

Serendipitous syzygies of scattering amplitudes

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(Received 28 July 2025; accepted 8 September 2025; published 23 October 2025)

We study linear relations between color-ordered all-plus amplitudes at one loop in Yang-Mills theory. We show that on general grounds, there are $(n-1)!/2 - 2$ relations for $n \geq 5$, leaving only two independent color-ordered amplitudes. We present two complementary approaches to finding such relations; one using numerical linear algebra and the other using syzygies in computational algebraic geometry. We obtain explicit forms for all relations through $n = 7$. We also study relations for the tree-level maximally helicity violating amplitudes through $n = 8$. The latter relations include the well-known color and Bern-Carrasco-Johansson identities.

DOI: [10.1103/cypj-74sj](https://doi.org/10.1103/cypj-74sj)

I. INTRODUCTION

Scattering amplitudes in Yang-Mills theories are functions of the colors and momenta of the external particles. The amplitudes are most conveniently written as sums over products of color traces and color-ordered or partial amplitudes, separating the dependence on the different kinds of degrees of freedom. The underlying Lie symmetry then implies linear relations between different partial amplitudes with constant coefficients. The best known of these are the Kleiss-Kuijf (KK) identities [1], which reduce the number of independent tree-level all-gluon amplitudes to $(n-2)!/2$. Bern, Carrasco, and Johansson (BCJ) discovered an additional set of identities [2] for tree-level partial amplitudes, with coefficients linear in momentum invariants. The BCJ relations reduce the number of independent amplitudes to $(n-3)!$. Both kinds of relations are all independent of the helicity configuration of the external particles.

Looking beyond tree level, Bern, Dixon, Dunbar, and one of the authors found group-theory relations for general one-loop amplitudes. Edison and Naculich classified [3] all relations of purely group-theory origin to all loop orders for amplitudes with up to six external gluons. Dunbar studied [4] relations between two-loop all-plus amplitudes, with some all- n results. At one loop and beyond, nearly all

helicity amplitudes have branch cuts, making it more challenging to find linear relations beyond those implied by group theory. At one loop, however, there are two classes of nontrivial amplitudes that are still purely rational; the all-plus and the single-minus (along with their helicity conjugates). A decade ago, Bjerrum-Bohr, Damgaard, Johansson, and Søndergaard (BBDJS) [5] observed additional relations for these two classes of amplitudes. They motivated the relations via a diagrammatic representation for these amplitudes. Some of the BBDJS relations have constant coefficients, and some have coefficients polynomial in the invariants. More recently, Dixon and Morales (DM) [6] used an observation by Costello [7] to give an interpretation of the constant-coefficient BBDJS identities as having partly a group-theory origin, and also to give a proof of completeness. (Both BBDJS and DM also discuss the one-minus amplitudes, which contain multiparticle poles, and thus have a more complicated structure of relations.)

In this paper, we explore such relations more fully, focusing on those with nonconstant coefficients. We show in Sec. III that they must exist in general, by constructing a related set of functions with the same relations. We compute the full sets of relations for the five-, six-, and seven-point amplitudes, and isolate generating vectors for the five- and six-point relations. We present two different approaches to systematically searching for such relations. One is based on numerical linear algebra, presented in Sec. IV; the other uses factorization and computational algebraic geometry, and is presented in Secs. V and VI. We present results for the all-plus amplitudes in Sec. VII, and for the tree-level maximally helicity violating (MHV) amplitudes in Sec. VIII. We record an interesting observation about the all-plus relations in Sec. IX. We summarize in Sec. X.

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II. COLOR DECOMPOSITION

In Yang-Mills theories, we can write amplitudes in a color-decomposed form,

$$\mathcal{A}_n^{(L)} = \sum_{c=1}^{d_n(L)} A_{n;c}^{(L)} t_c^{(L)}, \quad (2.1)$$

where $d_n(L)$ is the dimension of the trace basis at L loops for n particles. We focus on pure gluon amplitudes with $SU(N)$ color group, but the approach is general. We adopt the notation of Edison and Naculich [3]. We will be interested in linear relations between the color-ordered amplitudes $A_{n;c}$.

Each element t_c of the trace basis has the form,

$$N^k [\text{Tr}(\alpha_1) \text{Tr}(\alpha_2) \dots \text{Tr}(\alpha_m) + (-1)^n \text{Tr}(\alpha_1^T) \text{Tr}(\alpha_2^T) \dots \text{Tr}(\alpha_m^T)], \quad (2.2)$$

where m is the number of color traces and k denotes the number of explicit factors of N (which can also be interpreted as traces containing the identity). The α_i are products of $SU(N)$ generators, which together form a partition of all generators corresponding to the external particles. At two loops or beyond, there are subleading-color contributions with fewer traces, so generally $k \neq L - m + 1$. The color-ordered amplitude $A_{n;c}^{(L)}$ corresponding to t_c can also be labeled as follows:

$$A_{n;c}^{(L)} = A_{n;n_1;\dots;n_m}^{(L,k)}(\alpha_1; \alpha_2; \dots; \alpha_m), \quad (2.3)$$

where $n_i = |\alpha_i|$. These amplitudes are invariant under exchange of any two of the sets α_i , as well as the reflection operation,

$$A_{n;n_1;\dots;n_m}^{(L,k)}(\alpha_1; \alpha_2; \dots; \alpha_m) = (-1)^n A_{n;n_1;\dots;n_m}^{(L,k)}(\alpha_1^T; \alpha_2^T; \dots; \alpha_m^T), \quad (2.4)$$

justifying the combination of both cyclic and anti-cyclic color traces in the basis elements in Eq. (2.2). In addition, each element of the trace basis and associated amplitude are invariant under cyclic permutations of each of the α_i . At one loop, we have leading-color terms, with single-trace color factors (and a corresponding factor of N); and subleading-color terms, with double-trace color factors. The subleading-color amplitudes, coefficients of the latter in Eq. (2.1), are related by color algebra relations [8] to the leading-color ones. Accordingly, in this article we shall consider only relations between leading-color amplitudes. Both at tree level and at one loop, cyclicity and (anti)reflection properties of the color traces leave us with $(n-1)!/2$ distinct permutations of the amplitude arguments to consider. We take this set to be the permutations σ on legs $2, \dots, n$ with $\sigma_2 < \sigma_n$. To organize

our calculations and results, let us order these permutations lexically.

III. SYZYGIES OF RATIONAL FUNCTIONS

We are interested in the existence of relations,

$$\sum_{\sigma \in S_n^*} P_n^{[r]}[\sigma] A_n(\sigma) = 0, \quad (3.1)$$

where the $P_n^{[r]}[\sigma]$ are polynomials of degree r in the Mandelstam invariants of the external particles, so that we may call these relations *syzygies*; and where S_n^* is the set of $n_A = (n-1)!/2$ distinct permutations. We fix the helicities of the external particles.

Our definition includes the known identities originating in the color algebra. In general, the polynomials for different permutations σ will be *different* functions of the invariants, and not merely the same function with arguments related by a relabeling of invariants' indices. This is similar to the case of the BCJ relations. Using the (lexical) ordering of the permutations, we can also write these relations as vectors of dimension n_A .

How many identities should we expect? Consider, for example, the set of $(n-1)!/2$ all-plus amplitudes [9] given by permutations of,

$$A_{n;1}(1, 2, \dots, n) = \frac{i}{48\pi^2} \frac{H_n}{\langle 12 \rangle \langle 23 \rangle \dots \langle (n-1)n \rangle \langle n1 \rangle}, \quad (3.2)$$

where

$$H_n = - \sum_{1 \leq i_1 < i_2 < i_3 < i_4 \leq n} \langle i_1 i_2 \rangle [i_2 i_3] \langle i_3 i_4 \rangle [i_4 i_1]. \quad (3.3)$$

We can freely multiply each element of the set by the same function, and retain the same identities. Start by multiplying by the common spinor denominator; this is the product of angle products of all pairs,

$$\prod_{i < j}^n \langle i j \rangle. \quad (3.4)$$

The expressions we are considering are now polynomials in the spinor products. We can multiply them by a factor chosen to remove the spinor phase weight, such as

$$\frac{1}{[12][23] \dots [(n-1)n][n1]} \prod_{i < j}^n [i j], \quad (3.5)$$

which is also a polynomial in the spinor products. The resulting expressions have no spinor phase weight, and accordingly can be reexpressed in terms of invariants s_{ij}

and contracted Levi-Civita tensors $\epsilon_{ijkl} = \epsilon_{\mu\nu\lambda\rho} p_i^\mu p_j^\nu p_k^\lambda p_l^\rho$ by organizing the spinors into traces, and reducing those traces.

Next, as we are working in four spacetime dimensions, when we have five or more external particles, we can choose a basis of four momenta (for example, k_1, k_2, k_3, k_4). We can express every contracted Levi-Civita tensor in terms of that of the basis vectors (ϵ_{1234} for the example) by expanding its momenta in terms of the basis. The coefficients of that basis are ratios of Gram determinants, which in turn are polynomials in the invariants. The Gram determinant G is defined by

$$G \begin{pmatrix} p_1, \dots, p_m \\ q_1, \dots, q_m \end{pmatrix} = \det_{i,j} (2p_i \cdot q_j). \quad (3.6)$$

(When both rows are identical, we write a single list of arguments.)

The coefficients' denominator is the Gram determinant of the basis vectors, $G(1234)$. Multiply by up to four powers of this Gram determinant to clear the newly introduced denominators. We finally have a set of polynomial expressions, each in the form,

$$\text{Poly}_a(\{s_{ij}\}) + \text{Poly}_b(\{s_{ij}\})\epsilon_b, \quad (3.7)$$

which have exactly the same identities as the original set of amplitudes. (In this expression, ϵ_b is the Levi-Civita tensor of the basis momenta.) It is now easy to see that any triplet of expressions, and hence any triplet of amplitudes (A_1, A_2, A_3), must have a polynomial identity. We can see this by constructing it explicitly. Define,

$$\begin{aligned} \xi_1 &= \text{Poly}_{3,a} \text{Poly}_{2,b} - \text{Poly}_{2,a} \text{Poly}_{3,b}, \\ \xi_2 &= \text{Poly}_{1,a} \text{Poly}_{3,b} - \text{Poly}_{3,a} \text{Poly}_{1,b}, \\ \xi_3 &= \text{Poly}_{2,a} \text{Poly}_{1,b} - \text{Poly}_{1,a} \text{Poly}_{2,b}. \end{aligned} \quad (3.8)$$

Then (ξ_1, ξ_2, ξ_3) is a syzygy, that is,

$$\xi_1 A_1 + \xi_2 A_2 + \xi_3 A_3 = 0. \quad (3.9)$$

Each triplet of amplitudes will have an identity, and m amplitudes will have $m - 2$ identities, so that we have two independent amplitudes. In the four-point case, the Levi-Civita tensor vanishes identically, so that we have two identities for the three independent amplitudes, and one independent amplitude.

The *existence* of syzygies for rational amplitudes is thus not surprising. Indeed, as the above argument shows, any collection of rational functions of spinors with the same spinor phase weight will exhibit such relations. For generic functions of the same engineering dimensions of numerators and denominators as amplitudes, we would expect the

Poly_{*i,x*} to be of degree $\mathcal{O}(n^2)$ (or higher), and hence the polynomials in a relation (3.1) to be of degree $\mathcal{O}(n^4)$ or higher. What is surprising is the low polynomial degree at which they arise, and at least for the all-plus amplitude, the simple structure of the coefficients. As already shown by BBDJS, there are relations of degree zero and two for the all-plus amplitudes. For this specific helicity amplitude these relations turn out to exhaust all possible relations [6].

It is possible to search for syzygies by parametrizing them order by order in the degree of invariants, and solving for coefficients numerically (or showing that no solution exists). We discuss this brute-force approach in the next section. What the general argument above fails to consider are the special properties of amplitudes, in particular factorization. We discuss how to derive equations relying on factorization in Sec. V, and how to solve the resulting equations in subsequent sections.

IV. FINDING SYZYGIES NUMERICALLY

The simplest approach to finding relations between a given vector of amplitudes is via linear algebra performed numerically. For example, to find constant (degree-zero) relations, we can evaluate the vector of n_A amplitudes on n_A different momentum configurations, obtaining a $n_A \times n_A$ matrix V of values. It is important to do so using exact arithmetic—that is, either rational or finite-field momentum configurations—to avoid errors due to floating-point arithmetic in large matrices. It is straightforward to generate (complex) rational momentum configurations using the momentum-twistor approach described in Appendix E of Ref. [10]. As we have sampled the vector of amplitudes n_A times, the only nontrivial relations of the numeric vectors are those inherited from the original amplitudes. Thus, we can recover the set of degree-zero relations by simply computing the null space $\ker V$. The number of linearly independent relations between the vector of amplitudes is given by $\dim \ker V = \dim V - \text{rank } V$.

The momentum twistor parametrization of Ref. [10] represents the space of n massless on shell momenta in a momentum-conserving configuration using $3n - 10$ parameters z_i . The parametrization implicitly fixes the freedom associated with little group transformations, the spinor phase weights, in an arbitrary way. As the amplitudes have a nonzero spinor phase weight, each numerical sample of the vector will have a different phase associated to it. To obtain relations from numerical evaluations, all evaluations should have the same phase associated to them. We can achieve this by multiplying all amplitudes by the same overall factor to absorb this phase information. We define the adjusted vector of amplitudes,

$$\begin{aligned} &(\hat{A}_n(\sigma^{(1)}), \hat{A}_n(\sigma^{(2)}), \dots, \hat{A}_n(\sigma^{(n_A)})) \\ &= \phi(1, \dots, n)(A_n(\sigma^{(1)}), A_n(\sigma^{(2)}), \dots, A_n(\sigma^{(n_A)})), \end{aligned} \quad (4.1)$$

where the function ϕ is chosen such that the $\hat{A}$ s are free of spinor phase weight. In the case of all-plus amplitudes we can choose,

$$\phi(1^+, \dots, n^+) = \langle 12 \rangle \langle 23 \rangle \dots \langle n1 \rangle, \quad (4.2)$$

while for MHV amplitudes with the two negative helicities $m_{1,2}$ and remaining indices I_+ , we can choose

$$\phi(1, \dots, n) = [m_1 m_2]^2 \prod_{i \in I_+} \langle i \text{ succ}(i) \rangle. \quad (4.3)$$

(succ(i) is the succeeding element of I_+ , understood cyclicly.) As we are modifying all amplitudes by the same overall factor, any relation we obtain for the vector of $\hat{A}$ s is also a relation for the original vector of amplitudes.

Amplitudes over real kinematics are typically complex. Using a parametrization based on momentum twistors allows us to restrict the values of amplitudes. In particular, amplitudes which are rational functions of spinors, like those we are considering here, will take on rational values¹ for rational values of the parameters z_i ; $A_n(z_i) \in \mathbb{K}$ when $z_i \in \mathbb{K}$ for some field $\mathbb{K}$. To generate the matrix $V \in \mathbb{K}^{n_A \times n_A}$, we evaluate the vector of $\hat{A}$ s on n_A random sets of $\{z_1, \dots, z_{3n-10}\} \in \mathbb{K}^{3n-10}$. If we are interested in obtaining the explicit expressions for the relations and the dimension n_A is small—up to $\mathcal{O}(2000)$ —we can use $\mathbb{K} = \mathbb{Q}$. If we are instead interested just in counting the number of relations or are dealing with larger dimensions, say n_A of $\mathcal{O}(10000)$, we instead use a finite field $\mathbb{K} = \mathbb{F}_p$ for some suitably large prime for greater computational efficiency. In the latter case we would have to reconstruct the explicit (rational) relations from evaluations over a sufficient number of different prime fields $\mathbb{F}_p$. To obtain the null space of the matrix V over such a finite field, we use the C++ library **FFLAS – FFPACK** [11], which provides parallelized functions for modular linear algebra operations, and performs significantly better than the corresponding *Mathematica* built-in functions.²

We also want to go beyond degree-zero relations, and allow for factors of Mandelstam invariants multiplying the color-ordered amplitudes. We take the usual invariants,

$$s_{i_1 \dots i_k} = \left(\sum_{j=1}^k p_{i_j} \right)^2, \quad (4.4)$$

¹We divide out an overall factor of i .

²Due to technical constraints, the library is limited to finite fields with 32-bit primes. However, we observed significantly reduced efficiency close to this limit, consistent with the expected increase in modular operations. For this reason we perform computations over fields with $p = 2^{24} - 3 = 16,777,213$ or nearby primes, which provides a balance between performance and a sufficiently sized prime required for reconstructing rational numbers from the finite field values.

and consider the ring of polynomials in the invariants with coefficients over the field $\mathbb{K}$ (which will again be either $\mathbb{Q}$ or $\mathbb{F}_p$). Momentum conservation and Gram-determinant constraints,

$$G \begin{pmatrix} i_1 & i_2 & i_3 & i_4 & i_5 \\ j_1 & j_2 & j_3 & j_4 & j_5 \end{pmatrix} = 0, \quad (4.5)$$

reflecting the four-dimensionality of momenta, lead to relations between polynomials in this ring. (We have abbreviated the notation here by using the indices of the momenta as arguments to the Gram determinant.)

In order to ensure that the relations we find are indeed syzygies of the amplitudes and not a result of identities between invariants, we should establish a linearly independent basis $\mathcal{B}_m^{(n)}$ of monomials of any given degree m . For general n , a degree $m = 1$ basis is given by

$$\mathcal{B}_1^{(n)} = \{s_{12}, \dots, s_{n1}\} \cup \bigcup_{i=2}^{n-4} \{s_{1, \dots, 1+i}, \dots, s_{n-1-i, \dots, n-1}\}. \quad (4.6)$$

This basis has

$$|\mathcal{B}_1^{(n)}| = n + \sum_{i=2}^{n-4} n - i - 1 = \frac{n(n-3)}{2} \quad (4.7)$$

elements. This basis is valid in general spacetime dimension, and hence has more degrees of freedom than the usual $3n - 10$ of a four-dimensional momentum-conserving configuration of n massless momenta. The difference is accounted for by the $\frac{1}{2}(n-4)(n-5)$ Gram determinant relations. These relations are of degree five, and so are irrelevant for any analysis with $m \leq 4$.

For monomials of degree m , we can generate a basis $\mathcal{B}_m^{(n)}$ by taking symmetric products of the corresponding $m = 1$ basis,

$$\mathcal{B}_m^{(n)} = \{s_1 \cdots s_m | s_i \in \mathcal{B}_1^{(n)}\}, \quad (4.8)$$

which gives us

$$|\mathcal{B}_m^{(n)}| = \frac{(|\mathcal{B}_1^{(n)}|)_m}{m!} = \frac{1}{m!} \left(\frac{n(n-3)}{2} \right)_m \quad (4.9)$$

unique monomials. Here $(x)_n = x(x+1)\dots(x+n-1)$ is the Pochhammer symbol. For $n \geq 6$ and $m \geq 5$ this basis of monomials is overcomplete, and requires the imposition of the Gram-determinant relations. Once this is done, we are left with a reduced basis, which we denote by $\tilde{\mathcal{B}}_m^{(n)}$. Just as for the amplitudes, we can evaluate these bases of monomials in terms of the same momentum twistor parameters already used for the vector of amplitudes.

In order to obtain syzygies with polynomial coefficients of degree m numerically, we form the tensor product of the vector of adjusted amplitudes and the basis of degree- m invariants,

$$T_m = \tilde{\mathcal{B}}_m^{(n)} \otimes (\hat{A}_n(\sigma^{(1)}), \hat{A}_n(\sigma^{(2)}), \dots, \hat{A}_n(\sigma^{(n_A)})), \quad (4.10)$$

of dimension $n_{Am} = n_A \times |\tilde{\mathcal{B}}_m^{(n)}|$. We then proceed as before, evaluating T_m over n_{Am} numerical samples to obtain a sample matrix V . We then find the null space $\ker V$ to obtain the set of relations between the set of amplitudes times invariants. If we are only interested in the relations themselves we can stop here. However, we would ordinarily want to know the number of linearly independent relations as well. This number does *not* correspond to the matrix rank of V as just computed. Instead, we have to turn the null space $\ker V$ into relations between the vector of $\hat{A}_s$,

$$(\ker V) \cdot T_m = M \cdot (\hat{A}_n(\sigma^{(1)}), \hat{A}_n(\sigma^{(2)}), \dots, \hat{A}_n(\sigma^{(n_A)})), \quad (4.11)$$

where the matrix elements of M are now polynomials of degree m in the invariants. The rank of M then gives us the number of independent syzygies with coefficients of degree m . These syzygies fall into two categories. There are relations which are genuinely of degree m , and cannot be expressed as linear combinations of lower-degree syzygies. The matrix M also contains all syzygies of lower degree, multiplied by polynomials raising their degree to m . We can call the latter *inherited* syzygies. We can separate the inherited from the true relations by computing the quotient space of degree m relations by all relations with degree $m' < m$.

The numerical approach just described has some obvious limitations. It has diminishing performance in both runtime and especially in memory usage with increasing dimension n_{Am} , scaling roughly as $\mathcal{O}(n_{Am}^3)$ and $\mathcal{O}(n_{Am}^2)$ respectively because of the underlying linear algebra. In practice, using $\mathbb{K} = \mathbb{F}_{16777213}$ we are able to work out cases with $n_{Am} \sim \mathcal{O}(100000)$. Furthermore, the resulting relations are generally unstructured, as they are semirandomly chosen through the null space computation, with additional mixing of inherited and true relations. However, a significant benefit lies in its simplicity; very little additional information is required aside from deriving a suitable rational function representation of the underlying amplitudes and invariants, at which point it is agnostic to any additional structure. This makes it very versatile in quickly testing for relations for different sets of (rational) amplitudes, such as tree-level MHV as well as one-loop all-plus amplitudes. Furthermore, it is well-suited to obtaining the number of independent amplitudes at any specified polynomial degree in the invariants (or equivalently the number of independent relations). We display the counting of syzygies and of

TABLE I. Number of syzygies of different degrees (syz) and of resulting independent amplitudes (A) after imposition of all syzygies up to the given degree, for the one-loop all-plus amplitude.

n	Degree A	0		1		2	
		Syz	A	Syz	A	Syz	A
4	3	2	1	...	1	...	1
5	12	6	6	...	6	4	2
6	60	25	35	...	35	33	2
7	360	135	225	...	225	223	2

TABLE II. Number of syzygies of different degrees (syz) and of resulting independent amplitudes (A) after imposition of all syzygies up to the given degree, for the tree-level MHV amplitude. The “0” column corresponds to the Kleiss-Kuijff relations, while the “1” column corresponds to the BCJ identities.

n	Degree A	0		1		2	
		Syz	A	Syz	A	Syz	A
4	3	1	2	1	1	...	1
5	12	6	6	4	2	...	2
6	60	36	24	18	6	4	2
7	360	240	120	96	24	22	2
8	2520	1800	720	600	120	118	2

independent amplitudes for the one-loop all-plus and the tree-level MHV at various degrees in Tables I and II. Curiously, we observe that in both cases a sufficient number of relations quadratic in Mandelstams exist to reduce all partial amplitudes to just two independent ones, i.e., the minimal number allowed. In the case of the all-plus we have verified this up to seven and in the MHV case up to eight points. In all cases we have exact—i.e., $\mathbb{K} = \mathbb{Q}$ —expressions for these relations. We provide them in the form of generators over permutations, as discussed in Sec. VI.

V. SYZYGIES FROM FACTORIZATION

The amplitudes in the sum in Eq. (3.1) will have poles in the invariants. On each pole, the amplitudes that have a pole will factorize into a product of lower-point amplitudes. For the all-plus amplitude, poles arise only at spinor invariants $\langle ij \rangle$. Examine the residue at such a pole,

$$\sum_{\sigma \in S_n^*} \text{Res}_{\langle ij \rangle=0} P_n^{[r]}[\sigma] A_n(\sigma) = 0. \quad (5.1)$$

[The set S_n^* is defined below Eq. (3.1).] A given amplitude $A(\sigma)$ will have a pole at $\langle ij \rangle$ only if i and j are (cyclicly) adjacent in the permutation σ . Denote this by $(ij) \in \sigma$; and denote the subset of a set of permutations S which contain ij cyclicly adjacent by $F_{ij}(S)$,

$$F_{ij}(S) = \{\rho \in S | (ij) \in \rho\}. \quad (5.2)$$

Amplitudes without a pole will have zero residue.

Taking the residue in Eq. (5.1), and dropping overall factors, we obtain,

$$\sum_{\sigma \in B_{ij}} P_n^{[r]}[\sigma] \tau_\sigma^{(ij)} \omega_\sigma^{(ij)} A_{n-1}(\check{\sigma}) A_3((-P)^{-i^+ j^+}) = 0. \quad (5.3)$$

In this equation, $B_{ij} = F_{ij}(S_n^*)$; $\check{\sigma}$ denotes the reduction of σ to a permutation of the $n-1$ symbols $\{1, 2, i-1, P, j+1, \dots, n\}$ obtained by replacing “ ij ” by “ P ” and using cyclic symmetry and reflection (anti)symmetry to put it into canonical form; $\tau_\sigma^{(ij)}$ denotes the sign that may arise from putting the three-point amplitude into the given form from the original ordering of ij in σ ; and $\omega_\sigma^{(ij)}$ denotes the sign that may arise from reducing σ to $\check{\sigma}$ and putting the latter into canonical form. (Recall that we use a canonical form for σ where 1 always appears first, and where $\sigma_2 < \sigma_n$.)

To lighten the notation, define

$$\zeta_\sigma^{(ij)} = \tau_\sigma^{(ij)} \omega_\sigma^{(ij)}. \quad (5.4)$$

The factor of $A_3((-P)^{-i^+ j^+})$ is common to all terms, and nonvanishing for generic momenta, so we can divide it out. We are left with the following equation:

$$\sum_{\sigma \in B_{ij}} \zeta_\sigma^{(ij)} P_n^{[r]}[\sigma] A_{n-1}(\check{\sigma}) = 0. \quad (5.5)$$

In general, there are two syzygy entries $P_n^{[r]}$ that appear in the coefficient of each $(n-1)$ -point amplitude $A_{n-1}(\check{\sigma})$, so that we can rewrite our equation as

$$\sum_{\xi \in \check{B}_{ij}} \left(\sum_{\sigma | \check{\sigma} = \xi} \zeta_\sigma^{(ij)} P_n^{[r]}[\sigma] \right) A_{n-1}(\xi) = 0, \quad (5.6)$$

where the notation “ $\sigma | \check{\sigma} = \xi$ ” denotes all permutations σ for which the reduced permutation $\check{\sigma}$ is ξ , and where we denote the set of $\check{\sigma}$ corresponding to $\sigma \in B_{ij}$ by $\check{B}_{ij}$. There are two permutations σ for every reduced permutation $\check{\sigma}$.

As we are proceeding recursively, we assume that we know all syzygies for the $(n-1)$ -point amplitudes. That is, we know all the solutions to Eq. (3.1) for $(n-1)$ -point amplitudes. These solutions are expressed in terms of $(n-1)$ -point invariants. Equation (5.6) tells us that on the pole, the given sum of $P_n^{[r]}$ s must be a sum of $(n-1)$ -point syzygies,

$$\sum_{\sigma | \check{\sigma} = \xi} \zeta_\sigma^{(ij)} P_n^{[r]}[\sigma] |_{\langle ij \rangle \text{ pole}} = \sum_{(t,v) \in Z_{ij}} f_{t,v}^{(ij)} \tilde{P}_{n-1|v}^{[t]}[\xi], \quad \forall \xi \in \check{B}_{ij}, \quad (5.7)$$

where Z_{ij} denotes the set of indices (t, v) of the known syzygies $\tilde{P}$ of the $(n-1)$ -point amplitudes ($t \leq r$). The first index denotes the degree of the syzygy, and the second runs over all syzygies of the same degree. This equation represents a set of equations indexed by $\xi \in \check{B}_{ij}$, as the polynomials come in sets corresponding to a set of scattering amplitudes. This is reflected in the fact that the coefficients $f_{t,v}$ are independent of ξ .

We seek the solution to the simultaneous sets of equations for all ij [of which there are of course $n(n-1)/2$ choices at n points]. We are interested in solutions in terms of n -point invariants, away from the poles as well as on the poles.

In order to solve the equations, we must first extend the equations to general kinematics. This poses the first of several subtleties we must confront. We can promote the $(n-1)$ -point invariants to n -point invariants by replacing the momentum k_P by the sum $k_i + k_j$, but we must also allow for terms that vanish on the pole and hence do not contribute to the residue. Given a set of polynomials Y_{ij} which vanish on the pole, we could extend Eq. (5.7) by adding such polynomials,

$$\sum_{\sigma | \check{\sigma} = \xi} \zeta_\sigma^{(ij)} P_n^{[r]}[\sigma] = \sum_{(t,v) \in Z_{ij}} f_{t,v}^{(ij)} \tilde{P}_{n-1|v}^{[t]}[\xi] + \sum_{g \in G_{ij}} c_g^{(ij)}[\xi] g, \quad \forall \xi \in \check{B}_{ij}, \quad (5.8)$$

where G_{ij} is the Gröbner basis of Y_{ij} , which gives the complete set of independent polynomials we could add.

The relevant kinematics are *complex* (or holomorphic) collinear kinematics, with $|i\rangle||j\rangle$, rather than ordinary collinear kinematics, with $k_i||k_j$ with $|i\rangle|||j\rangle$ as well. We do not require the invariants to be real, and the computational algebraic geometry tools we shall employ implicitly take them to be complex.

The only invariant which vanishes in the complex collinear kinematics we are using is the pole invariant s_{ij} . This is problematic, because it does not fully parametrize the space of (spinorial) functions which vanish on the pole. As a result, using $G_{ij} = Y_{ij} = \{s_{ij}\}$ will not give us a complete set of solutions. As we discuss in the next section, this is not a problem as long as we obtain at least a few solutions. As we explain there, we can generate all solutions from a limited number of generators.

An alternative would be to extend the set of polynomials to include all those which vanish for ordinary collinear kinematics. In addition to s_{ij} , this set of polynomials consists of Gram determinants,

$$G \begin{pmatrix} k_i & k_j \\ k_a & k_b \end{pmatrix}, \quad (5.9)$$

for all pairs (a, b) . These are not all independent, however that lack of independence is taken care of by computing the

Gröbner basis. If we use this extended set, we now parametrize the required kinematics away from the pole, at the price of more complicated equations and the possibility of spurious “solutions” which we would need to prune later. If we are considering solutions of degree five or higher, we must also include Gram determinant constraints enforcing the four-dimensionality of momenta.

The set of equations we would have in this case would have the form of syzygy equations if the coefficients f and c were themselves polynomial. This turns out to be true for some n -point syzygies, but not necessarily for all. Were we to take f and c to be polynomial, we would still obtain solutions, but they would not all be of minimal degree. For the f coefficients, we can proceed as follows. The coefficients $f_{(0,v)}$ of degree-zero syzygies of the $(n-1)$ -point amplitudes necessarily appear with constant coefficients, and we can simply solve for them to eliminate them. This would anyway be good practice, as it would simplify the set of equations. For the coefficients $f_{(t>0,v)}$, we can eliminate them, again using tools from computational algebraic geometry (as implemented, for example, in the `Singular` function `eliminate`). This elimination increases the number of lower-degree solutions we obtain, but at the cost of increasing the computational complexity of solving the equations, because the polynomial coefficients of the resulting equations are typically of much higher degree. The c coefficients cannot necessarily be eliminated, however. They start out each appearing only once, and their elimination would swallow equations.

The equations either before or after elimination of the f coefficients can be solved using standard computational algebraic geometry tools, for example using `Singular`’s `syz` function. This yields a set of candidate solutions. We still need to impose the original Eq. (3.1) on the pole at infinity (if any). While one can do this analytically by using a BCFW shift [12] and taking the large-parameter limit, it is easiest and most practical to do this numerically. The number of candidate solutions is typically between the number of final solutions and a small multiple thereof, so that the size of numerical matrices one must handle is much more modest than in the purely numerical approach of Sec. IV.

A somewhat different approach to solving the equations solves them for a subset of syzygy-vector components (“dependent”) in terms of other components (“independent”) and terms proportional to f and c functions. These solutions are then supplemented by constraint equations involving only invariants alongside f and c functions. The independent syzygies parametrize the different degree-zero solutions. We can obtain them by setting all f and c functions to zero. We could again treat the constraint equations as syzygy equations, solving for f and c polynomials. It turns out that in some cases, if we simply solve the constraint equations for a subset of c functions, we obtain rational solutions for these functions, but the

equations for the syzygy-vector components remain polynomial after substitution of the values for the cs . When this holds, the remaining c and f functions parametrize vectors whose linear combinations (with the cs themselves taken to be linear in the invariants) are true syzygies.

All the solution approaches above leave the question of determining whether the solutions obtained are in fact of minimal degree or not. In the case of the all-plus amplitudes, we find that all solutions (through seven points) are either of degree zero or degree two. Hence, we expect only degree-two solutions beyond the set of degree-zero syzygies found by BBDJS and DM. It is straightforward to rule out degree-one solutions numerically, showing that the degree-two ones are in fact of minimal degree. More generally, one could in principle perform a so-called *primary decomposition* to either establish minimality, or to reduce solutions to those of minimal degree. In practice, however, available algorithms (for example, `Singular`’s `GTZmod`) are too slow to use for reduction, though they may be adequate to prove minimality.

VI. SOLUTION GENERATORS

Let us suppose that we have some solutions to the syzygy equation (3.1), but not a complete set. Can we generate additional solutions?

Define the following operation on a vector $v[\sigma](s_{ij})$ of polynomials in the s_{ij} , where the vector is indexed by the permutation σ ,

$$\text{Perm}[\rho](v[\sigma](s_{ij})) = \omega_{\rho^{-1} \circ \sigma} v[C(\rho^{-1} \circ \sigma)](s_{\rho(ij)}). \quad (6.1)$$

Here ω_σ is the sign (if any) that arises from putting σ into canonical form, and C is the combination of cyclic transformations and possibly a reflection required to put the permutation into canonical form. In *Mathematica* notation, given the fixed set of momentum labels A , the action of a permutation ρ on a subset $a \subset A$ is given by

$$\rho(a) = a/.Thread[A \rightarrow A[[\rho]]], \quad (6.2)$$

while the composition of permutations $[\rho_1 \circ \rho_2(a) = \rho_1(\rho_2(a))]$, which in *Mathematica* notation is `PermutationProduct[\rho_2, \rho_1]` is given by

$$\rho_1 \circ \rho_2 = \rho_1[[\rho_2]]. \quad (6.3)$$

Then,

$$\text{Perm}[\rho](P_n^{[r]}[\sigma]) \quad (6.4)$$

is also a solution.

As an example, consider a syzygy vector v ,

$$v = (v_1(s_{12}, s_{23}), v_2(s_{12}, s_{23}), v_3(s_{12}, s_{23})), \quad (6.5)$$

of a vector of four-point amplitudes

$$(A(1234), A(1243), A(1324)). \quad (6.6)$$

Then for example the permutation $\rho = (4, 2, 3, 1)$ generates the equally valid syzygy

$$\text{Perm}[\rho](v) = (v_3(s_{13}, s_{23}), v_2(s_{13}, s_{23}), v_1(s_{13}, s_{23})). \quad (6.7)$$

As the number of legs is even, there is no relative sign ω required in this case.

VII. SYZYGIES OF ALL-PLUS AMPLITUDES

Let us apply the techniques discussed in previous sections to the leading-color all-plus amplitude [9] in pure Yang-Mills theory. For the four-point amplitude, we have three permutations organized as a vector,

$$S_4^* = ((1\ 2\ 3\ 4), (1\ 2\ 4\ 3), (1\ 3\ 2\ 4)). \quad (7.1)$$

As expected from the general arguments given in Sec. III, we have two basis syzygies, both of degree zero,

$$\mathcal{S}_4^{(+)} = \langle (1, -1, 0), (1, 0, -1) \rangle. \quad (7.2)$$

These provide the initial conditions for an inductive calculation (or equivalently the stopping conditions for a recursive one). The order of components in each element of $\mathcal{S}_4^{(+)}$ follows Eq. (7.1):

$$v \cdot A(S_n^*) = \sum_{j=1}^{|S_n^*|} v_j A((S_n^*)_j), \quad (7.3)$$

for any syzygy $v \in \mathcal{S}_n^{(+)}$.

For the five-point amplitude, there are twelve permutations to consider,

$$\begin{aligned} S_5^* = & ((1\ 2\ 3\ 4\ 5), (1\ 2\ 3\ 5\ 4), (1\ 2\ 4\ 3\ 5), (1\ 2\ 4\ 5\ 3), (1\ 2\ 5\ 3\ 4), (1\ 2\ 5\ 4\ 3), \\ & (1\ 3\ 2\ 4\ 5), (1\ 3\ 2\ 5\ 4), (1\ 3\ 4\ 2\ 5), (1\ 3\ 5\ 2\ 4), (1\ 4\ 2\ 3\ 5), (1\ 4\ 3\ 2\ 5)). \end{aligned} \quad (7.4)$$

As expected from the general arguments in Sec. III, we have ten relations. These turn out to be six degree-zero relations, as found by BBDJS and DM, and four degree-two relations, as also found by BBDJS. They can be generated using Eq. (6.1) from just two relations, one of degree zero, and one of degree two:

$$\begin{aligned} & (0, 0, 0, 0, 0, 0, 1, 1, -1, -1, 1, -1), \\ & (0, 0, 0, Q_1(1, 2, 3, 4, 5), Q_1(3, 4, 5, 1, 2), 0, 0, Q_1(3, 2, 1, 5, 4), Q_1(5, 1, 2, 3, 4), 0, \\ & Q_1(2, 3, 4, 5, 1), 0), \end{aligned} \quad (7.5)$$

where

$$Q_1(i_1, \dots, i_5) = s_{i_1 i_2} s_{i_2 i_3} + s_{i_1 i_5} s_{i_2 i_3} + s_{i_1 i_2} s_{i_3 i_4} - s_{i_1 i_5} s_{i_3 i_4} - s_{i_2 i_3} s_{i_4 i_5} - s_{i_3 i_4} s_{i_4 i_5}. \quad (7.6)$$

Although we have not searched exhaustively, these generating solutions are likely to be minimal both in number and complexity. In generating the complete set of syzygies from these two, it is important to use all permutations of the five external legs, and not just those in Eq. (7.4). The complete set of syzygies is given in the auxiliary *Mathematica* file `syzygies5 – ap.m`.

For the six-point amplitude, the 60 permutations are given in the auxiliary file. Here, the basic factorization equations do not yield the lowest-degree form of all the degree-two syzygies; but we can generate them from a small set of vectors. For this amplitude, we find 58 syzygies in total, as expected from Sec. III. We find the 25 degree-zero syzygies

found by BBDJS and DM, as well as 33 additional degree-two syzygies (one more than reported by BBDJS).

The number of required generators depends on whether we want to generate the degree-zero relations directly with constant coefficients, or whether we are content to generate a set of degree-two identities whose linear combinations are multiples of the degree-zero ones. In the latter case, one degree-two generator suffices; in the former case, we need two degree-zero generators. The latter-case degree-two generator is not the simplest one possible, but it is the simplest single generator we have found. (There is a simpler generator that yields 15 relations.) All these generators, along with the complete set of relations, are given in the auxiliary

Mathematica file `syzygies6 – ap.m`. The results given there are unlikely to be in the simplest form possible.

In the case of the seven-point amplitude, we are able to find a single degree-two generator using the numerical linear algebra method of Sec. IV. Using all permutations of this generator in Eq. (6.1) yields all 358 relations, including [invariant-dependent multiples of] the degree zero ones. We give this generator and the complete set of relations in an auxiliary *Mathematica* file `syzygies7 – ap.m`.

VIII. SYZYGIES OF TREE-LEVEL MHV AMPLITUDES

The all-plus one-loop leading-color amplitude is special in that the only complex factorization channels it has are at zeros of $\langle ij \rangle$ products. There is another amplitude that shares this property; the tree-level MHV amplitude. This means that its syzygies also satisfy Eq. (5.3) and hence Eq. (5.6). The difference is in the initial conditions; the

four-point amplitude has only one degree-zero relation, the photon decoupling identity,

$$(1, 1, 1), \quad (8.1)$$

while the second identity is the simplest BCJ identity,

$$(0, s_{12}, -s_{23}). \quad (8.2)$$

The process of solving the equations is similar, however. For the five-point amplitudes, we expect no additional relations beyond the KK and BCJ ones, and indeed do not find any. For the six-point amplitudes, we ultimately find four additional degree-two relations beyond the KK and BCJ ones, for a total of 58 relations (as also found by BBDJS). These relations are given in the auxiliary *Mathematica* file `syzygies6 – mhv.m`. The degree-two relations are generated by a simple vector $v^{(6)}$, with

$$v_1^{(6)} = s_{16}s_{45}, \quad v_2^{(6)} = s_{15}(s_{14} + s_{46}), \quad v_5^{(6)} = s_{14}s_{15}, \quad v_6^{(6)} = s_{14}(s_{15} + s_{45}), \quad v_{17}^{(6)} = s_{12}s_{14}, \quad (8.3)$$

and remaining components zero. Amusingly, the same vector generates all 58 independent relations, though not in their simplest form.

We similarly find generators of degree-two relations for the seven- and eight-point amplitudes. They are vectors $v^{(7)}$ and $v^{(8)}$ with nonzero entries,

$$\begin{aligned} v_7^{(7)} &= s_{17}(s_{14} + s_{46} + s_{47}), & v_9^{(7)} &= v_{15}^{(7)} = s_{17}(s_{14} + s_{47}), & v_{10}^{(7)} &= v_{16}^{(7)} = s_{14}s_{17}v_{17}^{(7)} = s_{15}s_{47}, \\ v_{18}^{(7)} &= s_{14}(s_{17} + s_{57}), & v_{31}^{(7)} &= s_{17}s_{24}, \end{aligned} \quad (8.4)$$

and

$$\begin{aligned} v_1^{(8)} &= s_{67}(s_{15} + s_{56} + s_{57} + s_{58}), & v_7^{(8)} &= s_{67}(s_{15} + s_{57} + s_{58}), & v_9^{(8)} &= s_{67}(s_{15} + s_{58}), \\ v_{10}^{(8)} &= v_{52}^{(8)} = v_{62}^{(8)} = v_{65}^{(8)} = s_{67}s_{15}, & v_{49}^{(8)} &= s_{67}(s_{15} + s_{57} + s_{58}) + s_{46}s_{57}, \\ v_{51}^{(8)} &= s_{67}(s_{15} + s_{58}), & v_{55}^{(8)} &= s_{47}s_{56}, & v_{61}^{(8)} &= s_{67}(s_{15} + s_{58}), & v_{265}^{(8)} &= s_{67}s_{25}. \end{aligned} \quad (8.5)$$

Just as in the six-point case, these vectors generate all respectively 358 and 2518 relations between the tree-level seven- and eight-point MHV partial amplitudes, including those of degree-zero and degree-one.

We provide all syzygies as well as the generators above for six-, seven- and eight-point amplitude relations in the Supplemental Material [13] `syzygies6 – mhv.m`, `syzygies7 – mhv.m`, and `syzygies8 – mhv.m`.

IX. AN OBSERVATION

We can define a complete symmetrization operation on syzygies of n -point amplitudes,

$$\text{Sym}(v[\sigma](s_{ij})) = \sum_{\rho \in S_n} \text{Perm}[\rho](v[\sigma](s_{ij})). \quad (9.1)$$

For the four- and five-point all-plus amplitudes, we find that,

$$\text{Sym}v = 0, \quad (9.2)$$

for all syzygies. This is no longer generically true for the six-point amplitudes in a representation with 25 degree-zero syzygies and 33 degree-two ones. It is, however, possible to find a representation in the six-point case where all but one degree-zero syzygies vanish under this

symmetrization operation, and where all degree-two syzygies vanish. As the vanishing under symmetrization is not invariant under multiplication by a polynomial of the invariants, it is also possible to find a nonminimal-degree representation where all syzygies vanish under symmetrization.

X. CONCLUSIONS

The internal structure of Yang-Mills scattering amplitudes is expressed through linear relations between color-ordered amplitudes. The best-known relations [1] between amplitudes with constant coefficients emerge directly from Lie-algebra relations. The work of DM suggests that other degree-zero relations also have a partly group-theoretic origin. Additional relations have coefficients which are polynomial in the momentum invariants. The BCJ identities are well-known examples of such relations. Their origin is still unclear, but it is beyond group theory.

We showed in Sec. III that such relations must exist on general grounds, leaving two independent amplitudes for five or more external particles (one independent amplitude in the four-point case). The general argument does not force these relations to be of low polynomial degree; but explicit forms obtained by BBDJS and DM, and extended here, are of quadratic degree or lower for the one-loop all-plus amplitude and the tree-level MHV amplitude. We have checked through $n = 7$ for the all-plus amplitude and $n = 8$ for the MHV one that these exhaust all relations required by the general argument. We conjecture that this statement holds to all n .

We presented two techniques for solving for the relations, one numerical and one mostly analytic, based on the factorization properties of amplitudes. The latter are presumably responsible for the surprisingly low degree of the relations. The factorization equations we derived are special to amplitudes with only three-point amplitudes in

all factorization channels. They can be extended to more general factorizations, a subject we leave to future study. The generality of the arguments suggests that similar relations should also exist for rational terms in higher-loop amplitudes, and for the rational coefficients of integral functions as well. They may prove useful in reducing computational complexity in numerical calculations, and in understanding new symmetry structures underlying Yang-Mills theories. A generalization of the relations studied here also describes the redundancy in Kawai-Lewellen-Tye kernels [14,15] relating tree-level Yang-Mills and gravity amplitudes.

ACKNOWLEDGMENTS

We thank CERN, the GGI, the KITP, and the Perimeter Institute for their hospitality during phases of this work. We thank Lance Dixon and Anthony Morales for helpful discussions and comments, and Roland Bittleston and Kevin Costello for helpful discussions. This research was supported by the European Research Council (ERC) under the European Union's research and innovation programme Grant Agreement No. ERC-AdG-885414 ("Ampl2Einstein"); in part by Grant NSF No. PHY-2309135 to the Kavli Institute for Theoretical Physics (KITP); and in part by a grant from the Simons Foundation (No. 1034867, Dittrich). Research at Perimeter Institute is supported in part by the Government of Canada through the Department of Innovation, Science and Economic Development and by the Province of Ontario through the Ministry of Colleges and Universities.

DATA AVAILABILITY

The data that support the findings of this article are openly available [16].

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