

Nonperturbative origin of the electroweak scale: RGE in strongly-coupled dark gauge theories via the Dyson-Schwinger approach

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(Received 12 August 2024; accepted 21 May 2025; published 16 June 2025)

We propose a novel pathway to generate the electroweak scale (EW) via nonperturbative dynamics of a dark gauge sector based on the $SU(N)$ gauge group. Imposing the scale invariance of the theory, we investigate the electroweak symmetry breaking (EWSB) which is triggered dynamically via the condensation of gauge fields. Instead of the usual dimension-4 triggered breaking, a dimension 6 term (with Wilson coefficient compatible with Standard Model Effective Field Theory (SMEFT) bounds) coupling the Higgs boson and the Yang-Mills field quadratic term in the Lagrangian provides feedback from the gauge to the Higgs sector. We provide a novel method to estimate a nonperturbative EW scale generation using the exact solution of a truncated set of the background equations of motion in Yang-Mills theory in terms of Jacobi elliptic functions and the exact beta-function valid in the strongly coupled regimes via the Dyson-Schwinger approach. Particularly, we find an analytical result for the renormalization group equation (RGE) of the gauge coupling in the $SU(N)$ sector in the strongly coupled regime. The dynamics studied in this paper pave the way to a more realistic model building with possible resolution to the hierarchy problem and, in general, dynamical generation of scales.

DOI: [10.1103/czm7-1ccc](https://doi.org/10.1103/czm7-1ccc)

I. INTRODUCTION

With null results at the Large Hadron Collider (LHC) for any evidence of supersymmetry, alternatives to the solution of the gauge hierarchy problem are being explored. An intriguing possibility is that the hierarchy problem might be related to the scale invariance at the classical level, which is only broken at the quantum level due to quantum anomalies. As has been first shown, the dynamical generation of the gauge symmetry breaking is realized via radiative symmetry breaking by the Coleman-Weinberg mechanism [1]. However, when the mechanism is applied to the Standard Model (SM), the mass of the gauge bosons is predicted to be greater than that of the Higgs boson, $m_{Z,W} > m_H$. In the framework of Beyond the SM (BSM), realistic dynamical generations of the electroweak (EW) scale have been explored extensively in the literature [2–11]. Moreover,

in the context of nonminimally coupled gravity, scale invariant models naturally provide flat inflationary potentials [3,12–18], and also the mass scale of dark matter (DM) can be dynamically generated [15,19–22]. This has always been seen as a direction of model-building toward the dynamically generated solution to the gauge hierarchy problem in the SM [3,4,7,11,15,19,21,23–27]. See Refs. [28–34] for other studies of conformal invariance and dimensional transmutation of energy scales [35,36].

Besides the Coleman-Weinberg pathway, dynamical generation of mass scales in scale-invariant theories can be achieved via dimensional transmutation in a strongly interacting sector based on hidden gauge groups [26,37–39], including some examples from standard quantum chromodynamics (QCD) (see, e.g., [40]). This is purely due to nonperturbative effects. Already in the 1980s, the authors of [41–44] suggested to dynamically break the EW symmetry by the condensation of chiral fermions in high color representations, see Ref. [45] for recent consideration.

In this paper, we propose a scale-invariant scenario with a dark sector based on the $SU(N)$ gauge group, in which the electroweak symmetry breaking (EWSB) is triggered dynamically via the condensation of gauge fields. This mechanism enables us to understand the dynamic generation of the EW scale. The main part of this work is devoted

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to the development of a novel method to estimate the nonperturbative EW scale generation based on the exact solution of the background equations of motion in Yang-Mills theory in terms of Jacobi elliptic functions and the exact beta-function valid in the strongly coupled regimes via Dyson-Schwinger approach. Particularly, we find an analytical result for the renormalization group (RG) running of the gauge coupling in the classically conformal SU(N) sector in the strongly coupled regime. To study the nonperturbative regime, we utilize the exact solutions found in terms of the Jacobi elliptic functions following the analytic approach of Dyson-Schwinger equations, originally devised by Bender, Milton, and Savage in Ref. [46]. In this case, Green's functions of the theory are represented analytically, and therefore, it is straightforward to understand the effect of the background on the interactions that remain valid even in the strongly coupled regime [47]. This technique has been recently applied to QCD in Refs. [48–52] and to the SM Higgs sector in Ref. [53], as well as to other types of models over the past two decades in Refs. [47,52–68]. Recently, some of the authors have employed this technique to study nonperturbative hadronic contributions to the muon anomalous magnetic moment $(g-2)_\mu$ [48], QCD in the nonperturbative regime [49–51], nonperturbative false vacuum decay [69], as well as to explore the mass gap and confinement in string-inspired infinite-derivative and Lee-Wick theories [70–72].

Given the above *proviso*, we investigate, in this paper, nonperturbative properties of a simple Yang-Mills theory and show a derivation of the mass gap in the SU(N) gauge sector, condensation of gauge fields and a gap equation for the gauge bosons. We emphasize that our approach does not address the hierarchy problem that, probably, will involve some serious rethinking of fundamental physics.

The paper is organized as follows: in Sec. II, we briefly review the nonperturbative technique to understand the strongly coupled gauge sector collecting the main equations we need. Following these results, we study in Sec. III the dynamical generation of the EW scale. In Sec. IV, we study the mass spectrum of the theory. Section V is devoted to conclusions and discussions.

II. STRONGLY COUPLED YANG-MILLS: SHORT REVIEW

The aim of this review is to make the paper self-contained presenting already published material and to

obtain the main equations we will use in the following for a strongly coupled Yang-Mills theory. Details of the computations are given in the Appendixes.

A. Correlation functions

For a Yang-Mills theory, one has the Lagrangian

$$\mathcal{L} = \mathcal{L}_{\text{inv}} + \mathcal{L}_{\text{gf}} + \mathcal{L}_{\text{FP}}, \quad (1)$$

where $\mathcal{L}_{\text{inv}}$ denotes the classical gauge-invariant part, $\mathcal{L}_{\text{gf}}$ the gauge-fixing terms, and $\mathcal{L}_{\text{FP}}$ the Faddeev-Popov (FP) ghost term characteristic of non-Abelian gauge theories [73]

$$\begin{aligned} \mathcal{L}_{\text{inv}} &= -\frac{1}{4} F_{\mu\nu} \cdot F^{\mu\nu}, \\ \mathcal{L}_{\text{gf}} &= -\frac{1}{2\xi} (\partial \cdot A)^2, \\ \mathcal{L}_{\text{FP}} &= -\bar{c} \cdot \partial_\mu D^\mu c, \end{aligned} \quad (2)$$

where $F_{\mu\nu}^a$ is the field tensor (defined below), A_μ^a are the potentials, $\bar{c}^a, c^a$ the Faddeev-Popov ghosts and, in the usual notation, ξ is the gauge parameter, and D_μ represents the covariant derivative whose explicit form is given by

$$D_\mu c^a = \partial_\mu c^a + g(T^a)^{bc} A_\mu^b c^c \quad (3)$$

with $(T^a)^{bc}$ being the gauge group generators and g being the gauge coupling constant. The field strength is given by

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c, \quad (4)$$

where f^{abc} are the structure constants of the gauge group. Then, the partition function can be written down as

$$Z[j, \eta, \bar{\eta}] = \int [dA][d\bar{c}][dc] e^{-\int d^4x (\mathcal{L}_{\text{inv}} + \mathcal{L}_{\text{gf}} + \mathcal{L}_{\text{FP}} + j \cdot A + \bar{\eta} \cdot c + \bar{c} \cdot \eta)}, \quad (5)$$

where we have introduced a set of arbitrary bosonic currents j_μ^a and fermionic sources $\bar{\eta}^a, \eta^a$ that are needed to evaluate the correlation functions. Given the partition function, the correlation functions for $n > 1$ are defined by

$$\begin{aligned} G_{n\mu_1\mu_2\ldots\mu_n}^{a_1a_2\ldots a_n}(x_1, x_2, \ldots, x_n) &= \frac{\delta^n \ln Z}{\delta j_{\mu_1}^{a_1}(x_1) \delta j_{\mu_2}^{a_2}(x_2) \ldots \delta j_{\mu_n}^{a_n}(x_n)} \Big|_{j=0}, \\ K_{n\mu_1\mu_2\ldots\mu_n}^{a_1a_2\ldots a_n}(x_1, x_2, \ldots, x_n) &= \frac{\delta^n \ln Z}{\delta \eta_{\mu_1}^{a_1}(x_1) \delta \eta_{\mu_2}^{a_2}(x_2) \ldots \delta \eta_{\mu_n}^{a_n}(x_n)} \Big|_{\bar{\eta}, \eta=0}, \\ \bar{K}_{n\mu_1\mu_2\ldots\mu_n}^{a_1a_2\ldots a_n}(x_1, x_2, \ldots, x_n) &= \frac{\delta^n \ln Z}{\delta \bar{\eta}_{\mu_1}^{a_1}(x_1) \delta \bar{\eta}_{\mu_2}^{a_2}(x_2) \ldots \delta \bar{\eta}_{\mu_n}^{a_n}(x_n)} \Big|_{\bar{\eta}, \eta=0}. \end{aligned} \quad (6)$$

We have also used occasionally the mixed correlation functions that will be dubbed with the letters W , J , and L but appear to be 0 in our context limited to $n = 2$. These correspond to mixed vertexes in the theory for interactions between ghosts and gluons.

To explore a gauge theory in a nonperturbative regime, we apply the technique with the Dyson-Schwinger equation in differential form. This approach was devised in [46] and later on studied in several UV-completion scenarios involving string-field and higher-derivative theories [70–72]. We briefly review the technique in Appendix A for the scalar field. For a Yang-Mills field, we get for the 1P-correlation function

$$\square G_{1\mu}^{(j)a} + g f^{abc} \langle \partial_\nu \bar{A}_\mu^b A^{c\nu} \rangle + g f^{abc} \langle A^{b\nu} (\partial_\mu A_\nu^c - \partial_\nu A_\mu^c) \rangle \\ \times g^2 f^{abc} f^{cde} \langle A^{b\nu} A_\nu^d A_\mu^e \rangle + g f^{abc} \langle \bar{c}^b \partial_\mu c^c \rangle = j_\mu^a. \quad (7)$$

Similarly, for the ghost field is

$$-\square P_1^{(\eta)a} + g f^{abc} \langle (A_\mu^c) \partial^\mu c^b \rangle = \eta^a. \quad (8)$$

For our needs, we consider the following functions

$$G_{1\mu}^{(j)a}(x) = Z^{-1} \langle A_\mu^a(x) \rangle \\ P_1^{(\eta)a}(x) = Z^{-1} \langle c^a(x) \rangle, \quad (9)$$

and similarly for $\bar{c}^a$ yielding $\bar{P}_1^{(\eta)a}(x)$. We can evaluate the averages as

$$Z[j, \eta, \bar{\eta}] G_{1\mu}^{(j)a}(x) = \langle A_\mu^a(x) \rangle, \\ Z[j, \eta, \bar{\eta}] P_1^{(\eta)a}(x) = \langle c^a(x) \rangle. \quad (10)$$

We derive one time again with respect to $j(x)$ on the first equation yielding

$$Z[j, \eta, \bar{\eta}] G_{2\mu\nu}^{(j)ab}(x, x) + Z[j, \eta, \bar{\eta}] G_{1\mu}^{(j)a}(x) G_{1\nu}^{(j)b}(x) \\ = \langle A_\mu^a(x) A_\nu^b(x) \rangle, \quad (11)$$

and apply the derivative ∂^ν

$$Z[j, \eta, \bar{\eta}] \partial^\nu G_{2\mu\nu}^{(j)ab}(x, x) + Z[j, \eta, \bar{\eta}] \partial^\nu G_{1\mu}^{(j)a}(x) G_{1\nu}^{(j)b}(x) \\ = \langle \partial^\nu A_\mu^a(x) A_\nu^b(x) \rangle. \quad (12)$$

Then, a further derivation of Eq. (11) with respect to $j^{c\nu}$ yields

$$Z[j, \eta, \bar{\eta}] G_{2\mu\nu}^{(j)ab}(x, x) G_1^{(j)\nu c}(x) + Z[j, \eta, \bar{\eta}] G_{3\mu\nu}^{(j)abc\nu}(x, x, x) + Z[j, \eta, \bar{\eta}] G_{1\mu}^{(j)a}(x) G_{1\nu}^{(j)b}(x) G_1^{(j)\nu c}(x) \\ + Z[j, \eta, \bar{\eta}] G_{2\nu}^{(j)bc\nu}(x) G_{1\mu}^{(j)a}(x) = \langle A_\mu^a(x) A_\nu^b(x) A^{c\nu}(x) \rangle, \quad (13)$$

and the same to be done on the ghost field. Equation (9) yields

$$Z[j, \eta, \bar{\eta}] P_1^{(\eta)a}(x) = \langle c^a(x) \rangle. \quad (14)$$

The following equation is obtained after deriving with respect to ∂_μ and then with respect to $\bar{\eta}$,

$$Z[j, \eta, \bar{\eta}] \bar{P}_1^{(\eta)b}(x) \partial^\mu P_1^{(\eta)a}(x) + Z[j, \eta, \bar{\eta}] \partial^\mu K_2^{(\eta)ab}(x, x) \\ = \langle \bar{c}^b \partial^\mu c^a(x) \rangle. \quad (15)$$

A new 2P-function has been introduced and defined as

$$K_2^{(\eta)ab}(x, y) = \frac{1}{Z[j, \eta, \bar{\eta}]} \frac{\delta P_1^{(\eta)a}(x)}{\delta \eta^b(y)}, \quad (16)$$

and the other 2P-functions

$$J_{2\mu}^{(\eta,j)ab}(x, y) = \frac{1}{Z[j, \eta, \bar{\eta}]} \frac{\delta P_1^{(\eta)a}(x)}{\delta j^{b\mu}(y)}. \quad (17)$$

Then, we derive Eq. (14) with respect to $j^{b\mu}(x)$ and obtain

$$Z[j, \eta, \bar{\eta}] G_{1\mu}^{(j)b}(x) \partial^\mu P_1^{(\eta)a}(x) + Z[j, \eta, \bar{\eta}] \partial^\mu J_{2\mu}^{(\eta,j)ab}(x, x) \\ = \langle A_\mu^b(x) \partial^\mu c^a(x) \rangle. \quad (18)$$

Gathering everything together yields

$$\begin{aligned}
& \square G_{1\mu}^{(j)a} + g f^{abc} \partial^\nu [G_{2\mu\nu}^{(j)bc}(x, x) + G_{1\mu}^{(j)b}(x) G_{1\nu}^{(j)c}(x)] - g f^{abc} [\partial^\nu G_{2\mu\nu}^{(j)bc}(x, x) + \partial^\nu G_{1\mu}^{(j)b}(x) G_{1\nu}^{(j)c}(x)] \\
& - g f^{abc} [\partial_\mu G_{2\nu}^{(j)bc\nu}(x, x) + \partial_\mu G_{1\nu}^{(j)b}(x) G_1^{(j)c\nu}(x)] + g^2 f^{abc} f^{cde} [G_{2\mu\nu}^{(j)bd}(x, x) G_1^{(j)\nu e}(x) + \partial^\nu G_{3\mu\nu}^{(j)bde\nu}(x, x, x) \\
& + G_{1\mu}^{(j)b}(x) G_{1\nu}^{(j)d}(x) G_1^{(j)\nu e}(x) + G_{2\mu}^{(j)be\nu}(x, x) G_{1\nu}^{(j)d}(x) + G_{2\nu}^{(j)de\nu}(x, x) G_{1\mu}^{(j)b}(x)] \\
& - g f^{abc} \{ \bar{P}_1^{(\eta)b}(x) [\partial_\mu P_1^{(\eta)c}(x)] + \partial_\mu [K_2^{(\eta)bc}(x, x)] \} = j_\mu^a, \tag{19}
\end{aligned}$$

For the ghost field, it is

$$-\square P_1^{(\eta)c} - g f^{abc} G_{1\mu}^{(j)a}(x) \partial^\mu P_1^{(\eta)b}(x) - g f^{abc} \partial^\mu J_{2\mu}^{(\eta,j)ab}(x, x) = \eta^c. \tag{20}$$

The final step is given by setting all the currents to zero.

From Eq. (19), we derive it with respect to $j^{\lambda h}(y)$ and get

$$\begin{aligned}
& \square G_{2\mu\lambda}^{(j)ah}(x, y) + g f^{abc} \partial^\nu [G_{3\mu\nu\lambda}^{(j)bch}(x, x, y) + G_{2\mu\lambda}^{(j)bh}(x, y) G_{1\nu}^{(j)c}(x) + G_{1\mu}^{(j)b}(x) G_{2\nu\lambda}^{(j)ch}(x)] \\
& - g f^{abc} [\partial^\nu G_{2\mu\nu\lambda}^{(j)bch}(x, x, y) + \partial^\nu G_{2\mu\lambda}^{(j)bh}(x, y) G_{1\nu}^{(j)c}(x) + \partial^\nu G_{1\mu}^{(j)b}(x) G_{2\nu\lambda}^{(j)ch}(x, y)] \\
& - g f^{abc} [\partial_\mu G_{3\nu\lambda}^{(j)bch\nu}(x, x, y) + \partial_\mu G_{2\nu\lambda}^{(j)bh}(x, y) G_1^{(j)c\nu}(x) + \partial_\mu G_{1\nu}^{(j)b}(x) G_{2\lambda}^{(j)ch\nu}(x, y)] \\
& + g^2 f^{abc} f^{cde} [G_{3\mu\nu\lambda}^{(j)bdh}(x, x, y) G_1^{(j)\nu e}(x) + G_{2\mu\nu}^{(j)bd}(x, x) G_{2\lambda}^{(j)\nu eh}(x, y) + \partial^\nu G_{4\mu\nu\lambda}^{(j)bdeh\nu}(x, x, x, y) \\
& + G_{2\mu\lambda}^{(j)bh}(x, y) G_{1\nu}^{(j)d}(x) G_1^{(j)\nu e}(x) + G_{1\mu}^{(j)b}(x) G_{2\nu\lambda}^{(j)dh}(x, y) G_1^{(j)\nu e}(x) \\
& + G_{1\mu}^{(j)b}(x) G_{1\nu}^{(j)d}(x) G_{2\lambda}^{(j)\nu eh}(x, y) + G_{3\mu\lambda}^{(j)beh\nu}(x, x, y) G_{1\nu}^{(j)d}(x) + G_{2\mu}^{(j)be\nu}(x, x) G_{2\nu\lambda}^{(j)dh}(x, y) \\
& + G_{3\nu\lambda}^{(j)deh\nu}(x, x, y) G_{1\mu}^{(j)b}(x) + G_{2\nu}^{(j)de\nu}(x, x) G_{2\mu\lambda}^{(j)bh}(x, y)] - g f^{abc} \{ \bar{J}_{2\lambda}^{(\eta,j)bh}(x, y) [\partial_\mu P_1^{(\eta)c}(x)] \\
& + \bar{P}_1^{(\eta)b}(x) [\partial_\mu J_{2\lambda}^{(\eta,j)ch}(x, y)] + \partial_\mu [W_{3\lambda}^{(\eta,j)bch}(x, x, y)] \} = \delta^{ah} g_{\mu\lambda} \delta^4(x - y), \tag{21}
\end{aligned}$$

after the introduction of the 3P-function

$$W_{3\lambda}^{(\eta,j)abc}(x, y, z) = Z^{-1} \frac{\delta K_2^{(\eta)ab}(x, y)}{\delta j^{\lambda c}(z)}. \tag{22}$$

Taking the 1P-function for the ghost, we derive it with respect to $\eta^h(y)$ and obtain

$$-\square K_2^{(\eta)ch}(x, y) - i g f^{abc} L_{2\mu}^{(\eta,j)ah}(x, y) \partial^\mu P_1^{(\eta)b}(x) - i g f^{abc} G_{1\mu}^{(j)a}(x) \partial^\mu K_2^{(\eta)bh}(x, y) - i g f^{abc} \partial^\mu W_{3\mu}^{(\eta,j)abh}(x, x, y) = \delta^{ch} \delta^4(x - y). \tag{23}$$

We have introduced the 2P-function

$$L_{2\mu}^{(\eta,j)ab}(x, y) = \frac{\delta G_1^{(j)a}(x)}{\delta \eta^b(y)}. \tag{24}$$

We derive with respect to $j^{h\nu}(y)$ and this yields the equation for J_2 as

$$-\square J_2^{(\eta)ch\nu}(x, y) - i g f^{abc} G_{2\mu\nu}^{(j)ah}(x, y) \partial^\mu P_1^{(\eta)b}(x) - i g f^{abc} G_{1\mu}^{(j)a}(x) \partial^\mu J_2^{(\eta,j)bh\nu}(x, y) - i g f^{abc} \partial^\mu J_{3\mu}^{(\eta,j)abh}(x, x, y) = 0, \tag{25}$$

with the introduction of the 3P-function

$$J_{3\mu}^{(\eta,j)abc}(x, y, z) = \frac{\delta J_{2\mu}^{(\eta,j)ab}(x, y)}{\delta j^{\mu c}(z)}. \tag{26}$$

This set of equations is amenable to an exact solution [47] that is of a Gaussian kind [74] with all higher order correlation functions obtained by 1P- and 2P-correlation functions, provided the following conditions are satisfied:

- (1) The mapping theorem, presented in [62,64], is used to solve for G_1 and the remaining equations.
- (2) All the nP-correlation functions for $n > 2$ are 0 when two or more independent variables coincide. This arises from the property of the support of G_2 entering in the solutions in the same way as the product $\theta(x)\theta(-x)$ under integration, where $\theta(x)$ is the Heaviside step function.

One can verify that the solution obtained in this way trivially satisfies the Slavnov-Taylor identities [75] at the nonperturbative level.

In the following, we will use the solutions of the 2P-correlation functions. The propagator in the Landau gauge is given by (see Appendix C for a derivation)

$$G_2(p) = \frac{\pi^3}{4K^3(i)} \sum_{n=0}^{\infty} \frac{e^{-(n+\frac{1}{2})\pi}}{1 + e^{-(2n+1)\pi}} (2n+1)^2 \frac{1}{p^2 - m_n^2 + i\epsilon}. \quad (27)$$

We observe that the gauge field 2P-function is in agreement with the Källen-Lehman representation with a spectrum of particles given by

$$m_n = (2n+1) \frac{\pi}{2K(i)} \left(\frac{Ng^2}{2} \right)^{\frac{1}{4}} \mu, \quad (28)$$

where $K(i)$ is the complete elliptical integral of the first kind and μ is one of the integration constants of the theory [47]. At this stage we have neglected the quantum corrections inducing a gap equation for the gauge field [55], so that the gauge field propagator is a good approximation to the exact one provided that the mass shift induced by quantum fluctuations is small. This is what one sees for SU(N) Yang-Mills theory [55]. In the Landau gauge, the gauge field 2P-function can be written down in the form

$$D_{\mu\nu}^{ab}(p) = \delta_{ab} \left(\eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) G_2(p), \quad (29)$$

where $\eta_{\mu\nu}$ is the Minkowski metric tensor.

B. β -function for strongly coupled Yang-Mills theory

Following an exact solution of a truncated set of the background equations of motion to the Yang-Mills fields using elliptic integral equation, β -function for Yang-Mills theory has been obtained recently, in an almost exact form, in Ref. [52] and then later on extended onto various other extensions of QFT in the context of UV-completion [70–72]. We present this argument here in order to make the paper self-contained as much as possible. The idea is to apply the

Kugo-Ojima criterion for confinement starting from the exact solutions obtained for the Dyson-Schwinger equations in [47]. Kugo-Ojima criterion requires [76,77]

$$\int d^d x e^{ipx} \langle D_\mu \bar{c}^a(x), D_\nu c^b(y) \rangle = \delta^{ab} \left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2 - i\epsilon} \right) u(p^2) - \delta^{ab} \frac{p_\mu p_\nu}{p^2 - i\epsilon}, \quad (30)$$

where $D_\mu = \partial_\mu - igT^a A_\mu^a$ is the covariant derivative with T^a the group generators and A_μ^a the gauge potential and the u -function derived below. We expect that no massless particle propagates in our theory and we have the no-pole condition that yields here

$$1 + u(p^2 = 0) = 0, \quad (31)$$

which is the Kugo-Ojima condition for confinement [76,77].¹ In this equation, D_μ is the covariant derivative and $c, \bar{c}$ are the ghost fields. Therefore, BRST invariance plays an essential role in this argument.

Thus, the u function will be given by [52]

$$u(p^2) = -\frac{(N^2 - 1)^2}{2N} g^2 \int \frac{d^4 p'}{(2\pi)^4} \frac{1}{|p - p'|^2} G_2(p'). \quad (32)$$

Then, we have to evaluate the integral

$$\begin{aligned} u(0) &= -\frac{(N^2 - 1)^2}{2N} g^2 \int \frac{d^4 p}{(2\pi)^4} \frac{1}{p^2} \sum_{n=0}^{\infty} B_n \frac{1}{p^2 + m_n^2} \\ &= -\frac{(N^2 - 1)^2}{2N} g^2 \sum_{n=0}^{\infty} \frac{B_n}{m_n^2} \int \frac{d^4 p}{(2\pi)^4} \left(\frac{1}{p^2} - \frac{1}{p^2 + m_n^2} \right), \end{aligned} \quad (33)$$

with $B_n = \frac{\pi^3}{4K^3(-1)} \frac{e^{-(n+\frac{1}{2})\pi}}{1 + e^{-(2n+1)\pi}} (2n+1)^2$. This integral diverges and needs to be renormalized. We get after dimensional regularization

$$u(0) = \frac{(N^2 - 1)^2}{2N} \frac{\alpha}{4\pi} \left[-1 + \gamma + \sum_{n=0}^{\infty} B_n \ln \left(\frac{m_n^2}{4\pi\mu^2} \right) \right], \quad (34)$$

where use has been made of the identities $\sum_{n=0}^{\infty} B_n = 1$ and $\alpha = g^2/4\pi$. Finally, using the Kugo-Ojima criterion $u(0) = -1$, we get an equation for the running coupling

$$\frac{(N^2 - 1)^2}{2N} \frac{\alpha(\mu^2)}{4\pi} \left[-1 + \gamma + \sum_{n=0}^{\infty} B_n \ln \left(\frac{m_n^2}{4\pi\mu^2} \right) \right] = -1. \quad (35)$$

¹Equation (30) and the confinement condition are derived through the anticommuting charges of BRST symmetry (see [52] for a simplified derivation).

It should be emphasized that also m_n depends on α and this is a nonlinear equation to be solved. Indeed, it can be cast into a differential form yielding for the β -function for a SU(N) gauge theory

$$\frac{d\alpha}{dl} = -\beta_0 \frac{\alpha^2}{1 - \frac{1}{2}\beta_0\alpha}, \quad (36)$$

with $\beta_0 = (N^2 - 1)^2/8\pi N$. We have set $l = \ln(\mu^2/\sigma_0)$ as an independent variable. Such a β -function is almost exact because in the derivation we neglected a correction to the propagator arising from quantum fluctuations (see Eq. (C4) in Appendix C). The equation we obtained has the same functional form of the exact β -function obtained in supersymmetric Yang-Mills theory [78,79] and the corresponding one, inspired by the supersymmetric solution, for pure Yang-Mills theory [80].

III. GENERATING THE ELECTROWEAK SCALE DYNAMICALLY

A. Higgs coupled to a gauge field

Let us consider the following Higgs Lagrangian [81]

$$L = \partial^\mu h^\dagger \partial_\mu h - \lambda(h^\dagger h)^2 - \frac{1}{4} \left(1 + \frac{h^\dagger h}{M^2} \frac{\beta}{2g} \right) G^2, \quad (37)$$

where h is a Higgs field and a singlet representation of the SU(N) group, β the beta function we introduced above, M a mass scale, and $G_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc}A_\mu^b A_\nu^c$ the field tensor. Here, the mass term is not present. However, it is also possible to consider a positive mass term $(h^\dagger h)$. As long as the negative mass term dominates the EWSB dynamics via nonperturbative physics, what we describe below remains the same. The field h is not charged to avoid terms like $|h^2|A_\mu^a A^{\mu a}$ and we avoid to break explicitly the gauge invariance. Besides, we work in a confining phase of the gauge field where a renormalization group analysis, as per a UV-regime, cannot apply giving rise to such kind of breaking terms. Then, the idea is to use the mean-field approximation on the equation of motion

$$\square h = -\lambda(h^\dagger h)h - \frac{h}{M^2} \frac{\beta}{2g} G^2 \quad (38)$$

and to evaluate the gluon condensate $\langle G^2 \rangle$. Our aim is to show how an evaluation of the gluon condensate can be obtained directly from Yang-Mills theory with our approach. The gluon condensate can be computed starting from the following solutions for the gauge field [47] (for an extended explanation of the constants η_μ^a and the scalar field ϕ see Appendixes B and C)

$$A_\mu^a(x) = \eta_\mu^a \phi(x). \quad (39)$$

This is an application of a mapping theorem [62] providing a set of exact solutions for the gauge field starting from a ϕ^4 theory. For our aims, ϕ has just a mathematical meaning here. It should be emphasized that in some gauge such a theorem just warrants asymptotic and not an exact solution in the limit of the coupling running to infinity. This would not change our argument if we would change our gauge's choice. Therefore, one has for SU(N) (see Appendix B)

$$\langle G^2 \rangle = 2(D-2)(N^2-1)\langle (\partial\phi)^2 \rangle + 4\pi N(N^2-1)\alpha_s \langle \phi^4 \rangle. \quad (40)$$

To evaluate such a condensate we have to compute (p_E is the Euclidean momentum)

$$\langle (\partial\phi)^2 \rangle = - \int \frac{d^4 p_E}{(2\pi)^4} p_E^2 G_2(p_E), \quad (41)$$

where $G_2(p_E)$ is given in Eq. (27). Using this propagator, we are neglecting the effect of mass renormalization on the spectrum of the theory that we consider negligibly small [55]. Finally, we evaluate the integrals by moving the integration path to complex plane changing the metric to Euclidean. This will yield, after we introduce a cutoff Λ , (see Appendix C for a derivation)

$$\langle (\partial\phi)^2 \rangle = \frac{1}{8\pi^2} \frac{\Lambda^4}{4}. \quad (42)$$

This yields the value for the condensate

$$\left\langle \frac{\alpha}{\pi} G^2 \right\rangle = \alpha \frac{N^2-1}{8\pi^3} \Lambda^4 \quad (43)$$

The β -function is given by Eq. (36) and can be written as [52]

$$\beta(\alpha) = -\beta_0 \frac{\alpha^2}{1 - \frac{1}{2}\beta_0\alpha}, \quad (44)$$

with $\beta_0 = (N^2 - 1)^2/8\pi N$, whose plot is shown in Fig. 1.

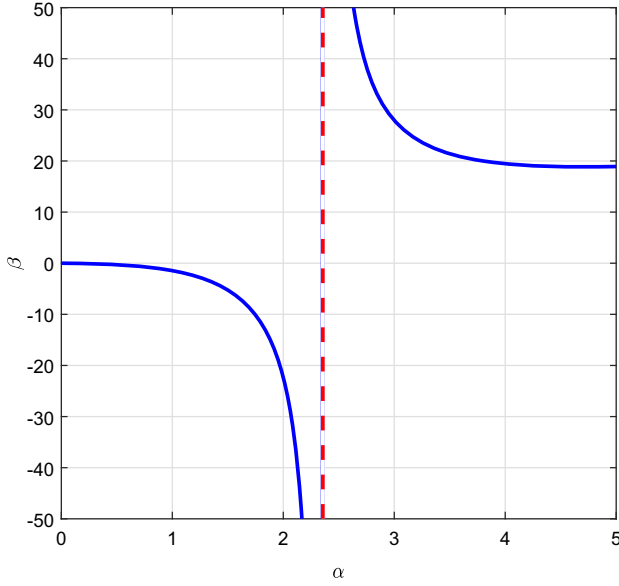
We can introduce an effective potential by noting that the gauge term in the Lagrangian can have the form,

$$L_g = -\frac{1}{4} \left(1 + \frac{h^\dagger h}{M^2} \frac{\beta}{2g} \right) G^2 = -\frac{1}{4g_s^2} \tilde{G}^2, \quad (45)$$

with $\tilde{G} = G/g$ and having set

$$\frac{1}{g_s^2} = \frac{1}{g^2} \left(1 + \frac{h^\dagger h}{M^2} \frac{\beta}{2g} \right). \quad (46)$$

Here, g_s should be understood as an effective coupling due to its dependence on h . This definition is needed to

FIG. 1. β -function for Yang-Mills theory given for SU(3).

complete the mathematical steps to derive the potential for h itself. We are assuming an infrared fixed point at $\alpha_s = \pi$. From the β function in Eq. (44), one gets the general solution

$$\alpha(l) = -\frac{2}{\beta_0} \frac{1}{W\left(-C e^{-2l} \frac{2}{\beta_0}\right)}, \quad (47)$$

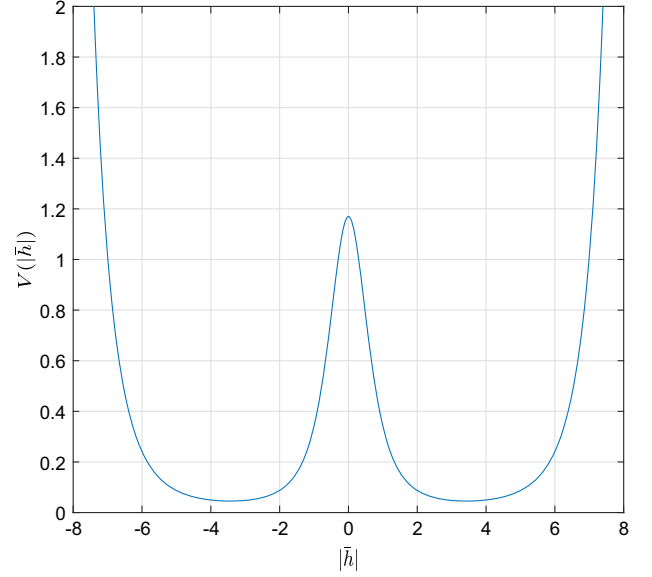
where C is an integration constant, W is the Lambert function that solves the equation $z = W(z)e^{W(z)}$ and $l = \ln(\Lambda/M)$ the renormalization group scaling with the cutoff Λ normalized to the mass M . Conventionally, we consider the value of β for $\alpha = \pi$ and we notice that this must be a fixed point for the renormalization group of the theory. This gives

$$\frac{1}{\alpha_s} = \frac{1}{\alpha} \left(1 + \frac{|h|^2}{M^2} \frac{\beta(\pi)}{4\sqrt{\pi}\alpha}\right) = \frac{1}{\pi} \left(1 + \frac{|h|^2}{M^2} \frac{\beta(\pi)}{4\pi}\right), \quad (48)$$

where $\alpha_s = g_s^2/4\pi$ and similarly for α and g . We can invert our solution for the running coupling to get the cutoff Λ at the value $\alpha = \pi$ as

$$\begin{aligned} \Lambda &= M(\alpha_s C)^{\frac{1}{2}} \exp\left(\frac{1}{\beta_0 \alpha_s}\right) \\ &= M(\pi C)^{\frac{1}{2}} \frac{1}{\left(1 + \frac{|h|^2}{M^2} \frac{\beta(\pi)}{4\pi}\right)^{\frac{1}{2}}} \exp\left(\frac{1}{\beta_0 \pi} \left(1 + \frac{|h|^2}{M^2} \frac{\beta(\pi)}{4\pi}\right)\right). \end{aligned} \quad (49)$$

Therefore, the potential will become [82]

FIG. 2. General behavior of the effective potential obtained from Eq. (50). In this case we have normalized $V(|\bar{h}|)$ to V_0 to make the potential dimensionless.

$$\begin{aligned} V_{\text{vac}} &= \frac{1}{4} \langle T_\mu^\mu \rangle = \left\langle \frac{\beta}{8g_s} \tilde{G}^2 \right\rangle \\ &= \frac{N^2 - 1}{64\pi^3} \frac{M^4 C^2}{\left(1 + \frac{|h|^2}{M^2} \frac{\beta(\pi)}{4\pi}\right)^2} \exp\left(\frac{4}{\beta_0 \pi} \left(1 + \frac{|h|^2}{M^2} \frac{\beta(\pi)}{4\pi}\right)\right). \end{aligned} \quad (50)$$

This potential is characterized by two scales: The Higgs scale given by

$$h_0^2 = \frac{4\pi M^2}{\beta(\pi)}, \quad (51)$$

and a general scale given by

$$V_0 = \frac{N^2 - 1}{64\pi^3} M^4 C^2. \quad (52)$$

Therefore, the form of the potential will be,

$$V(|\bar{h}|) = V_0 \frac{1}{(1 + |\bar{h}|^2)^2} \exp\left(\frac{4}{\beta_0 \pi} (1 + |\bar{h}|^2)\right), \quad (53)$$

where we have introduced the new field $\bar{h} = h/h_0$. This kind of potential has the behavior depicted in Fig. 2.

We can map the potential in Eq. (53) to the SM Higgs potential that has the parameters experimentally constrained. We get the following approximation for $\bar{h} \ll 1$,

$$V(|\bar{h}|) = V_0 e^{\frac{4}{\beta_0 \pi}} - \frac{2V_0}{\beta_0 \pi} e^{\frac{4}{\beta_0 \pi}} (-2 + \beta_0 \pi) |\bar{h}|^2 + \frac{V_0}{\beta_0^2 \pi^2} (8 - 8\beta_0 \pi + 3\beta_0^2 \pi^2) e^{\frac{4}{\beta_0 \pi}} |\bar{h}|^4 + O(|\bar{h}|^6). \quad (54)$$

We can now undo the normalization obtaining

$$V(|h|) = \frac{N^2 - 1}{64\pi^3} M^4 C^2 e^{\frac{4}{\beta_0 \pi}} - \frac{N\beta(\pi)}{16\pi^4 (N^2 - 1)} M^2 C^2 e^{\frac{4}{\beta_0 \pi}} (-2 + \beta_0 \pi) |h|^2 + \frac{N^2 \beta^2(\pi)}{16\pi^5 (N^2 - 1)^3} C^2 (8 - 8\beta_0 \pi + 3\beta_0^2 \pi^2) e^{\frac{4}{\beta_0 \pi}} |h|^4 + O(|h|^6). \quad (55)$$

So, we see that we have two arbitrary constants, M and C , that can fix the parameters for the two terms in the series

(omitting the constant at the leading order as inessential), granting agreement with the Higgs potential. We can compare this effective potential, by accounting for the h^6 contribution, to SMEFT studies to check for consistencies [83,84]. For our considerations, we omit the initial constant having no physical meaning and use Warsaw basis for the Wilson coefficients [83]

$$\begin{aligned} \lambda &= -\mu_H^2 C_{HD}, \\ C_H &= 2\lambda C_{HD}, \end{aligned} \quad (56)$$

where μ_H^2 is the coefficient of quadratic term in the Higgs potential, λ is the coefficient of the quartic term in the Higgs potential and C_H is the Wilson coefficient of the order six operator in the SMEFT. We see that, in our case, we have to bound the Wilson coefficient C_{HD} as given in Ref. [84]. By a direct comparison we can write

$$\begin{aligned} \mu_H^2 &= -\frac{M^4 C^2 e^{\frac{32N}{(N^2-1)^2}} N}{2(N^2-1)\pi^3 h_0} + \frac{(N^2-1)M^4 C^2 e^{\frac{32N}{(N^2-1)^2}}}{32\pi^3 h_0}, \\ \lambda &= \frac{8M^4 C^2 e^{\frac{32N}{(N^2-1)^2}} N^2}{(N^2-1)^3 \pi^3 h_0^2} - \frac{M^4 C^2 e^{\frac{32N}{(N^2-1)^2}} N}{(N^2-1)\pi^3 h_0^2} + \frac{3(N^2-1)M^4 C^2 e^{\frac{32N}{(N^2-1)^2}}}{64\pi^3 h_0^2}, \\ C_H &= -\frac{256M^4 C^2 e^{\frac{32N}{(N^2-1)^2}} N^3}{3(N^2-1)^5 \pi^3 h_0^3} + \frac{16M^4 C^2 e^{\frac{32N}{(N^2-1)^2}} N^2}{(N^2-1)^3 \pi^3 h_0^3} - \frac{3M^4 C^2 e^{\frac{32N}{(N^2-1)^2}} N}{2(N^2-1)\pi^3 h_0^3} + \frac{(N^2-1)M^4 C^2 e^{\frac{32N}{(N^2-1)^2}}}{16\pi^3 h_0^3}. \end{aligned}$$

From these, we derive

$$C_{HD} = \frac{C_H}{2\lambda} \stackrel{N \rightarrow 3}{=} \frac{19}{36h_0^2}, \quad (57)$$

and

$$v = \frac{\mu_H}{\sqrt{\lambda}} \stackrel{N \rightarrow 3}{=} \frac{2}{3} h_0 = \frac{4}{3} \frac{\sqrt{\pi} M}{\sqrt{\beta(\pi)}}. \quad (58)$$

Consistency reasons yield $\beta(\pi) \gg 16\pi/9$ for $M = O(\text{TeV})$ that agrees fairly well with Fig. 1 for Eq. (36). This also provides a physical interpretation for the mass scale M as the new physics energy scale. Finally, our lower bound for C_{HD} is about $0.235/M^2$ from the consistency limit for $\beta(\pi)$, well in the range given in Ref. [84].

IV. MASS SPECTRUM OF THE THEORY

The mass spectrum can be evaluated in a standard way as seen in [55]. If we are permitted to neglect the mass renormalization shift, that we can assume to be small, the spectrum can be approximated by, for SU(N),

$$m_n = (2n+1) \frac{\pi}{2K(i)} [2\pi N \alpha_s(h)]^{\frac{1}{4}} \mu, \quad (59)$$

where $K(i)$ is the complete elliptic integral of the first kind, $\alpha_s(h)$ evaluated at the fixed point and μ an integration constant of the theory [47]. Therefore, we can take this formula to evaluate the spectrum at the breaking of the symmetry $|h| = v$. Therefore, by using Eq. (48), we get

$$m_n(\bar{h}) = (2n+1)m_0(1+\bar{h}^2)^{-\frac{1}{4}}. \quad (60)$$

After having normalized to $(2n+1)m_0$, we get the plot in Fig. 3.

The spectrum is dampened out at increasing values of the Higgs vev.

V. CONCLUSIONS AND DISCUSSIONS

In this paper, we have investigated a generic Higgs field coupled to SU(N) gauge fields and then dynamically receiving a vacuum expectation value (vev) due to the condensation of the gauge fields below the confinement scale of the strongly coupled SU(N) dark sector. Following a novel technique developed by *Bender et al.*, we were able to compute this vev analytically as shown in Eq. (43). We also computed the exact RGE of the scalar-gauge dynamics and found the β -function of theory [Eq. (44)]. Such gauge

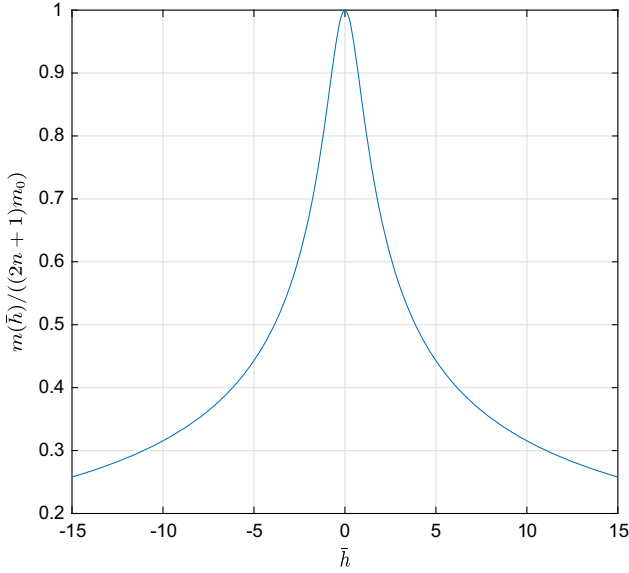


FIG. 3. Behavior of the mass spectrum as a function of $\bar{h}$.

field condensation and confining dynamics lead to an effective scalar potential of the Higgs field that mimics the standard Higgs potential with the negative mass squared term. We have found the full mass spectrum of the theory in Eq. (60). We have shown that the potential arising from the coupling with the gauge fields exhibits proper minima that can give rise to electroweak scale breaking (see Fig. 2) while the mass spectrum can be dampened out depending on the position of the vev of the Higgs field (see Fig. 3).

Such dynamically generated scales due to condensation of gauge fields, with novel mass spectrum of the theory (excited states) and the RGE could help to envisage a SM $\times$ SU(2)_D model, where the Higgs vev which breaks the SU(2)_D may have its origin due to strongly coupled gauge sector and the symmetry breaking scale can be very high in general and involve very interesting dark matter physics [8,39,85].

Finally, we envisage that no scales are special in nature including the EW scale or the Planck scale and entertain the possibility of some dynamical explanation for the generation of scales in nature. So starting from classically scale-invariant theories and scale generation via nonperturbative phase transition (just like QCD) provides an interesting avenue to understand why different interactions (for example, gravity and EW) behave differently. Such a study involving detailed generation of Planck and EW scales will require deeper investigation and is beyond the scope of the present paper, so we leave it for future studies.

ACKNOWLEDGMENTS

This work is supported in part by the United States, Department of Energy Grant No. DE-SC0012447 (N. O.).

APPENDIX A: DYSON-SCHWINGER EQUATIONS AND BENDER-MILTON-SAVAGE TECHNIQUE

In this appendix, we briefly present the Bender-Milton-Savage technique [46]. We start from the following partition function for a scalar field

$$Z[j] = \int [D\phi] e^{iS(\phi) + i \int d^4x j(x)\phi(x)}. \quad (\text{A1})$$

The equation for the 1P-function is given by

$$\left\langle \frac{\delta S}{\delta \phi(x)} \right\rangle = j(x) \quad (\text{A2})$$

where

$$\langle \dots \rangle = \frac{\int [D\phi] \dots e^{iS(\phi) + i \int d^4x j(x)\phi(x)}}{\int [D\phi] e^{iS(\phi) + i \int d^4x j(x)\phi(x)}}. \quad (\text{A3})$$

One has

$$\langle \phi(x_1)\phi(x_2)\dots\phi(x_n) \rangle = \frac{\delta^n \ln(Z[j])}{\delta j(x_1)\delta j(x_2)\dots\delta j(x_n)}, \quad (\text{A4})$$

and

$$\frac{\delta G_k(\dots)}{\delta j(x)} = G_{k+1}(\dots, x). \quad (\text{A5})$$

We apply all this to a ϕ^4 theory and get

$$S = \int d^4x \left[\frac{1}{2} (\partial\phi)^2 - \frac{\lambda}{4} \phi^4 \right], \quad (\text{A6})$$

so that,

$$\partial^2 \langle \phi \rangle + \lambda \langle \phi^3(x) \rangle = j(x). \quad (\text{A7})$$

For the 1P-function is

$$Z[j] \partial^2 G_1^{(j)}(x) + \lambda \langle \phi^3(x) \rangle = j(x), \quad (\text{A8})$$

and by definition

$$Z[j] G_1^{(j)}(x) = \langle \phi(x) \rangle. \quad (\text{A9})$$

After derivation with respect to $j(x)$, we get

$$Z[j] [G_1^{(j)}(x)]^2 + Z[j] G_2^{(j)}(x, x) = \langle \phi^2(x) \rangle, \quad (\text{A10})$$

and after another derivation step it one has

$$Z[j][G_1^{(j)}(x)]^3 + 3Z[j]G_1^{(j)}(x)G_2(x, x) + Z[j]G_3^{(j)}(x, x, x) = \langle \phi^3(x) \rangle. \quad (\text{A11})$$

We insert this into Eq. (A7) to obtain

$$\partial^2 G_1^{(j)}(x) + \lambda[G_1^{(j)}(x)]^3 + 3\lambda G_2^{(j)}(x, x)G_1^{(j)}(x) + G_3^{(j)}(x, x, x) = Z^{-1}[j]j(x) \quad (\text{A12})$$

Setting $j = 0$, we get the first Dyson-Schwinger equation

$$\partial^2 G_1(x) + \lambda[G_1(x)]^3 + 3\lambda G_2(x, x)G_1(x) + G_3(x, x, x) = 0, \quad (\text{A13})$$

where we realize that quantum corrections induced a mass term.

Our next step is to derive Eq. (A12) again with respect to $j(y)$. This gives the result

$$\begin{aligned} & \partial^2 G_2^{(j)}(x, y) + 3\lambda[G_1^{(j)}(x)]^2 G_2^{(j)}(x, y) + 3\lambda G_3^{(j)}(x, x, y) \\ & \times G_1^{(j)}(x) + 3\lambda G_2^{(j)}(x, x)G_2^{(j)}(x, y) + G_4^{(j)}(x, x, x, y) \\ & = Z^{-1}[j]\delta^4(x - y) + j(x) \frac{\delta}{\delta j(y)} (Z^{-1}[j]). \end{aligned}$$

By setting $j = 0$, one has for the 2P-function

$$\begin{aligned} & \partial^2 G_2(x, y) + 3\lambda[G_1(x)]^2 G_2(x, y) + 3\lambda G_3(x, x, y) \\ & \times G_1(x) + 3\lambda G_2(x, x)G_2(x, y) + G_4(x, x, x, y) = \delta^4(x - y). \end{aligned} \quad (\text{A14})$$

In principle, one can iterate such a procedure to whatever desired order yielding the full hierarchy of Dyson-Schwinger equations into PDE form.

APPENDIX B: GLUON CONDENSATE

We have to evaluate

$$\begin{aligned} \langle G_{\mu\nu} G^{\mu\nu} \rangle &= \langle (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc} A_\mu^b A_\nu^c) \\ &\times (\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu} + gf^{ade} A^{d\mu} A^{e\nu}) \rangle, \end{aligned} \quad (\text{B1})$$

using our exact solution for the equations of motion and the Green's function presented in Appendix C. We have seen that these kind of solutions are obtained by applying a mapping theorem on a quartic scalar field writing $A_\mu^a(x) = \eta_\mu^a \phi(x)$, where η_μ^a are constants and $\phi(x)$ the solution for the scalar field equation of motion. After some algebra, we get

$$\begin{aligned} \langle G_{\mu\nu} G^{\mu\nu} \rangle &= \langle (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)(\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu}) \rangle \\ &+ 2gf^{ade} \langle (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a) A^{d\mu} A^{e\nu} \rangle \\ &+ g^2 f^{abc} f^{ade} \langle A_\mu^b A_\nu^c A^{d\mu} A^{e\nu} \rangle \\ &= 2\eta_\mu^a \eta^{a\nu} \langle (\partial\phi)^2 \rangle - 2\eta_\mu^a \eta_\nu^c \langle \partial^\mu \phi \partial^\nu \phi \rangle \\ &+ g^2 f^{abc} f^{ade} \eta_\mu^b \eta_\nu^c \eta^{d\mu} \eta^{e\nu} \langle \phi^4 \rangle, \end{aligned} \quad (\text{B2})$$

where we have used the antisymmetry of f^{abc} that removes the mixed term linear in g . Now, we use the properties of the η -symbols

$$\eta_\mu^a \eta^{\mu b} = D\delta^{ab}, \quad \eta_\mu^a \eta_\nu^a = (N^2 - 1)g_{\mu\nu}, \quad (\text{B3})$$

to obtain

$$\begin{aligned} \langle G_{\mu\nu} G^{\mu\nu} \rangle &= 2(D - 2)(N^2 - 1) \langle (\partial\phi)^2 \rangle \\ &+ 4\pi DN(N^2 - 1) \alpha_s \langle \phi^4 \rangle, \end{aligned} \quad (\text{B4})$$

where D is the number of space-time dimensions that in our case is 4 and the identities $f^{abc} f^{abd} = N\delta^{cd}$ and $f^{abc} f^{abc} = N(N^2 - 1)$ have been used. In order to evaluate the gluon condensate, we observe that the scalar field that maps the Yang-Mills theory should have the partition function

$$Z[j] = \int [d\phi] e^{4i(N^2 - 1) \int d^4x [\frac{1}{2}(\partial\phi)^2 - \frac{Ng^2}{4}\phi^4 + j\phi]} \quad (\text{B5})$$

and we are able to get an explicit expression for it using just the corresponding 1P- and 2P-correlation functions. We can write

$$Z[j] = \mathcal{N} e^{i \int d^4x G_1(x) j(x) + i \int d^4x d^4x' j(x) G_2(x - x') j(x') + \dots}, \quad (\text{B6})$$

and

$$\begin{aligned} \langle \partial_\mu \phi(x_1) \partial^\mu \phi(x_2) \rangle &= \partial_\mu \frac{\delta}{\delta j(x_1)} \partial^\mu \frac{\delta \ln Z}{\delta j(x_2)} \Big|_{j=0}, \\ \langle \phi(x_1) \phi(x_2) \phi(x_3) \phi(x_4) \rangle &= \frac{\delta^4 \ln Z}{\delta j(x_1) \delta j(x_2) \delta j(x_3) \delta j(x_4)} \Big|_{j=0}, \end{aligned} \quad (\text{B7})$$

from which we derive

$$\begin{aligned} \langle (\partial\phi)^2 \rangle &= i\partial^2 G_2(x, x), \\ \langle \phi^4 \rangle &= iG_4(x, x, x, x), \end{aligned} \quad (\text{B8})$$

but $G_4(x, x, x, x)$ is 0 for our Gaussian solution being a higher-order correlation function evaluated in one single point x . This yields

$$\langle(\partial\phi)^2\rangle = \int \frac{d^4 p_E}{(2\pi)^4} p_E^2 \sum_{n=0}^{\infty} \frac{B_n}{p_E^2 + m_n^2}, \quad (\text{B9})$$

where the subscript E just means that we moved to Euclidean momentum. The integral needs to be regularized and we use a cutoff Λ for this purpose. Taking Λ large enough, we are left with

$$\langle(\partial\phi)^2\rangle = \frac{\Lambda^4}{32\pi^2}, \quad (\text{B10})$$

where use has been mad of the identity $\sum_{n=0}^{\infty} B_n = 1$. This yields for the gluon condensate

$$\left\langle \frac{\alpha_s}{\pi} G_{\mu\nu} G^{\mu\nu} \right\rangle = \alpha_s \frac{N^2 - 1}{8\pi^3} \Lambda^4. \quad (\text{B11})$$

APPENDIX C: EXACT PROPAGATOR FOR YANG-MILLS THEORY

We derive the propagator for Yang-Mills theory by assuming that the vacuum of the theory is well described by a Fubini instanton [86,87] that breaks translation invariance. Such a breaking would be a problem if some experiment could be devised to measure the 1P-correlation function but it is well-known that neither the potentials nor the fields are observable in non-Abelian gauge theories. On the other hand, we will show how the propagator recovers translation invariance and so, when decay rates or cross sections are computed using LSZ decomposition, there is no way to detect breaking of such a symmetry. This kind of 1P-correlation function grants the existence of a mass gap for the theory but this choice is not unique. In our case, it appears to give results in agreement with lattice data.

We want to solve the Eq. (19) and (21) using a Fubini instanton. We take an ansatz that makes all (dark) color fields proportional to a single scalar one with proportionality constants carrying the color and Minkowski indices [47,62,64]

$$G_{1\mu}^a(x) = \eta_\mu^a \phi(x), \quad (\text{C1})$$

where η_μ^a are such a set of constants (vacuum impedances) and $\phi(x)$ a scalar field for a quartic potential. Similarly, we can take

$$G_{2\mu\nu}^{ab}(x, y) = \delta_{ab} \left(g_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\partial^2} \right) G_2(x, y), \quad (\text{C2})$$

where we wrote the gauge projector using the metric $g_{\mu\nu}$ and the derivatives ∂_μ and now $G_2(x, y)$ is the propagator of the ϕ field. By a direct substitution into Eq. (19) and (21), we get [47]

$$\begin{aligned} \partial^2 \phi + \delta m^2 \phi + N g^2 \phi^3 &= 0, \\ \partial^2 G_2(x, y) + \delta m^2 G_2(x, y) + 3 N g^2 \phi^2(x) G_2(x, y) &= \delta^4(x - y), \end{aligned} \quad (\text{C3})$$

where

$$\delta m^2 = 2 N g^2 G_2(x, x), \quad (\text{C4})$$

is the mass shift arising from quantum effects and in need for regularization being divergent. After renormalization, we know that such a shift has a very small effect on the spectrum of the theory [55] and we do not consider it here and in the rest of the paper. Anyway, we observe that the field gains mass simply due to quantum fluctuations but a different choice of the G_1 function being a constant fails to yield a mass gap [88]. The choice in Eq. (C1) grants that the ghost field decouples and behaves as a free field and this will yield the propagator

$$K_2(p) = -\frac{1}{p^2 + i\epsilon}. \quad (\text{C5})$$

This is consistent with the decoupling solution observed in lattice computation [89–91] where the ghost propagator is seen to be the same of a free massless particle with properly changed sign.

The solutions are obtained by noting that the equation for ϕ can be solved by taking²

$$\phi(x) = \mu \left(\frac{2}{N g^2} \right) \text{sn}(p \cdot x + \theta, -1), \quad (\text{C6})$$

sn being a Jacobi elliptical function and provided that

$$p^2 = \sqrt{\frac{\lambda}{2}} \mu^2, \quad (\text{C7})$$

and μ and θ are arbitrary integration constants. Then, the equation for G_2 can be solved, maintaining the same approximation to neglect δm^2 in Eq. (C3) given by Eq. (C4), by taking for the homogeneous equation

$$\partial^2 G_{2h}(x, y) + 3 N g^2 \phi^2(x) G_{2h}(x, y) = 0, \quad (\text{C8})$$

and the solution can be written as (after a redefinition of the phase θ)³

²This can be checked by direct substitution and using the properties of the Jacobi elliptic functions.

³We are permitted to do so and this can also be seen approaching the problem differently by choosing $p = 0$ in Eq. (C6), this being an arbitrary parameter again, and solving in t instead, the presence of the phase being inessential for the solution of the homogeneous equation.

$$G_{2h}(x, y) = acn(p \cdot (x - y) + \theta, -1) dn(p \cdot (x - y) + \theta, -1), \quad (C9)$$

where cn and dn are Jacobian elliptic functions. One has

$$\frac{d}{dz} sn(z, i) = cn(z, i) dn(z, i) \quad (C10)$$

and

$$sn(z, i) = \frac{2\pi}{K(i)} \sum_{n=0}^{\infty} (-1)^n \frac{e^{-(n+\frac{1}{2})\pi}}{1 + e^{-(2n+1)\pi}} \sin \left((2n+1) \frac{\pi z}{2K(i)} \right). \quad (C11)$$

Such a solution can be cast in a Fourier series,

$$G_{2h}(x, y) = \frac{\pi^2}{K^2(i)} \sum_{n=0}^{\infty} (-1)^n (2n+1) \frac{e^{-(n+\frac{1}{2})\pi}}{1 + e^{-(2n+1)\pi}} \cos \left((2n+1) \frac{\pi}{2K(i)} (p \cdot (x - y) + \theta) \right). \quad (C12)$$

The choice $\theta = (4m+1)K(i)$ with $m \in \mathbb{Z}$ for the phase grants that the function $\phi_0(x)$ is zero on the light cone. Using the Laplace-Fourier transform

$$\tilde{G}_{2h}(\omega, \mathbf{x} - \mathbf{y}) = \int_0^{\infty} dt e^{-i(\omega+i\epsilon)t} G_{2h}(x, y), \quad (C13)$$

where $\epsilon > 0$ is a small quantity which is sent to zero in the end in order to obtain the correct (Feynman) integration path. After integration, we get

$$G_{2h}(\omega, \mathbf{x} - \mathbf{y}) = \frac{1}{2} \sum_{n=0}^{\infty} A_n \left[\frac{e^{i(2n+1)\frac{\pi}{2K(i)} p \cdot (x-y)}}{\left(\omega - (2n+1) \frac{\pi}{2K(i)} p_0 \right) + i\epsilon} - \frac{e^{-i(2n+1)\frac{\pi}{2K(i)} p \cdot (x-y)}}{\left(\omega + (2n+1) \frac{\pi}{2K(i)} p_0 \right) + i\epsilon} \right], \quad (C14)$$

where

$$A_n = \frac{\pi^2}{K^2(i)} (2n+1) \frac{e^{-(n+\frac{1}{2})\pi}}{1 + e^{-(2n+1)\pi}}. \quad (C15)$$

We can perform a Fourier transform in space and use the dispersion relation (C7) to obtain

$$G_{2h}(\omega, \mathbf{k}) = \frac{1}{2} \sum_{n=0}^{\infty} A_n \left[\frac{\delta^3(\mathbf{k} + (2n+1) \frac{\pi}{2K(i)} \mathbf{p})}{\left(\omega - (2n+1) \frac{\pi}{2K(i)} \sqrt{\mathbf{p}^2 + m^2} \right) + i\epsilon} - \frac{\delta^3(\mathbf{k} - (2n+1) \frac{\pi}{2K(i)} \mathbf{p})}{\left(\omega + (2n+1) \frac{\pi}{2K(i)} \sqrt{\mathbf{p}^2 + m^2} \right) + i\epsilon} \right]. \quad (C16)$$

Finally, we integrate out the arbitrary three-momentum p in order to get to the Feynman propagator (which is explicitly given only in full momentum space), using the covariant integration $\int d^3 p / 2E_p$ with $E_p = \sqrt{\mathbf{p}^2 + m^2}$, to obtain

$$G_{2h}(\omega, \mathbf{k}) = \frac{1}{4\sqrt{\mathbf{k}^2 / \left((2n+1) \frac{\pi}{2K(i)} \right)^2 + m^2}} \sum_{n=0}^{\infty} A_n \left[\frac{1}{\left(\omega - (2n+1) \frac{\pi}{2K(i)} \sqrt{\mathbf{k}^2 / \left((2n+1) \frac{\pi}{2K(i)} \right)^2 + m^2} \right) + i\epsilon} - \frac{1}{\left(\omega + (2n+1) \frac{\pi}{2K(i)} \sqrt{\mathbf{k}^2 / \left((2n+1) \frac{\pi}{2K(i)} \right)^2 + m^2} \right) + i\epsilon} \right]. \quad (C17)$$

Collecting the terms, we obtain back the formula presented in the main text. We see that the translation invariance is fully recovered for G_2 .

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