Symmetry topological field theory in superspace

Federico Ambrosino,[1,*] Alessandro Duci,[2,†] Pietro Antonio Grassi,[3,‡] and Silvia Penati[2,§]

[1]*Perimeter Institute for Theoretical Physics, Waterloo, Ontario N2L 2Y5, Canada*

[2]*Dipartimento di Fisica, Università degli Studi di Milano-Bicocca and INFN, Sezione di Milano-Bicocca, Piazza della Scienza 3, 20126 Milano, Italy*

[3]*Università del Piemonte Orientale, viale T. Michel, 11, 15121 Alessandria, Italy, and INFN, Sezione Torino, via P. Giuria, 1, 10125, Torino, Italy*

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We propose a manifestly supersymmetric formulation of the symmetry topological field theory for theories with supersymmetry. Our construction provides a superspace description in which symmetry and anomaly data, together with the relations among them imposed by supersymmetry, are manifestly organized into supermultiplets. Using the supergeometric formalism, we develop the general framework as a supersymmetric extension of BF theory and illustrate how familiar symmetry data are repackaged within it. As a first application, we study the two-dimensional compact superboson and its chiral counterpart, providing simple examples of the general construction.

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I. INTRODUCTION

The modern notion of symmetry in quantum field theory (QFT) goes well beyond ordinary unitary grouplike operators acting on local operators. The seminal paper [1] initiated a profound change in our understanding of symmetries by associating them with topological defects. This approach naturally accommodates higher-form symmetries, noninvertible symmetries and, more generally, categorical (non-group-like) structures. For recent reviews on these developments, see [2–7].

A particularly efficient way of packaging generalized symmetry data is provided by the symmetry topological field theory (SymTFT) [8–17]. This is a topological theory in one dimension higher which encodes the generalized symmetries of a QFT, their defects and their 't Hooft anomalies. It provides a unified framework in which charge and charged operators can be treated on the same footing, anomalies are incorporated through anomaly inflow, and operations such as gauging or stacking with invertible phases are translated into topological operations in the bulk. In this way, the SymTFT isolates the symmetry sector of a QFT from its local dynamics.

This framework was first developed extensively for finite or discrete symmetry data (see, e.g., [18]) and was subsequently realized in a variety of geometric and string-theoretic constructions [19–22]. More recently, the SymTFT paradigm has been extended to continuous internal symmetries, first in the Abelian case and then for continuous non-Abelian groups [23–25], and very recently to continuous spacetime and flavor symmetries [26,27]. For broad reviews of generalized symmetries and SymTFTs, see for instance [2,4,6].

Fermionic theories and symmetry structures involving a choice of spin structure have a long history and are by now well developed. Fermionic symmetry-protected-topological phases, spin TFTs, fermion condensation and categorical descriptions of fermionic topological phases provide well-developed frameworks for describing fermionic systems [28–31]. Closely related topological constructions have long played a central role in bosonization and fermionization dualities [32,33]. In particular, the description of bosonic and fermionic orbifolds in terms of three-dimensional topological quantum field theories provides an early realization of the bulk topological approach underlying the modern SymTFT framework [11]. More recently, these structures have also been investigated from the perspective of generalized symmetries and topological defects in fermionic QFTs [34–36]. Importantly, the existence of fermionic degrees of freedom at the boundary does not require the bulk SymTFT itself to be fermionic. For instance, fermionic condensation can lead to fermionic boundary theories starting from bosonic bulk data.

*Contact author: federicoambrosino25@gmail.com
†Contact author: a.duci@campus.unimib.it
‡Contact author: pietro.grassi@uniupo.it
§Contact author: silvia.penati@mib.infn.it

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The question we address here is different. A prominent example of a fermionic (spacetime) symmetry is supersymmetry. Therefore, given a supersymmetric QFT at the boundary, we ask whether its SymTFT can be formulated directly in superspace, so that supersymmetry is manifest also in the bulk description. Our construction provides an intrinsically supersymmetric realization of the SymTFT (we dub it SuSymTFT) in which currents, background fields, defects and anomaly inflow are organized into supersymmetric structures from the outset.

This also clarifies what is gained from such a formulation. The symmetry data of a supersymmetric theory may, when appropriate, be described by an ordinary bosonic or spin SymTFT. However, in a supersymmetric theory bosonic and fermionic currents belong to common supermultiplets, couple to background superfields, and obey Ward identities and anomaly relations constrained by supersymmetry. A superspace formulation keeps these relations manifest instead of describing the different components separately. The SuSymTFT may thus be viewed as a supersymmetric organization of the corresponding topological data, specifying how their different components are tied together by supersymmetry. This perspective arises naturally from supergeometry. Supersymmetric theories admit a natural formulation on supermanifolds (see [37] and references therein), where supersymmetry acts geometrically along the odd directions and component fields are assembled into superfields. It is therefore natural to ask whether this geometric organization can be retained upon passing to the topological theory in one dimension higher.

In this paper we develop such a framework using the geometry of supermanifolds and their integration theory [37–39]. Given a supersymmetric theory living on a supermanifold $\mathcal{M}^{(d|m)}$, we construct a supersymmetric SymTFT on a bulk supermanifold $\mathcal{M}^{(d+1|n)}$ whose boundary contains the physical superspace. The odd bulk dimension n is determined by the supersymmetry chosen in the bulk together with the amount preserved by the boundary. For example, in the standard half-Bogomol'nyi-Prasad-Sommerfield (BPS) realization one expects $n = 2m$. Concretely, we implement this framework through a supersymmetric BF-type theory. Its purpose is to retain manifestly the supersymmetry relations among the fields encoding its symmetry sector. Upon reduction to components, the superspace formulation reproduces the corresponding topological sectors together with the relations among them imposed by supersymmetry. We test our construction in two well-known examples: the two-dimensional $\mathcal{N} = (1, 1)$ supercompact boson and its chiral counterpart, for which chirality leads to a different anomaly structure. These familiar theories provide controlled laboratories in which the construction can be made explicit and compared with its component description.

The present work is primarily aimed at establishing the general framework. Potentially, the formulation of a manifestly supersymmetric SymTFT provides a natural setting where to determine which topological boundary conditions and manipulations are compatible with supersymmetry. Exploring this structure, and whether it leads to new supersymmetric boundary conditions or relations among different supersymmetric theories, is an important direction that we leave for future work.

In the rest of the paper we focus mainly on global symmetries, although the same supergeometric framework is naturally suited to incorporating the supersymmetry algebra itself [40].

Outline of the paper. The rest of the paper is organized as follows. In Sec. II we develop the general construction of SymTFT in superspace and introduce the notion of SuSymTFT for a theory on $\mathcal{SM}^{(d|m)}$ in terms of a bulk topological theory on $\mathcal{SM}^{(d+1|n)}$. In Sec. III we apply the formalism to the $\mathcal{N} = (1, 1)$ supercompact boson, while in Sec. IV we discuss the chiral case. We conclude with a discussion of open questions and possible future directions. The paper is supported by four Appendixes, where we first recall notations and conventions for the two-dimensional $\mathcal{N} = (1, 1)$ supermanifold together with main information about picture changing operators in supergeometry and then provide details for the evaluation of the mixed anomaly of Sec. III, and finally we discuss the supersymmetry invariance of the SuSymTFT for the compact boson.

II. SYMTFT IN SUPERSPACE

The symmetry sector of a QFT defined on a d-dimensional smooth manifold $\mathcal{M}^{(d)}$ can be encoded in a topological theory in one dimension higher: its SymTFT. If the physical theory on $\mathcal{M}^{(d)}$ has a symmetry group G (this may be internal or even spacetime), then its SymTFT is a theory of flat connection for the G group on $\mathcal{M}^{(d+1)}$. If G is finitely generated, then such a topological theory is a BF theory, possibly completed by Chern-Simons terms to capture the full set of global anomalies. The topological BF theory is described by the following general action:

$$S_{\mathrm{BF}} = \int_{\mathcal{M}^{(d+1)}} b^{(d-p)} \wedge da^{(p)}, \tag{1}$$

where $b^{(d-p)}$ and $a^{(p)}$ are continuous Abelian $(d-p)$-form and p-form, respectively, with $p = 0, \ldots, d$.

S_{BF} is invariant under gauge transformations

$$\delta a^{(p)} = d\lambda^{(p-1)}, \qquad \delta b^{(d-p)} = d\rho^{(d-p-1)}. \tag{2}$$

If $p > 1$ and/or $(d-p-1) > 0$, there is a residual gauge symmetry acting on the gauge parameters as well. In addition, the BF action enjoys a $(d-p-1)$-form symmetry with conserved current $J^{(p+1)} = da^{(p)}$ and a $(p-1)$-form symmetry generated by $\tilde{J}^{(d-p+1)} = db^{(d-p)}$.

In order to realize the sandwich construction encoded in a SymTFT, the $(d+1)$-dimensional manifold has the structure $\mathcal{M}^{(d+1)} = \mathcal{M}^{(d)} \times \mathcal{I}$, where the two extremes of the $\mathcal{I}$ interval define the symmetry and the physical spaces of the original d-dimensional theory. The construction has to be completed with suitable boundary conditions on $a^{(p)}$ and $b^{(d-p)}$ at the physical and topological symmetry boundaries. According to the choice of Neumann or Dirichlet boundary conditions at the symmetry boundary, topological Wilson-like operators of the BF theory give rise to the symmetry generators or nontopological operators charged under that symmetry. In the present example, the d-dimensional physical theory exhibits a global $(p-1)$-form symmetry generated by a conserved current $J^{(d-p)}$ that couples to $a^{(p)}$ and a dual $(d-p-1)$-form symmetry whose conserved current $\tilde{J}^{(p)}$ is gauged by $b^{(d-p)}$.

In the $p=1$ case, this construction admits a straightforward non-Abelian generalization.[1] The non-Abelian BF theory is simply obtained by covariantizing the action in (1) as

$$S_{\mathrm{BF}} = \mathrm{Tr} \int_{\mathcal{M}^{(d+1)}} b^{(d-1)} \wedge F^{(2)}, \quad \text{with}$$
$$F^{(2)} = da^{(1)} + a^{(1)} \wedge a^{(1)} \equiv \mathcal{D}a^{(1)}, \tag{3}$$

where $b^{(d-1)}$ is valued in the adjoint representation of G. The covariant differential $\mathcal{D}$ makes this expression invariant under non-Abelian gauge transformations

$$\delta a^{(1)} = \mathcal{D}\lambda^{(0)}, \qquad \delta b^{(d-1)} = \mathcal{D}\rho^{(d-2)}. \tag{4}$$

Complemented with suitable boundaries conditions at the two boundaries of $\mathcal{M}^{(d+1)}$, the action in (3) provides the SymTFT for a d-dimensional physical theory with a non-Abelian continuous 0-form symmetry.

A. BF theory on supermanifolds

We now move to the main problem of constructing the SymTFT for supersymmetric QFTs. This requires first generalizing the BF action to a supersymmetric one.

The natural geometric setting for studying supersymmetric theories in d dimensions is a supermanifold $\mathcal{SM}^{(d|m)}$, where the fermionic dimension m is given by the number of spinor components, that is the number of supersymmetries times the adapted dimensions of a spinor in the given bosonic dimension d and with a suitable signature.[2] We will consider a genus zero, real, smooth, oriented supermanifold, with a pseudo-Riemannian metric and a spin (more generally spin^c) structure.[3] Locally, it can be parametrized by a set of d real even coordinates x^a and m odd ones θ^α. Its geometry is described by supervielbeins (V^a, ψ^α), in terms of which one can define the superdifferential $d = V^a\partial_a + \psi^\alpha D_\alpha$, with D_α being supersymmetry covariant derivatives.

On a smooth supermanifold one can define pseudoforms $\omega^{(p|q)}$, $0 \le q \le m$, which are featured by a form number p and a picture number q. While p is the analog of the ordinary form degree, q can be changed by multiplying the pseudoform by a *picture changing operator* (PCO). We give details in Appendix A.

For top forms $\omega^{(p|q=m)}$ a consistent integration theory has been developed [38,42].[4] These "integral forms" are dual to superforms $\omega^{(d-p|q=0)} \equiv \star\omega^{(p|q=m)}$, where in the presence of a metric, the Hodge dual defines isomorphisms.

In order to write a BF action in the $\mathcal{SM}^{(d+1|m)}$ supermanifold, it is sufficient to promote ordinary forms in the action (1) to pseudoforms, such that their product saturates both the even and the odd dimensions. Taking into account that the differential has picture zero, the most general expression for a super-BF action is then

$$S_{\mathrm{sBF}} = \int_{\mathcal{SM}^{(d+1|m)}} b^{(d-p|m-q)} \wedge da^{(p|q)}. \tag{5}$$

In the absence of boundaries the equations of motion imply $da^{(p|q)} = db^{(d-p|m-q)} = 0$. Moreover, the action is invariant under Abelian supergauge transformations of the form

$$\delta a^{(p|q)} = d\lambda^{(p-1|q)}, \qquad \delta b^{(d-p|m-q)} = d\rho^{(d-p-1|m-q)}. \tag{6}$$

Special cases correspond to $q=0$ and $q=m$, where one of the two gauge forms is an integral form and the other one is a superform.

We note that, making use of suitable PCOs, we can also construct a well-defined action for pairs of pseudoforms whose picture number does not saturate the odd dimensions of the supermanifold. In fact, the action

$$S_{\mathrm{sBF}} = \int_{\mathcal{SM}^{(d+1|m)}} b^{(d-p|r)} \wedge da^{(p|q)} \wedge \mathbb{Y}^{(0|m-q-r)} \tag{7}$$

is perfectly well defined and coincides with (5) under the identification $b^{(d-p|r)} \wedge \mathbb{Y}^{(0|m-q-r)} \equiv b^{(d-p|m-q)}$ (up to a possible sign). The role of the PCO is to project the integral on a $(d|q+r)$ submanifold.

In the absence of boundaries, the action in (7) is independent of its embedding. In fact, changing the

[1] When $p>1$ one may still realize a non-Abelian connection via a higher-group structure [41].

[2] For instance, a four-dimensional $\mathcal{N}=1$ theory can be formulated in $\mathcal{SM}^{(4|4)}$, with four bosonic dimensions and four fermionic dimensions. Analogously, a four-dimensional $\mathcal{N}=2$ theory requires $\mathcal{SM}^{(4|8)}$, and so on.

[3] Topological notions for supermanifolds are by definition the ones for the bosonic submanifold [38].

[4] For a short summary, see for example Appendix A of [36].

embedding accounts for a shift of $\mathbb{Y}^{(0|m-q-r)}$ by a d-exact term. However, since the integrand $b^{(d-p|r)} \wedge da^{(p|q)}$ is closed[5] this shift is integrated to zero.

1. The 3d $\mathcal{N}=1$ super-BF theory

In order to make the previous construction more concrete, we study in details the action for the simple case of $\mathcal{N}=1$ super-BF theory in three dimensions (i.e., $d=2$, $m=2$). To shorten the notation, in (5) we set $b^{(1|2)} \equiv B$ and $a^{(1|0)} \equiv \tilde{B}$. We work out the action in components by looking at the component expansion of $\tilde{B}$, its superfield strength $\tilde{F} = d\tilde{B}$ and the integral form B, separately. The $\tilde{B}$ superform and its field strength $\tilde{F}$ can be expanded on the basis of the (V^a, ψ^α) superdreibeins as follows:

$$\tilde{B} = \tilde{B}_a V^a + \tilde{B}_\alpha \psi^\alpha, \qquad \tilde{F} = \tilde{F}_{ab} V^a V^b + (\tilde{W}\gamma_a \psi) V^a, \tag{8}$$

where, in order to select only the vector multiplet in its expansion, we imposed some constraints (also known as *conventional constraints*) on $\tilde{F}$. In the above expansion, $\tilde{F}_{ab}$ is the superfield strength in superspace, whereas the $\tilde{W}$ superfield has the gaugino as its lowest component. Bianchi identities together with conventional constraints imply $D^\alpha \tilde{W}_\alpha = 0$ and $\tilde{F}_{ab} = D\gamma_{ab}\tilde{W}$.

In order to write the expansion for B we first introduce the $(1|0)$ superform $\mathcal{B}$ and its superform field strength $\mathcal{F} = d\mathcal{B}$, whose component expansion is given by

$$\mathcal{B} = B_a V^a + B_\alpha \psi^\alpha, \qquad \mathcal{F} = F_{ab} V^a V^b + (W\gamma_a \psi) V^a, \tag{9}$$

where conventional constraints have been imposed, in analogy with the $\tilde{F}$ superform. Now, upon a suitable gauge fixing, the $(1|2)$-integral form B can be written in terms of these pseudoforms as

$$B^{(1|2)} = \left(\mathcal{B} + \frac{1}{3!} V^a V^b W \gamma_{ab} \iota\right) \wedge \mathbb{Y}^{(0|2)} \tag{10}$$

for an arbitrary PCO $\mathbb{Y}^{(0|2)}$. The presence of the W–term is crucial for ensuring the closure of B.

Inserting the previous expansions in Eq. (5), the action of the super-BF model can be cast in the following form (see [43] for details):

$$\begin{aligned} S_{\mathrm{sBF}} &= \int_{\mathcal{SM}^{(3|2)}} B \wedge \tilde{F} \\ &= \int_{\mathcal{SM}^{(3|2)}} (\mathcal{B} \wedge \tilde{F} + \epsilon^{\alpha\beta} W_\alpha \tilde{W}_\beta V_3) \wedge \mathbb{Y}^{(0|2)}. \end{aligned} \tag{11}$$

This is the super-BF theory written in the supergeometry language. The integrand factorizes into a closed $(3|0)$-superform Lagrangian and a generic $(0|2)$ PCO.

Given the closure of the Lagrangian, in (11) we can choose different $\mathbb{Y}^{(0|2)}$ without modifying the action. For example, if we choose $\mathbb{Y}^{(0|2)} = \theta^2\delta^2(\psi)$, the super-BF action reduces to its component form

$$S_{\mathrm{sBF}} = \int d^3x (\epsilon^{abc} b_a \partial_b \tilde{b}_c + \lambda^B_\alpha \epsilon^{\alpha\beta} \lambda^{\tilde{B}}_\beta), \tag{12}$$

where $b_a = B_a|_{\theta=0}$ and $\tilde{b}_a = \tilde{B}_a|_{\theta=0}$ are the component gauge fields of the superforms B and $\tilde{B}$, respectively, while $\lambda^B = W|_{\theta=0}$ and $\lambda^{\tilde{B}} = \tilde{W}|_{\theta=0}$ are their supersymmetric partners.

We conclude this section by observing that, as in the bosonic case, for $p=1$ we can generalize the Abelian BF theory on supermanifolds to be non-Abelian.

Taking inspiration from (3), we write the most general non-Abelian BF theory in supermanifold as

$$\begin{aligned} &S_{\mathrm{sBF}} = \mathrm{Tr} \int_{\mathcal{SM}^{(d+1|m)}} b^{(d-1|m)} \wedge F^{(2|0)} \quad \text{with} \\ &F^{(2|0)} = da^{(1|0)} + a^{(1|0)} \wedge a^{(1|0)}, \end{aligned} \tag{13}$$

where again $b_a = B_a^{(1|0)}|$ takes values in the adjoint representation of G. This action is invariant under the following gauge transformations:

$$\delta a^{(1|0)} = \mathcal{D}\lambda^{(0|0)}, \qquad \delta b^{(d-1|m)} = \mathcal{D}\rho^{(d-2|m)}, \tag{14}$$

where $\mathcal{D} = d + [a^{(1|0)}, \cdot]$.

We note that in the non-Abelian case the nonlinear dependence on the gauge superconnection forces the field strength $F^{(2|0)}$ to be necessarily a superform ($q=0$). However, we can always use a PCO to change its picture. Therefore, defining a generalized field strength

$$F^{(2|q)} = F^{(2|0)} \wedge \mathbb{Y}^{(0|q)}, \tag{15}$$

a more general non-Abelian BF action takes the form

$$S_{\mathrm{sBF}} = \mathrm{Tr} \int_{\mathcal{SM}^{(d+1|m)}} b^{(d-1|m-q)} \wedge F^{(2|q)}. \tag{16}$$

B. SuSymTFT

Having introduced the topological BF theory on supermanifolds, we can use it to construct the supersymmetric version of the SymTFT (SuSymTFT) for supersymmetric theories defined in superspace. This formulation has the

[5] As a consequence of the fact that the total integrand is a top form—then closed—and the PCO is closed as well.

advantage of exhibiting manifest supersymmetry, thus conveniently packaging the full set of bosonic and fermionic anomalies.

Generalizing the construction reviewed above for bosonic theories, in the supersymmetric setup the SuSymTFT for a theory living in a supermanifold $\mathcal{SM}^{(d|m)}$ is a topological field theory on the supermanifold $\mathcal{SM}^{(d+1|n)}$ such that $\partial\mathcal{SM}^{(d+1|n)} = \mathcal{SM}^{(d|m)}$. More precisely, the sandwich construction requires the SuSymTFT to have two boundaries, still dubbed *physical* and *topological*, the former being the one where the physical theory lives and the second one being a topological boundary condition for the bulk fields. Boundary conditions of the bulk fields at the boundaries guarantee gauge invariance of the bulk theory, at the same time topological boundary conditions specify which absolute theory lives on the physical boundary after interval compactification.

So far, this looks like a straightforward generalization of the bosonic construction. However, in the supersymmetric case special attention has to be devoted to the relation between the odd dimensions of the bulk and the boundary supermanifolds. In fact, since boundaries (partially) break supersymmetry, in general the dimensions of the fermionic support of the $\mathcal{SM}^{(d|m)}$ and $\mathcal{SM}^{(d+1|n)}$ supermanifolds are not the same. Typically, a codimension-one boundary is 1/2-BPS, preserving half of the supersymmetry charges. In fact, given the location of the boundary at $x^\perp = 0$, preserved Killing spinors are the ones which satisfy $\delta_\epsilon x^\perp = \epsilon\gamma^\perp\theta = 0$. The maximal solution is obtained by splitting $\epsilon\gamma^\perp\theta = \epsilon_-\theta_+ - \epsilon_+\theta_-$ with $\theta_\pm = P_\pm\theta, \epsilon_\pm = P_\pm\epsilon$, $P_\pm = \frac{1}{2}(1 \pm \gamma^\perp)$, and taking $\epsilon_- = \theta_- = 0$ or $\epsilon_+ = \theta_+ = 0$ or possibly mixed conditions. Therefore, the correct boundary to consider in the SuSymTFT construction can be sketchy defined as

$$\begin{aligned}\mathcal{SM}^{(d|m)} &\equiv \partial(\mathcal{SM}^{(d+1|2m)}) \\ &= \{x^\perp = 0, \theta_+ = 0 \text{ or } \theta_- = 0, \text{or mixed conds}\}\end{aligned} \tag{17}$$

where half of the supersymmetries present in $(d+1)$ dimensions survive.

At the level of the action, since in components the supersymmetry variation of the bulk Lagrangian is always a spacetime derivative, half of the supersymmetries can be preserved by imposing suitable boundary conditions on the fields and their derivatives or adding a suitable boundary term to the bulk action, whose variation cancels the residual term. The second approach is the "F+A" mechanism (or "supersymmetry without BC") introduced in [44] to preserve the right amount of supersymmetry in theories with boundaries. However, in the present context it is more natural to impose boundary conditions on the fields, as we are going to discuss in the explicit examples studied in the next sections.

III. SCALAR SUPERMULTIPLET

As a first simple example, we consider the free theory for an $\mathcal{N} = (1,1)$ supercompact boson living in a flat supermanifold $\mathcal{SM}^{(2|2)}$. We collect notations and conventions for $\mathcal{SM}^{(2|2)}$ in Appendix B.

In the nonsupersymmetric case, the construction of the SymTFT for the noninvertible T-duality symmetry of the compact boson has been considered in [45].

A. The supercompact boson

In the two-dimensional $\mathcal{N} = (1,1)$ superspace, we consider a real $(0|0)$-superform Φ whose θ expansion is given by

$$\begin{aligned}\Phi \equiv \Phi(z,\bar{z},\theta,\bar{\theta}) &= \phi(z,\bar{z}) + \theta\lambda(z,\bar{z}) \\ &+ \overline{\theta\lambda}(z,\bar{z}) + \theta\bar{\theta}F(z,\bar{z}).\end{aligned} \tag{18}$$

In order to implement compactness, we restrict Φ to be defined as a section valued in a target supermanifold $\mathcal{S}S^1$ which possesses one compact bosonic direction:

$$\Phi\colon \mathcal{SM}^{(2|2)} \to \mathcal{S}S^1.$$

The section of $\mathcal{S}S^1$ of radius R is defined in such a way that the lowest component of the superfield $\Phi|_{\theta=0} = \phi$ is a map from $\mathcal{SM}^{(2|2)}$ to the circle S^1, periodically identified as

$$\phi(z,\bar{z}) = \phi(z,\bar{z}) + 2\pi R.$$

The higher components of Φ are not periodically identified, being maps to the (super)tangent bundle of S^1.

The physical degrees of freedom—one compact scalar plus two Majorana-Weyl fermions—are governed by the following action [37]:

$$S = \frac{1}{4\pi}\int_{\mathcal{SM}^{(2|2)}} d\Phi \wedge \star d\Phi, \tag{19}$$

where $\star$ is the super-Hodge dual [39]. Defining the extra spinorial superfield

$$W_\alpha \equiv (W, \bar{W}) = (D\Phi, \bar{D}\Phi) \tag{20}$$

from the explicit expression (B8) for the differential we find (see Appendix B)

$$\begin{aligned}d\Phi &= V^a\partial_a\Phi + \psi^\alpha W_\alpha, \\ \star d\Phi &= \epsilon^{ab}V_b\delta^2(\psi)\partial_a\Phi + V\bar{V}\epsilon^{\alpha\beta}\iota_\alpha\delta^2(\psi)W_\beta,\end{aligned} \tag{21}$$

where $V^a \equiv (V, \bar{V})$ and $\psi^\alpha \equiv (\psi, \bar{\psi})$ are the supervielbeins and ι_α is the contraction along the $D_\alpha \equiv (D, \bar{D})$ vector field.

Since the action in (19) is the integral of a top form, it is manifestly supersymmetric as long as the $\mathcal{SM}^{(2|2)}$

supermanifold has no boundary. In fact, taking into account that its supersymmetry variation is defined as $\delta_\epsilon S \equiv \int[\mathcal{L}_\epsilon, d\Phi \wedge \star d\Phi]$, where $\mathcal{L}_\epsilon = [d, \imath_\epsilon]$ is the Lie derivative in the ϵ direction, we are left with a d-exact term that integrates to zero on a trivial boundary.

The equation of motion from this action is $d \star d\Phi = 0$. Using expansions (21) it can be rewritten as

$$V\bar{V}\delta^2(\psi)(1 + \epsilon^{\gamma\delta}D_\gamma D_\delta)(\epsilon^{\alpha\beta}D_\alpha D_\beta \Phi) = 0, \tag{22}$$

which implies $D\bar{D}\Phi = \bar{D}D\Phi = 0$. In fact, it is easy to see that on shell the action (19) reduces to the ordinary superspace action [46]

$$S = \frac{1}{4\pi}\int d^2z d^2\theta D\Phi \bar{D}\Phi. \tag{23}$$

This action defines a superconformal field theory at the quantum level.

1. Global symmetries

The system possesses two (on-shell) closed quantities, a $(1|0)$-superform $J = d\Phi$ and its dual $(1|2)$-integral form $\tilde{J} = \star d\Phi$, which lead to two conserved $U(1)$ currents. They correspond to two $U(1)$ symmetries, one $(0|2)$-integral form and one $(0|0)$-superform symmetries, respectively. In particular, the $(0|2)$-integral form symmetry corresponds to constant shifts, $\Phi \to \Phi + \omega$, where ω is a constant superfield. The topological charges implementing the action of these symmetries on local operators are

$$\begin{aligned} U_g(\Gamma^{(1|0)}) &= \exp\left(\frac{ig}{2\pi R}\int_{\Gamma^{(1|0)}} d\Phi\right) \\ &= \exp\left(\frac{ig}{2\pi R}\int_{\mathcal{SM}^{(2|2)}} d\Phi \wedge \mathbb{Y}^{(1|2)}_{\Gamma^{(1|0)}}\right), \\ V_h(\Sigma^{(1|2)}) &= \exp\left(ihR\int_{\Sigma^{(1|2)}} \star d\Phi\right) \\ &= \exp\left(ihR\int_{\mathcal{SM}^{(2|2)}} \star d\Phi \wedge \mathbb{Y}^{(1|0)}_{\Sigma^{(1|2)}}\right), \end{aligned} \tag{24}$$

where g and h are Grassmann even constant parameters satisfying $g, h \sim g, h + 2\pi$. Moreover, $\mathbb{Y}^{(1|2)}_{\Gamma^{(1|0)}}$ and $\mathbb{Y}^{(1|0)}_{\Sigma^{(1|2)}}$ are two PCOs which localize the integrals on the corresponding hypersurfaces. Topological invariance of these charges easily follows from the closure of the currents, together with the crucial property that under smooth deformations of the hypersurfaces the PCOs transform by a d-exact term.

The two operators in (24) are associated, respectively, to the momentum and winding charges of the supersymmetric vertex operator $:\exp(in\Phi):$ of the compact boson conformal field theory.

Because of the compactness of the target space, the holonomies are quantized, the action of the charges on the Hilbert space has a nontrivial kernel, and it is generated by the 2-torus

$$(\mathbb{R}/2\pi\mathbb{Z})^{(0|0)} \times (\mathbb{R}/2\pi\mathbb{Z})^{(0|2)} \simeq U(1)^{(0|0)} \times U(1)^{(0|2)}$$

exactly as in the bosonic case.

It is interesting to observe that in the presence of supersymmetry J and $\tilde{J}$ do not exhaust the spectrum of global $U(1)$ currents. In fact, they belong to two complete supermultiplets, whose components are still superform and integral form currents, respectively. For instance, if we apply a supersymmetry transformation to J, we obtain a closed spinorial current $\mathcal{J}_\alpha = (\mathcal{J}, \bar{\mathcal{J}})$, with

$$\epsilon\mathcal{J} \equiv \delta_\epsilon J = \mathcal{L}_\epsilon J, \qquad \bar{\epsilon}\,\bar{\mathcal{J}} \equiv \delta_{\bar{\epsilon}} J = \mathcal{L}_{\bar{\epsilon}} J, \tag{25}$$

where $\mathcal{L}_{\epsilon,\bar{\epsilon}}$ are the Lie derivatives in the ϵ and $\bar{\epsilon}$ directions, respectively. The closure of this new current easily follows from $[d, \mathcal{L}_{\epsilon,\bar{\epsilon}}] = 0$.

Analogously, applying a supersymmetry transformation to $\tilde{J}$ we generate another spinorial closed current $\tilde{\mathcal{J}}_\alpha = (\tilde{\mathcal{J}}, \bar{\tilde{\mathcal{J}}})$, where $\tilde{\mathcal{J}} = \star\mathcal{J}$ and $\bar{\tilde{\mathcal{J}}} = \star\bar{\mathcal{J}}$. A simple calculation shows that

$$\mathbb{Z}\bar{\mathbb{Z}}[\tilde{\mathcal{J}}] = \mathcal{J}, \qquad \mathbb{Z}\bar{\mathbb{Z}}[\bar{\tilde{\mathcal{J}}}] = \bar{\mathcal{J}}. \tag{26}$$

Additional supercurrents can be constructed as κ-symmetry superpartners, by acting on J with the Lie derivative with respect to the vector $\kappa = \kappa^\alpha D_\alpha$. We obtain another spinor of currents $(\mathcal{J}_\kappa, \bar{\mathcal{J}}_\kappa)$, where

$$\mathcal{J}_\kappa = dW, \qquad \bar{\mathcal{J}}_\kappa = d\bar{W}. \tag{27}$$

They are trivially closed, consistently with the fact that in the present geometric framework κ symmetry is a superdiffeomorphism.

The κ-symmetry transformed currents generate constant shifts of the fermions, $W \to W + c$, $\bar{W} \to \bar{W} + \bar{c}$, where $(c, \bar{c})$ are constant superfields. It is important to note that the fermion shift is not independent of the boson one, $\Phi \to \Phi + \omega$. In fact, since $(W, \bar{W}) = (D\Phi, \bar{D}\Phi)$, we trivially have $(c, \bar{c}) = (D\omega, \bar{D}\,\bar{\omega})$. In other words, the parameters are related by supersymmetry transformations as

$$\delta_\epsilon\omega = \epsilon c, \qquad \delta_\epsilon c = 0; \qquad \delta_{\bar{\epsilon}}\bar{\omega} = \overline{\epsilon c}, \qquad \delta_{\bar{\epsilon}}\bar{c} = 0. \tag{28}$$

2. Spacetime symmetries

We now discuss the spacetime symmetries of the supercompact boson. Although in two dimensions the $\mathcal{N} = (1,1)$ superconformal group is enhanced to the infinite-dimensional super-Virasoro algebra, here we focus only on supertranslations. Since supersymmetry and spacetime symmetries are naturally identified as $(0|0)$-superform symmetries [36], they should correspond to $(1|2)$-form

closed currents. Indeed, a supersymmetry spinorial $(1|2)$-current can be constructed, whose components are given by

$$\mathcal{S}^{(1|2)} = W\partial\Phi V\delta(\psi)\wedge\bar{\mathbb{Y}}^{(0|1)},$$
$$\bar{\mathcal{S}}^{(1|2)} = \overline{W\partial}\Phi\bar{V}\delta(\bar{\psi})\wedge\mathbb{Y}^{(0|1)} \tag{29}$$

upon a suitable choice of the PCOs. These two components can be checked to be closed using the on-shell free equations $\partial\bar{\partial}\Phi = 0, D\bar{W} = 0$ and $\bar{D}W = 0$.

Supersymmetry currents with lower picture can be obtained by applying the $\mathbb{Z}$ and $\bar{\mathbb{Z}}$ PCOs defined in Appendix A. Precisely, we can consider the alternative but equivalent currents[6]

$$\mathcal{S}^{(1|1)} \equiv \bar{\mathbb{Z}}\mathcal{S}^{(1|2)} = W\partial\Phi V\delta(\psi),$$
$$\bar{\mathcal{S}}^{(1|1)} \equiv \mathbb{Z}\bar{\mathcal{S}}^{(1|2)} = \overline{W\partial}\Phi\bar{V}\delta(\bar{\psi}) \tag{30}$$

or, applying both PCOs, a third pair of closed currents (we insert a minus sign for convenience)

$$\mathcal{S}^{(1|0)} \equiv -\mathbb{Z}\bar{\mathbb{Z}}\mathcal{S}^{(1|2)} = W\partial\Phi\psi + ((\partial\Phi)^2 + W\partial W)V,$$
$$\bar{\mathcal{S}}^{(1|0)} \equiv -\mathbb{Z}\bar{\mathbb{Z}}\bar{\mathcal{S}}^{(1|2)} = \bar{W}\,\bar{\partial}\,\Phi\bar{\psi} + ((\bar{\partial}\Phi)^2 + \bar{W}\,\bar{\partial}\,\bar{W})\bar{V}. \tag{31}$$

Here, the $\psi, \bar{\psi}$ and $V, \bar{V}$ components are easily recognized to be the holomorphic and antiholomorphic parts of the supersymmetry current and the energy-momentum tensor, respectively, written in superfields. If we further expand $W\partial\Phi$ and $\bar{W}\,\bar{\partial}\,\Phi$ in components by the use of (18),

$$W\partial\Phi = [\lambda\partial\phi - \theta((\partial\phi)^2 + \lambda\partial\lambda)] = j_S - \theta T,$$
$$\bar{W}\,\bar{\partial}\,\Phi = [\bar{\lambda}\,\bar{\partial}\,\phi - \bar{\theta}((\bar{\partial}\phi)^2 + \bar{\lambda}\,\bar{\partial}\,\bar{\lambda})] = \bar{j}_S - \bar{\theta}\,\bar{T}, \tag{32}$$

we recognize the ordinary holomorphic component of supersymmetry current, $j_S = \lambda\partial\phi$, and the holomorphic energy-momentum tensor $T = (\partial\phi)^2 + \lambda\partial\lambda$, plus their antiholomorphic counterparts.

In order to determine the topological generators of supertranslations, it is convenient to use the $(1|2)$ picture of the S current [see Eq. (29)] and first make an explicit choice of the two PCOs. Here we choose $\mathbb{Y}^{(0|1)} = V\delta'(\psi), \bar{\mathbb{Y}}^{(0|1)} = \bar{V}\delta'(\bar{\psi})$. We then couple the currents to two superparameters with constant components, $A = a + \epsilon\theta$ and $\bar{A} = \bar{a} + \overline{\epsilon\theta}$, and integrate on the $(1|2)$ surfaces $\Sigma := \{\bar{z} = 0\}$ and $\bar{\Sigma} := \{z = 0\}$, respectively. The projection on these two surfaces is realized by the two PCOs $\bar{\mathbb{Y}}^{(1|0)} = d\bar{z}\delta(\bar{z})$ and $\mathbb{Y}^{(1|0)} = dz\delta(z)$. Performing the supermanifold and the residual Berezin integrals explicitly, we eventually obtain the infinitesimal transformation operators

$$\delta_A = \int_\Sigma A\mathcal{S}^{(1|2)} = a\int dz T(z) + \epsilon\int dz j_S(z) = aP + \epsilon Q,$$
$$\delta_{\bar{A}} = \int_{\bar{\Sigma}} \bar{A}\bar{\mathcal{S}}^{(1|2)} = \bar{a}\int d\bar{z}\,\bar{T}(\bar{z}) + \bar{\epsilon}\int d\bar{z}\bar{j}_S(\bar{z}) = \bar{a}\,\bar{P} + \bar{\epsilon}\,\bar{Q}. \tag{33}$$

The a and $\bar{a}$ charges are nothing but the usual translation parameters multiplying the translation operators in direction z and $\bar{z}$, whereas ϵ and $\bar{\epsilon}$ are two Killing spinors multiplying the supersymmetry charges in the two chiral sectors.

Under the action of the supersymmetry operator Q the components of the current multiplet transform as

$$\delta_\epsilon j_S = -\epsilon T, \qquad \delta_\epsilon T = \epsilon\partial j_S. \tag{34}$$

Analogous identities hold in the antichiral sector.

The construction of all Virasoro generators follows a similar pattern and will be reported elsewhere [40].

3. Super-Chern-Weil symmetries

As discussed in [36], we expect the emergence of additional bosonic and fermionic higher form symmetries. For example, given the currents $J = d\Phi$ and $\mathcal{J}_\kappa = dW, \bar{\mathcal{J}}_\kappa = d\bar{W}$ we can construct the following higher conserved currents:

$$J^{(2|0)}_{bf} = d\Phi\wedge dW, \qquad J^{(2|0)}_{b\bar{f}} = d\Phi\wedge d\bar{W},$$
$$J^{(2|0)}_{ff} = dW\wedge dW, \qquad J^{(2|0)}_{f\bar{f}} = dW\wedge d\bar{W},$$
$$J^{(2|0)}_{\bar{f}\bar{f}} = d\bar{W}\wedge d\bar{W}. \tag{35}$$

In contrast with the previous construction, these currents are not integral forms on the supermanifold, as they do not saturate the odd directions. Therefore, the definition of the corresponding topological charges requires performing a suitable projection in the fermionic directions. On the other hand, since these currents are spacetime filling, they are expected to give rise to some superversion of (-1)-form symmetries.

In addition, we can construct other $(2|0)$-form currents by exploiting the two Chevalley-Eilenberg cohomologies of $\mathcal{N} = (1,1)$ superspace (see [36,47]), namely $\omega^{(1|0)} = \psi$ and $\bar{\omega}^{(1|0)} = \bar{\psi}$. New higher-form currents are

$$J^{(2|0)}_{\omega b} = \psi\wedge d\Phi, \qquad J^{(2|0)}_{\bar{\omega}b} = \bar{\psi}\wedge d\Phi,$$
$$J^{(2|0)}_{\omega f} = \psi\wedge dW, \qquad J^{(2|0)}_{\omega\bar{f}} = \psi\wedge d\bar{W},$$
$$J^{(2|0)}_{\bar{\omega}f} = \bar{\psi}\wedge dW, \qquad J^{(2|0)}_{\bar{\omega}\bar{f}} = \bar{\psi}\wedge d\bar{W}. \tag{36}$$

Following this path, we find several new higher-form currents (even and odd) which might play an interesting

[6]We can choose $\mathbb{Y}^{(0|1)} = V\iota\delta(\psi)$ and use $\mathbb{Z}[V\delta(\psi)] = -\psi$, $\mathbb{Z}[V\delta'(\psi)] = 1$ and analogous relations for the antiholomorphic sector.

role, another example being $J^{(4|0)} = \psi \wedge \bar{\psi} \wedge dW \wedge dW$. However, at the moment their rôle in physics is not clear. We leave this issue to future analysis.

B. Mixed anomaly

As a preparatory work towards the construction of the SuSymTFT for the supercompact boson, we investigate the emergence of possible 't Hooft anomalies for the various currents.

To this end, we adopt a well-established approach. We first minimally couple the symmetry currents to background gauge fields. Contextually, we use one of the background fields to covariantize the currents with respect to the corresponding symmetry—in this case we choose the shift symmetry—then we require the background to transform nontrivially under gauge transformations in order to ensure gauge invariance of the total action, up to a possible term that depends only on the background fields and the gauge parameters. The appearance of such a leftover signals the presence of a 't Hooft anomaly which can be compensated by a suitable inflow action.

Focusing only on the global and spacetime symmetries, we minimally couple $J = d\Phi$ and $\tilde{J} = \star d\Phi$ to background gauge fields $B^{(1|0)} \equiv B$ and $\tilde{B}^{(1|2)} \equiv \tilde{B}$, respectively, and the supersymmetry current $\mathcal{S}^{(1|1)}$ in (30) to a gravitino background $\chi^{(1|1)} \equiv \chi$. Moreover, in order to make the action manifestly invariant under the shift symmetry we use the B field to covariantize J and $\mathcal{S}^{(1|1)}$.

The general expansion of the B superform in components reads

$$B = b_V V + \bar{b}_V \bar{V} + b_\psi \psi + \bar{b}_\psi \bar{\psi}, \tag{37}$$

where b_V, $\bar{b}_V$, b_ψ, and $\bar{b}_\psi$ are ordinary superfields. Imposing conventional constraints on its field strength $F^{(2|0)} = dB$, that is imposing the vanishing of its ψ^2 and $\bar{\psi}^2$ components, it is easy to see that the component superfields of B are not all independent. Precisely, they are related as $b_V = -Db_\psi, \bar{b}_V = -\bar{D}\bar{b}_\psi$.

The covariantized version of the supersymmetry current can be obtained by exploiting the B components. It reads

$$\begin{aligned} \mathcal{S}_{\text{cov}} &= (W - b_\psi)(\partial\Phi - b_V)V\delta(\psi), \\ \bar{\mathcal{S}}_{\text{cov}} &= (\bar{W} - \bar{b}_\psi)(\bar{\partial}\Phi - \bar{b}_V)\bar{V}\delta(\bar{\psi}). \end{aligned} \tag{38}$$

We then promote the original action (19) to

$$\begin{aligned} S[B, \tilde{B}, \chi] = \frac{1}{4\pi} \int_{\mathcal{SM}^{(2|2)}} &[(d\Phi - B) \wedge \star(d\Phi - B) \\ &+ \tilde{B} \wedge (d\Phi - B) + \chi(\mathcal{S}_{\text{cov}} + \bar{\mathcal{S}}_{\text{cov}})]. \end{aligned} \tag{39}$$

Note that the coupling to the background fields does not spoil the manifest supersymmetry of the original action, being $S[B, \tilde{B}, \chi]$ the integral of a top form on a supermanifold without boundary.

It is easy to see that the gauged action is manifestly invariant under local shifts

$$\Phi \to \Phi + \omega, \qquad B \to B + d\omega \tag{40}$$

Moreover, as detailed in Appendix C, applying gauge transformations of gravitational and $\tilde{B}$ backgrounds of the form

$$\chi \to \chi + d\Lambda, \qquad \tilde{B} \to \tilde{B} + d\tilde{\omega} - \Sigma(\Lambda, B) \tag{41}$$

with $\Sigma(\Lambda, B)$ determined in Appendix C [see Eqs. (C7) and (C8)], the action is invariant up to the following 't Hooft anomaly:

$$\delta S[B, \tilde{B}, \chi] = \frac{1}{4\pi} \int_{\mathcal{SM}^{(2|2)}} \tilde{\omega} \wedge dB. \tag{42}$$

This is the supersymmetric version of the $U(1)^{(0|0)} \times U(1)^{(0|2)}$ mixed anomaly, already present in the bosonic case. It prevents the two global symmetries from being simultaneously gauged. We note that there is no gravitational one-loop anomaly, as expected for a nonchiral theory. Therefore, expression (42) exhausts the spectrum of anomalies of the theory.

The 't Hooft anomaly can be canceled by adding to $S[B, \tilde{B}, \chi]$ the following inflow action:

$$S_{\text{inflow}}[B, \tilde{C}] = -\frac{1}{4\pi} \int_{\mathcal{SM}^{(3|4)}} \tilde{C} \wedge dB \tag{43}$$

defined in the supermanifold $\mathcal{SM}^{(3|4)}$ with one bosonic dimension higher and twice the number of supercharges, whose boundary is $\mathcal{SM}^{(2|2)}$. $\tilde{C}$ is a gauge $(1|4)$ superform in the bulk subject to gauge transformations $\tilde{C} \to \tilde{C} + d\tilde{\omega}_B$, with boundary condition $\tilde{\omega}_B| = \tilde{\omega}$.

The inflow is exactly the super-BF action introduced in Sec. II A, specialized to the $(3|4)$-dimensional case.

C. SuSymTFT for the supercompact boson

In this section we construct the SuSymTFT for the two-dimensional nonchiral $\mathcal{N} = (1, 1)$ compact boson. We consider only the global $U(1)^{(0|0)} \times U(1)^{(0|2)}$ symmetries, postponing the construction for spacetime (super)symmetries to a future paper [40].

In order to define a consistent bulk three-dimensional topological theory coupled to this model, we follow the general prescription proposed in Sec. II B. As already mentioned there, the only nontrivial question concerns the number of supersymmetries that we need to realize in three dimensions. Since the boundary halves the number of supersymmetries, the correct supermanifold where defining

the SuSymTFT in this case is $\mathcal{SM}^{(3|4)}$. In other words, the SuSymTFT is going to be a three-dimensional $\mathcal{N}=2$ super-BF theory.

The continuous global symmetry sector is encoded into a SuSymTFT consisting of a bulk Abelian $\mathcal{N}=2$ supersymmetric BF theory, whose action in the $\mathcal{SM}^{(3|4)}$ supermanifold reads (conventions in three dimensions are given in Appendix D)

$$S_{\mathrm{sBF}}=\frac{1}{4\pi}\int_{\mathcal{SM}^{(3|4)}}\mathcal{A}_1\wedge d\mathcal{A}_2, \tag{44}$$

where the fields $\mathcal{A}_I$ are flat $\mathbb{R}$-valued superform connections:

$$\mathcal{A}_1\in\Omega^{(1|4)}(\mathcal{SM}^{(3|4)},\mathbb{R}),\qquad \mathcal{A}_2\in\Omega^{(1|0)}(\mathcal{SM}^{(3|4)},\mathbb{R})$$

subject to the gauge transformations $\delta\mathcal{A}_I=da_I$, for appropriate form degrees. The bulk action needs then be completed with the boundary term $S[B,\tilde{B},\chi]$ in (19).

The only nontrivial observables of the theory are $\mathcal{A}_I$ holonomies over nontrivial cycles in $\mathcal{SM}^{(3|4)}$, measured by the topological operators

$$\begin{aligned}\mathcal{U}_x(\Gamma)&=\exp\left(ix\int_{\Gamma^{(1|4)}}\mathcal{A}_1\right),\\ \mathcal{V}_y(\Sigma)&=\exp\left(iy\int_{\Sigma^{(1|0)}}\mathcal{A}_2\right),\end{aligned} \tag{45}$$

where, for the sake of notation, we dropped the superscript $(\cdot|\cdot)$. Because of the pairing in (44), the operators braid nontrivially:

$$\mathcal{U}_x(\Gamma)\mathcal{V}_y(\Sigma)=e^{4\pi ixy\mathcal{SL}(\Gamma,\Sigma)}\mathcal{V}_y(\Sigma)\mathcal{U}_x(\Gamma), \tag{46}$$

where $\mathcal{SL}(\Gamma,\Sigma)$ is the superlinking number between Γ and Σ; cf. [36].

To make this super-BF theory the SuSymTFT of the supercompact boson, we have to fix appropriate physical and topological boundary conditions. To this end, we take $\mathcal{SM}^{(3|4)}$ to be a supermanifold of the form

$$\mathcal{SM}^{(3|4)}=\mathcal{I}^{(1|2)}\times(\mathcal{SM}^{(2|2)}_{\mathrm{phys}}\cup\overline{\mathcal{SM}}^{(2|2)}_{\mathrm{top}}),$$

where $\mathcal{I}$ is a superinterval (a supermanifold whose bosonic component is an interval) and $\mathcal{SM}^{(2|2)}_{\mathrm{phys}}=\mathcal{SM}^{(2|2)}$ is the supermanifold where the supercompact boson is defined.

At the physical boundary we impose the following boundary conditions for the bulk gauge superforms:

$$\mathcal{A}_1|_{\partial\mathcal{SM}^{(3|4)}}=\tilde{B}\wedge\mathbb{Y}^{(0|2)},\qquad \mathcal{A}_2|_{\partial\mathcal{SM}^{(3|4)}}=B, \tag{47}$$

which ensure that the boundary term arising from the gauge variation of S_{sBF} in (44) gets exactly canceled at the physical boundary by the anomalous variation (42) of the physical theory.

The PCO $\mathbb{Y}^{(0|2)}$ necessarily introduced in (47) for matching the form picture localizes the expression on two out of four fermionic directions. Therefore, it breaks $\mathcal{N}=2$ supersymmetry in three dimensions (four real supercharges) to $\mathcal{N}=1$ in two dimensions (two real supercharges). This is how in supergeometry we formalize the fact that the presence of a boundary halves supersymmetry.

However, this is not the end of the story, as we have also to investigate how the bulk action (44) behaves under a supersymmetry transformation in the bulk. As we discuss in detail in Appendix D, the supersymmetry variation produces a boundary term, which has to vanish in order to restore supersymmetry. Remarkably, the boundary conditions in (47), required for matching the 't Hooft anomaly at the boundary, are also sufficient for canceling the boundary term coming from the supersymmetry variation of S_{sBF}. Therefore, the full supersymmetry invariance in the bulk is preserved, though at the price of imposing half-preserving boundary conditions.

At the topological boundary, we fix an absolute theory by either fixing a polarization for the $\mathbb{R}\times\mathbb{R}$ symmetry group or by coupling to topological edge modes. We will take here the former approach. To retrieve the supercompact boson, we specify a Lagrangian submanifold by picking a maximal torus $(\mathbb{R}/2\pi\mathbb{Z})^{(0|0)}\times(\mathbb{R}/2\pi\mathbb{Z})^{(0|2)}$ of mutually local operators at the boundary [this maximal torus trivializes the braiding phase (46)]. Different choices of boundary conditions are associated to distinct theories at the physical boundaries. Fixing a maximal torus for which $\mathcal{A}_2|_{\mathcal{SM}^{(2|2)}_{\mathrm{top}}}=0$ realizes a theory with a $\mathbb{R}$ $(0|0)$-global symmetry, and vice versa, $\mathcal{A}_1|_{\mathcal{SM}^{(2|2)}_{\mathrm{top}}}=0$ one with $\mathbb{R}$ $(0|2)$-global symmetry.

We close this section by discussing the reduction to components of S_{sBF} in (44), in particular how to link it to the conventional $\mathcal{N}=1$ supersymmetric BF action [46].

To this end, it is convenient to reexpress (44) as follows:

$$\int_{\mathcal{M}^{(3|4)}}\mathcal{L}^{(3|0)}\wedge\mathbb{Y}^{(0|4)}, \tag{48}$$

where the PCO is given by

$$\mathbb{Y}^{(0|4)}=(V\wedge V)^a(\theta_1^2\iota_2\gamma_a\iota_2+\theta_2^2\iota_1\gamma_a\iota_1)\delta^2(\psi_1)\delta^2(\psi_2). \tag{49}$$

This expression is obtained by the supergeometry setting in $D=4$ $\mathcal{N}=1$ framework and it shows a phenomenon discussed in [37] leading to the chiral and antichiral splitting of supersymmetric actions (a complete discussion is found also in [48]). A remark is needed: the expression of the PCO in (49) is not manifestly supersymmetric, but its supersymmetry variation is d-exact, and since the Lagrangian $\mathcal{L}^{(3|0)}$ is closed this fact is harmless.

The CS $\mathcal{N}=2$ theory in geometric formalism was obtained in [43,49,50] and using the PCO (49) the CS action is divided in two pieces the $\mathcal{N}=1$ CS and a mass term for a Wess-Zumino multiplet.

IV. CHIRAL SUPERMULTIPLET

We now consider another simple example, that is the model describing the free dynamics of a chiral supermultiplet in two-dimensional $N=(1,1)$ superspace. In the present context, this is described by a $(0|0)$ superform Φ satisfying $\bar{D}\Phi=0$ (and then $\bar{\partial}\Phi=0$). We will rely on notations and conventions used in the previous sections, with the addition of this extra condition.

The chirality constraint implies that the degrees of freedom reduce to a single holomorphic boson and a single holomorphic fermion. In fact, given the general expansion (18) for the Φ superfield, the chirality constraint reduces it to

$$\Phi=\phi+\theta\lambda, \qquad W\equiv D\Phi=\lambda-\theta\partial\phi \tag{50}$$

with $\bar{\partial}\phi=\bar{\partial}\lambda=0$. The action of the model has been written in [51–53], and in [54] within the framework of supergeometry.

The results found above for the chiral sector of the compact boson can be easily adapted to the present model. However, as we are going to discuss, this model exhibits two additional features. First of all, compared to the previous example, it possesses only one conserved $U(1)$ current, due to its self-duality. As we are going to show, this current is not anomalous and does not exhibit any mixed 't Hooft anomaly with the supersymmetry current. Moreover, the model suffers from a gravitational anomaly, due to the presence of a chiral multiplet. The gravitational anomaly has been computed long ago in [55]. Here, we look for the structure of the anomaly inflow that can be introduced to compensate it, formulated in the language of supergeometry. This extra term modifies the form of the SymTFT which is no longer represented simply by the BF model associated with the mixed 't Hooft anomaly.

A. Self-duality of the $U(1)$ currents

In this section we prove that the chirality constraint $\bar{D}\Phi=0$ implies self-duality of the $U(1)$ current $J=d\Phi$. To this end, we need first to introduce the definition of self-duality in supermanifolds.

In conventional bosonic manifolds, the (anti-)self-duality of a two-dimensional boson is expressed by the condition

$$\star d\phi=\pm d\phi. \tag{51}$$

Bianchi identity then implies the equation of motion $d\star d\phi=0$. Conditions (51) are satisfied for $\bar{\partial}\phi=0$ (or $\partial\phi=0$ in the anti-self-dual case).

In a supermanifold, the situation is more involved since we have to take into account the presence of fermions in the multiplet. In this case, the (anti-)self-duality condition (51) is promoted to an analog condition for the superform Φ [54]:

$$\star d\Phi=\pm d\Phi\wedge\mathbb{Y}^{(0|2)}, \tag{52}$$

where the PCO $\mathbb{Y}^{(0|2)}$ has been inserted to match the order and the picture of the two integral or superforms on the two sides. Applying one extra d operator, this identity leads to the equations of motion $d(\star d\Phi)=0$.

Alternatively, the self-duality condition can be written as

$$\mathbb{Z}\bar{\mathbb{Z}}[\star d\Phi]=\pm d\Phi, \tag{53}$$

where $\mathbb{Z}$ and $\bar{\mathbb{Z}}$ are the two lowering PCOs introduced in Appendix A.

Upon choosing the following PCO[7]:

$$\mathbb{Y}^{(0|2)}=\delta^2(\psi)-V\bar{V}\delta'(\psi)\delta'(\bar{\psi}) \tag{54}$$

and expanding $d\Phi$ and $\star d\Phi$ as in (21), constraint (52) enforces and is enforced by the usual (anti-)self-duality condition $\bar{D}\Phi=0$ (or $D\Phi=0$ in the anti-self-dual case), which is nothing but the chiral constraint.

Therefore, we can conclude that for a chiral multiplet the two conserved currents $J=d\Phi$ and $\tilde{J}=\star d\Phi$ satisfy the self-duality identity $J=\mathbb{Z}\bar{\mathbb{Z}}[\tilde{J}]$; thus, they describe the same $U(1)^{(0|0)}$ symmetry. Since supersymmetry and κ symmetry are diffeomorphisms of the supermanifold, the self-duality in (52) easily implies the same self-duality condition for $\mathcal{J}=\mathcal{L}_\epsilon J$ and $\mathcal{J}_\kappa=\mathcal{L}_\kappa J$ introduced in (25) and (27), respectively.

The chirality—or equivalently the self-duality—constraint implies that the expansion of the $(1|0)$ supercurrent J is shortened to

$$J=V\partial\Phi+\psi D\Phi \tag{55}$$

with holomorphic components $\bar{D}(\partial\Phi)=\bar{D}(D\Phi)=0$ [then $\bar{\partial}(\partial\Phi)=\bar{\partial}(D\Phi)=0$]. Generalizing, any self-dual $(1|0)$ superform has a chiral expansion of this form.

B. Mixed and gravitational anomalies

We now study the possible emergence of 't Hooft anomalies associated to the global symmetry, and mixed gauge-supersymmetry anomalies, in analogy with the previous example.

We first consider coupling the theory to a self-dual, chiral background

[7]This PCO is closed, but not exact, and differs from the more standard $\mathbb{Y}^{(0|2)}=\theta\bar{\theta}\delta(\psi)\delta(\bar{\psi})$ by exact terms.

$$B = b_V V + b_\psi \psi, \tag{56}$$

where B is a $(1|0)$ superform with $\bar{D} b_V = \bar{D} b_\psi = 0$ (and then $\bar{\partial} b_V = \bar{\partial} b_\psi = 0$).[8] We use B to covariantize J with respect to the chiral shift, $\Phi \to \Phi + \omega, B \to B + d\omega$, with $\bar{D}\omega = 0$.

The supersymmetry current is given by the chiral component of one of the currents (29)–(31), according to the choice of the form picture. As in the previous example, we can use the B components to covariantize $\mathcal{S}^{(1|1)}$ respect to the $U(1)$ shift symmetry, obtaining $\mathcal{S}_{\rm cov}$ in Eq. (38). We then consider the linear couplings with a chiral $U(1)$ and a gravitino background:

$$S_{\rm linear} = \int_{\mathcal{SM}^{(2|2)}} ((d\Phi - B) \wedge \tilde{B} + \chi \mathcal{S}_{\rm cov}) \tag{57}$$

and study its variation under gauge transformations $B \to B + d\omega$, $\tilde{B} \to \tilde{B} + d\tilde{\omega}$ and $\chi \to \chi + d\Lambda$. By construction, the action is manifestly invariant under B transformations. It is also invariant under $\tilde{B}$ transformations because of the chirality of $(d\Phi - B)$, whereas under χ variations, we obtain

$$\begin{aligned}\int_{\mathcal{SM}^{(2|2)}} \Lambda d\mathcal{S}_{\rm cov} = \int_{\mathcal{SM}^{(2|2)}} \Lambda [& db_\psi (\partial\Phi - b_V) \\ & + db_V (W - b_\psi)] V \delta(\psi).\end{aligned} \tag{58}$$

However, also this expression vanishes, thanks to the chirality of B. We then conclude that in the chiral model there is no mixed gauge-supersymmetry anomaly.

This does not exhaust the spectrum of anomalies of the chiral boson. In fact, it is well known that this model possesses a gravitational anomaly. In order to take it into account, we linearly couple the supersymmetry current $\mathcal{S}^{(1|1)}$ to a gravitational background field $H^{(1|1)}$.

A generic $(1|1)$ pseudoform has the following structure:

$$\begin{aligned}H^{(1|1)} = & \sum_{p\geq 0} A_p \psi^{p+1} \delta^{(p)}(\bar{\psi}) + \sum_{p\geq 0} \bar{A}_p \bar{\psi}^{p+1} \delta^{(p)}(\psi) \\ & + \sum_{p\geq 0} B_p V \psi^p \delta^{(p)}(\bar{\psi}) + \sum_{p\geq 0} \bar{B}_p \bar{V} \psi^p \delta^{(p)}(\bar{\psi}) \\ & + \sum_{p\geq 0} C_p V \bar{\psi}^p \delta^{(p)}(\psi) + \sum_{p\geq 0} \bar{C}_p \bar{V} \bar{\psi}^p \delta^{(p)}(\psi) \\ & + \sum_{p\geq 0} D_p V \bar{V} \psi^p \delta^{(p+1)}(\bar{\psi}) + \sum_{p\geq 0} \bar{D}_p V \bar{V} \bar{\psi}^p \delta^{(p+1)}(\psi),\end{aligned} \tag{59}$$

where the coefficients of the expansion are superfields. Here, the upper indices of the deltas denote the order of their derivatives with respect to ψ or $\bar{\psi}$. Powers of ψ or $\bar{\psi}$ are intentionally chosen to compensate the negative form degrees of $\delta^{(p)}(\bar{\psi})$ and $\delta^{(p)}(\psi)$.

The $H^{(1|1)}$ pseudoform is defined up to the following infinite chain of gauge transformations:

$$\delta H^{(1|1)} = d\Sigma^{(0|1)}, \qquad \delta\Sigma^{(-p|1)} = \Sigma^{(-p-1|1)}, \qquad \forall \ p \geq 0 \tag{60}$$

for generic pseudoform parameters Σ.

Evaluating the linear coupling between the supersymmetry current and the background, we obtain

$$\begin{aligned}\int_{\mathcal{SM}^{(2|2)}} \mathcal{S}^{(1|1)} \wedge H^{(1|1)} &= \int_{\mathcal{SM}^{(2|2)}} (W\partial\Phi) V \delta(\psi) \wedge \bar{B}_0 \bar{V} \delta(\bar{\psi}) \\ &= \int d^2z d^2\theta (W\partial\Phi) \wedge \bar{B}_0.\end{aligned} \tag{61}$$

The reason why only the $\bar{B}_0$ component of the expansion (59) survives can be traced back to the anticommuting nature of V and $\bar{V}$ and the obvious identities $\bar{\psi}\delta(\bar{\psi}) = 0$ and $\delta(\psi)\delta(\psi) = 0$.

The action in (61) is manifestly invariant under gauge transformations (60) thanks to the closure of $\mathcal{S}^{(1|1)}$.

In order to perform the Berezin integral in (61) and obtain the gravitational coupling in components, it is convenient to work in the Wess-Zumino gauge, $\bar{\theta}\bar{B}_0 = 0$. This is equivalent to choosing

$$\bar{B}_0 = \bar{\theta}(h + \theta\chi), \tag{62}$$

where h and χ are bosonic and fermionic ordinary fields, respectively. Inserting this expansion together with the one for $W\partial\Phi$ [see Eq. (32)] and evaluating the Berezin integral, we eventually obtain

$$\int_{\mathcal{SM}^{(2|2)}} \mathcal{S}^{(1|1)} \wedge H^{(1|1)} = \int d^2z [\lambda\partial\phi\chi + (\lambda\partial\lambda + (\partial\phi)^2)h]. \tag{63}$$

We recognize h playing the role of the vielbein coupled to the stress-energy tensor and χ the gravitino coupled to the supersymmetry current. Given transformations (34) for the current components, the coupling is supersymmetric if

$$\delta_\epsilon h = \epsilon\chi, \quad \delta_\epsilon\chi = -\partial(\epsilon h). \tag{64}$$

These are the well-known supergravity transformation rules with local supersymmetry parameter ϵ.

The linear coupling (63) leads to an anomalous Ward identity [55,56] which represents the easiest example of gravitational anomaly for the chiral supermultiplet. To account for this anomaly the SuSymTFT needs to be complemented by a Chern-Simons term, which in the present context reads

[8]We note that the holomorphy conditions together with the conventional constraints on the field strength lead to $dB = 0$.

$$S_{CS} = \alpha \int_{\mathcal{SM}^{(3|4)}} H^{(1|2)} \wedge dH^{(1|2)}. \tag{65}$$

The α coefficient has been determined in [56] by evaluating the supersymmetry variation of the one-loop effective action. It is subject to suitable quantization conditions to ensure invariance under background diffeomorphisms (see [57] for a general discussion).

The relation between the integral forms $H^{(1|2)}$ and the boundary $H^{(1|1)}$ is via a suitable gauge fixing of the extra θ coordinates. As previously described, this can be done by using the PCO along the additional fermionic variables, without affecting the physics.

Note that the expansion of $H^{(1|2)}$ differs from (59) since the supermanifold has acquired a new bosonic direction. This construction has been studied in [43,49,58] and for 3D supergravity in [59].

V. OUTLOOK

In this paper we have proposed an intrinsically supersymmetric formulation of the SymTFT—the SuSymTFT. By formulating the SymTFT directly into superspace, one gains the advantage of organizing the symmetries and anomalies of a supersymmetric theory directly into their natural multiplets conveniently packaged in superfields. In this work we have illustrated the construction explicitly in the case of two-dimensional (chiral and nonchiral) $\mathcal{N} = (1,1)$ theories.

We have advanced a formal construction of the SymTFT in a supersymmetric environment, but more work is required to embed our construction within the most general systematic study of spacetime and superconformal symmetries [26]. Furthermore, the SuSymTFT has to be better investigated close to the symmetry boundary by constructing the quantum algebra of topological and charged operators.

This work opens many interesting future directions of investigation.

(i) In the usual SymTFT framework, much of its power comes from the relation between different topological forms of a d-dimensional theory and different topological boundary conditions of the same $(d+1)$-dimensional bulk theory. The construction presented here provides a supersymmetric setting in which one could formulate the analogous question while keeping supersymmetry manifest. A natural next step is therefore to study the space of supersymmetry-preserving topological boundary conditions and classify which of them admit a supersymmetric completion compatible with the bulk supergeometry and the preserved boundary supercharges. This may provide a systematic way of relating different supersymmetric boundary theories associated with the same bulk SuSymTFT and, potentially, uncovering symmetry structures that are less transparent in a component formulation.

(ii) We have considered the case of minimal supersymmetry in two dimensions. To do that, we have placed the physical theory on a 1/2-BPS boundary of the bulk supermanifold $\mathcal{M}^{(3|4)}$ supporting an $\mathcal{N} = 2$ topological theory. For example, our construction could be straightforwardly generalized to the chiral $\mathcal{N} = (2,0)$ superspace. Beyond that, it would be interesting to explore more general cases where already the bulk exhibits extended supersymmetry.

(iii) Here we limited ourselves to constructing explicitly the SuSymTFT for the global symmetries. Following the recent construction of the SymTFT for spacetime symmetries [26], it would be very natural to also include the global supersymmetry algebra in such a construction. This would naturally entail coupling the bulk to (topological) supergravity. In this direction, we have done a preliminary discussion in the case of the chiral boson, but a systematic investigation is still missing.

(iv) While we have presented our proposal in full generality, as a first application we have considered only the case of two-dimensional theories. It would then be natural to investigate the application of our construction to the case of supersymmetric theories in higher dimensions, where the structure of superspace, as well as the choices of boundary conditions and the selection of topological bulk theories, would be much richer.

(v) As an aside problem, it would be interesting to investigate the alternative construction of half-BPS boundary supersymmetric theories without boundary conditions, exploiting the F+A construction of [44] rephrased in the supergeometry language. It would be interesting to understand whether such a construction may play a role in the symmetry TFT approach.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: PICTURE CHANGING OPERATORS: A PRIMER

Here we depict a few ingredients regarding PCOs, which we use in the main text. For the general construction of PCOs in supergeometry we refer to [60], whereas for an exhaustive review on PCOs in supermanifolds see [36].

As in string theory, there two types of PCOs: picture raising operators and picture lowering operators. In the present context the picture number is given by the number of Dirac deltas function $\delta(\psi)$ in a given expression, where ψ is the odd vielbein ($\psi = d\theta$ in flat superspace). Lowering means removing deltas; raising means increasing the number of deltas.

We recall that a raising PCO $\mathbb{Y}^{(0|q)}$ (increasing the picture by q units, without changing the form degree) has the following properties:

$$\begin{aligned} &d\mathbb{Y}^{(0|q)} = 0, \qquad \mathbb{Y}^{(0|q)} \neq dK^{(-1|q)}, \\ &\omega^{(r|s)} \to \omega^{(r|s+q)} = \omega^{(r|s)} \wedge \mathbb{Y}^{(0|q)}, \end{aligned} \tag{A1}$$

where $\omega^{(r|s)}$ is generic $(r|s)$ pseudoform.

The PCO is an isomorphism on the cohomology of d, namely $H^{(r|s)}$. Since the PCO is a representative of the cohomology class $H^{(0|q)}$, one can choose different but equivalent expressions. For instance, in a $\mathcal{SM}^{(2|2)}$ supermanifold with $(z, \bar{z}, \theta, \bar{\theta})$ coordinates (see Appendix B for details), taking $q = 2$ we can consider the following different PCO realizations:

$$\begin{aligned} \mathbb{Y}^{(0|2)} &= \theta\bar{\theta}\delta(\psi)\delta(\bar{\psi}), \\ \mathbb{Y}^{(0|2)} &= V\delta'(\psi)\bar{V}\delta'(\bar{\psi}), \\ \mathbb{Y}^{(0|2)} &= \delta(\psi)\delta(\bar{\psi}) + V\delta'(\psi)\bar{V}\delta'(\bar{\psi}), \end{aligned} \tag{A2}$$

where V and $\bar{V}$ are the bosonic vielbeins and $\delta'(\psi) \equiv \frac{\partial}{\partial\psi}\delta(\psi)$ stands for $\iota_D\delta(\psi)$, ι_D being the contraction operator (inner derivation) along the supersymmetry covariant derivative D. The first expression is not manifestly supersymmetric, whereas the second and the third ones are manifestly supersymmetry invariant. The first and the second PCOs have a nontrivial kernel; the last one has no kernel.

Lowering PCOs, denoted as $\mathbb{Z}^{(0|-q)}$, instead remove q deltas. They have the following properties:

$$\begin{aligned} &d\mathbb{Z}^{(0|-q)} = 0, \qquad \mathbb{Z}^{(0|-q)} \neq dK^{(-1|-q)}, \\ &\omega^{(r|s)} \to \omega^{(r|s-q)} = \mathbb{Z}^{(0|-q)}[\omega^{(r|s)}] \end{aligned} \tag{A3}$$

with $s \geq q$. Again, these PCOs are isomorphisms in the cohomology. An explicit realization for $q = -1$ is for example [60]

$$\mathbb{Z}^{(0|-1)} = [d, \Theta(\iota_D)] = \mathcal{L}_D\delta(\iota_D) + \iota_{\bar{\partial}}\delta'(\iota_D), \tag{A4}$$

where Θ is the Heaviside theta function and $\mathcal{L}_D$ and ι_D are the Lie derivative and the contraction operator along the vector field D, respectively, while ι_∂ is the contraction along the vector field ∂. The expression $\delta(\iota_D)$ is the Dirac delta of the contraction operator. It acts on $\delta(\psi)$ as $\delta(\iota_D)\delta(\psi) = 1$ (up to normalization factors).

Given $\mathbb{Z}^{(0|-1)}$ in (A4) we have the freedom to add exact terms without modifying the cohomology class. It follows that expression (A4) is supersymmetric and "almost" universal, in the sense that the superderivative D can always be replaced by any other odd vector field X, having the replacement the only effect of generating extra exact terms.

Finally, we note that being $\mathbb{Z}^{(0|-1)}$ expressed in terms of the Heaviside theta function—which is not a compact-support distribution—it is an element of the cohomology but as a representative is defined up to exact terms.

In general, the following identities hold:

$$\begin{aligned} \mathbb{Z}^{(0|-1)}\mathbb{Y}^{(0|1)}\mathbb{Z}^{(0|-1)} &= \mathbb{Z}^{(0|-1)}, \\ \mathbb{Y}^{(0|1)}\mathbb{Z}^{(0|-1)}\mathbb{Y}^{(0|1)} &= \mathbb{Y}^{(0|-1)}, \end{aligned} \tag{A5}$$

and for particular choices of the odd vector field in the definition of the PCO, one can find the inverse $\mathbb{Z}^{(0|-1)}\mathbb{Y}^{(0|1)} = 1$ in the full space of superforms or integral forms.

APPENDIX B: THE $\mathcal{SM}^{(2|2)}$ SUPERMANIFOLD

We parametrize the $\mathcal{SM}^{(2|2)}$ Minkowskian superspace by two bosonic light-cone real coordinates $(x^{++}, x^{--}) \equiv (z, \bar{z})$ and one Majorana spinor $\theta^\alpha = (\theta^+, \theta^-) \equiv (\theta, \bar{\theta})$ with real entries.

The corresponding differential 1-forms are $(dz, d\bar{z}, d\theta, d\bar{\theta})$ and the flat supervielbeins read

$$\begin{aligned} &V = dz + \theta d\theta, \qquad \bar{V} = d\bar{z} + \bar{\theta}d\bar{\theta}, \\ &\psi = d\theta, \qquad \bar{\psi} = d\bar{\theta}. \end{aligned} \tag{B1}$$

They are supersymmetric invariant under $\delta\theta = \epsilon, \delta\bar{\theta} = \bar{\epsilon}$ and $\delta z = \epsilon\theta, \delta\bar{z} = \bar{\epsilon}\,\bar{\theta}$. In fact, for rigid supersymmetry ($\epsilon, \bar{\epsilon}$ constant)

$$\delta V = d(\delta z) + \delta\theta d\theta + \theta d(\delta\theta) = -\epsilon d\theta + \epsilon d\theta + \theta d\epsilon = 0,$$
$$\delta\psi = d(\delta\theta) = d\epsilon = 0. \tag{B2}$$

The superzweibeins in (B1) satisfy the Maurer-Cartan equations

$$dV = \psi \wedge \psi, \qquad d\bar{V} = \bar{\psi} \wedge \bar{\psi}, \qquad d\psi = d\bar{\psi} = 0. \tag{B3}$$

If we set $V^a \equiv (V, \bar{V})$ and $\psi^\alpha \equiv (\psi, \bar{\psi})$, the action of the super-Hodge dual reads [39]

$$\star V^a = \epsilon^{ab} V_b \delta^{(2)}(\psi), \qquad \star\psi^\alpha = V\bar{V}\epsilon^{\alpha\beta}\iota_\beta \delta^{(2)}(\psi), \tag{B4}$$

where we have defined the antisymmetric tensors $\epsilon^{z\bar{z}} = \epsilon_{\bar{z}z} = 2$, $\epsilon^{+-} = \epsilon_{-+} = 1$, and $\delta^{(2)}(\psi) \equiv \delta(\psi)\delta(\bar{\psi})$.

Covariant derivatives in superspace are defined as

$$\partial_z \equiv \partial, \qquad \partial_{\bar{z}} \equiv \bar{\partial}, \qquad D = \partial_\theta - \theta\partial, \qquad \bar{D} = \partial_{\bar{\theta}} - \bar{\theta}\,\bar{\partial}\,. \tag{B5}$$

They form a representation of the Lie superalgebra

$$D^2 = -\partial, \qquad \bar{D}^2 = -\bar{\partial}, \qquad \{D, \bar{D}\} = 0. \tag{B6}$$

Correspondingly, supersymmetry charges are given by

$$Q = \partial_\theta + \theta\partial, \qquad \bar{Q} = \partial_{\bar{\theta}} + \bar{\theta}\,\bar{\partial} \tag{B7}$$

and satisfy the following superalgebra:

$$Q^2 = \partial, \qquad \bar{Q}^2 = \bar{\partial}, \qquad \{Q, \bar{Q}\} = 0,$$
$$\{D, Q\} = \{D, \bar{Q}\} = \{\bar{D}, Q\} = \{\bar{D}, \bar{Q}\} = 0.$$

In terms of superspace covariant derivatives, the differential acting on superfields is defined as

$$d = V\partial + \bar{V}\,\bar{\partial}\, + \psi D + \bar{\psi}\,\bar{D}\,. \tag{B8}$$

According to the general construction, in $\mathcal{SM}^{(2|2)}$ we can define $\omega^{(p|q)}$ pseudoforms at different pictures, with $q = 0, 1, 2$. A zero-picture form $\omega^{(p|0)}$ has an expansion in terms of $dz, d\bar{z}, d\theta, d\bar{\theta}$ differentials, while the expansion of a maximal-picture form $\omega^{(p|2)}$ is expressed in terms of $dz, d\bar{z}$ and the product of Dirac deltas $\delta(\psi), \delta(\bar{\psi})$ and derivatives thereof. In the main text, the expansions are made explicit whenever this is instrumental for the comprehension.

APPENDIX C: MIXED ANOMALY FOR THE COMPACT BOSON: THE DETAILS

In this appendix we detail the discussion of the gauge invariance of the action in (39) for the supercompact boson and determine explicitly the background gauge transformations that leave the action invariant, up to the 't Hooft anomaly (42).

In (39), let us focus on the coupling of the supersymmetry current to the gravitino background

$$\int_{\mathcal{SM}^{(2|2)}} \chi \wedge (\mathcal{S}_{\rm cov} + \bar{\mathcal{S}}_{\rm cov}), \tag{C1}$$

where $\mathcal{S}$ and $\bar{\mathcal{S}}$ are given in (38). If we now perform a gauge transformation $\chi \to \chi + d\Lambda$ with arbitrary $(0|1)$-form gauge parameter Λ, the action varies by the following expression:

$$-\Lambda\{[db_\psi(\partial\Phi - b_V) + db_V(W - b_\psi)]V\delta(\psi)$$
$$+ [d\bar{b}_\psi(\bar{\partial}\Phi - \bar{b}_V) + d\bar{b}_V(\bar{W} - \bar{b}_\psi)]\bar{V}\delta(\bar{\psi})\} \tag{C2}$$

integrated on $\mathcal{SM}^{(2|2)}$. The most general expansion of Λ in components is given by an infinite sum:

$$\begin{aligned}\Lambda = &\sum_{p\geq 0} \bar{a}_p \psi^p \delta^{(p)}(\bar{\psi}) + \sum_{p\geq 0} a_p \bar{\psi}^p \delta^{(p)}(\psi)\\ &+ \sum_{p\geq 0} b_p V \bar{\psi}^p \delta^{(p+1)}(\psi) + \sum_{p\geq 0} \overline{b_p V}\, \psi^p \delta^{(p+1)}(\bar{\psi})\\ &+ \sum_{p\geq 0} c_p V \psi^p \delta^{(p+1)}(\bar{\psi}) + \sum_{p\geq 0} \overline{c_p V}\, \bar{\psi}^p \delta^{(p)}(\psi)\\ &+ \sum_{p\geq 0} d_p V\bar{V}\psi^p \delta^{(p+2)}(\bar{\psi}) + \sum_{p\geq 0} \bar{d}_p V\bar{V}\bar{\psi}^p \delta^{(p+2)}(\psi),\end{aligned} \tag{C3}$$

where the coefficients are ordinary superfields and $\delta^{(p)}(\psi)$ means the p derivative of the delta function. However, when Λ appears as in (C2) multiplied by terms proportional to $V\delta(\psi)$ and $\bar{V}\delta(\bar{\psi})$, only a finite number of terms in the expansion survive. Precisely, expression (C2) reduces to

$$-\bar{\lambda}[db_\psi(\partial\Phi - b_V) + db_V(W - b_\psi)]V\delta(\psi)$$
$$-\lambda[d\bar{b}_\psi(\bar{\partial}\Phi - \bar{b}_V) + d\bar{b}_V(\bar{W} - \bar{b}_\psi)]\bar{V}\delta(\bar{\psi}), \tag{C4}$$

where we have defined

$$\lambda = a_0\delta(\psi) + b_0 V\delta'(\psi), \qquad \bar{\lambda} = \bar{a}_0\delta(\bar{\psi}) + \overline{b_0 V}\delta'(\bar{\psi}). \tag{C5}$$

Now, it is easy to see that expression (C4) can be recast in a very compact form, as

$$\Sigma(\Lambda, B) \wedge (d\Phi - B), \tag{C6}$$

where $\Sigma(\Lambda, B)$ is the $(1|2)$ superform

$$\Sigma(\Lambda, B) = \bar{\sigma}^{(1|1)}\delta(\psi) + \bar{\tau}^{(1|1)} V \delta'(\psi) + \sigma^{(1|1)}\delta(\bar{\psi}) + \tau^{(1|1)} \bar{V} \delta'(\bar{\psi}) \tag{C7}$$

whose form components read

$$\bar{\sigma}^{(1|1)} = \bar{\lambda}(\bar{V}\,\bar{\partial}\, b_\psi + \bar{\psi}\,\bar{D}\, b_\psi), \qquad \sigma^{(1|1)} = \lambda(V\partial\bar{b}_\psi + \psi D\bar{b}_\psi),$$
$$\bar{\tau}^{(1|1)} = \bar{\lambda}(\bar{V}\,\bar{\partial}\, b_V + \bar{\psi}\,\bar{D}\, b_V), \qquad \tau^{(1|1)} = \lambda(V\partial\bar{b}_V + \psi D\bar{b}_V). \tag{C8}$$

In conclusion, we have found that under a gravitino gauge transformation featured by parameter Λ the gravity part of the action varies as

$$\delta \int_{\mathcal{SM}^{(2|2)}} \chi \wedge (\mathcal{S}_{\text{cov}} + \bar{\mathcal{S}}_{\text{cov}}) = \int_{\mathcal{SM}^{(2|2)}} \Sigma(B, \Lambda) \wedge (d\Phi - B). \tag{C9}$$

It is then straightforward to see that this variation can be canceled by modifying the gauge transformation of $\tilde{B}$ as

$$\tilde{B} \to \tilde{B} + d\tilde{\omega} - \Sigma(\Lambda, B) \tag{C10}$$

so that, referring to Eq. (39) we obtain

$$\begin{aligned}\delta S[B, \tilde{B}, \chi] &= \frac{1}{4\pi} \int_{\mathcal{SM}^{(2|2)}} [(d\tilde{\omega} - \Sigma(\Lambda, B)) \wedge (d\Phi - B) \\ &\quad + \Sigma(\Lambda, B) \wedge (d\Phi - B)] \\ &= \frac{1}{4\pi} \int_{\mathcal{SM}^{(2|2)}} \tilde{\omega} \wedge dB.\end{aligned} \tag{C11}$$

This is the expected mixed anomaly that we discussed in the main text.

APPENDIX D: SUPERSYMMETRY WITH BOUNDARY

In this appendix we discuss the supersymmetry invariance of the SuSymTFT action for the supercompact boson introduced in Sec. III C.

In supergeometry, an infinitesimal supersymmetry transformation is captured by the Lie derivative with respect to the vector field $\epsilon = \epsilon^\alpha Q_\alpha$, where Q_α are the real supercharges of the theory.

The integral of a top form ω on a manifold without boundary is automatically supersymmetric, as

$$\int_{\mathcal{SM}} \mathcal{L}_\epsilon \omega \equiv \int_{\mathcal{SM}} \iota_\epsilon(d\omega) + d(\iota_\epsilon \omega) = 0. \tag{D1}$$

In fact, the first term vanishes because ω is a top form, while the second term vanishes by applying Stokes theorem in the absence of a boundary.

Instead, if the supermanifold has a boundary, the last term in (D1) survives as a boundary term

$$\int_{\partial\mathcal{SM}} \iota_\epsilon \omega. \tag{D2}$$

This term has to be removed in some way in order to restore supersymmetry. One possibility would be adding a boundary action whose variation cancels exactly (D2). This would preserve supersymmetry without imposing any boundary condition. However, in the present context this approach is not necessary since in any case we have to impose boundary conditions to match the boundary 't Hooft anomaly. In fact, in this appendix we show that boundary conditions (47) on the bulk fields are already sufficient to make the supersymmetry variation of S_{sBF} in (44) vanish, upon a suitable choice of the PCO.

To this end, we parametrize the Minkowskian supermanifold $\mathcal{SM}^{(3|4)}$ with three bosonic coordinates $x^a = (z, \bar{z}, \tau)$ $(a = 1, 2, 3)$ and four real fermionic coordinates arranged into two Majorana $3d$ spinors θ_i^α $(i = 1, 2$ and $\alpha = \pm)$. The corresponding supervielbeins are

$$V^a = dx^a + \theta_i \gamma^a d\theta_i, \qquad \psi_i^\alpha = d\theta_i^\alpha, \tag{D3}$$

which are invariant under rigid supersymmetry transformations. Their corresponding dual vector fields are given by

$$\partial_a, \qquad D_\alpha^i = \partial_\alpha^i - (\theta^i \gamma^a)_\alpha \partial_a, \tag{D4}$$

and they satisfy the following algebra:

$$\{D_\alpha^i, D_\beta^j\} = -2\delta^{ij}\gamma_{\alpha\beta}^a \partial_a. \tag{D5}$$

In this basis, the differential operator can be written as

$$d = V^a \partial_a + \psi_i^\alpha D_\alpha^i, \tag{D6}$$

while the supercharges implementing supersymmetry transformations are

$$Q_\alpha^i = \partial_\alpha^i + (\theta^i \gamma^a)_\alpha \partial_a \tag{D7}$$

satisfying the following algebra:

$$\{Q_\alpha^i, Q_\beta^j\} = 2\delta^{ij}\gamma_{\alpha\beta}^a \partial_a, \qquad \{Q_\alpha^i, D_\beta^j\} = 0. \tag{D8}$$

As a consistency check, one can reduce the $3d$ $\mathcal{N} = 2$ superalgebra to the coordinates $(z, \bar{z}, \theta_1^+, \theta_2^-)$ (setting $\tau = 0$ and $\theta_1^- = \theta_2^+ = 0$) and see that it reproduces the $\mathcal{N} = (1, 1)$ superalgebra of Appendix B, upon identifying

$\theta \equiv \theta_1^+, \bar{\theta} \equiv \theta_2^-$. This is an explicit realization of supersymmetry breaking at the boundary.

Referring to the field content of the bulk action S_{sBF} in (44), the most general expansions of $(1|0)$ and $(1|4)$ forms on the basis of supervielbeins (D3) read

$$\mathcal{A}_1 = (\tilde{A}_a V^a + \tilde{A}^{\alpha}_{i,ab} V^a V^b \iota^i_\alpha + \tilde{A}^{(\alpha\beta)}_{(ij)} V^3 \iota^i_\alpha \iota^j_\beta)\delta^4(\psi),$$
$$\mathcal{A}_2 = A_a V^a + A^i_\alpha \psi^\alpha_i. \tag{D9}$$

Using the conventional constraints, one can write the field strength $d\mathcal{A}_2$ as

$$d\mathcal{A}_2 = F_{ab} V^a V^b + (W^i \gamma_a)_\alpha V^a \psi^\alpha_i, \tag{D10}$$

where $F_{ab} = \partial_{[a} A_{b]}$ and $(W^i\gamma_a)_\alpha = \partial_a A^i_\alpha - D^i_\alpha A_a$.

Inserting these expansions in the bulk action (44), it is easy to work out that S_{sBF} has the following form:

$$S_{\text{sBF}} = \frac{1}{4\pi}\int_{\mathcal{SM}^{(3|4)}} \mathcal{L}_{\text{sBF}}(A,\tilde{A}) V^3 \delta^4(\psi), \quad \text{with}$$
$$\mathcal{L}_{\text{sBF}}(\tilde{A}, A) = \epsilon^{abc}(\tilde{A}_a F_{bc} - \tilde{A}^i_{ab}\gamma_c W_i). \tag{D11}$$

Now, if we compute its supersymmetry variation, we obtain a boundary contribution of the form (D2), where the contraction acts on the supervolume form $V^3\delta^4(\psi)$. Exploiting the following identity:

$$\iota_\epsilon(V^3\delta^4(\psi)) = (\epsilon_{abc}(\epsilon_i\gamma^a\theta_i)V^b V^c - V^3 \epsilon^\alpha_i \iota^i_\alpha)\delta^4(\psi), \tag{D12}$$

we are eventually left with

$$\delta_\epsilon S_{\text{sBF}} = \frac{1}{2\pi}\int_{\partial\mathcal{SM}^{(3|4)}} \mathcal{L}_{\text{sBF}}(\tilde{A}|_\partial, A|_\partial)(\epsilon_i\gamma^3\theta_i) V^2\delta^4(\psi). \tag{D13}$$

Here the notation $\tilde{A}|_\partial$ and $A|_\partial$ stands schematically for the components of $\mathcal{A}_1$ and $\mathcal{A}_2$ at the boundary $\tau = 0$, according to the boundary conditions in (47).

To work them out explicitly, we have to specify the choice of the PCO in (47). It should encode the information about half of the supersymmetries that get broken by the boundary. In other words, the PCO should manifestly break supertranslations along half of the fermionic directions.

In order to preserve $\mathcal{N} = (1,1)$ supersymmetry at the boundary, without loss of generality we can choose

$$\mathbb{Y}^{(0|2)} = \theta_1^- \theta_2^+ \delta(\psi_1^-)\delta(\psi_2^+) \tag{D14}$$

and identify the other fermionic directions with the ones at the boundary, namely $(\theta_1^+, \theta_2^-) \equiv (\theta, \bar{\theta})$.

Recalling that the most general $(1|0)$ and $(1|2)$ forms on $\mathcal{SM}^{(2|2)}$ can be expanded, respectively, as (now $a = 1, 2$)

$$B = b_a V^a + b_\psi \psi + \bar{b}_\psi \bar{\psi},$$
$$\tilde{B} = \tilde{b}_a V^a \delta^2(\psi) + \tilde{b}_\psi V^2 \iota\delta^2(\psi) + \bar{\tilde{b}}_\psi V^2 \bar{\iota}\delta^2(\psi) \tag{D15}$$

and using the explicit expression for the PCO, the boundary conditions in (47) written in superfield components read

$$\begin{aligned}
&A_a|_\partial = b_a, && \tilde{A}_a|_\partial = (\theta_1^-\theta_2^+)\tilde{b}_a,\\
&A_3|_\partial = 0, && \tilde{A}_3|_\partial = 0,\\
&A^1_+|_\partial = b_\psi, && (\tilde{A}_1)^+_{ab}|_\partial = \frac{1}{2}(\theta_1^-\theta_2^+)\epsilon_{ab}\tilde{b}_\psi,\\
&A^2_-|_\partial = \bar{b}_\psi, && (\tilde{A}_2)^-_{ab}|_\partial = \frac{1}{2}(\theta_1^-\theta_2^+)\epsilon_{ab}\bar{\tilde{b}}_\psi,\\
&A^1_-|_\partial = 0, && (\tilde{A}_1)^-_{ab}|_\partial = 0,\\
&A^2_+|_\partial = 0, && (\tilde{A}_2)^+_{ab}|_\partial = 0,\\
&(\tilde{A}_i)^\alpha_{a3}|_\partial = 0, && (\tilde{A}_{(ij)})^{\alpha\beta}|_\partial = 0.
\end{aligned} \tag{D16}$$

One can easily check that with such boundary conditions the expression $\mathcal{L}_{\text{sBF}}(A|_\partial, \tilde{A}|_\partial)$ is identically vanishing; therefore, the SymTFT is manifestly supersymmetric in $\mathcal{SM}^{(3|4)}$.

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