, C. Biggio, F. Bonnet, M. B. Gavela, and T. Hambye, Low energy effects of neutrino masses, *J. High Energy Phys.* **12** (2007) 061.
- [96] A. Giarnetti, J. Herrero-Garcia, S. Marciano, D. Meloni, and D. Vatsyayan, Neutrino masses from new Weinberg-like operators: Phenomenology of TeV scalar multiplets, *J. High Energy Phys.* **05** (2024) 055.
- [97] C. Hagedorn, J. Herrero-García, E. Molinaro, and M. A. Schmidt, Phenomenology of the generalised scotogenic model with fermionic dark matter, *J. High Energy Phys.* **11** (2018) 103.
- [98] Y. Cai, J. D. Clarke, M. A. Schmidt, and R. R. Volkas, Testing radiative neutrino mass models at the LHC, *J. High Energy Phys.* **02** (2015) 161.
- [99] P. S. B. Dev and A. Pilaftsis, Minimal radiative neutrino mass mechanism for inverse seesaw models, *Phys. Rev. D* **86**, 113001 (2012).
- [100] S. Centelles Chuliá, R. Srivastava, and A. Vicente, The inverse seesaw family: Dirac and Majorana, *J. High Energy Phys.* **03** (2021) 248.
- [101] I. M. Ávila, V. De Romeri, L. Duarte, and J. W. F. Valle, Phenomenology of scotogenic scalar dark matter, *Eur. Phys. J. C* **80**, 908 (2020).
- [102] A. C. B. Machado, J. Montaña, P. Pasquini, and V. Pleitez, Analytical solution for the Zee mechanism, *arXiv:1707.06977*.
- [103] P. S. B. Dev, J. Heeck, and A. Thapa, Neutrino mass models at μ TRISTAN, *Eur. Phys. J. C* **84**, 148 (2024).
- [104] A. Herrero-Brocal and A. Vicente, Minimal Majorana neutrino mass models, *arXiv:2510.00113*.
- [105] J. Ellis, TikZ-Feynman: Feynman diagrams with TikZ, *Comput. Phys. Commun.* **210**, 103 (2017).