

Low-lying light tetraquark states with quantum numbers $J^P = 0^+, 1^+$ and 2^+

Hao Wu, Mao-Jun Yan[✉], Chun-Sheng An^{✉,*}, and Cheng-Rong Deng[✉]

School of Physical Science and Technology, Southwest University, Chongqing 400715, China



(Received 18 July 2025; accepted 2 October 2025; published 22 October 2025)

The low-lying light tetraquark states are investigated in the nonrelativistic quark model including the pseudoscalar meson exchange, where two different confinement potential schemes, the Cornell potential and the linear potential, are employed, along with the instanton-induced interaction serving as the residual spin-dependent interaction. The numerical results show agreement with masses of $f_0(500)$, $f_0(1370)$, $f_0(1500)$, $f_0(2020)$, $f_0(2200)$, $h_1(1170)$, $h_1(1595)$, $h_1(1900)$, $h_1(1965)$, $h_1(2215)$, $f_2(1430)$, $f_2(1640)$, $f_2(1810)$, $f_2(2010)$, $f_2(2150)$, $a_0(980)$, $a_0(1450)$, $a_0(1950)$, $a_1(1260)$, $a_1(1640)$, $a_2(1700)$, $K_0^*(1430)$, $K_0^*(1950)$, $K_1(1270)$, $K_1(1440)$, $K_1(1650)$, and $K_2^*(1980)$. The results shed light on the spectrum of these mesons and offer guidance to search for the tetraquarks in the future.

DOI: 10.1103/cxs1-h9j1

I. INTRODUCTION

Since $X(3872)$ [1] and $D_{s0}^*(2317)$ [2] have been reported in 2003, the physical observation of many new hadronic states is challenging our current understanding of hadrons as conventional mesons and baryons with valence contents of quark-antiquark and three quarks, respectively, since most of them do not fit in the well-known quark model. This difficulty brought back a long-standing discussion on the exotic hadronic structures, i.e., multiquark configurations that might have quantum numbers beyond those assigned to the conventional mesons and baryons.

$X(3872)$ and $D_{s0}^*(2317)$ have intrigued the particle physics community due to their unique properties and the insights into the nature of strong interactions. Therefore, tetraquark candidates in the light sector, cousins of $X(3872)$ and $D_{s0}^*(2317)$, play a crucial role in understanding the strong interaction in the nonperturbative region. There are dozens of light mesons with masses below 2000 MeV, where the rich spectrum covers the genuine mesons, glueballs, hybrids, molecules, compact tetraquarks, and admixtures. The candidates of tetraquarks are essential to explore the dynamics of multiquark states.

The concept of tetraquarks ($qq\bar{q}\bar{q}$) arose from discrepancies in the meson spectrum, such as the overpopulation of states and anomalous properties (e.g., mass, decay width). Light tetraquarks propose bound states of two quarks and two antiquarks, extending beyond the $q\bar{q}$

paradigm. These states may exhibit exotic quantum numbers (e.g., $J^{PC} = 0^{--}$, $J^{PC} = 1^{-+}$) forbidden by conventional $q\bar{q}$ picture, or manifest as resonant structures in decay channels [e.g., $f_0(500)$, $a_0(980)$].

Jaffe firstly pioneered the systematic study of light tetraquarks, proposing the $qq\bar{q}\bar{q}$ structure for scalar mesons [3]. Subsequently, Weinstein and Isgur proposed the meson-meson molecular interpretations [4], then Maiani *et al.* further extended the diquark-antidiquark model to the light and heavy sectors [5]. In addition, 't Hooft *et al.* have also discussed scalar mesons as tetraquarks in the context of chiral symmetry [6]. Thereafter, in order to get better understanding of light tetraquark states, numerous studies have been conducted to calculate their spectrum using different theoretical approaches, such the QCD sum rule [7–9], the relativistic diquark-antidiquark picture [10], the flux-tube model with a multibody confinement interaction x[11], the modified Godfrey-Isgur quark model and quark-pair creation (QPC) model [12], and the lattice QCD [13].

In the scalar sector, there are low-lying $f_0(500)$, $\kappa(700)$, $f_0(980)$, and $a_0(980)$, which are widely studied in models. The details of those states can be referred to in Refs. [14–19] and the references therein. In the region of 1700–2000 MeV, $a_0(1710)$, with mass and width of $1704 \pm 5 \pm 2$ MeV and $110 \pm 15 \pm 11$ MeV, is reported in $\eta\pi$ invariant mass distributions in $\eta_c \rightarrow \eta\pi\pi$ [20]. $a_0(1710)$, $f_0(1710)$, and $f_0(1770)$ are confirmed in $K\bar{K}$ invariant mass distributions in $D_s \rightarrow K_s K\pi$ [21]. Additionally, $f_0(2020)$ and $f_0(2470)$ are reported in the $J/\psi \rightarrow \gamma\eta'\eta'$ [22].

The light axial vector particles listed on the PDG [23] are in the region of 1100–1600 MeV, where $f_1(1285)$, $f_1(1420)$, and $f_1(1510)$ are reported in $K_s^0 K_s^0 \pi^0$ invariant mass in $J/\psi \rightarrow \gamma K_s^0 K_s^0 \pi^0$ [24]. $h_1(1170)$ is observed in the

*Contact author: ancs@swu.edu.cn

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3π invariant mass in $\pi p \rightarrow 3\pi n$ [25,26], $h_1(1415)$ is observed in the $K\bar{K}\pi$ invariant mass in $K^-p \rightarrow K_s^0 K^\pm \pi^\mp \Lambda$ [27] as well as in the $\eta'\gamma$ invariant mass distributions in $J/\psi \rightarrow \gamma\eta'\eta'$ [22], and $h_1(1595)$ is reported in the $\omega\eta$ invariant mass distributions in $\pi^-p \rightarrow \omega\eta n$ [28]. $a_1(1260)$ and $a_1(1640)$ are observed in diffractive dissociation of 190 GeV pions into $\pi^-\pi^+\pi^-$ [29]. $b_1(1235)$ is reported in the $\omega\pi$ channel in $p\pi$ reaction [30]. Additionally, BESIII reports a new $h_1(1900)$ meson by analyzing the $\eta\pi$ invariant mass distributions in $J/\psi \rightarrow \omega\eta$ [31]. COMPASS collaboration identifies a new $a_1(1420)$ meson with mass and width of 1414_{-13}^{+15} MeV and 153_{-23}^{+8} MeV in diffractive dissociation of 190 GeV pions into $\pi^-\pi^+\pi^-$ [32]. Meanwhile, Crystal Barrel collaboration first observes $h_1(1965)$ and $h_1(2215)$ in $\omega\eta$ invariant mass distributions in $p\bar{p} \rightarrow \omega\eta$ [33].

In the tensor sector, there are 12 light isoscalar tensor mesons with $I = 0$ listed in PDG [23], *i.e.*, $f_2(1270)$, $f_2(1430)$, $f_2(15525)$, $f_2(1565)$, $f_2(1640)$, $f_2(1810)$, $f_2(1910)$, $f_2(1950)$, $f_2(2010)$, $f_2(2150)$, $f_2(2300)$, $f_2(2340)$. These resonances are intensively studied in radiative J/ψ decays [34–43]. Tensor mesons with $I = 1$, $a_2(1320)$ and $a_2(1700)$, are reported in diffractive dissociation of πp collisions [44], in two-photon collisions [45], and in $p\bar{p}$ collisions [46–48].

To understand the mesons above, the spectrum is widely studied in phenomenological methods. In addition to the studies on the compact tetraquarks in the classical quark potential model [49–55], the QCD sum rule approach [56–59], and the diquark-diquark framework [60–62], the interactions driven by instanton are introduced to the S-wave tetraquark spectrum in the nonrelativistic quark model. The Cornell potential [63–65] and linear potential [66] are considered, where the instanton-induced interaction serves as the residual spin-dependent interaction [6,64–72].

This article is organized as follows. Section II presents the formalism of our model, including effective Hamiltonian, wave function configurations, and model parameters. Numerical results and discussions for the spectrum of fully light tetraquark states in two models are shown in Sec. III. Finally, Sec. IV contains a brief conclusion.

II. THEORETICAL FRAMEWORK

A. Effective Hamiltonian

In this study, the nonrelativistic quark potential model is utilized to explore the spectrum of the tetraquark states. In said model, the effective Hamiltonian reads

$$H_{\text{eff}} = \sum_{i=1}^4 (m_i + T_i) - T_{c.m.} + V_{\text{Conf}} + V_{\text{Ins}} + V_{\text{OBE}}, \quad (1)$$

where m_i and T_i are the constituent mass and kinetic energy of the i -th quark, respectively. $T_{c.m.}$ is the energy in the

center of mass frame (c.m.). V_{Conf} stands for the quark confinement potential, corresponding to the Cornell potential [63–65] and the linear potential [66], respectively,

$$V_{\text{Cornell}} = \sum_{i<j} -\frac{3}{16} (\vec{\lambda}_i^c \cdot \vec{\lambda}_j^c) \left(br_{ij} - \frac{4}{3} \frac{\alpha_{ij}}{r_{ij}} + C_0 \right) \quad (2)$$

and

$$V_{\text{Linear}} = \sum_{i<j} -\frac{3}{16} (\vec{\lambda}_i^c \cdot \vec{\lambda}_j^c) (a_{ij} + br_{ij}), \quad (3)$$

where $\vec{\lambda}_{i(j)}^c$ is Gell-Mann matrix in $SU(3)$ color space acting on the $i(j)$ th quark, b , α_{ij} , and C_0 are strength of quark confinement, QCD running coupling constant between two quarks, and zero point energy, respectively. a_{ij} , b_{ij} are the offset and slope of relative quark-(anti)quark distance. Notice that the choices of a_{ij} (b_{ij}) are different with respect to the diquarks of qq and $q\bar{q}$.

For the residual spin-dependent interaction, the 't Hooft's instanton-induced interaction [73] is employed,

$$V_{\text{Ins}} = V_{\text{Ins}}^{qq} + V_{\text{Ins}}^{q\bar{q}}, \quad (4)$$

with

$$V_{\text{Ins}}^{qq} = \sum_{i<j} -\hat{g}_{ij}^{qq} (P_{ij}^{S=1} P_{ij}^{C,6} + 2P_{ij}^{S=0} P_{ij}^{C,\bar{3}}) \delta^3(\vec{r}_{ij}), \quad (5)$$

$$V_{\text{Ins}}^{q\bar{q}} = \sum_{i<j} \hat{g}_{ij}^{q\bar{q}} \left[\frac{3}{2} P_{ij}^{S=1} P_{ij}^{C,8} + P_{ij}^{S=0} \left(\frac{1}{2} P_{ij}^{C,8} + 8P_{ij}^{C,1} \right) \right] \delta^3(\vec{r}_{ij}), \quad (6)$$

where V_{Ins}^{qq} acts both on quark-quark and antiquark-antiquark pairs in addition to $V_{\text{Ins}}^{q\bar{q}}$ acting on the quark-antiquark pair. $\hat{g}_{ij}^{qq}$ and $\hat{g}_{ij}^{q\bar{q}}$ are flavor-dependent coupling strength operators [64]. The spin projection operators $P_{ij}^{S=0}$, $P_{ij}^{S=1}$ correspond to spin-singlet and spin-triplet states, respectively. So do $P_{ij}^{C,\bar{3}}$, $P_{ij}^{C,6}$, $P_{ij}^{C,1}$, and $P_{ij}^{C,8}$ acting on the color space with the superscripts labeling color antitriplet $\bar{3}_c$, color sextet 6_c , color singlet 1_c , and color octet 8_c , respectively.

Concerning the contact interaction in the form of the Dirac delta function $\delta^3(\vec{r}_{ij})$ causes problems in numerically solving the eigenequation of the Hamiltonian, and the delta function is regularized by a Gaussian regulator [66,74,75],

$$\delta^3(\vec{r}_{ij}) \rightarrow \left(\frac{\sigma}{\sqrt{\pi}} \right)^3 \exp(-\sigma^2 r_{ij}^2). \quad (7)$$

where σ is a regularization parameter originating from the finite sizes of the constituent quarks.

In addition to the gluon exchange between quarks in short range, the Goldstone boson (pseudoscalar meson) exchange potentials provide the long-range quark-quark interactions, with the following explicit functional forms [75–77],

$$V_{\text{OBE}} = \sum_{i<j} (V_{ij}^\pi + V_{ij}^K + V_{ij}^\eta), \quad (8)$$

with

$$V_{ij}^\pi = \frac{g_{ch}^2}{4\pi} \frac{m_\pi^2}{12m_i m_j} \frac{\Lambda_\pi^2 m_\pi}{\Lambda_\pi^2 - m_\pi^2} \vec{\sigma}_i \cdot \vec{\sigma}_j \left[Y(m_\pi \vec{r}_{ij}) - \frac{\Lambda_\pi^3}{m_\pi^3} Y(\Lambda_\pi \vec{r}_{ij}) \right] \sum_{a=1}^3 (\vec{\lambda}_i^a \cdot \vec{\lambda}_j^a), \quad (9)$$

$$V_{ij}^K = \frac{g_{ch}^2}{4\pi} \frac{m_K^2}{12m_i m_j} \frac{\Lambda_K^2 m_K}{\Lambda_K^2 - m_K^2} \vec{\sigma}_i \cdot \vec{\sigma}_j \left[Y(m_K \vec{r}_{ij}) - \frac{\Lambda_K^3}{m_K^3} Y(\Lambda_K \vec{r}_{ij}) \right] \sum_{a=4}^7 (\vec{\lambda}_i^a \cdot \vec{\lambda}_j^a), \quad (10)$$

$$V_{ij}^\eta = \frac{g_{ch}^2}{4\pi} \frac{m_\eta^2}{12m_i m_j} \frac{\Lambda_\eta^2 m_\eta}{\Lambda_\eta^2 - m_\eta^2} \vec{\sigma}_i \cdot \vec{\sigma}_j \left[Y(m_\eta \vec{r}_{ij}) - \frac{\Lambda_\eta^3}{m_\eta^3} Y(\Lambda_\eta \vec{r}_{ij}) \right] [\cos \theta_p (\vec{\lambda}_i^8 \cdot \vec{\lambda}_j^8) - \sin \theta_p], \quad (11)$$

where the $\vec{\lambda}_{i(j)}$ is the SU(3) flavor Gell-Mann matrices. The $\Lambda_{\pi(K,\eta)}$ and θ_p are the cutoff parameters and mixing angle, which are quoted from Ref. [75]. The m_π , m_K , and m_η are the physical masses of the SU(3) Goldstone bosons. $Y(x)$ is defined by $Y(x) = e^{-x}/x$. The chiral coupling constant g_{ch} can be obtained from the πNN coupling constant through Ref. [78]

$$\frac{g_{ch}^2}{4\pi} = \left(\frac{3}{5}\right)^2 \frac{g_{\pi NN}^2}{4\pi} \frac{m_{u,d}^2}{m_N^2}. \quad (12)$$

B. Wave functions

The wave function of a tetraquark state is a product of the orbital, color, spin, and flavor wave functions,

$$\Psi = \psi_{\text{orbital}} \otimes \psi_{\text{color}} \otimes \psi_{\text{spin}} \otimes \psi_{\text{flavor}}. \quad (13)$$

The orbital wave function of a few-body system is expanded in terms of a set of Gaussian basis functions, forming an approximated complete set in a finite coordinate space [79]. The wave function of the ground four-quark states in the coordinate space reads,

$$\psi_{\text{orbital}}(\{\vec{r}_i\}) = \prod_{i=1}^4 \sum_j^n C_{ij} \left(\frac{1}{\pi \beta_{ij}^2} \right)^{3/4} \exp \left[-\frac{1}{2\beta_{ij}^2} r_i^2 \right], \quad (14)$$

where $\{\beta_{ij}\}$ are the harmonic oscillator length parameters, and originated from the angular frequencies $\{\omega_j\}$ of the harmonic oscillator by $1/\beta_{ij}^2 = m_i \omega_j$. Meanwhile, we assume that $\{\omega_j\}$ are independent of the quark masses, m_i , i.e., then the spatial wave function can be simplified as,

$$\begin{aligned} \psi_{\text{orbital}}(\{\vec{r}_i\}) &= \sum_j^n C_j \prod_{i=1}^4 \left(\frac{m_i \omega_j}{\pi} \right)^{3/4} \exp \left[-\frac{m_i \omega_j}{2} r_i^2 \right] \\ &= \sum_j^n C_j \varphi(\omega_j, \{\vec{r}_i\}), \end{aligned} \quad (15)$$

which is always adopted for the calculations of multi-quark systems [80–82].

Following the work in Ref. [79], we define $1/\beta_{ij}^2 = 1/\beta_j^2 = m_q \omega_j$ (m_q denotes the constituent mass of the corresponding quark), thus the parameters β_j are set to be geometric series,

$$\beta_j = \beta_1 a^{j-1} \quad (j = 1, 2, \dots, n), \quad (16)$$

where the parameters $\{\beta_1, \beta_n, n\}$ are the Gaussian size parameters in geometric progression for numerical calculations, and the final results are stable and independent of these parameters within an approximated complete set in a large space.

In addition, to remove the influence of the center-of-mass kinetic energy, here we rewrite the tetraquark wave function using the coordinates defined as follows,

$$\vec{\xi}_1 = \vec{r}_1 - \vec{r}_2, \quad (17)$$

$$\vec{\xi}_2 = \vec{r}_3 - \vec{r}_4, \quad (18)$$

$$\vec{\xi}_3 = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2} - \frac{m_3 \vec{r}_3 + m_4 \vec{r}_4}{m_3 + m_4}, \quad (19)$$

$$\vec{\xi}_4 = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2 + m_3 \vec{r}_3 + m_4 \vec{r}_4}{m_1 + m_2 + m_3 + m_4}, \quad (20)$$

with these one can rewrite Eq. (15) as

$$\psi_{\text{orbital}}(\{\vec{\xi}_i\}) = \sum_j^n C_j \prod_{i=1}^4 \left(\frac{\mu_i \omega_j}{\pi} \right)^{3/4} \exp \left[-\frac{\mu_i \omega_j}{2} \xi_i^2 \right], \quad (21)$$

where $\mu_1 \equiv \frac{m_1 m_2}{m_1 + m_2}$, $\mu_2 \equiv \frac{m_3 m_4}{m_3 + m_4}$, $\mu_3 \equiv \frac{(m_1 + m_2)(m_3 + m_4)}{M}$, $\mu_4 = M \equiv m_1 + m_2 + m_3 + m_4$. With the trial wave

function defined in Eq. (21), the kinetic energy matrix element is worked out to be

$$\left\langle \sum_{i=1}^4 T_i - T_{c.m.} \right\rangle = \frac{9}{4} \sum_j^n \sum_k^n C_j C_k \frac{(\omega_j \omega_k)^4}{\left(\frac{\omega_j + \omega_k}{2}\right)^7}. \quad (22)$$

Note that the coordinate $\vec{\xi}_4$ should indicate motion of the center of mass of the studied tetraquark state, and of course $\vec{\xi}_4 = 0$ in the c.m. frame. While in the present case, the above equations including the Hamiltonian and wave functions are not written as the forms in the tetraquark c.m. frame, and contributions from motions of the center of mass are finally eliminated. As shown in Eq. (1), the kinetic energy of the c.m. can be subtracted by the term $-T_{c.m.}$, which leads to the final kinetic energy given in Eq. (22). On the other hand, one may doubt that the nonzero coordinate $\vec{\xi}_4$ should result in additional nonvanishing matrix elements of the inner quark potential, but note that all the potential terms in Eqs. (2)–(11) are functions of the relative coordinates of the interacting quark pairs, namely $\vec{r}_{ij} = \vec{r}_i - \vec{r}_j$, and by resolving Eq. (20), one can get that $\{\vec{r}_i\}$ can be expressed by $\{\vec{\xi}_i\}$ as follows:

$$\vec{r}_i = -\vec{\xi}_4 + \sum_{m=1}^3 a_{im} \vec{\xi}_m, \quad (23)$$

with a_{im} being corresponding constants that can be expressed by the quark masses. Thus, all the potential terms for any interacting quark pairs in the tetraquark system are independent on the coordinate $\vec{\xi}_4$ although $\vec{\xi}_4 \neq 0$, since $\vec{r}_{ij}$ doesn't depend on $\vec{\xi}_4$. Consequently, contributions from motions of c.m. of the studied tetraquark systems could be eliminated completely by the presently used scheme.

Stemming from the Pauli principle, color confinement for the tetraquark system and the ground state is characterized by a symmetric spatial wave function; we construct eight symmetric configurations for $qq\bar{q}\bar{q}$ in the space of color $\otimes$ spin $\otimes$ flavor,

$$\begin{aligned} |1\rangle &= [qq]_{\bar{6}_c 1_s \bar{6}_f} [\bar{q}\bar{q}]_{\bar{6}_c 1_s \bar{6}_f}, & |2\rangle &= [qq]_{\bar{3}_c 1_s \bar{3}_f} [\bar{q}\bar{q}]_{\bar{3}_c 1_s \bar{3}_f}, \\ |3\rangle &= [qq]_{\bar{3}_c 3_s \bar{6}_f} [\bar{q}\bar{q}]_{\bar{3}_c 1_s \bar{3}_f}, & |4\rangle &= [qq]_{\bar{3}_c 1_s \bar{3}_f} [\bar{q}\bar{q}]_{\bar{3}_c 3_s \bar{6}_f}, \\ |5\rangle &= [qq]_{\bar{6}_c 3_s \bar{3}_f} [\bar{q}\bar{q}]_{\bar{6}_c 1_s \bar{6}_f}, & |6\rangle &= [qq]_{\bar{6}_c 1_s \bar{6}_f} [\bar{q}\bar{q}]_{\bar{6}_c 3_s \bar{3}_f}, \\ |7\rangle &= [qq]_{\bar{3}_c 3_s \bar{6}_f} [\bar{q}\bar{q}]_{\bar{3}_c 3_s \bar{6}_f}, & |8\rangle &= [qq]_{\bar{6}_c 3_s \bar{3}_f} [\bar{q}\bar{q}]_{\bar{6}_c 3_s \bar{3}_f}, \end{aligned} \quad (24)$$

where the q stands for u, d, s quarks. The color structures **6** and $\bar{3}$ (**6** and **3**) of diquarks (antidiquarks) are represented by the c subscript, the spin structures **3** and **1** of the diquarks (antidiquarks) are represented by the s subscript, and the flavor structures **6** and $\bar{3}$ (**6** and **3**) of diquarks (antidiquarks) are represented by the f subscript,

respectively. The explicit flavor wave functions of $nn\bar{n}\bar{n}$, $nn\bar{n}\bar{s}$, $ns\bar{n}\bar{s}$, $ns\bar{s}\bar{s}$, $ss\bar{s}\bar{s}$ are presented in the Appendix.

C. Strategy

A basic assumption of the nonrelativistic quark potential model is that quarks, trapped inside hadrons, are nonrelativistic, obeying the Schrödinger equation [83]. According to the matrix elements in Secs. II A and II B, the mass spectra can be obtained by solving the generalized matrix eigenvalue problem,

$$\sum_j^n \sum_k^n C_j^i (H_{jk}^d - E_i^d N_{jk}) C_k^i = 0, \quad (25)$$

where $i = 1 - n$ and

$$H_{jk}^d = \langle \psi(\omega_j)(CSF) | H_{\text{eff}} | \psi(\omega_k)(CSF) \rangle, \quad (26)$$

$$N_{jk} = \langle \psi(\omega_j)(CSF) | \psi(\omega_k)(CSF) \rangle. \quad (27)$$

The H_{jk}^d are the matrix elements in the total spin-color-flavor-spatial bases, E_i^d stands for the eigenvalues, C_k^i are the corresponding eigenvectors, and (CSF) stands for the color $\otimes$ spin $\otimes$ flavor wave function. Concerning the Gaussian function associated with different harmonic oscillator frequencies that are not orthogonal, N_{jk} is not an identity matrix.

The procedures of the numerical calculations are organized as follows: firstly, we derive n eigenenergies E_i^d and n corresponding eigenvectors C^i by solving Eq. (25) to get the masses of single configurations, and then we calculate the off-diagonal effects between different configurations. Secondly, according to the Rayleigh-Ritz variational principle, the lowest eigenenergy E_m^d should correspond to the steady state energy. Therefore, we select a set of $\{\beta_1, \beta_n, n\}$, then systematically vary and refine the harmonic oscillator length parameter until the energy E_m^d converges to a stable minimum.

The parameters $\{\beta_1, \beta_n, n\} = \{0.02 fm, 6 fm, 40\}$ are adopted in the Models I (V_{Cornell}) and II (V_{Linear}). There are additional parameters with the same choices in the models, where the parameters are the coupling strengths in INS hyperfine interaction g_{nb} and g_{sb} , the regulation parameter of the δ function σ , and the constituent quark masses m_n and m_s . Alternatively, there are differences in the behavior of the confinement in models. In Model I, the parameters add the QCD effective coupling constants α_{nn} , α_{ns} , α_{ss} , the quark confinement strength b , and the zero-point energy C_0 . For Model II, the parameters add the slope of the relative quark-(anti)quark distance b_{qq} , $b_{q\bar{q}}$, the offset of the relative quark-(anti)quark distance a_{qq} , $a_{q\bar{q}}$.

In the current model, the interaction between quarks and antiquarks in tetraquark states is treated as a simple sum of two-body confinement and two-body residual interactions,

TABLE I. Inputs in two models.

Model I		Model II	
Parameter(Unit)	Value	Parameter(Unit)	Value
m_n (MeV)	340	m_n (MeV)	340
m_s (MeV)	511	m_s (MeV)	511
σ (MeV)	200	σ (MeV)	200
$g_{nn}(10^{-4} \text{ MeV}^{-2})$	1.090	$g_{nn}(10^{-4} \text{ MeV}^{-2})$	1.090
$g_{ns}(10^{-4} \text{ MeV}^{-2})$	0.657	$g_{ns}(10^{-4} \text{ MeV}^{-2})$	0.662
$b(\text{MeV} \cdot \text{fm}^{-1})$	177	$b_{qq}(\text{MeV} \cdot \text{fm}^{-1})$	218
		$b_{q\bar{q}}(\text{MeV} \cdot \text{fm}^{-1})$	395
C_0 (MeV)	-150	a_{qq} (MeV)	-393
		$a_{q\bar{q}}$ (MeV)	-537
α_{nn}	0.485		
α_{ns}	0.488		
α_{ss}	0.500		

TABLE II. Masses of the pseudoscalar and the vector mesons on ground states. Rows denoted by “Model I/II” are numerical results estimated in Models I/II, and rows denoted by “Exp.” are physical masses in PDG [23].

States	π	ρ	ω	ϕ	K	K^*	η	η'
Exp. (MeV)	139	770	782	1019	497	892	548	957
Model I (MeV)	139	749	739	1019	497	892	570	964
Model II (MeV)	139	750	749	1019	496	890	572	972

which is analogous to the description of mesons in constituent-quark models. Consequently, we use the masses of a set of ground-state mesons listed in the PDG [23] to determine the parameters in the models, the explicit numbers of all the parameters are shown in Table I, where the spectrum of the low lying pseudoscalar and vector mesons are displayed in Table II.

III. RESULTS AND DISCUSSIONS

According to the configurations in Eq. (24), the wave functions of the tetraquarks are represented by the eight configurations, with details organized in Tables III, VI, and IX. With the parameters of two models in Table I, the mass spectra of light tetraquark systems are calculated and presented in Tables IV, VII, and X. Consider that a physical tetraquark state should be composed of two types of components with $6_c \otimes \bar{6}_c$ and $\bar{3}_c \otimes 3_c$ color structures; configuration mixing is taken into account during the numerical calculation. Additionally, the mass spectra of $nn\bar{n}\bar{n}$, $ns\bar{n}\bar{s}$, $ss\bar{s}\bar{s}$, $nn\bar{n}$, and $ns\bar{s}\bar{s}$ are displayed in Figs. 1–3, where the horizontal lines are two-meson thresholds.

A. $nn\bar{n}\bar{n}$ system

The configurations of $nn\bar{n}\bar{n}$ are organized in the Table III, while the energy eigenstates are presented in

TABLE III. Configurations of $nn\bar{n}\bar{n}$ system.

I^G	J^{PC}	Configurations			
0^+	0^{++}	$ 1\rangle$	$ 2\rangle$	$ 7\rangle$	$ 8\rangle$
0^-	1^{+-}	$ 7\rangle$	$ 8\rangle$		
0^+	2^{++}	$ 7\rangle$	$ 8\rangle$		
1^-	0^{++}	$ 1\rangle$	$ 7\rangle$		
1^-	1^{++}	$ 3'\rangle = \frac{1}{\sqrt{2}}(3\rangle + 4\rangle)$	$ 5'\rangle = \frac{1}{\sqrt{2}}(5\rangle + 6\rangle)$		
1^+	1^{+-}	$ 4'\rangle = \frac{1}{\sqrt{2}}(3\rangle - 4\rangle)$	$ 6'\rangle = \frac{1}{\sqrt{2}}(5\rangle - 6\rangle)$	$ 7\rangle$	
1^-	2^{++}	$ 7\rangle$			
2^+	0^{++}	$ 1\rangle$	$ 7\rangle$		
2^-	1^{+-}	$ 7\rangle$			
2^+	2^{++}	$ 7\rangle$			

Table IV. Additionally, the mass spectrum of $nn\bar{n}\bar{n}$ is displayed in Fig. 1.

For the $nn\bar{n}\bar{n}$ states, there are four energies predicted in the $0^+(0^{++})$ sector, where the lowest energy corresponds to $f_0(500)$. $f_0(500)$ is observed in the low energy $\pi\pi$ scattering [84] as well as in the $\pi\pi$ invariant mass distributions in $J/\psi(1S) \rightarrow \gamma(\pi\pi)$ [85]. It has been considered as a candidate for scalar glueball [85–88] and a dynamically generated pole in the $\pi\pi - K\bar{K}$ scattering [89], where the average mass and width of $f_0(500)$ are $M_{f_0(500)} = 400\text{--}800$ MeV, $\Gamma_{f_0(500)} = 100\text{--}800$ MeV [23]. In Model I, the lowest energy of $nn\bar{n}\bar{n}$ state with $I^G(J^{PC}) = 0^+(0^{++})$ is 493.67 MeV, and in Model II it is 468.31 MeV. The predictions of both models agree well with the physical mass of $f_0(500)$, suggesting that $f_0(500)$ may contain a sizable tetraquark component. Additionally, the highest energies predicted in both models are higher than the two-meson thresholds and may decay into $\eta\omega$, $\eta'\omega$, and $\eta'\eta'$.

In the $I^G(J^{PC}) = 0^-(1^{+-})$ sector, Model I yields two states with the energies of 1116.17 and 1822.33 MeV, while Model II yields 1116.52.19 and 1818.72 MeV, respectively. The lower state is a plausible candidate for $h_1(1170)$, whose mass is $1166 \pm 5 \pm 3$ MeV [25,26,90]. Concerning $h_1(1170)$ being interpretable as $[q\bar{q}]$ [91], the physical h_1 state might be a hybrid and decay into $\rho\pi$.

In the $I^G(J^{PC}) = 0^+(2^{++})$ sector, Model I yields two states with energies of 1423.99 and 1766.76 MeV, while Model II yields 1429.79 and 1726.97 MeV, respectively. The BABAR collaboration reports the observation of the $f_2(1430)$ resonance in the processes $\gamma\gamma \rightarrow \eta_c(1S) \rightarrow \eta'\pi^+\pi^-$ [20], where the mass and width of $f_2(1430)$ are $1440 \pm 11 \pm 3$ MeV and $46 \pm 15 \pm 5$ MeV, respectively. The lowest energy levels in the models match the $f_2(1430)$ mass. Moreover, the highest energies in these models correspond to $f_2(1640)$, whose mass is 1639 ± 6 MeV [23,92,93]. In Ref. [11], Deng *et al.* employ a novel flux-tube structure to calculate the energy spectrum of tetraquark states. $f_2(1640)$ is interpreted as a tetraquark with quantum numbers $N^{2S+1}L_J = 2^1D_2$, as seen in Table V. Therefore, $f_2(1640)$ may contain both S - and D -wave tetraquark components.

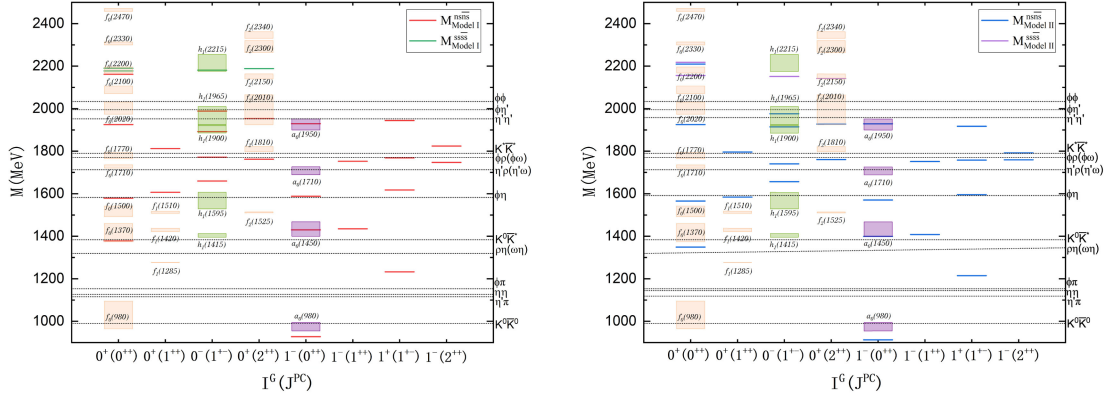


FIG. 2. Mass spectrum of S-wave $ns\bar{n}\bar{s}$ and $ss\bar{s}\bar{s}$ states. The data on the left correspond to Model I, whereas the data on the right pertain to Model II. The red and green solid lines depict the numerical results of the $ns\bar{n}\bar{s}$ and $ss\bar{s}\bar{s}$ states, respectively, as calculated within Model I. Conversely, the blue and purple lines represent the numerical results of these states as determined by Model II. The gray dotted lines are the two-body thresholds, the rectangles represent $ns\bar{n}\bar{s}$ and $ss\bar{s}\bar{s}$ states' physical masses quoted from PDG [23], and the bands of the rectangles stand for the uncertainties of masses.

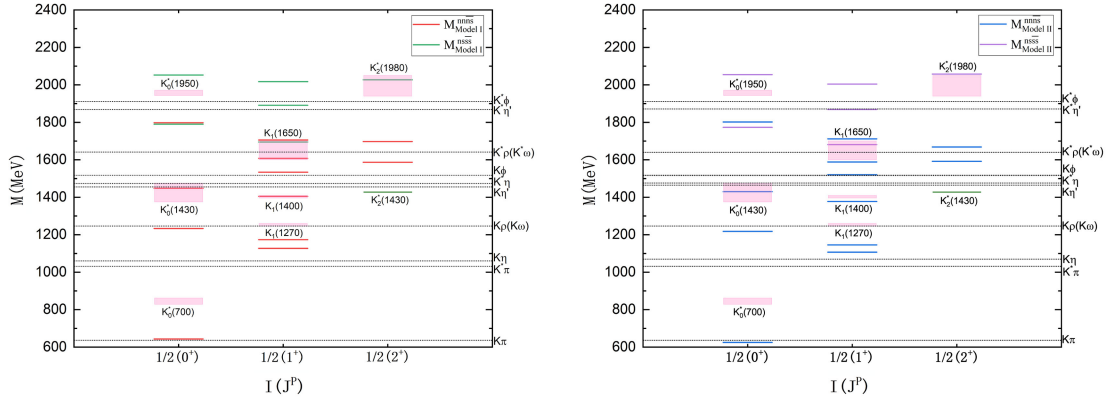


FIG. 3. Mass spectrum of S-wave $nn\bar{n}\bar{s}$ and $ns\bar{s}\bar{s}$ states. The data on the left corresponds to Model I, whereas the data on the right pertains to Model II. The red and green solid lines depict the numerical results of the $nn\bar{n}\bar{s}$ and $ns\bar{s}\bar{s}$ states, respectively, as evaluated in Model I. Conversely, the blue and purple lines represent the corresponding outcomes for these states as determined by Model II. The gray dotted lines are the two-body thresholds, the rectangles represent $nn\bar{n}\bar{s}$ and $ns\bar{s}\bar{s}$ states' physical masses taken from PDG [23], and the bands of the rectangles stand for the uncertainties of masses.

negative pions into the $\pi^-\pi^-\pi^+$ state [95], the $a_1(1420)$ observed in the diffractive dissociation of 190 GeV pions into the $\pi^-\pi^-\pi^+$ state [96], and the $a_1(1640)$ observed in the reaction $\pi^-p \rightarrow \pi^+\pi^-\pi^-p$ state [97], where the mass and widths are

$$(M_{a_1(1260)} = 1230 \pm 40, \Gamma_{a_1(1260)} = 250 \sim 600) \text{ MeV},$$

$$(M_{a_1(1420)} = 1414^{+15}_{-13}, \Gamma_{a_1(1420)} = 153^{+8}_{-23}) \text{ MeV},$$

$$(M_{a_1(1640)} = 1655 \pm 16, \Gamma_{a_1(1640)} = 250 \pm 40) \text{ MeV}.$$

The lowest energy predictions from both models agree with $a_1(1260)$, and the highest energy predictions respond to $a_1(1640)$. In contrast, the mass of $a_1(1420)$ exhibits significant deviations from both model predictions, effectively excluding the interpretation as compact tetraquark states. In Ref. [98], the mass of the axial-vector tetraquark

state with $I^G(J^{PC}) = 1^-(1^{++})$ is predicted to be approximately 1830 MeV in QCD sum rules. This result disfavors the interpretation of $a_1(1420)$ as a pure tetraquark state, which is consistent with the predictions in our models.

In the $I^G(J^{PC}) = 1^-(2^{++})$ sector, Model I yields a state with energy of 1654.10 MeV, and Model II gives 1612.13 MeV. These values suggest that this state is a plausible candidate for $a_2(1700)$, whose mass is 1706 ± 14 MeV [23,45,48]. Concerning $a_2(1700)$ being interpretable as $[q\bar{q}]$ [94], the physical a_2 state might be a hybrid and decay into $\pi\pi$. In addition, several isotensor states are also displayed; see Table IV.

B. $ns\bar{n}\bar{s}$ and $ss\bar{s}\bar{s}$ systems

The configurations of $ns\bar{n}\bar{s}$ and $ss\bar{s}\bar{s}$ are organized in the Table VI, where the energy eigenstates are presented in

TABLE V. Comparison of the calculated masses of $nn\bar{n}\bar{n}$ states for the observed exotic states. Columns 2 and 3 present the results from Model I and II. Columns 4 to 6 list the mass predictions from other representative studies of tetraquark interpretations.

State	PDG [23] (MeV)	Model I (MeV)	Model II (MeV)	Ref. [13] (MeV)	Ref. [11] (MeV)	Ref. [10] (MeV)
$f_0(500)$	400–800	494	468	...	587	596
$h_1(1170)$	1166	1116	1117	...	1304	...
$f_2(1430)$	1440	1424	1430	...	1465	...
$f_2(1640)$	1639	1767	1727	...	1641	...
$a_1(1260)$	1230	1238	1199	...	1839	1201
$a_1(1640)$	1655	1553	1594	...	2271	...
$a_2(1700)$	1706	1654	1612	...	1807	...

TABLE VI. Configurations of $ns\bar{n}\bar{s}$ and $ss\bar{s}\bar{s}$ systems.

For $ns\bar{n}\bar{s}$ states			
I^G	J^{PC}	Configurations	
0^+	0^{++}	$ 1\rangle$	$ 2\rangle$
0^+	1^{++}	$ 3'\rangle = \frac{1}{\sqrt{2}}(3\rangle + 4\rangle)$	$ 5'\rangle = \frac{1}{\sqrt{2}}(5\rangle + 6\rangle)$
0^-	1^{+-}	$ 4'\rangle = \frac{1}{\sqrt{2}}(3\rangle - 4\rangle)$	$ 6'\rangle = \frac{1}{\sqrt{2}}(5\rangle - 6\rangle)$
0^+	2^{++}	$ 7\rangle$	$ 8\rangle$
1^-	0^{++}	$ 1\rangle$	$ 2\rangle$
1^-	1^{++}	$ 3'\rangle = \frac{1}{\sqrt{2}}(3\rangle + 4\rangle)$	$ 5'\rangle = \frac{1}{\sqrt{2}}(5\rangle + 6\rangle)$
1^+	1^{+-}	$ 4'\rangle = \frac{1}{\sqrt{2}}(3\rangle - 4\rangle)$	$ 6'\rangle = \frac{1}{\sqrt{2}}(5\rangle - 6\rangle)$
1^-	2^{++}	$ 7\rangle$	$ 8\rangle$
For $ss\bar{s}\bar{s}$ states			
I^G	J^{PC}	Configurations	
0^+	0^{++}	$ 1\rangle$	$ 7\rangle$
0^-	1^{+-}	$ 7\rangle$	
0^+	2^{++}	$ 7\rangle$	

Table VII. Additionally, the mass spectra of $ns\bar{n}\bar{s}$ and $ss\bar{s}\bar{s}$ are displayed in Fig. 2.

For the $ns\bar{n}\bar{s}$ tetraquark states in the $0^+(0^{++})$ sector, four distinct mass predictions are obtained. Among these, the lowest masses were calculated to be 1378.43 and 1349.06 MeV, which align closely with $f_0(1370)$, rather than $f_0(980)$. Thus, $f_0(980)$ cannot be described as a compact tetraquark state in the present models. Due to the masses of $f_0(980)$ being close to the $K\bar{K}$ threshold, the $K\bar{K}$ channel plays a crucial role in this region [14,99]. In Ref. [100], the decay widths for both the process $f_0(980) \rightarrow \gamma\gamma$ and the interaction decay $f_0(980) \rightarrow \pi\pi$ are calculated using the phenomenological Lagrangian approach. The results provide strong evidence that $f_0(980)$ is mainly the hadronic molecular state of $K\bar{K}$.

$f_0(1370)$ is observed in the $\pi^+\pi^+\pi^-\pi^-$ invariant mass distributions in $\bar{p}n \rightarrow \pi^+\pi^+\pi^-\pi^-$ [101] as well as in the $K\bar{K}$ invariant mass distributions in $D^+ \rightarrow K^-K^+K^+$ [102]. The mass and width of $f_0(1370)$ are found to be $1422 \pm 15 \pm 9 \pm 28$ MeV and $324 \pm 38 \pm 18 \pm 38$ MeV. The lowest energy predictions align with the mass of $f_0(1370)$. Furthermore, the second-lowest energy predictions respond

to $f_0(1500)$, whose mass and width are 1522 ± 25 MeV and 108 ± 33 MeV [23,103–106]. In Ref. [107], Li *et al.* study the two-hadronic decays of $f_0(1370)$, $f_0(1500)$, $f_0(1710)$ in the admixture of the scalar glueball and quarkonia, where $f_0(1370)$, $f_0(1500)$, and $f_0(1710)$ are strong candidates for the glueball states. Therefore, $f_0(1370)$ and $f_0(1500)$ might be hybrid states with a sizable tetraquark component.

$f_0(2020)$ is observed in the invariant mass of the $\eta'\eta'$ system in $J/\psi \rightarrow \gamma\eta'\eta'$ [22,108], where the mass and width are $1982 \pm 3^{+54}_{-0}$ MeV and $436 \pm 4^{+46}_{-49}$ MeV, respectively. The second-highest energy predictions of the $ns\bar{n}\bar{s}$ state with $I^G(J^{PC}) = 0^+(0^{++})$ are 1925.20 and 1924.92 MeV, which are consistent with the physical mass of $f_0(2020)$. Thus, $f_0(2020)$ may contain a sizable tetraquark component. On the other hand, $f_0(2200)$ is reported in the $\eta\eta$ invariant mass from π^-p charge-exchange reactions at 32.5 GeV/c [85], where the mass and width are $M_{f_0(2200)} = 2187 \pm 14$ MeV, $\Gamma_{f_0(2200)} = 210 \pm 40$ MeV.

In Model I, the lowest energy of $ss\bar{s}\bar{s}$ state with $I^G(J^{PC}) = 0^+(0^{++})$ is 2192.51 MeV, which is very consistent with the mass of the $f_0(2200)$ state. The predictions with the highest energy of the $ns\bar{n}\bar{s}$ state with $I^G(J^{PC}) = 0^+(0^{++})$ are 2161.56 and 2209.63 MeV, which are consistent with the physical mass of $f_0(2200)$. Thus, $f_0(2200)$ may contain a sizable tetraquark component. As shown in Table VIII, Ref. [13] predicts that the 2S-wave light tetraquark states with $I = 0$, $J = 0$ calculated within non-relativistic quark models are associated with $f_0(2020)$ and $f_0(2200)$. Therefore, $f_0(2020)$ and $f_0(2200)$ may contain a hybrid structure both in the 1S- and 2S-isoscalar scalar mesons.

For the $ns\bar{n}\bar{s}$ system, four different energy levels are predicted in the $I^G(J^{PC}) = 0^-(1^{+-})$ sector. The lowest energy predictions are 1659.82 and 1656.33 MeV, which respond to $h_1(1595)$, whose mass is $1594 \pm 15^{+10}_{-60}$ MeV [28]. The second-highest energy predictions are 1891.92 and 1914.45 MeV, matching with the $h_1(1900)$ structure in the $\phi\eta$ invariant mass in $J/\psi \rightarrow \phi\pi^0\eta$ [109], where the mass and width are $1911 \pm 6(stat.) \pm 14(sys.)$ MeV and $149 \pm 12(stat.) \pm 23(sys.)$ MeV. The predictions with the highest energy are 1988.36 and 1975.80 MeV. These values

TABLE VII. Numerical results for the $ns\bar{n}\bar{s}$ and $ss\bar{s}\bar{s}$ states in two models. The quantum numbers $I^G(J^{PC})$, and configurations for each state are given in columns 1 and 2. The numerical results of pure configuration and the configurations mixing, and the corresponding configurations mixing coefficients in Model I, are presented in columns 3 to 5, while the calculation results of Model II are given in columns 6 to 8.

$I^G(J^{PC})$	Conf.	Model I			Model II		
		Pure	Configurations mixing		Pure	Configurations mixing	
		E (MeV)	E (MeV)	Mixing coefficients	E (MeV)	E (MeV)	Mixing coefficients
For $ns\bar{n}\bar{s}$ state							
$0^+(0^{++})$	1⟩	1894.70	1378.43	(0.35, 0.50, −0.46, 0.64)	1934.56	1349.06	(−0.36, −0.53, 0.46, 0.61)
	2⟩	1536.21	1579.18	(−0.25, 0.85, 0.13, −0.43)	1510.52	1565.37	(−0.25, 0.84, 0.18, 0.44)
	7⟩	1883.92	1925.20	(0.74, 0.10, 0.65, −0.02)	1846.75	1924.92	(0.65, 0.09, 0.74, −0.10)
	8⟩	1729.54	2161.56	(−0.50, 0.08, 0.58, 0.63)	1757.15	2209.63	(0.61, −0.06, −0.45, 0.65)
$0^+(1^{++})$	3′⟩	1730.19	1606.57	(−0.63, 0.77)	1685.11	1584.10	(0.72, 0.69)
	5′⟩	1688.66	1812.29	(0.77, 0.63)	1694.91	1795.92	(−0.69, 0.72)
$0^-(1^{+-})$	4′⟩	1800.60	1659.82	(−0.09, −0.02, −0.53, 0.84)	1757.73	1656.33	(0.16, −0.03, 0.61, 0.77)
	6′⟩	1862.86	1771.54	(0.87, −0.49, 0.00, 0.00)	1896.99	1740.27	(0.95, −0.31, 0.00, 0.00)
	7⟩	1893.28	1891.92	(−0.49, 0.87, 0.00, 0.00)	1852.63	1914.45	(0.31, 0.95, 0.00, 0.00)
	8⟩	1754.90	1988.36	(−0.11, 0.05, 0.83, 0.53)	1779.50	1975.80	(−0.10, −0.06, 0.78, 0.61)
$0^+(2^{++})$	7⟩	1911.59	1761.97	(0.47, 0.88)	1864.34	1760.91	(−0.61, 0.79)
	8⟩	1804.14	1953.75	(0.88, −0.46)	1823.96	1927.39	(0.79, 0.61)
$1^-(0^{++})$	1⟩	1725.26	927.46	(−0.07, 0.64, −0.09, 0.76)	1737.59	912.70	(−0.08, −0.62, 0.11, 0.77)
	2⟩	1310.28	1430.18	(0.62, 0.22, 0.74, −0.04)	1303.68	1399.10	(−0.58, 0.23, 0.78, 0.01)
	7⟩	1632.85	1588.32	(−0.10, 0.73, −0.17, 0.64)	1586.33	1570.22	(0.09, 0.74, −0.16, 0.64)
	8⟩	1206.58	1929.01	(0.77, −0.02, −0.63, 0.01)	1182.98	1928.55	(0.80, 0.02, 0.60, 0.02)
$1^-(1^{++})$	3′⟩	1554.51	1434.78	(0.79, −0.61)	1522.43	1407.96	(0.81, 0.58)
	5′⟩	1632.67	1752.40	(0.61, 0.79)	1637.14	1751.62	(−0.58, 0.81)
$1^+(1^{+-})$	4′⟩	1576.82	1232.35	(−0.49, −0.42, −0.07, 0.75)	1544.54	1214.53	(0.50, −0.42, 0.08, 0.75)
	6′⟩	1684.54	1617.34	(0.62, 0.35, 0.27, 0.64)	1693.79	1595.54	(0.67, −0.26, 0.28, −0.63)
	7⟩	1893.78	1768.38	(−0.60, 0.72, 0.35, 0.04)	1851.94	1758.10	(0.53, 0.69, −0.48, 0.08)
	8⟩	1407.08	1944.14	(0.00, −0.43, 0.89, −0.15)	1394.96	1917.06	(0.03, 0.52, 0.83, 0.19)
$1^-(2^{++})$	7⟩	1819.26	1747.17	(0.25, 0.97)	1777.91	1759.23	(−0.65, 0.76)
	8⟩	1752.08	1824.17	(0.97, −0.25)	1773.25	1791.92	(0.76, 0.65)
For $ss\bar{s}\bar{s}$ state							
$0^+(0^{++})$	1⟩	2189.93	2177.52	(0.00, 1.00)	2217.96	2155.79	(0.00, 1.00)
	7⟩	2177.52	2189.93	(1.00, 0.00)	2155.79	2217.96	(1.00, 0.00)
$0^-(1^{+-})$	7⟩	2181.24			2151.32		
$0^+(2^{++})$	7⟩	2188.25			2141.86		

suggest that this state is a plausible candidate for $h_1(1965)$, whose mass and width are 1965 ± 45 MeV and 345 ± 75 [110]. In Ref. [13], the $1S$ -wave light tetraquark state with $I = 0$, $J = 1$ predicted by nonrelativistic quark models aligns with the $h_1(1595)$, while the $2S$ -wave state corresponds to $h_1(1965)$. Further analysis by Wang *et al.* using the modified Godfrey-Isgur model and QPC approach suggests that $h_1(1595)$ is a promising candidate for the $2^1P_1 n\bar{n}$ state, $h_1(1900)$ is more likely to correspond to the $2^1P_1 s\bar{s}$, and $h_1(1965)$ may correspond to the $3^1P_1 q\bar{q}$ state [12]. These interpretations imply that these states may have hybrid structures: $h_1(1595)$ could be a mixture of a $2P - n\bar{n}$ and a $1S - ns\bar{n}\bar{s}$ tetraquark component; $h_1(1900)$ might incorporate both a $2P$ -wave $s\bar{s}$ component and a

$1S$ -wave $ns\bar{n}\bar{s}$ tetraquark component; while $h_1(1965)$ may feature a $3P$ -wave $n\bar{n}$ component mixed with both $1S$ -wave and $2S$ -wave $ns\bar{n}\bar{s}$ tetraquark components.

For the $I^G(J^{PC}) = 0^+(2^{++})$ sector, there are two energy states predicted in both models. The lowest energy predictions are 1761.97 and 1760.91 MeV. These values suggest that this state is a plausible candidate for $f_2(1810)$, whose mass is 1815 ± 12 MeV [23,111,112]. Concerning $f_2(1810)$ being interpretable as $[q\bar{q}]$ [112], the physical f_2 state might be a hybrid and decay into $\eta\eta$. In Refs. [113,114], Etkin *et al.* perform a partial wave analysis of the reaction $\pi^- p \rightarrow \phi\phi n$, and reveal three resonances with quantum numbers $I^G(J^{PC}) = 0^+(2^{++})$ in the $\phi\phi$ channel, named $g_T, g_{T'}, g_{T''}$. Later, they were named

TABLE VIII. Comparison of the calculated masses of $ns\bar{n}\bar{s}$ and $ss\bar{s}\bar{s}$ states for the observed exotic states. Columns 2 and 3 present the results from Model I and II. Columns 4 to 6 list the mass predictions from other representative studies of tetraquark interpretations.

State	PDG [23] (MeV)	Model I (MeV)	Model II (MeV)	Ref. [13] (MeV)	Ref. [11] (MeV)	Ref. [10] (MeV)
$f_0(1370)$	1422	1378	1349	1431	1318	1179
$f_0(2020)$	1982	1925	1925	1986	1925	...
$f_0(2200)$	2178	2162	2210	2252	2095	2203
$h_1(1595)$	1594	1660	1656	1678	1586	...
$h_1(1900)$	1911	1892	1914	...	...	...
$h_1(1965)$	1965	1988	1976	2081	...	1942
$f_2(1810)$	1815	1762	1761	...	1755	...
$f_2(2010)$	2011	1954	1927	...	1946	2097
$a_0(980)$	980	927	913	...	1318	992
$a_0(1450)$	1439	1430	1399	...	1590	1480
$a_0(1950)$	1931	1929	1929	...	...	...
$h_1(2215)$	2215	2181	2151	...	...	2267
$f_2(2150)$	2157	2188	2141	...	...	...

$f_2(2010)$, $f_2(2300)$, $f_2(2340)$, respectively, where the masses and widths are

$$(M_{f_2(2010)} = 2011^{+62}_{-76}, \Gamma_{f_2(2010)} = 202^{+67}_{-62}) \text{ MeV},$$

$$(M_{f_2(2300)} = 2297 \pm 28, \Gamma_{f_2(2300)} = 149 \pm 41) \text{ MeV},$$

$$(M_{f_2(2340)} = 2339 \pm 55, \Gamma_{f_2(2340)} = 319^{+81}_{-69}) \text{ MeV}.$$

In Model I, the highest energy of the S -wave $ns\bar{n}\bar{s}$ state with $I^G(J^{PC}) = 0^+(2^{++})$ is 1953.75 MeV, while in Model II it is 1927.39 MeV. Numerically, both are consistent with the mass of $f_2(2010)$, suggesting that $f_2(2010)$ may contain a significant $ns\bar{n}\bar{s}$ tetraquark state. In contrast, $f_2(2300)$ and $f_2(2340)$ are inconsistent with the predictions of two models and cannot be explained as a compact tetraquark state here.

For the $I^G(J^{PC}) = 1^-(0^{++})$ sector, Model I yields four states with energies of 927.46, 1430.18, 1588.32, and 1929.01 MeV, while Model II yields 912.70, 1399.10, 1570.22, and 1928.55 MeV, respectively. The lowest energy predictions respond to $a_0(980)$, whose mass and width are 980 ± 20 MeV and 50–100 MeV [21,23,115]. The second-lowest energy predictions respond to $a_0(1450)$, whose mass and width are 1439 ± 34 MeV and 258 ± 14 MeV [23,116–118]. The predictions with the highest energy respond to $a_0(1950)$, whose mass and width are 1931 ± 26 MeV and 270 ± 40 MeV [23,119]. In Ref. [11], the $1S$ light tetraquark states with $I^G(J^{PC}) = 1^-(0^{++})$ correspond to $a_0(980)$ and the $2S$ -wave light tetraquark states with $I^G(J^{PC}) = 1^-(0^{++})$ correspond to $a_0(1450)$. Therefore, the $a_0(980)$ state may contain a component of $1S$ tetraquark, while the $a_0(1450)$ is complicated.

Note that for the $ns\bar{n}\bar{s}$ tetraquark system, the present two models consistently predict the state with quantum numbers $I = 0$ to be higher in energy than the corresponding $I = 1$ state. This mass inversion results from the unique behavior of instanton-induced potentials in low-lying tetraquark configurations. The observation reveals:

- (1) Conventional interactions maintain isospin symmetry, contributing equally to all states;
- (2) The instanton potential exhibits a striking isospin dependence:
 - (a) For the interaction between ns or $\bar{n}\bar{s}$: identical negative contributions to both $I = 0$ and $I = 1$ channels
 - (b) For the interaction between $n\bar{n}$:
 - (i) Positive energy shift ($+\Delta E$) in $I = 0$ sector
 - (ii) Negative shift ($-\Delta E$) in $I = 1$ sector

This discrepancy pattern ultimately leads to the energy splitting between isospin eigenstates. To cancel the energy gap induced by instanton effects, a viable approach is to introduce $q\bar{q}$ mixing coupling, as detailed in Ref. [6].

For the $ss\bar{s}\bar{s}$ tetraquark states, both models predict a state in the $0^-(1^{+-})$ sector and a state in the $0^+(2^{++})$ sector. The predictions in the $0^-(1^{+-})$ sector are 2181.24 and 2151.32 MeV. These values suggest that this state is a plausible candidate for $h_1(2215)$, whose mass and width are 2215 ± 40 MeV and 325 ± 55 [110]. The predictions in the $0^+(2^{++})$ sector are 2188.25 and 2141.86 MeV. These values suggest that this state is a plausible candidate for $f_2(2150)$, whose mass and width are 2157 ± 12 MeV and 152 ± 30 MeV [23,120–123]. In Ref. [10], Ebert *et al.* employ the quasipotential approach within quantum chromodynamics to evaluate the spectrum of light tetraquarks. They predicted the mass of the $ss\bar{s}\bar{s}$ state with $J^{PC} = 1^{+-}$ is 2267 MeV, aligning with $h_1(2215)$. Additionally, there is a prediction of the $ns\bar{n}\bar{s}$ state with $J^{PC} = 2^{++}$ at 2097 MeV, which corresponds to $f_2(2150)$ [10]. Therefore, $h_1(2215)$ may contain a component of $ss\bar{s}\bar{s}$, while $f_2(2150)$ may contain components of $ns\bar{n}\bar{s}$ and $ss\bar{s}\bar{s}$.

C. $nn\bar{n}\bar{s}$ and $ns\bar{s}\bar{s}$ systems

The configurations of $nn\bar{n}\bar{s}$ and $ns\bar{s}\bar{s}$ are organized in the Table IX. The energy eigenstates are presented in Table X, where the component of the tetraquark state, the

TABLE IX. Configurations of $nn\bar{n}\bar{s}$ and $ns\bar{s}\bar{s}$ systems.

$nn\bar{n}\bar{s}$ states							
I	J^P	Configurations					
$\frac{1}{2}$	0^+	$ 1\rangle$	$ 2\rangle$	$ 7\rangle$	$ 8\rangle$		
$\frac{1}{2}$	1^+	$ 3\rangle$	$ 4\rangle$	$ 5\rangle$	$ 6\rangle$	$ 7\rangle$	$ 8\rangle$
$\frac{1}{2}$	2^+	$ 7\rangle$	$ 8\rangle$				
$\frac{3}{2}$	0^+	$ 1\rangle$	$ 7\rangle$				
$\frac{3}{2}$	1^+	$ 3\rangle$	$ 6\rangle$	$ 7\rangle$			
$\frac{3}{2}$	2^+	$ 7\rangle$					
For $ns\bar{s}\bar{s}$ states							
I	J^P	Configurations					
$\frac{1}{2}$	0^+	$ 1\rangle$	$ 7\rangle$				
$\frac{1}{2}$	1^+	$ 4\rangle$	$ 5\rangle$	$ 7\rangle$			
$\frac{1}{2}$	2^+	$ 7\rangle$					

corresponding quantum numbers $I(J^P)$, and the configurations for each state are given in columns 1 and 3, the pure configuration calculation results, the configurations mixing results, and the corresponding configurations mixing coefficients in Model I are presented in columns 4 to 6, while the calculation results of Model II are given in columns 7 to 9. Additionally, the mass spectra of $nn\bar{n}\bar{s}$ and $ns\bar{s}\bar{s}$ are shown in Fig. 3.

For the $nn\bar{n}\bar{s}$ system, four distinct energy levels are predicted in the $I(J^P) = \frac{1}{2}(0^+)$ sector. The second-highest energy predictions are 1448.06 and 1429.79 MeV. These values suggest that this state is a plausible candidate for $K_0^*(1430)$, whose mass and width are 1425 ± 50 MeV and 270 ± 80 MeV [20,23,124–126]. In Ref. [10], Ebert *et al.* predict the $nn\bar{n}\bar{s}$ state with $J^P = 0^+$ at 1332 MeV, which aligns with $K_0^*(1430)$. Meanwhile, Ref. [11] interprets $K_0^*(1430)$ as a tetraquark with quantum numbers $N^{2S+1}L_J = 2^1S_0$. Therefore, $K_0^*(1430)$ may exhibit a hybrid structure incorporating both 1S and 2S tetraquark components.

In the $I(J^P) = \frac{1}{2}(1^+)$ sector, there are six distinct energy levels for $nn\bar{n}\bar{s}$ states and three distinct energy levels for $ns\bar{s}\bar{s}$ states predicted in each model. The second-lowest energy predictions are 1173.66 and 1145.81 MeV. These values suggest that this state is a plausible candidate for $K_1(1270)$, whose mass and width are 1253 ± 7 MeV and 101 ± 12 MeV [23,127–131]. The third-lowest energy predictions are 1404.91 and 1377.45 MeV. These values suggest that this state is a plausible candidate for $K_1(1400)$, whose mass and width are 1403 ± 7 MeV and 174 ± 13 MeV [23,129,132,133]. The second-highest and highest energy predictions of $nn\bar{n}\bar{s}$ and the lowest energy predictions of $ns\bar{s}\bar{s}$ are aligned well with $K_1(1650)$, whose mass and width are 1650 ± 50 MeV and 150 ± 50 MeV [23,133–136]. Thus $K_1(1650)$ may contain both $nn\bar{n}\bar{s}$ and $ns\bar{s}\bar{s}$ tetraquark components. As shown in Table XI,

TABLE X. Numerical results for the $nn\bar{n}\bar{s}$ and $ns\bar{s}\bar{s}$ states in two models.

		Model I				Model II			
State	$I(J^P)$	Conf.	Pure E (MeV)	E (MeV)	Configurations mixing Mixing coefficients	Pure E (MeV)	E (MeV)	Configurations mixing Mixing coefficients	
$nn\bar{n}\bar{s}$	$\frac{1}{2}(0^+)$	[1]	1593.33	643.25	(−0.16, 0.68, 0.21, 0.68)	1609.33	624.73	(0.17, 0.66, 0.22, 0.69)	
		[2]	991.05	1233.22	(0.44, 0.64, −0.51, −0.37)	993.30	1217.90	(−0.44, 0.63, −0.56, −0.31)	
		[7]	1493.91	1448.06	(0.42, −0.36, −0.54, 0.63)	1448.73	1429.79	(−0.33, −0.41, −0.54, 0.65)	
		[8]	1040.78	1794.66	(0.77, −0.03, 0.63, 0.01)	1022.79	1801.74	(0.81, 0.03, −0.58, −0.05)	
$nn\bar{n}\bar{s}$	$\frac{1}{2}(1^+)$	[3]	1430.13	1127.41	(−0.46, −0.11, 0.10, 0.39, 0.17, 0.77)	1391.39	106.76	(0.49, −0.16, −0.12, −0.40, −0.14, 0.73)	
		[4]	1286.03	1173.66	(0.07, 0.82, −0.47, −0.08, −0.12, 0.29)	1259.77	1145.81	(−0.04, 0.82, −0.45, −0.04, −0.11, −0.33)	
		[5]	1503.15	1404.91	(0.61, −0.17, 0.04, −0.41, 0.46, 0.46)	1510.89	1377.45	(0.63, 0.14, 0.05, −0.33, 0.50, −0.47)	
		[6]	1526.85	1533.85	(0.02, 0.53, 0.66, 0.21, 0.46, −0.18)	1536.47	1520.11	(−0.14, −0.50, 0.53, 0.13, 0.60, 0.26)	
		[7]	1561.39	1607.98	(0.61, −0.01, 0.21, 0.54, −0.51, 0.15)	1515.66	1588.58	(0.55, 0.13, −0.43, −0.54, 0.43, 0.11)	
		[8]	1245.98	1705.71	(0.17, −0.11, −0.53, 0.58, 0.53, −0.23)	1236.29	1711.75	(−0.20, 0.13, −0.56, 0.64, 0.41, 0.23)	
$nn\bar{n}\bar{s}$	$\frac{1}{2}(2^+)$	[7]	1684.84	1587.09	(0.34, 0.94)	1639.61	1592.21	(0.61, 0.79)	
		[8]	1600.07	1697.62	(0.94, −0.34)	1621.03	1668.43	(0.79, −0.61)	
$nn\bar{n}\bar{s}$	$\frac{3}{2}(0^+)$	[1]	1493.27	1027.14	(0.64, 0.77)	1498.37	976.37	(−0.61, 0.79)	
		[7]	1344.40	1810.53	(0.77, −0.64)	1285.66	1807.67	(0.79, 0.61)	
$nn\bar{n}\bar{s}$	$\frac{3}{2}(1^+)$	[3]	1328.35	1128.27	(0.75, −0.63, −0.19)	1294.92	1101.95	(0.77, 0.60, 0.20)	
		[6]	1422.54	1453.77	(0.25, 0.01, 0.97)	1422.91	1403.28	(−0.24, 0.00, 0.97)	
		[7]	1446.70	1615.55	(0.60, 0.78, −0.16)	1395.96	1608.56	(−0.59, 0.79, −0.15)	
$nn\bar{n}\bar{s}$	$\frac{3}{2}(2^+)$	[7]	1631.45			1683.17			
$ns\bar{s}\bar{s}$	$\frac{1}{2}(0^+)$	[1]	1949.62	1794.42	(0.63, 0.76)	1965.72	1773.32	(0.23, 0.97)	
		[7]	1897.46	2052.66	(0.76, −0.63)	1862.30	2054.70	(0.97, −0.23)	
$ns\bar{s}\bar{s}$	$\frac{1}{2}(1^+)$	[4]	1790.52	1702.83	(0.73, 0.54, −0.41)	1761.46	1680.92	(0.76, 0.49, −0.42)	
		[5]	1888.27	1890.87	(0.68, −0.53, 0.50)	1897.84	1868.71	(0.64, −0.45, 0.62)	
		[7]	1932.54	2017.63	(−0.05, 0.65, 0.76)	1894.56	2004.23	(−0.12, 0.74, −0.66)	
$ns\bar{s}\bar{s}$	$\frac{1}{2}(2^+)$	[7]	1998.37			1956.38			

TABLE XI. Comparison of the calculated masses of $nn\bar{n}\bar{s}$ and $ns\bar{s}\bar{s}$ states for the observed exotic states. Columns 2 and 3 present the results from Models I and II. Columns 4 to 6 list the mass predictions from other representative studies of tetraquark interpretations.

State	PDG [23] (MeV)	Model I (MeV)	Model II (MeV)	Ref. [13] (MeV)	Ref. [11] (MeV)	Ref. [10] (MeV)
$K_0^*(1430)$	1425	1448	1430	...	1380	1332
$K_1(1270)$	1253	1174	1146	...	1233	1057
$K_1(1400)$	1403	1405	1377	...	1456	...
$K_1(1650)$	1650	1703	1681	...	1644	1855
$K_0^*(1950)$	1957	2053	2055	...	1968	...
$K_2^*(1980)$	1990	1998	1956	...	1968	2001

in Ref. [11], Deng *et al.* interpret $K_1(1270)$ as a $nn\bar{n}\bar{s}$ tetraquark state with quantum numbers $N^{2S+1}L_J = 1^3S_1$ and $K_1(1400)$ as its radial excitation with quantum numbers $N^{2S+1}L_J = 2^3S_1$. Incorporating predictions from the two models presented herein, $K_1(1270)$ is predicted to dominate as a $nn\bar{n}\bar{s}$ tetraquark component, while $K_1(1400)$ may exhibit a mixed structure incorporating both $1S$ - and $2S$ -tetraquark components.

For the $ns\bar{s}\bar{s}$ system, both models predict two states in the $I(J^P) = \frac{1}{2}(0^+)$ sector and one state in $I(J^P) = \frac{1}{2}(2^+)$. The predictions with the highest energy in the $\frac{1}{2}(0^+)$ sector are 2052.66 and 2054.70 MeV. These values suggest that this state is a plausible candidate for $K_0^*(1950)$, whose mass and width are 1957 ± 14 MeV and 170 ± 50 [20,23,126,137,138]. The predictions in the $\frac{1}{2}(2^+)$ sector are 1998.37 and 1956.38 MeV. These values suggest that this state is a plausible candidate for $K_2^*(1980)$, whose mass and width are 1990^{+60}_{-50} MeV and 348^{+50}_{-30} MeV [23,139–141]. In Ref. [11], Deng *et al.* interpret $K_0^*(1950)$ as a $nn\bar{n}\bar{s}$ tetraquark state with quantum numbers $N^{2S+1}L_J = 1^5D_0$ and $K_2^*(1980)$ as a $nn\bar{n}\bar{s}$ tetraquark state with quantum numbers $N^{2S+1}L_J = 2^1D_2$. Incorporating predictions from the two models presented herein, $K_0^*(1950)$ may contain a mixed structure incorporating both a $1S$ $ns\bar{s}\bar{s}$ component and $1D$ $nn\bar{n}\bar{s}$ component, while $K_2^*(1980)$ could exhibit a hybrid composition blending a $1S$ $ns\bar{s}\bar{s}$ component and $2D$ $nn\bar{n}\bar{s}$ component. Furthermore, several mesons in the $I = \frac{3}{2}$ sector are predicted, see Table X.

IV. SUMMARY AND DISCUSSION

Investigations on the structure and properties of the light tetraquark states have a long history, starting from the 1970s. Theoretically, the spectrum of the $qq\bar{q}\bar{q}$ states has been studied using different approaches, such as various quark models, the QCD sum rule, and the lattice QCD.

In present work, the nonrelativistic quark model is adapted to study the spectrum of low-lying light tetraquark states, where two confinement potential schemes and the instanton-induced interaction are employed. The numerical results are consistent with several physical light exotic hadron candidates, indicating that the corresponding compact tetraquark components may contribute to mesons.

For the $nn\bar{n}\bar{n}$ system, both models anticipate that $f_0(500)$ may contain a compact tetraquark component with $I^G(J^{PC}) = 0^+(0^{++})$. One state characterized by the quantum numbers $I^G(J^{PC}) = 0^-(1^{+-})$ is energetically located in close proximity to the resonance $h_1(1170)$. Two states with $I^G(J^{PC}) = 0^+(2^{++})$ are predicted to coincide with the resonances $f_2(1430)$ and $f_2(1640)$, respectively. Furthermore, two states with $I^G(J^{PC}) = 1^-(1^{++})$ are expected to align with $a_1(1260)$ and $a_1(1640)$, and a state with $I^G(J^{PC}) = 1^-(2^{++})$ may relate to $a_2(1700)$, respectively.

For the $ns\bar{n}\bar{s}$ system, both models anticipate that four states characterized by the quantum numbers $I^G(J^{PC}) = 0^+(0^{++})$ are predicted in close proximity to the masses of $f_0(1370)$, $f_0(1500)$, $f_0(2020)$, and $f_0(2200)$, respectively. Three states with $I^G(J^{PC}) = 0^-(1^{+-})$ are predicted to coincide with the masses of $h_1(1595)$, $h_1(1900)$, and $h_1(1965)$, respectively. Two states with $I^G(J^{PC}) = 0^+(2^{++})$ are predicted to coincide with the masses $f_2(1810)$ and $f_2(2010)$, respectively. Similarly, three states possessing $I^G(J^{PC}) = 1^-(0^{++})$ are expected to correlate with $a_0(980)$, $a_0(1450)$, and $a_0(1950)$. Moreover, in the $ss\bar{s}\bar{s}$ system, it is predicted that a single $0^-(1^{+-})$ state aligns with $h_1(2215)$ and a single $0^+(2^{++})$ state responds to $f_2(2150)$.

The investigation identifies a $nn\bar{n}\bar{s}$ tetraquark state with quantum numbers $I(J^P) = \frac{1}{2}(0^+)$ aligned with $K_0^*(1430)$ in the $nn\bar{n}\bar{s}$ system, along with two $nn\bar{n}\bar{s}$ tetraquark states with $I(J^P) = \frac{1}{2}(1^+)$ corresponding to $K_1(1270)$ and $K_1(1400)$. Meanwhile, in the $ns\bar{s}\bar{s}$ system, both models predict a single state with $I(J^P) = \frac{1}{2}(0^+)$ near $K_0^*(1950)$ and a single state with $I(J^P) = \frac{1}{2}(2^+)$ responds to $K_2^*(1980)$. Moreover, $K_1(1650)$ may contain both $nn\bar{n}\bar{s}$ and $ns\bar{s}\bar{s}$ tetraquark components with $I(J^P) = \frac{1}{2}(1^+)$.

Additionally, several predictions have not been detected yet. The isotensor state in the window of 1500–1700 MeV might be very interesting, and is potentially to be observed in 4π invariant mass distributions, which deserves further studies both from the theoretical and the physical sides.

In summary, the low-lying tetraquark spectrum, with $J^P = 0^+, 1^+, 2^+$, is explored in the nonrelativistic quark model including the Cornell potential (Model I) and

instanton interaction in the linear potential (Model II), where the pseudoscalar meson exchange is introduced as the long range interaction between quarks. The analysis of the pattern of the spectrum provides a hint to understanding the observations and inspires studies on the light meson spectrum.

In addition, one may notice that the instanton-induced interaction can provide us not only the information on mass splitting of the studied tetraquark states, but also the coupling between traditional $q\bar{q}$ states and the tetraquark states; this kind of coupling should lead to mixing of $q\bar{q}$ and $qq\bar{q}\bar{q}$ states. Corresponding investigations are now under consideration.

ACKNOWLEDGMENTS

This research is supported by the Fundamental Research Funds for the Central Universities under Grants No. SWU-

XDJH202304 and No. SWU-KQ25016, and the National Natural Science Foundation of China under Grant No. 12305096.

DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX: EXPRESSIONS OF FLAVOR WAVE FUNCTIONS

1. Flavor wave functions of $nn\bar{n}\bar{n}$

For the $nn\bar{n}\bar{n}$ system (n stands for u, d quarks), there are 16 flavor configurations for $nn\bar{n}\bar{n}$, that are $uu\bar{u}\bar{u}$, $uu\bar{u}\bar{d}$, $ud\bar{u}\bar{u}$, $ud\bar{u}\bar{d}$, $du\bar{u}\bar{u}$, $ud\bar{d}\bar{d}$, $ud\bar{u}\bar{d}$, $du\bar{u}\bar{d}$, $udd\bar{u}$, $udd\bar{d}$, $dd\bar{u}\bar{d}$, $dd\bar{d}\bar{u}$, $dd\bar{d}\bar{d}$, which are classified into isospin space,

(i) $I = 0$.

$$([nn]_{I=0}[\bar{n}\bar{n}]_{I=0})^0 = \frac{1}{2}(ud\bar{u}\bar{d} - du\bar{u}\bar{d} - ud\bar{d}\bar{u} + du\bar{d}\bar{u}), \quad (\text{A1})$$

$$([nn]_{I=1}[\bar{n}\bar{n}]_{I=1})^0 = \frac{1}{\sqrt{12}}(2uu\bar{u}\bar{u} + ud\bar{u}\bar{d} + ud\bar{d}\bar{u} + du\bar{u}\bar{d} + du\bar{d}\bar{u} + 2dd\bar{d}\bar{d}). \quad (\text{A2})$$

(ii) $I = 1$.

$$([nn]_{I=1}[\bar{n}\bar{n}]_{I=0})^+ = \frac{1}{\sqrt{2}}(uu\bar{u}\bar{d} - uu\bar{d}\bar{u}), \quad (\text{A3})$$

$$([nn]_{I=1}[\bar{n}\bar{n}]_{I=0})^0 = \frac{1}{2}(ud\bar{u}\bar{d} - ud\bar{d}\bar{u} + du\bar{u}\bar{d} - du\bar{d}\bar{u}), \quad (\text{A4})$$

$$([nn]_{I=1}[\bar{n}\bar{n}]_{I=0})^- = \frac{1}{\sqrt{2}}(dd\bar{u}\bar{d} - dd\bar{d}\bar{u}), \quad (\text{A5})$$

$$([nn]_{I=0}[\bar{n}\bar{n}]_{I=1})^+ = \frac{1}{\sqrt{2}}(ud\bar{u}\bar{u} - du\bar{u}\bar{u}), \quad (\text{A6})$$

$$([nn]_{I=0}[\bar{n}\bar{n}]_{I=1})^0 = -\frac{1}{2}(ud\bar{u}\bar{d} + ud\bar{d}\bar{u} - du\bar{u}\bar{d} - du\bar{d}\bar{u}), \quad (\text{A7})$$

$$([nn]_{I=0}[\bar{n}\bar{n}]_{I=1})^- = \frac{1}{\sqrt{2}}(ud\bar{d}\bar{d} - du\bar{d}\bar{d}), \quad (\text{A8})$$

$$([nn]_{I=1}[\bar{n}\bar{n}]_{I=1})^+ = -\frac{1}{2}(uu\bar{u}\bar{d} + uu\bar{d}\bar{u} + ud\bar{d}\bar{d} + du\bar{d}\bar{d}), \quad (\text{A9})$$

$$([nn]_{I=1}[\bar{n}\bar{n}]_{I=1})^0 = \frac{1}{\sqrt{2}}(uu\bar{u}\bar{u} - dd\bar{d}\bar{d}), \quad (\text{A10})$$

$$([nn]_{I=1}[\bar{n}\bar{n}]_{I=1})^- = \frac{1}{2}(ud\bar{u}\bar{u} + du\bar{u}\bar{u} + dd\bar{u}\bar{d} + dd\bar{d}\bar{u}). \quad (\text{A11})$$

(iii) $I = 2$.

$$([nn]_{I=1}[\bar{n}\bar{n}]_{I=1})^{++} = uu\bar{d}\bar{d}, \quad (\text{A12})$$

$$([nn]_{I=1}[\bar{n}\bar{n}]_{I=1})^+ = -\frac{1}{2}(uu\bar{u}\bar{d} + uu\bar{d}\bar{u} - u\bar{d}\bar{d}\bar{d} - d\bar{u}\bar{d}\bar{d}), \quad (\text{A13})$$

$$([nn]_{I=1}[\bar{n}\bar{n}]_{I=1})^0 = \frac{1}{\sqrt{6}}(uu\bar{u}\bar{u} - u\bar{d}\bar{u}\bar{d} - u\bar{d}\bar{d}\bar{u} - d\bar{u}\bar{d}\bar{u} - d\bar{u}\bar{u}\bar{d} + d\bar{d}\bar{d}\bar{d}), \quad (\text{A14})$$

$$([nn]_{I=1}[\bar{n}\bar{n}]_{I=1})^- = \frac{1}{2}(u\bar{d}\bar{u}\bar{u} + d\bar{u}\bar{u}\bar{u} - d\bar{d}\bar{u}\bar{d} - d\bar{d}\bar{d}\bar{u}), \quad (\text{A15})$$

$$([nn]_{I=1}[\bar{n}\bar{n}]_{I=1})^{--} = d\bar{d}\bar{u}\bar{u}. \quad (\text{A16})$$

2. Flavor wave functions of $ns\bar{n}\bar{s}$ and $ss\bar{s}\bar{s}$

For the $ns\bar{n}\bar{s}$ and $ss\bar{s}\bar{s}$ systems (n stands for u, d quarks), there are four flavor configurations with respect to $ns\bar{n}\bar{s}$. The tetraquark states are classified as,

(i) $I = 0$.

$$([ns][\bar{n}\bar{s}])^0 = \frac{1}{\sqrt{2}}(us\bar{u}\bar{s} + ds\bar{d}\bar{s}). \quad (\text{A17})$$

(ii) $I = 1$.

$$([ns][\bar{n}\bar{s}])^+ = -us\bar{d}\bar{s}, \quad (\text{A18})$$

$$([ns][\bar{n}\bar{s}])^0 = \frac{1}{\sqrt{2}}(us\bar{u}\bar{s} - ds\bar{d}\bar{s}), \quad (\text{A19})$$

$$([ns][\bar{n}\bar{s}])^- = ds\bar{u}\bar{s}. \quad (\text{A20})$$

3. Flavor wave functions of $nn\bar{n}\bar{s}$ and $ns\bar{s}\bar{s}$

Note that $[ns][\bar{n}\bar{n}]$, $[ss][\bar{n}\bar{s}]$ are the antiparticles of $nn\bar{n}\bar{s}$, $ns\bar{s}\bar{s}$, respectively. Therefore, we just consider the $nn\bar{n}\bar{s}$ and $ns\bar{s}\bar{s}$ systems in the strangeness $S = 1$ sector. There are eight configurations with respect to $nn\bar{n}\bar{s}$, namely $uu\bar{u}\bar{s}$, $u\bar{u}\bar{d}\bar{s}$, $u\bar{d}\bar{u}\bar{s}$, $u\bar{d}\bar{d}\bar{s}$, $d\bar{u}\bar{d}\bar{s}$, $d\bar{d}\bar{u}\bar{s}$, $d\bar{d}\bar{d}\bar{s}$, and two for

$ns\bar{s}\bar{s}$, that are $us\bar{s}\bar{s}$, $ds\bar{s}\bar{s}$. The tetraquark states are grouped into $I = 1/2$ and $3/2$ sectors,

(i) $I = \frac{1}{2}$.

$$([nn]_{I=0}[\bar{n}\bar{s}]_{I=\frac{1}{2}})^+ = -u\bar{d}\bar{d}\bar{s}, \quad (\text{A21})$$

$$([nn]_{I=0}[\bar{n}\bar{s}]_{I=\frac{1}{2}})^0 = u\bar{d}\bar{u}\bar{s}, \quad (\text{A22})$$

$$([nn]_{I=1}[\bar{n}\bar{s}]_{I=\frac{1}{2}})^+ = \sqrt{\frac{2}{3}}uu\bar{u}\bar{s} + \sqrt{\frac{1}{3}}u\bar{d}\bar{d}\bar{s}, \quad (\text{A23})$$

$$([nn]_{I=1}[\bar{n}\bar{s}]_{I=\frac{1}{2}})^0 = \sqrt{\frac{1}{3}}u\bar{d}\bar{u}\bar{s} + \sqrt{\frac{2}{3}}d\bar{d}\bar{d}\bar{s}, \quad (\text{A24})$$

$$([ns]_{I=\frac{1}{2}}[\bar{s}\bar{s}]_{I=0})^+ = us\bar{s}\bar{s}, \quad (\text{A25})$$

$$([ns]_{I=\frac{1}{2}}[\bar{s}\bar{s}]_{I=0})^0 = ds\bar{s}\bar{s}. \quad (\text{A26})$$

(ii) $I = \frac{3}{2}$.

$$([nn]_{I=1}[\bar{n}\bar{s}]_{I=\frac{1}{2}})^{++} = -uu\bar{d}\bar{s}, \quad (\text{A27})$$

$$([nn]_{I=1}[\bar{n}\bar{s}]_{I=\frac{1}{2}})^+ = \sqrt{\frac{1}{3}}uu\bar{u}\bar{s} - \sqrt{\frac{2}{3}}u\bar{d}\bar{d}\bar{s}, \quad (\text{A28})$$

$$([nn]_{I=1}[\bar{n}\bar{s}]_{I=\frac{1}{2}})^0 = \sqrt{\frac{2}{3}}u\bar{d}\bar{u}\bar{s} - \sqrt{\frac{1}{3}}d\bar{d}\bar{d}\bar{s}, \quad (\text{A29})$$

$$([nn]_{I=1}[\bar{n}\bar{s}]_{I=\frac{1}{2}})^- = d\bar{d}\bar{u}\bar{s}. \quad (\text{A30})$$

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