

QED photon-fermion vertex from its Dyson-Schwinger equation in 4D: The full vertex, the transverse form factors, and the perturbative solution

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We investigate the Dyson-Schwinger equation for the photon-fermion one-particle irreducible vertex in QED in linear covariant gauges. The longitudinal component of this vertex is described using the Ball-Chiu basis, while its transverse part is expressed with the Kızılersu-Reenders-Pennington basis. Combining the vertex Ward-Takahashi identity with the vertex equation, we derive a set of exact, nonlinear integral equations governing the transverse vertex. These equations hold for any linear covariant gauge and must be solved self-consistently. We discuss several approximations to the exact equations, generalizing results previously obtained at the perturbative one-loop level. Various kinematical configurations are also examined. Furthermore, we compute the perturbative solution of the transverse vertex Dyson-Schwinger equations for all transverse form factors and derive their perturbative asymptotic expressions.

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I. INTRODUCTION AND MOTIVATION

The traditional approach to solve quantum electrodynamics (QED) relies on perturbation theory (PT), which provides the standard diagrammatic representation of the solution, and is also at the heart of our understanding of quantum field theory (QFT). Although important, PT cannot tell the full story, as there are phenomena, such as chiral symmetry breaking, whose explanation lies beyond this approach. This is particularly true when the coupling is large enough, challenging the applicability of PT itself.

For QED the expansion parameter is the fine-structure constant that has the value $\alpha = 7.2973525693(11) \times 10^{-3}$ [1]. Its smallness supports the use of perturbative methods. However, one might question whether the perturbative expansion converges or define an asymptotic series, see, e.g., [2] and references therein. For large couplings, perturbation theory becomes unreliable and new phenomena emerge. For example, at sufficiently large couplings the theory predicts a phase transition, that can be associated with dynamical mass generation, see, e.g., [3–8] and references therein. The study of QED at large couplings is not a purely academic problem. It offers insights into the solutions and phenomena expected to occur within quantum chromodynamics (QCD), see, e.g., [9–12]. Exploring

either QED at strong coupling or QCD requires non-perturbative methods.

The infinite tower of Dyson-Schwinger equations (DSEs), which relate all the Green functions of the theory, provide a way to solve any QFT, regardless of the coupling strength. The DSEs allow to go beyond the traditional perturbative approach, offering a powerful alternative to analyze and solve quantum field theories.

This work aims to study QED in a general linear covariant gauge, defined by a gauge fixing parameter ξ , using the framework of the DSE. The goal is to combine simultaneously the integral equations for the two point functions, i.e., the fermion and photon propagators, and the equation for the photon-fermion vertex, to gather information on the vertex from first principles, and to explore other gauges than the Landau gauge. Hopefully, this approach will deliver insights to help understanding the DSE for non-Abelian gauge theories, that include the standard model of particle physics or QCD. A precise understanding of the photon-fermion vertex has implications extending far beyond QED. In the standard model of particle physics, a detailed knowledge of this vertex is essential to characterize other vector boson couplings, both to disentangle the contributions from different interactions and to probe potential signatures of physics beyond the standard model.

Another first-principles approach to QFT are Monte Carlo lattice simulations. These rely on the Euclidean formulation of the theories and are not suitable to be used in its Minkowski spacetime version. The lattice simulations requires a careful control of finite-volume and finite-spacing effects, together with the use of realistic physical parameters. For full QCD, the vertex longitudinal form factors were computed, in the soft gluon limit, recently in [13]. Previous

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lattice estimations of the form factors using the quenched approximation can be found in [14–16]. For QED, to the best knowledge of the author, the only lattice calculation of the photon-fermion vertex was reported in the proceeding [17]. While these results provide benchmarks, a proper separation of strong interaction effects from pure QED contributions must be performed before any meaningful comparison can be made.

The investigation of QED through DSEs presents numerous challenges, which must be addressed by building upon the extensive existing literature on DSE applications to gauge theories. While a comprehensive citation of all relevant works would be impractical, we have endeavored to maintain a focused bibliography and apologize for any omissions.

We will not reproduce the derivation of the integral equations here. The complete derivation, along with notational conventions, which we follow, can be found in [18]. The DSE formalism itself is well documented in the literature. For a general discussion of DSEs and their application to QED, we refer readers to [19], among other sources. The application of DSEs to QCD is discussed in, e.g., [20–32] (see also the references therein).

The vertex Dyson-Schwinger equation, together with the vertex Ward-Takahashi identity (WTI), provide a complete framework for calculating all twelve form factors required to fully describe this one-particle irreducible Green function. As is well established, the longitudinal form factors are determined by the WTI and depend solely on the fermion propagator functions. The computation of the transverse form factors builds on these results and, within the approach adopted herein, relies on the vertex integral equation.

From the vertex DSE, a set of exact nonlinear equations, that must be solved self-consistently, is derived. These equations apply to any general kinematical situation. However, some special kinematical situations will be discussed to help with the understanding of the vertex. In particular, the case with on-shell external fermions, leading to the on-shell vertex, from where one can read, for example, the fermion anomalous magnetic moments, is investigated. This anomalous magnetic moment will be expressed in terms of the fundamental form factors associated with the photon-fermion vertex.

In order to grasp the properties of the transverse form factors, approximations to the exact equations, that go beyond the usual perturbative approach, are discussed and, within the framework considered, it is shown that, in the chiral limit, the number of form factors describing the vertex is smaller than 12. This finding generalizes the conclusions coming from the one-loop perturbative solution of QED for a general covariant gauge. Moreover, our analysis facilitates the study of the various form factors for the full class of linear covariant gauges. Being a first step toward a complete computation of the QED photon-fermion vertex, the current study ends by looking at the perturbative solution, for all transverse form factors, of the

equations derived. Expressions for their asymptotic behavior are provided. Whenever possible, the solutions for the vertex are compared to the outcome of a one-loop perturbative calculation.

This work is organized as follows. In Sec. II the basic definitions are introduced, also to set the notation. In Sec. III the renormalization of QED is revisited and the integral equations to be studied are given in terms of physical quantities. A perturbative description of the renormalized equations, together with a brief discussion of infrared divergences, is given at the end of the section. The basis used to describe the vertex is introduced in Sec. IV, together with the solutions of the vertex Ward-Takahashi identity for its longitudinal components. In the current work, the Ball-Chiu set of operators [33] is used for the longitudinal vertex, while for the orthogonal components we rely on the operators introduced by Kizilersu-Reenders-Pennington [34]. In Sec. V we refer to the structure of the integral equations using the full vertex, that are given in the appendices. In Sec. VI the tensor decomposition of the vertex is discussed, together with the consideration of special kinematical configurations. In Sec. VII an on-shell vertex is defined and its form factors are given in terms of the form factors that are associated with the basis used to describe the vertex. In Sec. VIII the computation of the eight transverse form factors is discussed based on vertex Dyson-Schwinger integral equation. The exact equations derived have to be solved self-consistently, as they give the transverse form factor in terms of the full vertex itself, that is a function of all the longitudinal and transverse form factors. In Sec. IX an approximation scheme to the vertex equation, inspired in the setup built from perturbation theory, is discussed. In Sec. X, a perturbative analysis of the Dyson-Schwinger vertex equation is provided, including the discussion of the asymptotic behavior for all nonvanishing transverse form factors. Finally, in Sec. XI we summarize and conclude.

II. DEFINITIONS AND INTEGRAL EQUATIONS

Let us introduce the notations to be used throughout this work. Herein, all expressions are given in Minkowski spacetime. For linear covariant gauges, the photon propagator in momentum space reads

$$\begin{aligned} D_{\mu\nu}(k) &= -\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)D(k^2) - \frac{\xi}{k^2} \frac{k_\mu k_\nu}{k^2} \\ &= -P_{\mu\nu}^\perp(k)D(k^2) - \frac{\xi}{k^2} P_{\mu\nu}^L(k). \end{aligned} \quad (1)$$

The Lorentz invariant scalar function $D(k^2)$ will also be referred to as the photon propagator. The inverse of the fermion propagator is given by

$$S^{-1}(p) = A(p^2)\not{p} - B(p^2) + i\epsilon, \quad (2)$$

where $A(p^2)$ and $B(p^2)$ are Lorentz scalar functions and the limit $\epsilon \rightarrow 0^+$ is to be taken at the end of the calculations. In the following, unless clearly stated, the $i\epsilon$ term will be omitted from now on.

The photon-fermion one-particle irreducible Green function (1PI), also named photon-fermion vertex, will be written as $\Gamma^\mu(p, -p - k; k)$, where p is the incoming fermion momentum, $p + k$ is the outgoing fermion momentum, and k is the incoming photon momentum. This Green function has to comply with the QED gauge symmetry that relates its longitudinal part, relative to the photon momentum, to the fermion propagator via a WTI. The Ward-Takahashi vertex identity determines [33] (not uniquely) the longitudinal part of Γ^μ . We postpone the discussion on the vertex to later sections.

The bare DSEs to be considered are

$$S^{-1}(p) = (\not{p} - m) - ig^2 \int \frac{d^4 k}{(2\pi)^4} D_{\mu\nu}(k) [\gamma^\mu S(p - k) \times \Gamma^\nu(p - k, -p; k)] \quad (3)$$

for the fermion propagator and

$$\frac{1}{D(k^2)} = k^2 - i \frac{g^2}{3} \int \frac{d^4 p}{(2\pi)^4} \text{Tr}[\gamma_\mu S(p) \Gamma^\mu(p, -p + k; -k) \times S(p - k)] \quad (4)$$

for the photon propagator. The bare Dyson-Schwinger equation for the photon-fermion vertex reads

$$\begin{aligned} \Gamma^\mu(p, -p - k; k) = & \gamma^\mu + ig^2 \int \frac{d^4 q}{(2\pi)^4} D_{\zeta\zeta'}(q) \gamma^\zeta S(p - q) \\ & \times \{ \Gamma^\mu(p - q, -p - k + q; k) \\ & \times S(p + k - q) \Gamma^{\zeta'}(p + k - q, -p - k; q) \\ & + \Gamma^{\zeta'\mu}(p - q, -p - k; q, k) \}. \end{aligned} \quad (5)$$

The DSE for the vertex requires, besides the propagators and Γ^μ itself, the one-particle irreducible two-photon-two-fermion Green function $\Gamma^{\mu\nu}$. Gauge symmetry constrains df these two Green functions, with Γ^μ and $\Gamma^{\mu\nu}$ having to comply with WTIs. The vertex WTI has been discussed long ago [33]. The solution of the WTI for $\Gamma^{\mu\nu}$, for general kinematics, can be found in [18]. The solutions of the WTI determine, nonuniquely, the longitudinal components of the vertices—see the discussion in [19]. The 1PI function $\Gamma^{\mu\nu}$ is the solution of its own DSE, that calls for higher-order Green functions. An approximation to the DSE for $\Gamma^{\mu\nu}$ can be found in [18] but, given its complexity, it will not be considered in this work and, when building solutions of the above DSE, the contribution of $\Gamma^{\mu\nu}$ will be ignored. This is a drastic approximation that can be justified within a perturbative solution of the theory, given that, to lowest order in the coupling constant, $\Gamma^{\mu\nu}$ vanish. In perturbation

theory, to lowest order in the coupling constant, this 1PI Green function appears at the one-loop level and is of order $\mathcal{O}(g^4)$.

III. RENORMALIZED QED EQUATIONS

The renormalization constants Z_i relate bare and physical quantities and are defined as

$$\begin{aligned} A_\mu = \frac{Z_1}{Z_2 Z_3} A_\mu^{(\text{phys})}, \quad \psi = \frac{Z_1}{Z_2 Z_3} \psi^{(\text{phys})}, \quad g = \frac{Z_1}{Z_2 Z_3} g^{(\text{phys})}, \\ m = \frac{Z_0}{Z_2} m^{(\text{phys})}, \quad \text{and} \quad \xi = Z_3 \xi^{(\text{phys})}, \end{aligned} \quad (6)$$

and allow us to rewrite the DSE for the corresponding physical quantities. For QED the vertex WTI requires $Z_1 = Z_2$, see, e.g., [18,19], and the renormalized gap equation (3) reads

$$\begin{aligned} S^{-1}(p) = & Z_2 \not{p} - Z_0 m - ig^2 Z_2 \int \frac{d^4 k}{(2\pi)^4} D_{\mu\nu}(k) \\ & \times [\gamma^\mu S(p - k) \Gamma^\nu(p - k, -p; k)] \\ = & Z_2 \not{p} - Z_0 m - ig^2 Z_2 \Sigma(p), \end{aligned} \quad (7)$$

the renormalized photon gap equation (4) is

$$\begin{aligned} \frac{1}{D(k^2)} = & Z_3 k^2 - i \frac{g^2}{3} Z_2 \int \frac{d^4 p}{(2\pi)^4} \text{Tr}[\gamma_\mu S(p) \\ & \times \Gamma^\mu(p, -p + k; -k) S(p - k)] \\ = & Z_3 k^2 \left(1 - ig^2 \frac{Z_2}{Z_3} \Pi(k^2) \right), \end{aligned} \quad (8)$$

and the equation for the Green function (5) is given by

$$\begin{aligned} \Gamma^\mu(p, -p - k; k) = & Z_2 \gamma^\mu + ig^2 Z_2 \int \frac{d^4 q}{(2\pi)^4} D_{\zeta\zeta'}(q) \\ & \times \{ \gamma^\zeta S(p - q) \Gamma^\mu(p - q, -p - k + q; k) \\ & \times S(p + k - q) \Gamma^{\zeta'}(p + k - q, -p - k; q) \\ & + \gamma^\zeta S(p - q) \Gamma^{\zeta'\mu}(p - q, -p - k; q, k) \}. \end{aligned} \quad (9)$$

In the renormalized Eqs. (7) to (9), the index (phys) was omitted to simplify the notation. The functions $\Sigma(p)$ and $\Pi(k^2)$ are, respectively, the fermion and the photon physical self-energies. The renormalization constants can be computed requiring that

$$A(\mu_F^2) = 1, \quad B(\mu_F^2) = m, \quad \text{and} \quad D(\mu_B^2) = \frac{1}{\mu_B^2}, \quad (10)$$

where μ_F and μ_B are the renormalization mass scales for the fermion and boson fields. Writing, as usual, the fermion

self-energy as $\Sigma(p) = \Sigma_v(p^2)\not{p} + \Sigma_s(p^2)$, then

$$Z_2 = \frac{1}{1 - ig^2 \Sigma_v(\mu_F^2)}, \quad Z_0 = 1 - i \frac{Z_2}{m} g^2 \Sigma_s(\mu_F^2),$$

and $Z_3 = 1 + i Z_2 g^2 \Pi(\mu_B^2)$. (11)

In QED, the combination $g^2 D(k^2)$ is independent of the renormalization scale, see, e.g., [18,19] and references therein, and can be used to define an effective charge. The lowest-order perturbative solution for $D(k^2)$ is such that this effective charge diverges for small k^2 and vanishes at large momentum scales. This effective charge is nothing but the Coulomb potential if one assumes that there is no energy exchange.

A. On the perturbative analysis of the DSE and the photon self-energy

Let us assume that the coupling constant g is small enough in such a way that the theory can be solved using perturbation theory. Then, inserting the expressions for Z_i in the renormalized DSE, after some algebra, the integral equations can be approximated by

$$A(p^2) = \frac{1 - ig^2 \Sigma_v(p^2)}{1 - ig^2 \Sigma_v(\mu_F^2)} \approx 1 - ig^2 (\Sigma_v(p^2) - \Sigma_v(\mu_F^2)) + \mathcal{O}(g^4), \quad (12)$$

$$B(p^2) = m + ig^2 \frac{\Sigma_s(p^2) - \Sigma_s(\mu_F^2)}{1 - ig^2 \Sigma_v(\mu_F^2)} \approx m + ig^2 (\Sigma_s(p^2) - \Sigma_s(\mu_F^2)) + \mathcal{O}(g^4), \quad (13)$$

$$\frac{1}{D(k^2)} = k^2 \left(1 - ig^2 \frac{\Pi(k^2) - \Pi(\mu_B^2)}{1 - ig^2 \Sigma_v(\mu_F^2)} \right) \approx k^2 (1 - ig^2 (\Pi(k^2) - \Pi(\mu_B^2))) + \mathcal{O}(g^4). \quad (14)$$

These subtractions make the theory UV finite. Calling for dimensional regularization to perform the momentum integrations, for example, the photon self-energy is in first-order perturbation theory, in the chiral limit where $m = 0$, given by

$$\frac{1}{D(k^2)} = k^2 \left(1 - \frac{\alpha}{3\pi} N_\chi \ln \frac{k^2}{\mu_B^2} \right), \quad (15)$$

where N_χ is the number of chiral fermions and $\alpha = g^2/4\pi$. In the chiral limit, first-order perturbation theory predicts a pole for the photon propagator at $k^2 = 0$ together with a pole at $k^2 = \mu_B^2 \exp(3\pi/\alpha N_\chi)$, that moves to zero or to infinity as $\mu_B^2 \rightarrow 0$ or $\mu_B^2 \rightarrow +\infty$, respectively. In the limit $\mu_B^2 \rightarrow 0$, this later pole no longer shows up. The pole goes away from $k^2 = 0$ as the number of chiral fermions N_χ increases. On the other hand, for $k^2 = 0$ and for a finite

fermion mass, independently of the renormalization scale μ_B , the perturbative result reads

$$\frac{1}{D(0)} = -\frac{2\alpha}{3\pi} N_\chi m^2, \quad (16)$$

which appears as a photon mass term that grows linearly with the mass of the fermion within the loop. This effective photon mass vanishes in the limit of massless fermions, if one assumes that the theory behaves smoothly in the limit of a vanishing fermion mass.

The infrared problems in QED are understood as being associated with having a massless photon. Gauge symmetry does not allow for a mass term at the level of the Lagrangian density. However, perturbation theory requires an IR regulator, as the theory is plagued with divergences that are associated with the low-momentum regions. For the DSE, possible IR divergences can be investigated studying the renormalized equations at zero momentum, as considered in the last paragraph. This discussion becomes clear after the consideration of a tensor basis for the photon-fermion vertex, as is done in Sec. IV. In what concerns the IR properties of the propagators, the fermion gap shows no naive IR divergences. However, to have a finite and IR-safe equation for the photon equation, the replacement of $\Pi(k^2)$ by $\Pi(k^2) - \Pi(0)$ should be considered. The redefinition of the photon self-energy is equivalent to renormalize the theory at zero momentum, implying a $Z_3 = 1$.

The IR behavior of the vertex DSE is more involved, when compared to the IR properties of the two-point functions. This equation requires the knowledge of the two-photon-two-fermion vertex when one of the photon momenta vanishes. For this kinematics, the WTI for the two-photon-two-fermion vertex was solved in [18]. However, this solution of the WTI only gives the longitudinal component of $\Gamma^{\mu\nu}$, that is described in terms of the fermion propagator form factors $A(p^2)$ and $B(p^2)$, providing no information on its remaining components. In general, the tensor decomposition of $\Gamma^{\mu\nu}$ demands a basis with a large number of operators [35–37], making the comprehension of their IR properties and, therefore, the vertex behavior at lower momenta outside perturbation theory difficult to analyze. Anyway, by choosing to solve the integral equations within a perturbative approach, then $\Gamma^{\mu\nu}$ can be ignored in the first step, and the usual perturbative divergences show up. For the discussion of infrared problems within QED, see, e.g., [38–42] and references therein.

IV. THE PHOTON-FERMION VERTEX

The photon-fermion Green function Γ^μ appears in all the DSEs considered so far. Its description calls for twelve form factors [33] and it is common to write Γ^μ as a sum of a longitudinal Γ_L and a transverse Γ_T part, relative to the photon momentum, i.e.,

$$\Gamma^\mu(p_2, p_1; p_3) = \Gamma_L^\mu(p_2, p_1; p_3) + \Gamma_T^\mu(p_2, p_1; p_3), \quad (17)$$

where p_2 is the incoming fermion momentum, $-p_1$ is the outgoing fermion momentum, and p_3 is the incoming photon momentum. Given that all the momenta are incoming, they verify the relation $p_1 + p_2 + p_3 = 0$. From the definition of the transverse vertex, it follows that

$$p_{3\mu} \Gamma_T^\mu(p_2, p_1; p_3) = 0. \quad (18)$$

The analysis of the vertex DSE, see Eq. (5), becomes easier after the introduction of a tensor basis describing its longitudinal and transverse components. Let us write

$$\Gamma_{L\mu}(p_2, p_1; p_3) = \sum_{i=1}^4 \lambda_i(p_1^2, p_2^2, p_3^2) L_\mu^{(i)}(p_1, p_2, p_3), \quad (19)$$

$$\Gamma_{T\mu}(p_2, p_1; p_3) = \sum_{i=1}^8 \tau_i(p_1^2, p_2^2, p_3^2) T_\mu^{(i)}(p_1, p_2, p_3), \quad (20)$$

where $L_\mu^{(i)}$ and $T_\mu^{(i)}$ are the set of tensor operators that define the basis for the vertex, and λ_i and τ_i are Lorentz scalar form factors. Note the different ordering of the momenta in the lhs and rhs in Eqs. (19) and (20). For the longitudinal part of the vertex, we take the Ball-Chiu longitudinal set of operators [33] that are given by

$$L_\mu^{(1)}(p_1, p_2, p_3) = \gamma_\mu, \quad (21)$$

$$L_\mu^{(2)}(p_1, p_2, p_3) = (\not{p}_1 - \not{p}_2)(p_1 - p_2)_\mu, \quad (22)$$

$$L_\mu^{(3)}(p_1, p_2, p_3) = (p_1 - p_2)_\mu, \quad (23)$$

$$L_\mu^{(4)}(p_1, p_2, p_3) = \sigma_{\mu\nu}(p_1 - p_2)^\nu, \quad (24)$$

while for the orthogonal part of the vertex we rely on the Kızılersu-Reenders-Pennington basis [34],

$$T_\mu^{(1)}(p_1, p_2, p_3) = p_{1\mu}(p_2 \cdot p_3) - p_{2\mu}(p_1 \cdot p_3), \quad (25)$$

$$T_\mu^{(2)}(p_1, p_2, p_3) = -T_\mu^{(1)}(p_1, p_2, p_3)(\not{p}_1 - \not{p}_2), \quad (26)$$

$$T_\mu^{(3)}(p_1, p_2, p_3) = p_3^2 \gamma_\mu - p_{3\mu} \not{p}_3, \quad (27)$$

$$T_\mu^{(4)}(p_1, p_2, p_3) = T_\mu^{(1)}(p_1, p_2, p_3) \sigma_{\alpha\beta} p_1^\alpha p_2^\beta, \quad (28)$$

$$T_\mu^{(5)}(p_1, p_2, p_3) = \sigma_{\mu\nu} p_3^\nu, \quad (29)$$

$$T_\mu^{(6)}(p_1, p_2, p_3) = \gamma_\mu(p_1^2 - p_2^2) + (p_1 - p_2)_\mu \not{p}_3, \quad (30)$$

$$T_\mu^{(7)}(p_1, p_2, p_3) = -\frac{1}{2}(p_1^2 - p_2^2)[\gamma_\mu(\not{p}_1 - \not{p}_2) - (p_1 - p_2)_\mu] - (p_1 - p_2)_\mu \sigma_{\alpha\beta} p_1^\alpha p_2^\beta, \quad (31)$$

$$T_\mu^{(8)}(p_1, p_2, p_3) = -\gamma_\mu \sigma_{\alpha\beta} p_1^\alpha p_2^\beta + p_{1\mu} \not{p}_2 - p_{2\mu} \not{p}_1, \quad (32)$$

that is free of kinematical singularities. In Eqs. (21) to (32), we take $\sigma_{\mu\nu} = \frac{1}{2}[\gamma_\mu, \gamma_\nu]$.

The longitudinal form factors are determined by the WTI for the vertex, see, e.g., [18,33], and are given by

$$\lambda_1(p_1^2, p_2^2, p_3^2) = \frac{1}{2}(A(p_1^2) + A(p_2^2)), \quad (33)$$

$$\lambda_2(p_1^2, p_2^2, p_3^2) = \frac{1}{2(p_1^2 - p_2^2)}(A(p_1^2) - A(p_2^2)), \quad (34)$$

$$\lambda_3(p_1^2, p_2^2, p_3^2) = \frac{1}{p_1^2 - p_2^2}(B(p_1^2) - B(p_2^2)), \quad (35)$$

$$\lambda_4(p_1^2, p_2^2, p_3^2) = 0. \quad (36)$$

If A and B are smooth functions, then λ_2 and λ_3 are regular in the limit of $p_1^2 \rightarrow p_2^2$ and become proportional to the derivatives of A and B , respectively, at $p_1^2 = p_2^2$. A straightforward calculation shows that, for zero photon momentum,

$$\lambda_1(p^2, p^2, 0) = A(p^2), \quad \lambda_2(p^2, p^2, 0) = \frac{1}{2} \frac{dA(p^2)}{dp^2},$$

$$\text{and } \lambda_3(p^2, p^2, 0) = \frac{dB(p^2)}{dp^2}, \quad (37)$$

a result that complies with the Ward identity [43]

$$\Gamma^\mu(p, p; 0) = \frac{\partial S^{-1}(p)}{\partial p_\mu}. \quad (38)$$

As discussed, in QED the WTI determine the longitudinal form factors λ_i . For non-Abelian theories the WTI identities are replaced by Slavnov-Taylor identities (STI). In QCD, the gluon-fermion vertex verifies a STI that involves Green functions other than the fermion propagator, namely, the ghost dressing function and the quark-ghost scattering kernel. The solution of this STI for the λ_i was worked out in [44]. The expressions found in QCD for the λ_i form factors are not as simple as in QED. For example, in QCD the form factor λ_4 no longer vanishes, as happens for the solution of the vertex WTI in QED. The particular case of the solutions of STI in the soft gluon limit, defined by a vanishing gluon momentum, can be found in [45] and should be compared with the results of Eq. (37). Assuming that the form factors are smooth functions of the momenta and are not singular for vanishing gluon momentum, then

the transverse components of the vertex do not contribute to the soft quark-gluon vertex. Indeed, the transverse operators that define the vertex basis become null operators in the soft gluon limit.

V. THE FULL SET OF DYSON-SCHWINGER EQUATIONS

In the previous section, a tensor basis to describe the photon-fermion vertex was introduced. The basis allows us to write the DSEs (7)–(9) in terms of the form factors λ_i and τ_i . The expressions can be found in Appendix A. In particular, the scalar and vector components of the fermion

gap equation are given in Appendix A 1, the photon equation is reported in Appendix A 2, and the decomposition of the photon-fermion vertex in terms of Dirac bilinear forms is described in Appendix A 3.

VI. THE PHOTON-FERMION VERTEX

The analysis of the DSE for the photon-fermion vertex, see Eq. (5) for its bare version or (9) for the renormalized case, can be worked out with the help of the basis given in Eqs. (24) and (32). In particular, the lhs of the integral equation can be written highlighting the tensor decomposition in terms of Dirac bilinears. This decomposition reads

$$\begin{aligned} \Gamma^\mu(p, -p-k; k) = & p^\mu(k^2\tau_1 - 2\lambda_3) - k^\mu((pk)\tau_1 + \lambda_3) \quad (\text{Scalar Component}) \\ & + \gamma^\mu[\lambda_1 + k^2\tau_3 + (k^2 + 2(pk))\tau_6] \quad (\text{Vector Components}) \\ & + \not{k}[k^\mu(\lambda_2 - (pk)\tau_2 - \tau_3 - \tau_6) + p^\mu(2\lambda_2 + k^2\tau_2 - 2\tau_6)] + 2\not{p}[k^\mu(\lambda_2 - (kp)\tau_2) + p^\mu(2\lambda_2 + k^2\tau_2)] \\ & + i\tau_8\gamma_\sigma\gamma_5\epsilon^{\sigma\mu\alpha\beta}p_\alpha k_\beta \quad (\text{Axial Component}) \\ & + \frac{1}{2}\sigma^{\mu\alpha}k_\alpha[2\tau_5 + \tau_7(k^2 + 2(kp))] \quad (\text{Tensor Components}) \\ & + \sigma^{\mu\alpha}p_\alpha[\tau_7(k^2 + 2(kp))] + \sigma^{\alpha\beta}p_\alpha k_\beta[k^\mu(\tau_7 - (pk)\tau_4) + p^\mu(k^2\tau_4 + 2\tau_7)], \end{aligned} \quad (39)$$

where the form factors are short notations for

$$\lambda_i = \lambda_i((p+k)^2, p^2, k^2) \quad \text{and} \quad \tau_i = \tau_i((p+k)^2, p^2, k^2). \quad (40)$$

As can be read from the above expression, the Dirac scalar bilinear term requires the longitudinal λ_3 , that is fixed by the WTI, and the transverse form factor τ_1 . Likewise, the Dirac vector term calls for τ_2 , τ_3 , and τ_6 , the Dirac tensor involves τ_4 , τ_5 , and τ_7 , and the axial vector term is described by a

unique form factor τ_8 . As a cross-check of the decomposition given in Eq. (39), the contraction $k_\mu\Gamma^\mu$ can be computed and, as expected, it demands only the longitudinal form factors λ_i .

The photon-fermion vertex has been evaluated in perturbation up to one loop, for a generic linear covariant gauge, in [34]. In particular, the one-loop perturbative solution for the transverse vertex is known, and its tensor structure reproduces, for large photon momentum, that of the operator associated with τ_3 . According to Eq. (39), for large k , the asymptotic transverse vertex reads

$$\begin{aligned} \Gamma_T^\mu(p, -p-k; k) = & p^\mu k^2\tau_1 - k^\mu(pk)\tau_1 + \gamma^\mu(k^2\tau_3 + k^2\tau_6) \\ & + \not{k}[p^\mu(k^2\tau_2 - 2\tau_6) - k^\mu((pk)\tau_2 + \tau_3 + \tau_6)] + 2\not{p}[p^\mu k^2\tau_2 - k^\mu(kp)\tau_2] + i\tau_8\gamma_\sigma\gamma_5\epsilon^{\sigma\mu\alpha\beta}p_\alpha k_\beta \\ & + \frac{1}{2}\sigma^{\mu\alpha}k_\alpha[2\tau_5 + \tau_7k^2] + \sigma^{\mu\alpha}p_\alpha[\tau_7k^2] + \sigma^{\alpha\beta}p_\alpha k_\beta[k^\mu(\tau_7 - (pk)\tau_4) + p^\mu(k^2\tau_4 + 2\tau_7)] \\ = & \frac{\alpha\xi}{8\pi} \left(\frac{k^\mu\not{k}}{k^2} - \gamma^\mu \right) \ln \frac{k^2}{p^2}, \end{aligned} \quad (41)$$

where, in the last line, the asymptotic one-loop perturbative result is quoted. The recovering of the perturbative expression requires that, for large k ,

$$\tau_3 \approx -\frac{\alpha\xi}{8\pi k^2} \ln \frac{k^2}{p^2}, \quad \tau_{1,6} < \frac{1}{k^2}, \quad \tau_5 < \frac{1}{k}, \quad \tau_{2,4,7} < \frac{1}{k^3}, \quad \text{and} \quad \tau_8 < \frac{1}{k}. \quad (42)$$

In what concerns the form factors $\tau_{i \neq 3}$, their exact asymptotic scaling law is not determined exactly. The above relations define only upper limits on their asymptotic scaling with the photon momentum.

Profiting from the decomposition of the vertex in Dirac bilinears, the contraction of the vertex equation with the incoming fermion momentum p^μ results in another Lorentz scalar that can be used to access τ_1 . Indeed, this contraction gives

$$\begin{aligned} p_\mu \Gamma^\mu(p, -p - k; k) = & \tau_1(p^2 k^2 - (pk)^2) - \lambda_3(2p^2 + (pk)) \\ & + \not{k}[\lambda_2((kp) + 2p^2) + \tau_2(k^2 p^2 - (kp)^2) - \tau_3(kp) - \tau_6((kp) + 2p^2)] \\ & + \not{p}[\lambda_1 + 2\lambda_2((kp) + 2p^2) + 2\tau_2(k^2 p^2 - (kp)^2) + \tau_3 k^2 + \tau_6(k^2 + 2(kp))] \\ & + \sigma^{\alpha\beta} p_\alpha k_\beta \left[\tau_4(p^2 k^2 - (kp)^2) + \tau_5 + \tau_7 \left(\frac{1}{2} k^2 + 2(kp) + 2p^2 \right) \right]. \end{aligned} \quad (43)$$

Recalling that the longitudinal form factors are determined by the Ward-Takahashi identity for the vertex, then τ_1 can be evaluated solving the equation

$$\begin{aligned} \tau_1((p+k)^2, p^2, k^2) \\ = \frac{1}{p^2 k^2 - (pk)^2} \\ \times \left(\frac{1}{4} \text{Tr}(p_\mu \Gamma^\mu(p, -p - k; k)) + \lambda_3(2p^2 + (pk)) \right). \end{aligned} \quad (44)$$

This is an exact relation and is an example of the type of relations that will be considered to access the transverse form factors. These relations explore the tensorial properties of Γ^μ . Another example of such type for the photon-fermion vertex allows the computation of τ_8 ,

$$\begin{aligned} \tau_8((p+k)^2, p^2, k^2) \epsilon^{\zeta\mu\alpha\beta} p_\alpha k_\beta \\ = i \frac{1}{4} \text{Tr}(\Gamma^\mu(p, -p - k; k) \gamma^\zeta \gamma_5). \end{aligned} \quad (45)$$

This form factor vanishes for a tree level type of vertex. Indeed, for a tree level type of vertex where $\Gamma^\mu = \gamma^\mu$, $\lambda_2 = \lambda_3 = 0$, and the above equations imply that $\tau_1 = \tau_8 = 0$.

A. Special kinematics

The vertex decomposition in terms of Dirac bilinears given in Eq. (39) is valid for any kinematical configuration. However, for particular choices of the incoming quark momentum p and of the photon incoming momentum k , assuming that the form factors are not singular, the decomposition simplifies. For example, for a vanishing incoming fermion momentum $p = 0$, it follows that

$$\begin{aligned} \Gamma^\mu(0, -k; k) = & -k^\mu \lambda_3 + \gamma^\mu [\lambda_1 + k^2(\tau_3 + \tau_6)] \\ & + \not{k} k^\mu [\lambda_2 - \tau_3 - \tau_6] \\ & + \frac{1}{2} \sigma^{\mu\alpha} k_\alpha [2\tau_5 + k^2 \tau_7], \end{aligned} \quad (46)$$

and it is enough to know the combinations of transverse form factors $\tau_3 + \tau_6$ and $2\tau_5 + k^2 \tau_7$ to describe Γ^μ . Recall that the longitudinal form factors are the solutions of the vertex WTI.

Another kinematical configuration where Eq. (39) simplifies considerably is the case of a vanishing photon momentum (soft photon limit), where Γ^μ is described only by the longitudinal form factors λ_i . Indeed, in this kinematics, and as long as the τ_i do not have poles at zero momentum, the expression for the vertex becomes

$$\begin{aligned} \Gamma^\mu(p, -p; 0) = & 2\lambda_3(p^2, p^2, 0) p^\mu + \lambda_1(p^2, p^2, 0) \gamma^\mu + 4\lambda_2(p^2, p^2, 0) \not{p} p^\mu \\ = & -2B'(p^2) p^\mu + A(p^2) \gamma^\mu + 2A'(p^2) \not{p} p^\mu. \\ = & \gamma^\mu + i g^2 \int \frac{d^4 q}{(2\pi)^4} D_{\zeta\zeta'}(q) \gamma^\zeta S(p - q) \\ & \times \{ \Gamma^\mu(p - q, -p + q; 0) S(p - q) \Gamma^{\zeta'}(p - q, -p; q) + \Gamma^{\zeta'\mu}(p - q, -p; q, 0) \}. \end{aligned} \quad (47)$$

In the above expression, the result of Eq. (37) was used to write the longitudinal form factors in terms of the fermion propagator functions. The (bare) vertex DSE is also given explicitly to identify the various contributions to Γ^μ . The two vertices $\Gamma^\mu(p, -p; 0)$ and $\Gamma^\mu(p - q, -p + q; 0)$, which appear on the rhs of the DSE, require only the longitudinal form factors, see the first line of Eq. (47), and are determined by the vertex WTI. On the other hand, the

longitudinal component of the two-photon-two-fermion vertex for the kinematics also appears in the above equation. The solution of the corresponding WTI writes $\Gamma^{\mu\nu}$ in terms of the fermion propagator functions A and B and their derivatives, and can be found in [18]. For a general kinematics, the two-photon-two-fermion vertex $\Gamma^{\mu\nu}$ has a complex tensor structure that includes both longitudinal and transverse, relative to the photons' momenta,

tensor operators. For completeness, we provide the expression for the two-photon-two-fermion derived in [18] that solves the associated WTI when one of the photon momenta vanishes,

$$\begin{aligned}\Gamma^{\zeta\mu}(p-q, -p; q, 0) &= 2g^{\zeta\mu}B'(p^2) + 2\frac{q^{\zeta}p^{\mu} - q^{\zeta}q^{\mu}}{q^2}[B'(p^2) - B'((p-q)^2)] \\ &\quad + \frac{q^{\zeta}\gamma^{\mu}}{q^2}[A((p-q)^2) - A(p^2)] - 2\not{q}\frac{q^{\zeta}p^{\mu} - q^{\zeta}q^{\mu}}{q^2}A'((p-q)^2) \\ &\quad - 2\not{p}\left\{g^{\zeta\mu}A'(p^2) + \frac{q^{\zeta}p^{\mu} - q^{\zeta}q^{\mu}}{q^2}[A'(p^2) - A'((p-q)^2)]\right\},\end{aligned}\quad (48)$$

where the expressions derived previously for λ_i were used. If the inverse of the fermion wave function has a weak dependence on the momentum, an approximation that is often used by setting $A(p^2) = 1$, then the last terms in $\Gamma^{\zeta\mu}$ can be ignored and

$$\Gamma^{\zeta\mu}(p-q, -p; q, 0) \approx 2g^{\zeta\mu}B'(p^2) + 2\frac{q^{\zeta}p^{\mu} - q^{\zeta}q^{\mu}}{q^2}[B'(p^2) - B'((p-q)^2)], \quad (49)$$

and vanishes at high momenta, where an essentially constant $B(p^2)$ is expected to match the perturbative result.

Another kinematical configuration is when the incoming fermion momentum and the photon momentum are orthogonal, i.e., when $(pk) = 0$. In this case, the vertex becomes

$$\begin{aligned}\Gamma^{\mu}(p, -p-k; k) &= p^{\mu}(k^2\tau_1 - 2\lambda_3) - k^{\mu}\lambda_3 \\ &\quad + \gamma^{\mu}[\lambda_1 + k^2(\tau_3 + \tau_6)] + \not{k}[k^{\mu}(\lambda_2 - \tau_3 - \tau_6) + p^{\mu}(2\lambda_2 + k^2\tau_2 - 2\tau_6)] + 2\not{p}[p^{\mu}(2\lambda_2 + k^2\tau_2) - k^{\mu}\lambda_2] \\ &\quad + i\tau_8\gamma_5\epsilon^{\sigma\mu\alpha\beta}p_{\alpha}k_{\beta} + \frac{1}{2}\sigma^{\mu\alpha}k_{\alpha}[2\tau_5 + \tau_7k^2] + \sigma^{\mu\alpha}p_{\alpha}[\tau_7k^2] + \sigma^{\alpha\beta}p_{\alpha}k_{\beta}[k^{\mu}\tau_7 + p^{\mu}(k^2\tau_4 + 2\tau_7)]\end{aligned}\quad (50)$$

and, again, only particular combinations of the transverse form factors have to be considered.

B. Comment on vertex models

From a practical point of view, the use of the DSE to solve QED or QCD relies, many times, on modeling the fermion-boson vertex. The call for vertex models necessarily leads to a simplification of the Eq. (39) tensor structure. A subset of the available vertex models that can be found in the literature is in [19,46–51], see also the references therein. For example, the Curtis-Pennington vertex [50,51] considers

$$\begin{aligned}\Gamma^{\mu}(p, -p-k; k) &= (-2p^{\mu} - k^{\mu})\lambda_3 \\ &\quad + \gamma^{\mu}(\lambda_1 + (k^2 + 2(kp))\tau_6) \\ &\quad + \not{k}(k^{\mu} + 2p^{\mu})(\lambda_2 - \tau_6) \\ &\quad + 2\not{p}(k^{\mu} + 2p^{\mu})\lambda_2.\end{aligned}\quad (51)$$

This vertex model for QED is compatible with multiplicative renormalizability, for massless fermions and in the quenched approximation, and considers a single transverse form factor τ_6 . This form factor contributes to $B(0)$, see Eq. (A1), and, therefore, can be made compatible with the chiral symmetry breaking mechanism for a large enough coupling constant.

As a final comment, we note that the decomposition discussed for the vertex, or any of its simplifications, explores the tensor structure of Γ^{μ} . In general, it is valid both for the photon-fermion vertex and also for the quark-gluon vertex, module its color structure, and after including the contribution of λ_4 . The same rationale applies to the decomposition of the fermion gap equation in terms of the vertex form factors, see Eqs. (A1) and (A2). Indeed, the QCD fermion gap equation, after taking into account the color factors and λ_4 , that appears only in the scalar equation, is the same up to a proper replacement of the coupling constant and the renormalization constants. The main difference between QCD and QED is the bosonic DSE that, in QCD, requires gluon one-particle irreducible Green functions with three and four gluon legs together with ghost contributions.

VII. THE ON-SHELL PHOTON-FERMION VERTEX

The particular case that corresponds to the evaluation of the vertex diagram with fermions that are on-shell allows the use of the Dirac equation to simplify the writing of the vertex and build an effective on-shell vertex. Let us define the on-shell photon-fermion vertex $\tilde{\Gamma}$ by

$$[\bar{u}(p')\tilde{\Gamma}^\mu(p, -p' = -p - k; k)u(p)] = [\bar{u}(p')\Gamma^\mu(p, -p' = -p - k; k)u(p)]|_{p^2=p'^2=m^2}. \quad (52)$$

Assuming that the free Dirac equation can be applied to simplify the rhs of this equation, then, after some algebra, one arrives at

$$\begin{aligned} \tilde{\Gamma}^\mu(p, -p' = -p - k; k) &= (p' + p)^\mu \{-\lambda_3 + (m^2 - (p'p))\tau_1 + \tau_5 + 2m[\lambda_2 + (m^2 - (p'p))\tau_2]\} \\ &\quad + \gamma^\mu \{\lambda_1 + 2(m^2 - (p'p))\tau_3 - 2m\tau_5\} + i\tau_8\gamma_\sigma\gamma_5\epsilon^{\sigma\mu\alpha\beta}p_\alpha p'_\beta \\ &= \gamma^\mu \{\lambda_1 + 2(m^2 - (p'p))\tau_3 + 2m[2m\lambda_2 - \lambda_3 + (m^2 - (p'p))(\tau_1 + 2m\tau_2)]\} \\ &\quad + \sigma^{\mu\alpha}k_\alpha \{-\lambda_3 + (m^2 - (p'p))\tau_1 + \tau_5 + 2m[\lambda_2 + (m^2 - (p'p))\tau_2]\} + i\tau_8\gamma_\sigma\gamma_5\epsilon^{\sigma\mu\alpha\beta}p_\alpha p'_\beta, \end{aligned} \quad (53)$$

where the last expression was obtained from the first with the help of the Gordon identity

$$2m\bar{u}(p)\gamma^\mu u(p) = \bar{u}(p)((p' + p)^\mu - \sigma^{\mu\alpha}(p' - p)_\alpha)u(p). \quad (54)$$

An interesting limit of this vertex is the chiral on-shell photon-fermion vertex that corresponds to take the limit $m \rightarrow 0$ together with $\lambda_3 \rightarrow 0$, as the solutions of the vertex WTI imply. Assuming a smooth behavior of the form factors, it follows that

$$\begin{aligned} \tilde{\Gamma}^\mu(p, -p' = -p - k; k) &= \gamma^\mu(\lambda_1 - 2(p'p)\tau_3) + \sigma^{\mu\alpha}k_\alpha(\tau_5 - (p'p)\tau_1) \\ &\quad + i\tau_8\gamma_\sigma\gamma_5\epsilon^{\sigma\mu\alpha\beta}p_\alpha p'_\beta, \end{aligned} \quad (55)$$

i.e., the on-shell vertex in the chiral limit gets contributions only from λ_1 , τ_1 , τ_3 , τ_5 , and τ_8 . For the vertex models that do not consider these transverse form factors, the on-shell vertex in the chiral limit is reduced to its tree-level tensor structure.

The anomalous magnetic and electric fermion form factors can be read from Eq. (53) or, for their chiral limit, from Eq. (55). Note that the vertex Ward-Takahashi identity for the photon-fermion vertex implies

$$\begin{aligned} k_\mu[\bar{u}(p')\tilde{\Gamma}^\mu(p, -p' = -p - k; k)u(p)] &= k_\mu[\bar{u}(p')\Gamma^\mu(p, -p' = -p - k; k)u(p)] \\ &= [\bar{u}(p')(S^{-1}(-p') - S^{-1}(p))u(p)] \\ &= [\bar{u}(p')(B(p^2) - B(p'^2) - m(A(p'^2) \\ &\quad + A(p^2)))u(p)], \end{aligned} \quad (56)$$

which vanishes either in the chiral limit or when A and B are momentum-independent functions, as occurs for the tree-level solution of QED.

The results derived in this section are quite general and are valid also for QCD, after the correction of the color structure and after taking into consideration the

contribution associated with λ_4 . If for QED the on-shell condition seems to be a reasonable approximation, in QCD the on-shell condition raises conceptual difficulties due to quark confinement. There are also important differences between the longitudinal form factors for QED and QCD. The Ward-Takahashi identity for the vertex in QED is replaced in QCD by a STI that has contributions from the quark-ghost scattering kernel and ghost dressing function. The solution of the vertex STI for the λ_i has a higher complexity when compared to the Abelian vertex. However, in lowest order in the coupling constant, the perturbative solution of the Slavnov-Taylor identity for the vertex reproduces the results for the longitudinal vertex of the Abelian theory and, in this sense, it is tempting to use the Ball-Chiu vertex in QCD, despite its known limitations. A discussion on the longitudinal form factors in QCD can be found in, e.g., [20–22,44,45,52–57] and references therein. See also [58] for a recent discussion of the QCD corrections to the on-shell photon-quark vertex.

VIII. TRANSVERSE FORM FACTORS FROM THE PHOTON-FERMION VERTEX DSE

In this section, a computational framework for the evaluation of the transverse form factors τ_i from the vertex Dyson-Schwinger equation is built. Such procedure was already worked out for τ_1 , see Eq. (44), and for τ_8 in Eq. (45). It remains to derive exact expressions for the remaining six form factors. The procedure relies on the photon-fermion vertex DSE, i.e., Eq. (9), and the tensor decomposition of the vertex, as given in Eqs. (39) and (A6). The equations written below should be solved self-consistently for the transverse form factors and are generalizations of those derived for τ_1 and τ_8 . We start by reviewing the procedure for these two form factors.

A. The form factor τ_1

The transverse form factor τ_1 can be computed using Eq. (44) and it is given by the solution of equation

$$\begin{aligned} & \tau_1((p+k)^2, p^2, k^2) \\ &= \frac{1}{p^2 k^2 - (pk)^2} \left\{ \frac{1}{4} \text{Tr}(p_\mu \Gamma^\mu(p, -p-k; k)) \right. \\ & \quad \left. + \frac{2p^2 + (pk)}{k^2 + 2(pk)} (B((p+k)^2) - B(p^2)) \right\}. \end{aligned} \quad (57)$$

It is instructive to compare this expression for τ_1 with the result obtained in perturbation theory at the one-loop level for QED [34]. When reading the expressions from this work, the different momentum convention¹ should be taken into account. In both cases, i.e., in the exact expression given in (57) and in the outcome of one-loop perturbation theory, the overall factor $1/(p^2 k^2 - (pk)^2)$ is seen and defines a scaling law for τ_1 . The comparison also allows for an estimation of the trace term written in the above equation.

Oftentimes, in a practical situation, the vertex is modeled and a given tensor structure is considered explicitly. The overall factor mentioned previously is not always considered in the models for τ_1 . See, for example, [48,59,60] and references therein for models that consider the overall kinematical factor. An example where the kinematical factor is not used can be found, e.g., in [61]. Other vertex models do not even consider τ_1 at all, for example, in [46,50,51,57,62].

Note also that, at lowest order in perturbation theory, i.e., for vertex $\Gamma^\mu = \gamma^\mu$ and a B that is momentum independent, this form factor vanishes. This result agrees with the chiral limit of the one-loop perturbative vertex calculation, see [34], where τ_1 is proportional to the fermion mass. Also, it implies that, in perturbation theory, this form factor is, at large momenta, necessarily small for light fermions.

B. The form factor τ_8

For completeness we review the computation of τ_8 that is associated with the Dirac axial vector of the decomposition of Γ^μ , see Eq. (39) or (A6). It follows that

$$\tau_8 \epsilon^{\zeta\mu\alpha\beta} p_\alpha k_\beta = \frac{i}{4} \text{Tr}(\gamma^\zeta \gamma_5 \Gamma^\mu), \quad (58)$$

and after contraction with the Levi-Civita tensor and momenta, it is possible to arrive at a scalar equation for τ_8 . Further discussion on the computation of τ_8 can be found in Sec. X.

C. The form factors τ_2 , τ_3 , and τ_6

The transverse form factors τ_2 , τ_3 , and τ_6 can be computed from Eq. (39) combined with the vectorial part

¹For example, in [34] the quantity Δ^2 corresponds, in our conventions, to $(pk)^2 - p^2 k^2$ that later will be called $-\Delta$.

of Eq. (43). To disentangle the various form factors, let us define

$$\begin{aligned} \Lambda_1 &= \frac{1}{4} \text{Tr}(\gamma_\mu \Gamma^\mu) - 4\lambda_1 - (k^2 + 4(pk) + 4p^2)\lambda_2 \\ &= 2(p^2 k^2 - (pk)^2)\tau_2 + 3k^2\tau_3 + 3(k^2 + 2(pk))\tau_6, \end{aligned} \quad (59)$$

$$\begin{aligned} \Lambda_2 &= \frac{1}{4} \text{Tr}(k\gamma_\mu \Gamma^\mu) - (pk)\lambda_1 - (2p^2 + (pk))(k^2 + 2(pk))\lambda_2 \\ &= (p^2 k^2 - (pk)^2)[(k^2 + 2(pk))\tau_2 - 2\tau_6], \end{aligned} \quad (60)$$

$$\begin{aligned} \Lambda_3 &= \frac{1}{4} \text{Tr}(p\gamma_\mu \Gamma^\mu) - p^2\lambda_1 - (2p^2 + (pk))^2\lambda_2 \\ &= (p^2 k^2 - (pk)^2)[(2p^2 + (pk))\tau_2 + \tau_3 + \tau_6]. \end{aligned} \quad (61)$$

Solving these identities in terms of the transverse form factor, it follows that

$$\tau_2 = -\frac{1}{4\Delta} \left(\Lambda_1 + 3\frac{(pk)}{\Delta}\Lambda_2 - 3\frac{k^2}{\Delta}\Lambda_3 \right), \quad (62)$$

$$\begin{aligned} \tau_3 &= \frac{1}{8\Delta^2} (\Delta(k^2 + 4p^2 + 4(pk))\Lambda_1 \\ & \quad + (4\Delta + 3(pk)(k^2 + 4p^2 + 4(pk)^2))\Lambda_2 \\ & \quad - (4\Delta + 3k^4 + 12(pk)(k^2 + (pk)))\Lambda_3), \end{aligned} \quad (63)$$

$$\begin{aligned} \tau_6 &= \frac{1}{8\Delta^2} (-\Delta(k^2 + 2(pk))\Lambda_1 \\ & \quad - (4\Delta + 3(pk)(k^2 + 2(pk)))\Lambda_2 \\ & \quad + 3k^2(k^2 + 2(pk))\Lambda_3). \end{aligned} \quad (64)$$

As for the form factors τ_1 and τ_8 , the above relations are equations that have to be solved self-consistently with respect to τ_2 , τ_3 , and τ_6 , as Γ^μ , or its contractions, are functions of all the vertex form factors. Once more, we remind the reader that the longitudinal form factors λ_i are determined by the WTI and are functions of the fermion propagator form factors A and B .

As for τ_1 and τ_8 , all the form factors τ_2 , τ_3 , and τ_6 share the common factor multiplicative kinematical factor

$$\frac{1}{\Delta} = \frac{1}{p^2 k^2 - (pk)^2} \quad (65)$$

that can come as a higher power of Δ . This factor is also present in the one-loop perturbative solution for these transverse form factors. This common factor is not always taken into account in the models for the photon-fermion vertex. For a tree-level vertex type, the Λ_i vanish and, therefore, all the form factors τ_2 , τ_3 , and τ_6 vanish. From the above expression one can derive naive asymptotic dependencies on the photon momentum that should be read with some care, as the Λ_i are also functions of the form factors

themselves. A naive look at the above expressions suggests that for large photon momentum, the expressions are dominated by τ_2 and are proportional to Λ_1 .

D. The form factors τ_4 , τ_5 , and τ_7

Proceeding as in the previous section but looking at the tensor components of Eq. (39), one can access the transverse form factors τ_4 , τ_5 , and τ_7 from

$$\begin{aligned}\Omega_1 &= \frac{1}{4} \text{Tr}(\sigma^{\alpha\beta} p_\alpha k_\beta (p_\mu \Gamma^\mu)) \\ &= -(p^2 k^2 - (pk)^2) \left((p^2 k^2 - (pk)^2) \tau_4 + \tau_5 \right. \\ &\quad \left. + \frac{1}{2} (k^2 + 4p^2 + 4(pk)) \tau_7 \right),\end{aligned}\quad (66)$$

$$\begin{aligned}\Omega_2 &= \frac{1}{4} \text{Tr}(\sigma_{\alpha\mu} k^\alpha \Gamma^\mu) \\ &= k^2 (p^2 k^2 - (pk)^2) \tau_4 + 3k^2 \tau_5 \\ &\quad + \left(\frac{3}{2} k^4 + 2k^2 p^2 + 6k^2 (pk) + 4(pk)^2 \right) \tau_7,\end{aligned}\quad (67)$$

$$\begin{aligned}\Omega_3 &= \frac{1}{4} \text{Tr}(\sigma_{\alpha\mu} p^\alpha \Gamma^\mu) \\ &= (p^2 k^2 - (pk)^2) (pk) \tau_4 + 3(pk) \tau_5 \\ &\quad + \left(2p^2 k^2 + \frac{3}{2} k^2 (pk) + 6p^2 (pk) + 4(pk)^2 \right) \tau_7.\end{aligned}\quad (68)$$

Indeed, it follows that

$$\tau_4 = \frac{1}{2(k^2 + 2(pk))\Delta^2} (-3(k^2 + 2(pk))\Omega_1 + \Omega_2 - 2\Delta\Omega_3),\quad (69)$$

$$\tau_5 = \frac{1}{4\Delta} (2\Omega_1 + (2p^2 + (pk))\Omega_2 - (k^2 + 2(pk))\Omega_3),\quad (70)$$

$$\tau_7 = -\frac{1}{2(k^2 + 2(pk))\Delta} ((pk)\Omega_2 - k^2\Omega_3).\quad (71)$$

Again, as for $\tau_{1,2,3,6}$, the form factors are all proportional to $1/\Delta$. Furthermore, for a tree level type of vertex, one has that $\Omega_i = 0$ and, therefore, the $\tau_{4,5,7}$ vanish.

IX. THE VERTEX DSE, THE FORM FACTORS, AND APPROXIMATIONS

The rhs of this Dyson-Schwinger equation (9) has, besides a “tree level” type of contribution, two terms requiring momentum integration. In one of these two terms, the full vertex appears twice, with different momenta

setups, while in the other a contribution of the two-photon-two-fermion one-particle irreducible Green function has to be evaluated. The first term, that requires only Γ^μ itself, needs the following vertices:

$$\Gamma^\mu(U, -U - k; k) \quad \text{and} \quad \Gamma^\nu(V, -V - q; q), \quad (72)$$

with the momenta $U = p - q$, $V = p + k - q = U + k$, and where q is the loop momentum that appears in the integration. The full vertex Γ^μ requires twelve form factors. The consideration of its full structure in the integral equation results in a lengthy expression that is hard to understand and disentangle. To study the vertex DSE itself and to be able to solve it, approximations are welcome. The considerations below apply to the analysis only of the rhs of the vertex Dyson-Schwinger equation, i.e., to the terms inside the momentum integral. The approximations to be discussed are inspired in the perturbative program for QED.

To solve the vertex integral equation, on its rhs, the contributions of the transverse form factors τ_i are disregarded and only the longitudinal form factors are considered. The λ_i 's are the solutions given in Eqs. (33) to (36) of the vertex WTI. By taking into account all the longitudinal form factors, one goes beyond the conventional perturbation theory, whose starting point sets $\lambda_1 = 1$ and ignores $\lambda_{2,3}$. Despite this approximation, the rhs of the vertex equation still has a large number of terms and, to simplify further, the two-photon-two-fermion term will not be taken into account. In perturbation theory, the two-photon-two-fermion 1PI Green function vanishes at tree level and gets a first contribution from one-loop diagrams, that is of order g^4 . See [18] for the DSE that determines $\Gamma^{\mu\nu}$ and its diagrammatic representation. From this point of view, it seems reasonable to ignore $\Gamma^{\mu\nu}$ in a first approximation. Despite the approximations, they still result in large expressions that are difficult to handle and further simplifications will be considered to compute the τ_i 's. To reduce further the number of terms on the rhs of the equation we take, only on its rhs, the chiral limit defined by setting $m = B = 0$. Then, to be consistent with the WTI (35), on the rhs, we also ignore the contribution of the λ_3 form factor. According to one-loop perturbation theory, in the chiral limit only a subset of the transverse form factors has to be considered. Our analysis of the vertex integral equation finds the same set of nonvanishing transverse form factors as in the perturbative solution.

The approximations discussed to solve the vertex equation do not consider the full vertex on the rhs of the equation and, therefore, can compromise the multiplicative renormalizability of the theory. Despite the approximations, it turns out that the solution for Γ^μ results in transverse form factors that do not all vanish. The presence of the τ_i 's is crucial to recover such an important property for QED, see, e.g., [46,50] and references therein. Although we are not able to prove that the approximations introduced

do not lead to a violation of the multiplicative renormalizability, the presence of the various τ_i in the solution seems to suggest that the violation of multiplicative renormalizability should be small.

The results in this section can also be viewed in two ways. On one hand, the approximations on the rhs of the vertex equation identify the dominant tensor structures in Γ^μ that are expected to be dominant, i.e., identify the tensor structures that can be disregarded in a first approximation. On the other hand, assuming a power series solution, in terms of the coupling constant, of the vertex equation, it suggests that deviations should occur at larger order in g and, therefore, possible violations of multiplicative renormalizability should

take place at $\mathcal{O}(g^4)$ or at higher order. The perturbative analysis performed in Sec. X solves exactly the vertex equation up to g^2 and reproduces the gross features of the standard perturbative vertex calculation.

Let us proceed with the analysis of the vertex DSE to compute the transverse form factors using the above approximations in the evaluation of its rhs. The calculation relies on the main results of Sec. VIII. In the analysis of the Dirac structure of the vertex Dyson-Schwinger integral equation, our focus goes to the terms under the integral. Within the approximations discussed, this terms reads (recall that we are disregarding the contribution of $\Gamma^{\mu\nu}$)

$$\begin{aligned} D_{\xi\xi'}(q)\gamma^\xi S(p-q)\Gamma^\mu(p-q, -p-k+q; k)S(p+k-q)\Gamma^{\xi'}(p+k-q, -p-k; q) \\ = -D(q^2)[\gamma^\xi S(p-q)\Gamma^\mu(p-q, -p-k+q; k)S(p+k-q)\Gamma_\xi(p+k-q, -p-k; q)] \\ - \frac{1}{q^2}\left(\frac{\xi}{q^2} - D(q^2)\right)[\not{q}S(p-q)\Gamma^\mu(p-q, -p-k+q; k)S(p+k-q)(q^\xi\Gamma_\xi(p+k-q, -p-k; q))] \\ = -D(q^2)\tilde{\Gamma}_{(D)}^\mu - \frac{1}{q^2}\left(\frac{\xi}{q^2} - D(q^2)\right)\tilde{\Gamma}_{(\xi)}^\mu. \end{aligned} \quad (73)$$

A global factor of

$$ig^2 Z_2 \int \frac{d^4 q}{(2\pi)^4} \quad (74)$$

is omitted to simplify the notation. Before proceeding, we recall that in QED the combination $g^2 D(q^2)$ is renormalization scale independent and can be seen as an effective charge. For the Landau gauge ($\xi = 0$), this effective charge multiplies all terms in the vertex Dyson-Schwinger equation. For other gauges, the role of the effective charge is not so obvious. In this sense the Landau gauge is a special gauge. Note, however, that in the quenched approximation where $D(q^2) = 1/q^2$, the Feynman gauge ($\xi = 1$) shares this same property.

A. Getting τ_1

The evaluation of this form factor was discussed in Sec. VIII A and is summarized in Eq. (57). The evaluation of the traces

$$\text{Tr}(p_\mu \tilde{\Gamma}_{(D)}^\mu) \quad \text{and} \quad \text{Tr}(p_\mu \tilde{\Gamma}_{(\xi)}^\mu) \quad (75)$$

gives terms that are proportional to λ_3 , and if this form factor vanishes, then

$$\begin{aligned} \tau_1((p+k)^2, p^2, k^2) &= \frac{2p^2 + (pk)}{\Delta(k^2 + 2(pk))} (B((p+k)^2) \\ &\quad - B(p^2)). \end{aligned} \quad (76)$$

For a constant $B(p^2)$ or in the chiral limit, i.e., for massless fermions, this form factor vanishes.

B. Getting τ_8

For the evaluation of τ_8 we have to consider Eq. (58) and contract it with a Levi-Civita tensor, arriving at

$$\begin{aligned} \tau_8 \epsilon_{\zeta\mu\alpha\beta} p^\alpha k^\beta \epsilon^{\zeta\mu\alpha'\beta'} p_{\alpha'} k_{\beta'} &= 2\tau_8 ((pk)^2 - p^2 k^2) \\ &= \frac{i}{4} \text{Tr}(\gamma^\zeta \gamma_5 \Gamma^\mu) \epsilon_{\zeta\mu\alpha\beta} p^\alpha k^\beta. \end{aligned} \quad (77)$$

The corresponding contractions with $\tilde{\Gamma}_{(D)}^\mu$ and $\tilde{\Gamma}_{(\xi)}^\mu$ produce lengthy expressions that require the products $\lambda_1 \lambda_1$, $\lambda_1 \lambda_2$, and $\lambda_2 \lambda_2$, with the two form factors evaluated at different kinematical points. These expressions simplify in the limit of $\lambda_2 = 0$, that corresponds to a $A(p^2)$ that is essentially constant. In this simplified version, the traces read

$$\begin{aligned} \tilde{\Gamma}_{(D)}^\mu &\rightarrow 4\lambda_1((U+k)^2, U^2, k^2)\lambda_1((V+q)^2, V^2, q^2) \\ &\quad \times [(kU)(pV) - (kV)(pU)] \end{aligned} \quad (78)$$

and

$$\begin{aligned} \tilde{\Gamma}_{(\xi)}^\mu &\rightarrow 2\lambda_1((U+k)^2, U^2, k^2)\lambda_1((V+q)^2, V^2, q^2) \\ &\quad \times [(qU)((kq)(pV) - (kV)(pq)) \\ &\quad + (qV)((kU)(pq) - (kq)(pU))]. \end{aligned} \quad (79)$$

From these results, one can build the renormalized expression

$$\begin{aligned} \tau_8((p+k)^2, p^2, k^2) &= \frac{ig^2 Z_2}{\Delta} \int \frac{d^4 q}{(2\pi)^4} \lambda_1((U+k)^2, U^2, k^2) \lambda_1((V+q)^2, V^2, q^2) \\ &\times \frac{A(U^2)}{A^2(U^2)U^2 - B^2(U^2)} \frac{A(V^2)}{A^2(V^2)V^2 - B^2(V^2)} \left\{ 2D(q^2)[(kU)(pV) - (kV)(pU)] \right. \\ &\left. + \frac{1}{q^2} \left(\frac{\xi}{q^2} - D(q^2) \right) [(qU)((kq)(pV) - (kV)(pq)) + (qV)((kU)(pq) - (kq)(pU))] \right\} \end{aligned} \quad (80)$$

that will be explored later in a perturbative type of approach.

C. Getting τ_2 , τ_3 , and τ_6

The starting points for the computation of τ_2 , τ_3 , and τ_6 are the expressions for the Λ_i 's derived in Sec. VIII C, see Eqs. (62)–(64). The complete expression for each of these quantities is, once more, rather lengthy and, as in the calculation of τ_8 , requires the products $\lambda_i \lambda_j$ at different kinematical points. A simplified version can be obtained by ignoring λ_2 . For the trace appearing in Λ_1 it follows that

$$\tilde{\Gamma}_{(D)}^\mu \rightarrow 4\lambda_1((U+k)^2, U^2, k^2) \lambda_1((V+q)^2, V^2, q^2)(UV) \quad (81)$$

and

$$\tilde{\Gamma}_{(\xi)}^\mu \rightarrow 4\lambda_1((U+k)^2, U^2, k^2) \lambda_1((V+q)^2, V^2, q^2)(qU)(qV). \quad (82)$$

For the traces appearing in Λ_2 and Λ_3 , there is no contribution coming from $\tilde{\Gamma}_{(D)}^\mu$ and $\tilde{\Gamma}_{(\xi)}^\mu$, even if λ_2 is taken into account. Then, for Λ_2 and Λ_3 the traces in their definitions vanish in the chiral limit, i.e., for $B = 0$ and $\lambda_3 = 0$, and they are given by

$$\begin{aligned} \Lambda_2 &= -(pk)\lambda_1((p+k)^2, p^2, k^2) \\ &\quad - (2p^2 + (pk))(k^2 + 2(pk))\lambda_2((p+k)^2, p^2, k^2), \end{aligned} \quad (83)$$

$$\begin{aligned} \Lambda_3 &= -p^2\lambda_1((p+k)^2, p^2, k^2) \\ &\quad - (2p^2 + (pk))^2\lambda_2((p+k)^2, p^2, k^2), \end{aligned} \quad (84)$$

and with the help of Eqs. (33) and (34) they can be rewritten in terms of the fermion propagator function $A(p^2)$. On the other hand,

$$\begin{aligned} \Lambda_1 &= 4Z_2 - 4ig^2 Z_2 \int \frac{d^4 q}{(2\pi)^4} \lambda_1((U+k)^2, U^2, k^2) \\ &\quad \times \lambda_1((V+q)^2, V^2, q^2) \\ &\quad \times \frac{A(U^2)}{A^2(U^2)U^2 - B^2(U^2)} \frac{A(V^2)}{A^2(V^2)V^2 - B^2(V^2)} \\ &\quad \times \left\{ D(q^2)(UV) + \frac{1}{q^2} \left(\frac{\xi}{q^2} - D(q^2) \right) (qU)(qV) \right\}. \end{aligned} \quad (85)$$

Then, it follows from Eqs. (62) to (64) that the transverse form factors are given by

$$\begin{aligned} \tau_2 &= -\frac{1}{4\Delta} \Lambda_1 - \frac{3}{4\Delta} (\lambda_1 + 2(2p^2 + (pk))\lambda_2) \\ &= -\frac{1}{4\Delta} \{ \Lambda_1 + 3[\lambda_1 + 2(2p^2 + (pk))\lambda_2] \}, \end{aligned} \quad (86)$$

$$\begin{aligned} \tau_3 &= \frac{1}{8\Delta} \left\{ (k^2 + 4p^2 + 4(pk))\Lambda_1 \right. \\ &\quad + k^2 \left(\left[3 + 4\frac{p^2}{k^2} + 8\frac{(pk)}{k^2} \right] \lambda_1 \right. \\ &\quad \left. \left. + (2p^2 + (pk)) \left[2 + 8\frac{p^2}{k^2} + 8\frac{(pk)}{k^2} \right] \lambda_2 \right) \right\}, \end{aligned} \quad (87)$$

$$\begin{aligned} \tau_6 &= -\frac{1}{8\Delta} \left\{ (k^2 + 2(pk))\Lambda_1 \right. \\ &\quad + k^2 \left(\left[3 + 2\frac{(pk)}{k^2} \right] \lambda_1 \right. \\ &\quad \left. \left. + (2p^2 + (pk)) \left[2 + 4\frac{(pk)}{k^2} \right] \lambda_2 \right) \right\}. \end{aligned} \quad (88)$$

The form factors λ_1 and λ_2 are related to the fermion propagator form factor $A(p^2)$ and the above expressions can be rewritten in terms of $A(p^2)$. The form factors in Eqs. (86) to (88) are evaluated at the kinematical point $((p+k)^2, p^2, k^2)$. The above expressions for τ_3 and for τ_6

imply that, within the approximations considered, they comply with the equality

$$\begin{aligned}\tau_3 + \tau_6 &= \frac{1}{4\Delta}(2p^2 + (pk))\Lambda_1 + \frac{1}{4\Delta}((2p^2 + 3(pk))\lambda_1 \\ &\quad + 2(2p^2 + (pk))^2\lambda_2) \\ &= \frac{2p^2 + (pk)}{4\Delta} \left\{ \Lambda_1 + \frac{2p^2 + 3(pk)}{2p^2 + (pk)}\lambda_1 \right. \\ &\quad \left. + 2(2p^2 + (pk))\lambda_2 \right\}.\end{aligned}\quad (89)$$

This particular combination of the two form factors appears in the expression of the full vertex, see Eq. (39), and the last equation can be used to simplify the writing of the vertex.

D. Getting τ_4 , τ_5 , and τ_7

Of the full set of transverse form factors, it remains to discuss the evaluation of τ_4 , τ_5 , and τ_7 from the vertex equation. As a first step, the computation of the Ω_i 's defined in Eqs. (66) to (68) is required. In the chiral limit, as defined previously, it turns out that all the Ω_i 's vanish and, therefore, in this limit, $\tau_4 = \tau_5 = \tau_7 = 0$. In one-loop perturbation theory, these form factors are proportional to the fermion mass and vanish in the chiral limit.

E. On the transverse form factors and the chiral limit of the vertex equation

Our discussion for the transverse form factors τ_i using the vertex equation rely on an exact solution of the vertex WTI, given in Eqs. (33) to (36). On the rhs of the equation, the chiral limit inside the integral is taken, implying to set $m = B = 0$ and to take $\lambda_3 = 0$ to be consistent with the solution of the vertex WTI. Further, the contribution of the two-photon-two-fermion Green function to the solution is not taken into account. To arrive at manageable expressions, the contribution of λ_2 is, sometimes, ignored. Despite the approximations considered, the calculation using the vertex DSE confirms the estimation of the one-loop perturbative calculation [34] for the τ_i that, for massless fermions, the nonvanishing form factors to be considered are τ_2 , τ_3 , τ_6 , and τ_8 . Moreover, the derived expressions show that the combination of momenta Δ , see Eq. (65), defines a momentum scale for the transverse form factors, as in the perturbative calculation.

F. On the asymptotic scaling of the transverse form factors

The results just derived for the transverse form factor allow us to describe their asymptotic behavior, both in the limit where the incoming fermion momentum p or the incoming photon momentum k are large. The combination of momenta Δ defined in Eq. (65) defines a scale for all the transverse form factors. To study the asymptotic regime, it

is convenient to write

$$\Delta = p^2 k^2 \left(1 - \frac{(pk)^2}{p^2 k^2} \right) = p^2 k^2 (1 - \cos^2 \theta_{pk}). \quad (90)$$

In what concerns power counting, in the large p and/or k limit, one can set $\Delta \approx p^2 k^2$. Then, naive power counting gives

$$\tau_1(k^2, p^2, k^2) \approx \frac{(pk)}{p^2 k^4} (B(k^2) - B(p^2)). \quad (91)$$

The corresponding analysis for $\tau_8(k^2, p^2, k^2)$ is more involved due to the momentum integration that appears on its definition. A full answer requires being able to perform the integrations to understand the leading asymptotic behavior. This can certainly be done in perturbation theory and it will be considered in Sec. X.

For the study of the asymptotic behavior of $\tau_{2,3,6}$, the longitudinal form factors λ_2 and λ_3 are subleading relative to λ_1 and, therefore, the leading term is associated with the later longitudinal form factor. At large k , it follows that

$$\tau_2(k^2, p^2, k^2) \approx -\frac{Z_2}{p^2 k^2} - \frac{3}{8p^2 k^2} (A(k^2) + A(p^2)), \quad (92)$$

$$\tau_3(k^2, p^2, k^2) \approx \frac{Z_2}{2p^2} + \frac{3}{16p^2} (A(k^2) + A(p^2)), \quad (93)$$

$$\tau_6(k^2, p^2, k^2) \approx -\frac{Z_2}{2p^2} - \frac{3}{16p^2} (A(k^2) + A(p^2)), \quad (94)$$

and all these form factors, together with the sum $\tau_3 + \tau_6$, become irrelevant at large momenta.

Naive power counting suggests that, in the large photon momentum limit, the vertex is dominated by the longitudinal form factors. In this case, for large k the vertex reads

$$\begin{aligned}\Gamma^\mu(p, -p - k; k) &\approx -k^\mu \lambda_3(k^2, p^2, k^2) + \gamma^\mu \lambda_1(k^2, p^2, k^2) \\ &\quad + \not{k} k^\mu \lambda_2(k^2, p^2, k^2),\end{aligned}\quad (95)$$

where

$$\lambda_1(k^2, p^2, k^2) = \frac{1}{2} (A(k^2) + A(p^2)), \quad (96)$$

$$\lambda_2(k^2, p^2, k^2) = \frac{1}{2} \frac{1}{k^2 - p^2} (A(k^2) - A(p^2)), \quad (97)$$

$$\lambda_3(k^2, p^2, k^2) = \frac{1}{k^2 - p^2} (B(k^2) - B(p^2)). \quad (98)$$

On the other hand, for large p , the photon-fermion vertex is given by

$$\Gamma^\mu(p, -p - k; k) = -2p^\mu \lambda_3 + \gamma^\mu \lambda_1 + 4\not{p} p^\mu \lambda_2, \quad (99)$$

where the longitudinal form factors are

$$\lambda_1(p^2, p^2, k^2) = A(p^2), \quad \lambda_2(p^2, p^2, k^2) = \frac{1}{2} A'(p^2),$$

$$\text{and } \lambda_3(p^2, p^2, k^2) = B'(p^2). \quad (100)$$

If the terms with derivatives of A and/or B are subleading, then the above expression can be further simplified.

X. PERTURBATIVE ESTIMATION OF THE TRANSVERSE FORM FACTORS

The above expressions for the transverse form factors can be evaluated with perturbation theory. This calculation requires a discussion of the renormalization in QED. The renormalization constants can be computed requiring

$$A(\mu_F^2) = 1 \quad \text{and} \quad B(\mu_F^2) = m^{(\text{phys})}, \quad (101)$$

where μ_F is the renormalization mass scale and $m^{(\text{phys})}$ is the physical fermion mass. The bare functions, evaluated using dimensional regularization and in Minkowski space-time, that determine the inverse of the fermion propagator were computed to one-loop order in the coupling constant in [34] and are given by

$$A^{(0)}(p^2) = 1 + \frac{\alpha\xi}{4\pi} \left[C\mu^\epsilon + \left(1 + \frac{m_0^2}{p^2} \right) (1 - L(p^2)) \right], \quad (102)$$

$$B^{(0)}(p^2) = m_0 \left\{ 1 + \frac{\alpha\xi}{4\pi} [2 + C\mu^\epsilon - L(p^2)] + \frac{\alpha}{\pi} \left[1 + \frac{3}{4} C\mu^\epsilon - \frac{3}{4} L(p^2) \right] \right\}, \quad (103)$$

where m_0 is the bare fermion mass, the spacetime dimension is $D = 4 - \epsilon$, μ is the usual mass parameter introduced to ensure that the coupling α remains dimensionless for any D , and

$$C = \frac{2}{\epsilon} - \gamma_E + \ln 4\pi + \ln \frac{m^2}{\mu^2}, \quad (104)$$

$$L(p^2) = \left(1 - \frac{m^2}{p^2} \right) \ln \left(1 - \frac{p^2}{m^2} \right). \quad (105)$$

To first order in the coupling constant, the bare coupling α_0 is undistinguishable from the renormalized α . Further, the expressions for the form factors are all multiplied by g^2 and, for the level of precision under discussion, one can rely on the tree-level gauge boson propagator. It follows from the renormalization conditions (101) that the corresponding renormalized functions are

$$A(p^2) = 1 + \frac{\alpha\xi}{4\pi} \left[L(\mu_F^2) - L(p^2) + \frac{m^2}{p^2} (1 - L(p^2)) - \frac{m^2}{\mu_F^2} (1 - L(\mu_F^2)) \right], \quad (106)$$

$$B(p^2) = m \left\{ 1 + \frac{\alpha}{4\pi} (3 + \xi) [L(\mu_F^2) - L(p^2)] \right\}, \quad (107)$$

where m is the physical fermion mass. Further, the fermion field and the mass renormalization constants are

$$Z_2 = 1 - \frac{\alpha\xi}{4\pi} \left[C\mu^\epsilon + \left(1 + \frac{m^2}{\mu_F^2} \right) (1 - L(\mu_F^2)) \right], \quad (108)$$

$$Z_0 = 1 - \frac{\alpha\xi}{4\pi} [2 + C\mu^\epsilon - L(\mu_F^2)] - \frac{\alpha}{\pi} \left[1 + \frac{3}{4} C\mu^\epsilon - \frac{3}{4} L(\mu_F^2) \right], \quad (109)$$

respectively. A straightforward calculation shows that the perturbative solution for the transverse vertex in [34] reproduces the solution of the WTl.

The perturbative estimation of τ_1 , i.e., of Eq. (76), is

$$\tau_1((p+k)^2, p^2, k^2) = m \frac{\alpha}{4\pi} (3 + \xi) \frac{2p^2 + (pk)}{(p^2 k^2 - (pk)^2)(k^2 + 2(pk))}, \quad (110)$$

$$\left\{ \ln \left(\frac{m^2 - p^2}{m^2 - (p+k)^2} \right) + \frac{m^2}{(p+k)^2} \ln \left(1 - \frac{(p+k)^2}{m^2} \right) - \frac{m^2}{p^2} \ln \left(1 - \frac{p^2}{m^2} \right) \right\}. \quad (111)$$

In the small momentum limit where $p^2, k^2 \lesssim m^2$, this form factor is given by

$$\tau_1((p+k)^2, p^2, k^2) \approx \frac{1}{m} \frac{\alpha(3 + \xi)}{8\pi} \frac{2p^2 + (pk)}{p^2 k^2 - (pk)^2}. \quad (112)$$

The divergence at low momenta is not a problem for the DSE, as it is the combination $k^2 \tau_1$, multiplied by other factors of momenta coming from the integration measure, that appears in the integral equations, see Eqs. (A1), (A2), and (A4). In the large momentum limit, the expression for τ_1 simplifies but its exact expression depends on the relative importance of the momenta p^2 and k^2 . The above expression shares with the one-loop perturbative solution the common global factor

$$\frac{\alpha}{4\pi} \frac{(3 + \xi)}{p^2 k^2 - (pk)^2} m. \quad (113)$$

The perturbative solution can be found in [34] and is written in terms of Spence functions.

Let us now compute τ_8 within the approximation scheme considered so far. Its expression, see Eq. (80), is given by

$$\begin{aligned}\tau_8((p+k)^2, p^2, k^2) &= \frac{ig^2 Z_2}{\Delta} \int \frac{d^4 q}{(2\pi)^4} \frac{1}{q^2} \frac{1}{U^2 - m^2} \frac{1}{V^2 - m^2} \left\{ 2[(kU)(pV) - (kV)(pU)] \right. \\ &\quad \left. + \frac{1}{q^2} (\xi - 1)[(qU)((kq)(pV) - (kV)(pq)) + (qV)((kU)(pq) - (kq)(pU))] \right\} \\ &= \frac{ig^2 Z_2}{\Delta} \int \frac{d^4 q}{(2\pi)^4} \frac{1}{q^2} \frac{1}{(p-q)^2 - m^2} \frac{1}{(p+k-q)^2 - m^2} \left\{ 2[(pk)^2 - k^2 p^2 + k^2(pq) - (pk)(kq)] \right. \\ &\quad \left. + \frac{1}{q^2} (\xi - 1)[2(pk)(pq)(kq) - k^2(pq)^2 - p^2(kq)^2 + q^2(k^2(pq) - (pk)(kq))] \right\}. \quad (114)\end{aligned}$$

The momentum integrations can be performed within dimensional regularization, after the introduction of Feynman parameters. The relevant momentum integrations are summarized in Appendix B. After a straightforward algebra, this form factors reads

$$\begin{aligned}\tau_8((p+k)^2, p^2, k^2) &= -\frac{\alpha}{4\pi} \left\{ \int_0^1 dx \int_0^x dy \frac{x-2}{\mathcal{M}^2 - K^2} + \int_0^1 dx \int_0^x dy \frac{y\mathcal{M}^2}{(\mathcal{M}^2 - K^2)^2} \right. \\ &\quad \left. + \xi \left[\int_0^1 dx \int_0^x dy \frac{x}{\mathcal{M}^2 - K^2} + \int_0^1 dx \int_0^x dy \frac{y\mathcal{M}^2}{(\mathcal{M}^2 - K^2)^2} \right] \right\}, \quad (115)\end{aligned}$$

where $K = xp + yk$ and $\mathcal{M}^2 = x(p^2 - m^2) + y(k^2 + 2(pk)) + i\epsilon$. For large photon momentum, $K^2 = y^2 k^2$ and $\mathcal{M}^2 = k^2(y + i\epsilon/k^2)$, and it follows that

$$\begin{aligned}\tau_8((p+k)^2, p^2, k^2) &= -\frac{\alpha}{4\pi k^2} \left\{ \int_0^1 dx \int_0^x dy \frac{x-2}{y(1-y) + i\epsilon/k^2} + \int_0^1 dx \int_0^x dy \frac{y^2}{(y(1-y) + i\epsilon/k^2)^2} \right. \\ &\quad \left. + \xi \left[\int_0^1 dx \int_0^x dy \frac{x}{y(1-y) + i\epsilon/k^2} + \int_0^1 dx \int_0^x dy \frac{y^2}{(y(1-y) + i\epsilon/k^2)^2} \right] \right\}, \quad (116)\end{aligned}$$

while for large fermion momentum,

$$\begin{aligned}\tau_8((p+k)^2, p^2, k^2) &= -\frac{\alpha}{4\pi p^2} \left\{ \int_0^1 dx \frac{x(x-2)}{x(1-x) + i\epsilon/p^2} + \frac{1}{2} \int_0^1 dx \frac{x^3}{(x(1-x) + i\epsilon/p^2)^2} \right. \\ &\quad \left. + \xi \left[\int_0^1 dx \frac{x^2}{x(1-x) + i\epsilon/p^2} + \frac{1}{2} \int_0^1 dx \frac{x^3}{(x(1-x) + i\epsilon/p^2)^2} \right] \right\}. \quad (117)\end{aligned}$$

The integration over Feynman parameters results in pure numbers, see Appendix B 1, and, therefore, for large photon or fermion momentum τ_8 scale with the inverse of the momentum squared. The perturbative estimation of this form factor, see [34], shares with the above result the common factor $\alpha/4\pi\Delta$.

Let us proceed and look at the computation of τ_2 , τ_3 , and τ_6 . Within the approximations considered, their evaluation needs the quantity Λ_1 that is defined in Eq. (85). Following the same reasoning as in the previous cases, it follows that

$$\begin{aligned}\Lambda_1 &= 4Z_2 - 4ig^2 \int \frac{d^4 q}{(2\pi)^4} \frac{1}{q^2} \frac{1}{(p-q)^2 - m^2} \frac{1}{(p+k-q)^2 - m^2} \\ &\quad \times \left\{ p^2 + (pk) - \frac{(pq)^2}{q^2} - \frac{(pq)(kq)}{q^2} + \xi \left[q^2 - (kq) - 2(pq) + \frac{(pq)^2}{q^2} + \frac{(pq)(kq)}{q^2} \right] \right\}. \quad (118)\end{aligned}$$

The momentum integration has a divergent term, the ξq^2 term inside the integral. This divergence is exactly cancelled by the pole term of Z_2 , see Eq. (108), and Λ_1 is finite. Indeed, an evaluation of the divergent integral combined with the expression for Z_2 results in the finite expression

$$4 \left[1 + \frac{\alpha}{4\pi} \xi \int_0^1 dx \int_0^x dy \left\{ \frac{K^2}{\mathcal{M}^2 - K^2} + 2 \ln \frac{\mathcal{M}^2 - K^2}{m^2} \right\} - \frac{\alpha}{4\pi} \xi \left(1 + \frac{m^2}{\mu_F^2} \right) (1 - L(\mu_F^2)) \right] \quad (119)$$

that should be added to

$$4 \frac{\alpha}{4\pi} \int_0^1 dx \int_0^x dy \left\{ \frac{1 + \xi p^2 + (pk)}{2} \frac{p^2}{\mathcal{M}^2 - K^2} - \xi \frac{x((pk) + 2p^2) + y(k^2 + 2(pk))}{\mathcal{M}^2 - K^2} \right. \\ \left. + (\xi - 1) \frac{x^2 p^2 (p^2 + (pk)) + y^2 (pk)(k^2 + (pk)) + xy(p^2(k^2 + (pk)) + (pk)(p^2 + (pk)))}{(\mathcal{M}^2 - K^2)^2} \right\}. \quad (120)$$

For small fermion momentum, it follows that

$$\Lambda_1 = 4 \left\{ 1 + \frac{\alpha}{4\pi} \xi \int_0^1 dx \int_0^x dy \left(\frac{k^2}{m^2} \frac{y^2}{y(1-y) \frac{k^2}{m^2} - x + i\epsilon/m^2} + 2 \ln \left(y(1-y) \frac{k^2}{m^2} - x + i\epsilon/m^2 \right) \right) \right. \\ \left. + \frac{\alpha}{4\pi} \int_0^1 dx \int_0^x dy \left(\frac{1 + \xi(pk)}{2} \frac{k^2}{m^2} - \xi \frac{k^2}{m^2} + (\xi - 1) \frac{(pk)k^2}{m^4} \frac{y}{y(1-y) \frac{k^2}{m^2} - x + i\epsilon/m^2} \right) \right. \\ \left. \times \frac{y}{y(1-y) \frac{k^2}{m^2} - x + i\epsilon/m^2} - \frac{\alpha}{4\pi} \xi \left(1 + \frac{m^2}{\mu_F^2} \right) (1 - L(\mu_F^2)) \right\} \\ \longrightarrow 4 \left\{ 1 + \frac{\alpha}{4\pi} \xi \frac{k^2}{m^2} \int_0^1 dx \int_0^x dy \frac{y(y-1)}{y(1-y) \frac{k^2}{m^2} - x + i\epsilon/m^2} \right\} \sim 4 \left(1 + \frac{\alpha}{8\pi} \xi \right), \quad (121)$$

where in the last line only the leading term in k^2 was taken into account. The large photon momentum limit gives

$$\Lambda_1 = 4 \left\{ 1 + \frac{\alpha}{4\pi} \xi \int_0^1 dx \int_0^x dy \left(\frac{y^2}{y(1-y) + i\epsilon/k^2} \right. \right. \\ \left. \left. + 2 \ln \left(\frac{k^2}{m^2} (y(1-y) + i\epsilon/k^2) \right) \right) \right. \\ \left. - 4 \frac{\alpha}{4\pi} \xi \int_0^1 dx \int_0^x dy \frac{y}{y(1-y) + i\epsilon/k^2} \right. \\ \left. - \frac{\alpha}{4\pi} \xi \left(1 + \frac{m^2}{\mu_F^2} \right) (1 - L(\mu_F^2)) \right\} \\ \longrightarrow 4 \left\{ 1 + \frac{\alpha}{4\pi} \xi \ln \frac{k^2}{m^2} \right\}. \quad (122)$$

At small fermion momentum,

$$\lambda_1 = \frac{1}{2} (A(k^2) + A(p^2)), \quad \lambda_2 = \frac{1}{2k^2} (A(k^2) - A(p^2)), \quad (123)$$

and, therefore,

$$\tau_2 = -\frac{1}{\Delta} \left(1 + \frac{\alpha}{8\pi} \xi - \frac{3}{8} (A(k^2) + A(p^2)) \right. \\ \left. - \frac{3(pk)}{4k^2} (A(k^2) - A(p^2)) \right) \quad (124)$$

$$\tau_3 = \frac{k^2}{2\Delta} \left(1 + \frac{\alpha}{8\pi} \xi + \frac{3}{4} (A(k^2) + A(p^2)) \right. \\ \left. + \frac{1(pk)}{4k^2} (A(k^2) - A(p^2)) \right) \quad (125)$$

$$\tau_6 = -\tau_3, \quad (126)$$

while at large photon momentum the form factors are given by the same expressions with the substitution

$$1 + \frac{\alpha}{8\pi} \xi \longrightarrow 1 + \frac{\alpha}{4\pi} \xi \ln \frac{k^2}{m^2}. \quad (127)$$

For the Landau gauge where $A(k^2) = 1$ and $\xi = 0$, the above expressions simplify into

$$\tau_2 = -\frac{1}{4\Delta} \quad \text{and} \quad \tau_3 = \frac{5k^2}{4\Delta} = \frac{5}{4 \left(p^2 - \frac{(pk)^2}{k^2} \right)}. \quad (128)$$

In both limits considered and within the approximation discussed, $\tau_3 + \tau_6 = 0$, and the full vertex, see Eq. (39), simplifies into

$$\begin{aligned}
\Gamma^\mu(p, -p - k; k) = & p^\mu(k^2\tau_1 - 2\lambda_3) - k^\mu((pk)\tau_1 + \lambda_3) \\
& + \gamma^\mu[\lambda_1 - 2(pk)\tau_3] + \not{k}[k^\mu(\lambda_2 - (pk)\tau_2) \\
& + p^\mu(2\lambda_2 + k^2\tau_2 + 2\tau_3)] \\
& + 2\not{p}[k^\mu(\lambda_2 - (kp)\tau_2) + p^\mu(2\lambda_2 + k^2\tau_2)] \\
& + i\tau_8\gamma_\sigma\gamma_5\epsilon^{\sigma\mu\alpha\beta}p_\alpha k_\beta. \tag{129}
\end{aligned}$$

In the Landau gauge, where $\xi = 0$ and $A(k^2) = 1$, the above expressions can be simplified further.

XI. SUMMARY AND CONCLUSIONS

In the current work, the DSEs for the propagators together with the photon-fermion vertex are investigated. After the introduction of a tensor basis to describe the vertex, a set of exact nonlinear integral equations for each of the transverse form factors is derived. These equations are quite general and can be applied both to the QED photon-fermion vertex and to the QCD quark-gluon vertex, after taking into account the color structure and correcting for the contribution of λ_4 . The nonlinear integral equations allow, in principle, to compute the full vertex, assuming that the longitudinal part of the vertex is given by the solution of the vertex Ward-Takahashi identity. Indeed, as reported, the equations are solved, for any linear covariant gauge, within a perturbative approach. The comparison of the solutions of the integral equation with the result of one-loop perturbation theory show that the main features are shared by the two solutions of QED. For example, in the chiral limit both solutions coincide in the number of form factors that are needed to fully describe Γ^μ . Being a first step to access the full vertex, this result is encouraging and motivates the realization of a nonperturbative study of the vertex in QED. Nonperturbative studies can be performed, e.g., by going to the Euclidean spacetime or by calling for integral representations of the form factors.

Any solution of the DSE requires defining a truncation of the infinite tower of relations. For QED, assuming that perturbation theory makes sense, there is a “natural” way of approaching the infinite tower of integral equations. Herein, the truncation considered ignores the contribution of the two-photon-two-fermion one-particle irreducible Green function. This drastic approximation can be improved, as it can be found in the literature a solution of the corresponding WTI, that can be combined with the corresponding DSE. This integral equation is also a function of the full

vertex, making the realization of such an improvement not trivial. What changes by improving the solution of the DSE for, e.g., chiral symmetry breaking and the number of critical fermion flavors [63] remains to be seen. Using the information that can be read from the vertex integral equation can also help to improve the modeling of the photon-fermion vertex and its compatibility with the requirements of QFT.

A full study of the system of equations also requires a proper handling of the infrared divergences, which were briefly touched on when looking at the photon propagator equation. This issue is also important in the definition of an on-shell vertex and its relation to the fermion anomalous magnetic and electric moments. In particular, in the current manuscript the on-shell vertex is given in terms of the fundamental form factors that are associated with Γ^μ .

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: DYSON-SCHWINGER EQUATIONS WITH FULL BASIS OF OPERATORS

For completeness, in this appendix, we report on the DSE using the full basis discussed in the main text. The Dirac algebra was worked out with the help of FeynCalc [64–66], a Mathematica software package.

1. The fermion gap equation

The fermion gap equation (3) can be projected into a Dirac scalar and a Dirac vector term. The scalar component of the equation is obtained by taking the Dirac trace of Eq. (3) that, in a general linear covariant gauge, reads

$$\begin{aligned}
B(p^2) = & m - ig^2 \int \frac{d^4 k}{(2\pi)^4} \frac{1}{A^2((p-k)^2)(p-k)^2 - B^2((p-k)^2)} \left\{ D(k^2) \left(A((p-k)^2) \left[2\lambda_3 p^2 \left(\frac{(pk)^2}{p^2 k^2} - 1 \right) + \tau_1 k^2 p^2 \left(1 - \frac{(pk)^2}{k^2 p^2} \right) \right. \right. \right. \\
& + \tau_4 k^4 p^2 \left(1 - \frac{(pk)}{k^2} - \frac{(pk)^2}{p^2 k^2} + \frac{(pk)^3}{p^2 k^4} \right) + 3\tau_5 k^2 \left(1 - \frac{(pk)}{k^2} \right) + \tau_7 k^4 \left(\frac{3}{2} + 4\frac{p^2}{k^2} - \frac{15(pk)}{2k^2} - 6\frac{p^2(pk)}{k^2} + 8\frac{(pk)^2}{k^4} \right) \Big] \\
& + B((p-k)^2) \left[3\lambda_1 + 4\lambda_2 p^2 \left(1 - \frac{(pk)^2}{p^2 k^2} \right) + 2\tau_2 k^2 p^2 \left(1 - \frac{(pk)^2}{p^2 k^2} \right) + 3\tau_3 k^2 + 3\tau_6 k^2 \left(2\frac{(pk)}{k^2} - 1 \right) \Big] \right) \\
& + \frac{\xi}{k^2} \left(A((p-k)^2) \left[\lambda_3 k^2 \left(-\frac{2(pk)^2}{k^4} + 3\frac{(pk)}{k^2} - 1 \right) \right] + B((p-k)^2) \left[\lambda_1 + \lambda_2 k^2 \left(\frac{4(pk)^2}{k^4} - 4\frac{(pk)}{k^2} + 1 \right) \right] \right) \Big\}. \quad (A1)
\end{aligned}$$

The vector component is obtained after multiplying Eq. (3) by $\not{p}$ and then taking the Dirac trace

$$\begin{aligned}
A(p^2) = & 1 + ig^2 \int \frac{d^4 k}{(2\pi)^4} \frac{1}{A^2((p-k)^2)(p-k)^2 - B^2((p-k)^2)} \left\{ D(k^2) \left(A((p-k)^2) \left[\lambda_1 \left(-1 + 3\frac{(pk)}{p^2} - 2\frac{(pk)^2}{p^2 k^2} \right) \right. \right. \right. \\
& + 2\lambda_2 k^2 \left(1 + 2\frac{p^2}{k^2} - 2\frac{(pk)}{k^2} - \frac{(pk)^2}{p^2 k^2} - 2\frac{(pk)^2}{k^4} + 2\frac{(pk)^3}{p^2 k^4} \right) + \tau_2 k^4 \left(1 + 2\frac{p^2}{k^2} - 2\frac{(pk)}{k^2} - \frac{(pk)^2}{p^2 k^2} - 2\frac{(pk)^2}{k^4} + 2\frac{(pk)^3}{p^2 k^4} \right) \\
& + \tau_3 k^2 \left(-1 + 3\frac{(pk)}{p^2} - 2\frac{(pk)^2}{p^2 k^2} \right) + 3\tau_6 k^2 \left(1 - \frac{(pk)}{p^2} - 2\frac{(pk)}{k^2} + 2\frac{(pk)^2}{p^2 k^2} \right) + 2\tau_8 k^2 \left(1 - \frac{(pk)^2}{p^2 k^2} \right) \Big] \\
& + B((p-k)^2) \left[2\lambda_3 \left(\frac{(pk)^2}{p^2 k^2} - 1 \right) + \tau_1 k^2 \left(1 - \frac{(pk)^2}{p^2 k^2} \right) + \tau_4 k^2 (pk) \left(1 - \frac{(pk)^2}{p^2 k^2} \right) \right. \\
& + 3\tau_5 \frac{(pk)}{p^2} + \tau_7 k^2 \left(-2 + \frac{3(pk)}{2p^2} + 6\frac{(pk)}{k^2} - 4\frac{(pk)^2}{p^2 k^2} \right) \Big] + \frac{\xi}{k^2} \left(A((p-k)^2) \left[\lambda_1 \left(-1 - \frac{(pk)}{p^2} + 2\frac{(pk)^2}{p^2 k^2} \right) \right. \right. \\
& + \lambda_2 k^2 \left(1 - \frac{(pk)}{p^2} - 4\frac{(pk)}{k^2} + 4\frac{(pk)^2}{p^2 k^2} + 4\frac{(pk)^2}{k^4} - 4\frac{(pk)^3}{p^2 k^4} \right) \Big] + B((p-k)^2) \left[\lambda_3 \left(\frac{(pk)}{p^2} - 2\frac{(pk)^2}{p^2 k^2} \right) \right] \Big\}, \quad (A2)
\end{aligned}$$

where in these equations,

$$\lambda_i = \lambda_i(p^2, (p-k)^2, k^2) \quad \text{and} \quad \tau_i = \tau_i(p^2, (p-k)^2, k^2). \quad (A3)$$

The fermion self-energies Σ_s and Σ_v , see Eq. (7) for the definition, can be read from Eqs. (A1) and (A2). This decomposition of the gap equation in terms of vertex form factors holds both for QED and for QCD, after correcting for the color degrees of freedom and adding the contribution of λ_4 , that vanishes in QED.

2. The photon gap equation

In the basis considered, the equation for the photon propagator is given by

$$\begin{aligned}
\frac{1}{D(k^2)} = & k^2 - i\frac{g^2}{3} \int \frac{d^4 p}{(2\pi)^4} \frac{1}{A^2\left(\left(p + \frac{k}{2}\right)^2\right)\left(p + \frac{k}{2}\right)^2 - B^2\left(\left(p + \frac{k}{2}\right)^2\right)} \\
& \times \frac{1}{A^2\left(\left(p - \frac{k}{2}\right)^2\right)\left(p - \frac{k}{2}\right)^2 - B^2\left(\left(p - \frac{k}{2}\right)^2\right)}
\end{aligned}$$

$$\begin{aligned}
& \times \left\{ A \left(\left(p + \frac{k}{2} \right)^2 \right) A \left(\left(p - \frac{k}{2} \right)^2 \right) \left[2\lambda_1 \left(1 - 4 \frac{p^2}{k^2} \right) + 4\lambda_2 \left(\frac{4p^4 - 2(pk)^2}{k^2} + p^2 \right) \right. \right. \\
& \quad + 2\tau_2 \left(-4 \frac{p^2(pk)^2}{k^2} + k^2 p^2 + 4p^4 - (pk)^2 \right) \\
& \quad + \tau_3 \left(-8 \frac{(pk)^2}{k^2} + 3k^2 - 4p^2 \right) \\
& \quad \left. + 6\tau_6 \left(4p^2 \frac{(pk)}{k^2} - (pk) \right) + 8\tau_8 \left(p^2 - \frac{(pk)^2}{k^2} \right) \right] \\
& + A \left(\left(p + \frac{k}{2} \right)^2 \right) B \left(\left(p - \frac{k}{2} \right)^2 \right) \left[-4\lambda_3 \left(\frac{(pk)}{k^2} + 2 \frac{p^2}{k^2} \right) + 4\tau_1 \left(p^2 - \frac{(pk)^2}{k^2} \right) \right. \\
& \quad + 2\tau_4 (p^2 k^2 + 2p^2(pk) - 2(pk)^2 - 2 \frac{(pk)^3}{k^2}) + 6\tau_5 \left(1 + 2 \frac{(pk)}{k^2} \right) \\
& \quad \left. + 4\tau_7 \left(p^2 + 2 \frac{(pk)^2}{k^2} + 6p^2 \frac{(pk)}{k^2} \right) \right] \\
& + B \left(\left(p + \frac{k}{2} \right)^2 \right) A \left(\left(p - \frac{k}{2} \right)^2 \right) \left[4\lambda_3 \left(\frac{(pk)}{k^2} - 2 \frac{p^2}{k^2} \right) + 4\tau_1 \left(p^2 - \frac{(pk)^2}{k^2} \right) \right. \\
& \quad + 2\tau_4 (p^2 k^2 - 2p^2(pk) - 2(pk)^2 + 2 \frac{(pk)^3}{k^2}) + 6\tau_5 \left(1 - 2 \frac{(pk)}{k^2} \right) \\
& \quad \left. + 4\tau_7 \left(p^2 + 2 \frac{(pk)^2}{k^2} - 6p^2 \frac{(pk)}{k^2} \right) \right] \\
& + B \left(\left(p + \frac{k}{2} \right)^2 \right) B \left(\left(p - \frac{k}{2} \right)^2 \right) \left[16\lambda_1 \frac{1}{k^2} + 16\lambda_2 \frac{p^2}{k^2} + 8\tau_2 \left(p^2 - \frac{(pk)^2}{k^2} \right) + 12\tau_3 - 24\tau_6 \frac{(pk)}{k^2} \right] \Big\}, \tag{A4}
\end{aligned}$$

where the form factors are defined by

$$\lambda_i = \lambda_i((p - k/2)^2, (p + k/2)^2, k^2) \quad \text{and} \quad \tau_i = \tau_i((p - k/2)^2, (p + k/2)^2, k^2). \tag{A5}$$

As for the fermion gap equation, the photon DSE gets contributions from all the transverse form factors.

3. The photon-fermion vertex equation

The decomposition of the photon-fermion vertex in terms of Dirac bilinear forms within the basis defined by Eqs. (24) to (32) is given by

$$\begin{aligned}
\Gamma^\mu(p, -p - k; k) &= p^\mu(k^2\tau_1 - 2\lambda_3) - k^\mu((pk)\tau_1 + \lambda_3) \quad (\text{Scalar Component}) \\
&+ \gamma^\mu[\lambda_1 + k^2\tau_3 + (k^2 + 2(pk))\tau_6] \quad (\text{Vector Components}) \\
&+ \not{k}[k^\mu(\lambda_2 - (pk)\tau_2 - \tau_3 - \tau_6) + p^\mu(2\lambda_2 + k^2\tau_2 - 2\tau_6)] + 2\not{p}[k^\mu(\lambda_2 - (kp)\tau_2) + p^\mu(2\lambda_2 + k^2\tau_2)] \\
&+ i\tau_8\gamma_\sigma\gamma_5\epsilon^{\sigma\mu\alpha\beta}p_\alpha k_\beta \quad (\text{Axial Component}) \\
&+ \frac{1}{2}\sigma^{\mu\alpha}k_\alpha[2\tau_5 + \tau_7(k^2 + 2(kp))] \quad (\text{Tensor Components}) \\
&+ \sigma^{\mu\alpha}p_\alpha[\tau_7(k^2 + 2(kp))] + \sigma^{\alpha\beta}p_\alpha k_\beta[k^\mu(\tau_7 - (pk)\tau_4) + p^\mu(k^2\tau_4 + 2\tau_7)], \tag{A6}
\end{aligned}$$

with the form factors being

$$\lambda_i = \lambda_i((p + k)^2, p^2, k^2) \quad \text{and} \quad \tau_i = \tau_i((p + k)^2, p^2, k^2). \tag{A7}$$

The above decomposition is useful to rewrite the DSE for the vertex.

APPENDIX B: MOMENTUM INTEGRATION

The integrations over momentum that appear in the evaluation of the perturbative-like solution of the vertex equation can be performed, within dimensional regularization, with the help of Feynman parametrization. Indeed, one can write

$$\frac{1}{q^2} \frac{1}{(p-q)^2 - m^2} \frac{1}{(p+k-q)^2 - m^2} = \Gamma(3) \int_0^1 dx \int_0^x dy \frac{1}{(q^2 + 2(Kq) + \mathcal{M}^2)^3}, \quad (\text{B1})$$

together with

$$\frac{1}{q^4} \frac{1}{(p-q)^2 - m^2} \frac{1}{(p+k-q)^2 - m^2} = \Gamma(4) \int_0^1 dx \int_0^x dy \frac{y}{(q^2 + 2(Kq) + \mathcal{M}^2)^4}, \quad (\text{B2})$$

where $K = -xp - yk$ and $\mathcal{M}^2 = x(p^2 - m^2) + y(k^2 + 2(pk))$. It follows that the integrations over q to consider are

$$\begin{aligned} I_0 &= \int \frac{d^n q}{(2\pi)^n} \frac{1}{q^2} \frac{1}{(p-q)^2 - m^2} \frac{1}{(p+k-q)^2 - m^2} \\ &= \frac{i}{16\pi^2} \int_0^1 dx \int_0^x dy \frac{1}{\mathcal{M}^2 - K^2} \left[1 + \frac{\epsilon}{2} \left(\ln \frac{\mathcal{M}^2 - K^2}{\zeta^2} + \ln 4\pi - \gamma_E \right) + \dots \right], \end{aligned} \quad (\text{B3})$$

$$\begin{aligned} I_\mu &= \int \frac{d^n q}{(2\pi)^n} q_\mu \frac{1}{q^2} \frac{1}{(p-q)^2 - m^2} \frac{1}{(p+k-q)^2 - m^2} \\ &= \frac{i}{16\pi^2} \int_0^1 dx \int_0^x dy \frac{xp_\mu + yk_\mu}{\mathcal{M}^2 - K^2} \left[1 + \frac{\epsilon}{2} \left(\ln \frac{\mathcal{M}^2 - K^2}{\zeta^2} + \ln 4\pi - \gamma_E \right) + \dots \right], \end{aligned} \quad (\text{B4})$$

$$\begin{aligned} I_{\mu\nu} &= \int \frac{d^n q}{(2\pi)^n} q_\mu q_\nu \frac{1}{q^2} \frac{1}{(p-q)^2 - m^2} \frac{1}{(p+k-q)^2 - m^2} \\ &= \frac{i}{16\pi^2} \int_0^1 dx \int_0^x dy \frac{1}{\mathcal{M}^2 - K^2} \left((xp_\mu + yk_\mu)(xp_\nu + yk_\nu)(1 + \dots) \right. \\ &\quad \left. + \frac{1}{2} g_{\mu\nu} (\mathcal{M}^2 - K^2) \left[\frac{2}{\epsilon} + \ln \frac{\mathcal{M}^2 - K^2}{\zeta^2} + \ln 4\pi - \gamma_E + \dots \right] \right), \end{aligned} \quad (\text{B5})$$

where $n = 4 - \epsilon$, ζ is usual mass introduced in the dimensional regularization, and γ_E is the Euler-Mascheroni constant. We call the reader's attention to the change of notation in this appendix and that used to write the results in Eqs. (102) and (103) and subsequent results that follow the notation of [34]. Besides the above integrals, the perturbative solution of the vertex equation also needs the following finite integrals:

$$\begin{aligned} \tilde{I}_0 &= \int \frac{d^4 q}{(2\pi)^4} \frac{1}{q^4} \frac{1}{(p-q)^2 - m^2} \frac{1}{(p+k-q)^2 - m^2} \\ &= \frac{i}{16\pi^2} \int_0^1 dx \int_0^x dy \frac{1}{(\mathcal{M}^2 - K^2)^2}, \end{aligned} \quad (\text{B6})$$

$$\begin{aligned} \tilde{I}_\mu &= \int \frac{d^4 q}{(2\pi)^4} q_\mu \frac{1}{q^4} \frac{1}{(p-q)^2 - m^2} \frac{1}{(p+k-q)^2 - m^2} \\ &= \frac{i}{16\pi^2} \int_0^1 dx \int_0^x dy \frac{xp_\mu + yk_\mu}{(\mathcal{M}^2 - K^2)^2}, \end{aligned} \quad (\text{B7})$$

$$\begin{aligned} \tilde{I}_{\mu\nu} &= \int \frac{d^4 q}{(2\pi)^4} q_\mu \frac{1}{q^4} \frac{1}{(p-q)^2 - m^2} \frac{1}{(p+k-q)^2 - m^2} \\ &= \frac{i}{16\pi^2} \int_0^1 dx \int_0^x dy \frac{1}{(\mathcal{M}^2 - K^2)^2} \left((xp_\mu + yk_\mu)(xp_\nu + yk_\nu) + \frac{g_{\mu\nu}}{2} (\mathcal{M}^2 - K^2) \right), \end{aligned} \quad (\text{B8})$$

$$\begin{aligned}
\tilde{I}_{\mu\nu\eta} &= \int \frac{d^4 q}{(2\pi)^4} q_\mu \frac{1}{q^4} \frac{1}{(p-q)^2 - m^2} \frac{1}{(p+k-q)^2 - m^2} \\
&= \frac{i}{16\pi^2} \int_0^1 dx \int_0^x dy \frac{1}{(\mathcal{M}^2 - K^2)^2} \left[(xp_\mu + yk_\mu)(xp_\nu + yk_\nu)(xp_\eta + yk_\eta) \right. \\
&\quad \left. + \frac{1}{2} (g_{\mu\nu}(xp_\eta + yk_\eta) + g_{\mu\eta}(xp_\nu + yk_\nu) + g_{\nu\eta}(xp_\mu + yk_\mu))(\mathcal{M}^2 - K^2) \right]. \tag{B9}
\end{aligned}$$

1. Integration over Feynman parameters

After performing the momentum integrations and in the asymptotic regime, one is left with the integration over Feynman parameters of the type

$$I(n, a) = \int_0^a dx \frac{x^n}{x(1-x) + i\epsilon/k^2}. \tag{B10}$$

In order to perform the integration we rewrite the denominator as

$$\text{Den} = -(x - z_0)(x - z_1), \tag{B11}$$

where

$$\begin{aligned}
z_0 &= \frac{1 + e^{i\eta\pi/2}}{2} = \frac{1}{2} \left(1 + i \sin\left(\frac{\eta\pi}{2}\right) \right), \\
z_1 &= \frac{1 - e^{i\eta\pi/2}}{2} = \frac{1}{2} \left(1 - i \sin\left(\frac{\eta\pi}{2}\right) \right), \tag{B12}
\end{aligned}$$

and

$$\eta = \begin{cases} +1, & \text{for } k^2 > 0, \\ -1, & \text{for } k^2 < 0. \end{cases} \tag{B13}$$

With these definitions,

$$I(n, a) = -e^{-i\eta\pi/2} \int_0^a dx x^n \left(\frac{1}{x - z_0} - \frac{1}{x - z_1} \right). \tag{B14}$$

Then, the integrals read

$$I(0, a) = i \left(\ln \frac{a - z_0}{a - z_1} + \ln \frac{z_1}{z_0} \right), \tag{B15}$$

$$I(1, a) = i \left(z_0 \ln \frac{z_0 - a}{z_0} - z_1 \ln \frac{z_1 - a}{z_1} \right), \tag{B16}$$

$$I(2, a) = i \left(a(z_0 - z_1) + z_0^2 \ln \frac{z_0 - a}{z_0} - z_1^2 \ln \frac{z_1 - a}{z_1} \right), \tag{B17}$$

$$\begin{aligned}
I(3, a) &= i \left(\frac{a^2}{2} (z_0 - z_1) + a^3 (z_0^2 - z_1^2) \right. \\
&\quad \left. + z_0^3 \ln \frac{z_0 - a}{z_0} - z_1^3 \ln \frac{z_1 - a}{z_1} \right), \tag{B18}
\end{aligned}$$

and they are pure complex numbers.

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- [1] S. Navas *et al.* (Particle Data Group), *Phys. Rev. D* **110**, 030001 (2024).
 - [2] Jean Zinn-Justin, *Quantum Field Theory and Critical Phenomena*, 5th ed. (Oxford University Press, New York, 2021).
 - [3] R. Fukuda and T. Kugo, *Nucl. Phys.* **B117**, 250 (1976).
 - [4] V. A. Miransky, *Nuovo Cimento A* **90**, 149 (1985).
 - [5] V. A. Miransky, *Phys. Lett.* **165B**, 401 (1985).
 - [6] V. P. Gusynin and V. A. Miransky, *Phys. Lett. B* **191**, 141 (1987).
 - [7] V. A. Miransky and V. P. Gusynin, *Prog. Theor. Phys.* **81**, 426 (1989).
 - [8] O. Oliveira, T. Frederico, and W. de Paula, *Eur. Phys. J. C* **84**, 851 (2024).
 - [9] A. Grozin, *Lectures on QED and QCD* (World Scientific, Singapore, 2007), ISBN 978-981-256-914-1, 978-981-256-914-1.
 - [10] M. Q. Huber, *Phys. Rep.* **879**, 1 (2020).
 - [11] G. Eichmann, *Few Body Syst.* **63**, 57 (2022).
 - [12] M. N. Ferreira and J. Papavassiliou, *Prog. Part. Nucl. Phys.* **144**, 104186 (2025).
 - [13] A. Kizilersü, O. Oliveira, P. J. Silva, J. I. Skullerud, and A. Sternbeck, *Phys. Rev. D* **103**, 114515 (2021).
 - [14] A. Kizilersu, D. B. Leinweber, J. I. Skullerud, and A. G. Williams, *Eur. Phys. J. C* **50**, 871 (2007).
 - [15] J. I. Skullerud, P. O. Bowman, A. Kizilersu, D. B. Leinweber, and A. G. Williams, *J. High Energy Phys.* **04** (2003) 047.

- [16] J. Skullerud and A. Kizilersu, *J. High Energy Phys.* **09** (2002) 013.
- [17] A. Sternbeck, M. Leutnant, and G. Eichmann, *Proc. Sci., LATTICE2018* (2019) 068 [arXiv:1904.10705].
- [18] O. Oliveira, H. L. Macedo, and R. C. Terin, *Few Body Syst.* **64**, 67 (2023).
- [19] C. D. Roberts and A. G. Williams, *Prog. Part. Nucl. Phys.* **33**, 477 (1994).
- [20] Richard Williams, Schwinger-Dyson equations in QED and QCD the calculation of fermion-antifermion condensates, Durham theses, Durham University. Available at Durham E-Theses (2007), Online: <http://etheses.dur.ac.uk/2558/>.
- [21] R. Alkofer and L. von Smekal, *Phys. Rep.* **353**, 281 (2001).
- [22] C. S. Fischer, *J. Phys. G* **32**, R253 (2006).
- [23] M. N. Ferreira and J. Papavassiliou, *Particles* **6**, 312 (2023).
- [24] M. Ding, C. D. Roberts, and S. M. Schmidt, *Particles* **6**, 57 (2023).
- [25] R. Alkofer, *Symmetry* **15**, 1787 (2023).
- [26] D. Binosi, *Few Body Syst.* **63**, 42 (2022).
- [27] J. Papavassiliou, *Chin. Phys. C* **46**, 112001 (2022).
- [28] W. A. Bardeen, S. T. Love, and V. A. Miransky, *Phys. Rev. D* **42**, 3514 (1990).
- [29] J. Braun, C. S. Fischer, and H. Gies, *Phys. Rev. D* **84**, 034045 (2011).
- [30] O. Antipin, S. Di Chiara, M. Mojaza, E. Mølgaard, and F. Sannino, *Phys. Rev. D* **86**, 085009 (2012).
- [31] A. C. Aguilar, M. N. Ferreira, B. M. Oliveira, J. Papavassiliou, and L. R. Santos, *Eur. Phys. J. C* **83**, 889 (2023).
- [32] F. Gao, J. Papavassiliou, and J. M. Pawłowski, *Phys. Rev. D* **103**, 094013 (2021).
- [33] J. S. Ball and T. W. Chiu, *Phys. Rev. D* **22**, 2542 (1980).
- [34] A. Kizilersu, M. Reenders, and M. R. Pennington, *Phys. Rev. D* **52**, 1242 (1995).
- [35] W. A. Bardeen and W. K. Tung, *Phys. Rev.* **173**, 1423 (1968); **4**, 3229(E) (1971).
- [36] M. Perrottet, *Lett. Nuovo Cimento* **7S2**, 915 (1973).
- [37] R. Tarrach, *Nuovo Cimento A* **28**, 409 (1975).
- [38] F. Bloch and A. Nordsieck, *Phys. Rev.* **52**, 54 (1937).
- [39] D. R. Yennie, S. C. Frautschi, and H. Suura, *Ann. Phys. (N.Y.)* **13**, 379 (1961).
- [40] T. Kinoshita, *J. Math. Phys. (N.Y.)* **3**, 650 (1962).
- [41] T. D. Lee and M. Nauenberg, *Phys. Rev.* **133**, B1549 (1964).
- [42] J. Gamboa, *J. High Energy Phys.* **07** (2025) 184.
- [43] J. C. Ward, *Phys. Rev.* **78**, 182 (1950).
- [44] A. C. Aguilar and J. Papavassiliou, *Phys. Rev. D* **83**, 014013 (2011).
- [45] O. Oliveira, W. de Paula, T. Frederico, and J. P. B. C. de Melo, *Eur. Phys. J. C* **79**, 116 (2019).
- [46] A. Kizilersu and M. R. Pennington, *Phys. Rev. D* **79**, 125020 (2009).
- [47] A. Kizilersu, T. Sizer, M. R. Pennington, A. G. Williams, and R. Williams, *Phys. Rev. D* **91**, 065015 (2015).
- [48] J. R. Lessa, F. E. Serna, B. El-Bennich, A. Bashir, and O. Oliveira, *Phys. Rev. D* **107**, 074017 (2023).
- [49] P. Maris and P. C. Tandy, *Phys. Rev. C* **60**, 055214 (1999).
- [50] D. C. Curtis and M. R. Pennington, *Phys. Rev. D* **42**, 4165 (1990).
- [51] D. C. Curtis and M. R. Pennington, *Phys. Rev. D* **46**, 2663 (1992).
- [52] A. C. Aguilar, J. C. Cardona, M. N. Ferreira, and J. Papavassiliou, *Phys. Rev. D* **96**, 014029 (2017).
- [53] O. Oliveira, T. Frederico, W. de Paula, and J. P. B. C. de Melo, *Eur. Phys. J. C* **78**, 553 (2018).
- [54] D. Binosi, L. Chang, J. Papavassiliou, S. X. Qin, and C. D. Roberts, *Phys. Rev. D* **95**, 031501 (2017).
- [55] A. C. Aguilar, J. C. Cardona, M. N. Ferreira, and J. Papavassiliou, *Phys. Rev. D* **98**, 014002 (2018).
- [56] A. C. Aguilar, M. N. Ferreira, C. T. Figueiredo, and J. Papavassiliou, *Phys. Rev. D* **99**, 034026 (2019).
- [57] O. Oliveira, T. Frederico, and W. de Paula, *Eur. Phys. J. C* **80**, 484 (2020).
- [58] C. Mena and L. F. Palhares, [arXiv:2311.14178](https://arxiv.org/abs/2311.14178).
- [59] L. Albino, A. Bashir, B. El-Bennich, E. Rojas, F. E. Serna, and R. C. da Silveira, *J. High Energy Phys.* **11** (2021) 196.
- [60] B. El-Bennich, F. E. Serna, R. C. da Silveira, L. A. F. Rangel, A. Bashir, and E. Rojas, *Rev. Mex. Fis. Suppl.* **3**, 0308092 (2022).
- [61] A. Bashir, R. Bermudez, L. Chang, and C. D. Roberts, *Phys. Rev. C* **85**, 045205 (2012).
- [62] A. C. Aguilar, D. Binosi, and J. Papavassiliou, *Phys. Rev. D* **86**, 014032 (2012).
- [63] A. Bashir, C. Calcanéo-Roldan, L. X. Gutierrez-Guerrero, and M. E. Tejeda-Yeomans, *Phys. Rev. D* **83**, 033003 (2011).
- [64] R. Mertig, M. Böhm, and A. Denner, *Comput. Phys. Commun.* **64**, 345 (1991).
- [65] V. Shtabovenko, R. Mertig, and F. Orellana, *Comput. Phys. Commun.* **256**, 107478 (2020).
- [66] V. Shtabovenko, R. Mertig, and F. Orellana, *Comput. Phys. Commun.* **207**, 432 (2016).