Neutrino oscillations and PMNS matrix in gauge-Higgs unification

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In the $SO(5) \times U(1) \times SU(3)$ gauge-Higgs unification in the Randall-Sundrum (RS) warped space, neutrino oscillations and the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix in the W couplings originate from Majorana mass terms on the ultraviolet brane in the RS space. Tiny neutrino masses are generated by an inverse seesaw mechanism. The PMNS matrix in the normal ordering with $\delta_{\rm CP} = \pi$ naturally arises consistently with the NuFit-6.0 analysis of neutrino oscillation data.

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I. INTRODUCTION

The standard model (SM) of $SU(3)_C \times SU(2)_L \times U(1)_Y$ gauge theory has been successful in describing phenomena at low energies, but has a gauge hierarchy problem when embedded in a larger theory such as grand unification. Gauge-Higgs unification (GHU) models have been proposed to solve the gauge hierarchy problem by unifying the Higgs boson and gauge fields. The SM gauge symmetry $SU(2)_L \times U(1)_Y$ is dynamically broken by an Aharonov-Bohm (AB) phase, θ_H, in the fifth dimension. The Higgs boson appears as a 4D fluctuation mode of θ_H. A finite mass of the Higgs boson is generated by quantum effects of the AB phase θ_H [1–24].

Among various GHU models, the $SO(5) \times U(1) \times SU(3)$ GHU in the Randall-Sundrum (RS) warped space, inspired by $SO(11)$ gauge-Higgs grand unification models [25–28], has been extensively investigated as a realistic model [22–24]. The grand unified theory (GUT)-inspired GHU yields nearly the same phenomenology at low energies as the SM, and gives many predictions. GHU models in the RS warped space predict large parity violation in the couplings of quarks and leptons to Kaluza-Klein (KK) excited modes of gauge bosons, whose effects can be seen in forward–backward asymmetry in fermion pair production in electron–positron (e^-e^+) collisions [29–33]. Deviation from the SM is expected in the processes of W^-W^+ and single Higgs production as well [34,35]. Further, the GUT-inspired GHU predicts a larger W boson mass than the SM [36], just between the SM value and the value reported by the CDF Collaboration in 2022 [37]. The Higgs cubic and quartic self-couplings are predicted to be smaller than in the SM by about 10% and 40%, respectively [24,38].

GHU models on an orbifold such as the RS space exhibit the phenomenon of anomaly flow. The magnitude of gauge anomalies varies as the AB phase θ_H changes. Further, there holds a holographic formula for anomaly coefficients, expressed in terms of the values of the wave functions of gauge fields on the ultraviolet (UV) and infrared (IR) branes [39–41]. Extension of GHU models to grand unification has been intensively studied as well [25–28,42–49]. The status of GHU has been summarized in Ref. [38].

In a previous paper [50], the origin of the Cabibbo-Kobayashi-Maskawa (CKM) matrix in the W couplings in the GUT-inspired GHU was clarified. Flavor mixing in the quark sector arose in the down-type quark multiplets, and the observed CKM matrix was reproduced within experimental errors. Flavor-changing neutral currents (FCNCs) in the Z couplings were naturally suppressed as well.

In this paper, we investigate the problem of flavor mixing in the lepton sector, namely the origin of neutrino oscillations in GHU. In the GUT-inspired GHU, tiny neutrino masses are explained by an inverse seesaw mechanism in which Majorana mass terms on the UV brane play an important role. The mixing in the neutrino mass eigenstates in the W couplings arises from these Majorana mass terms. It is known that the mixing angles in the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix are much larger than those in the CKM matrix. Large mixing angles follow from Majorana mass terms.

In Sec. II, the GUT-inspired $SO(5) \times U(1)_X \times SU(3)_C$ GHU model with general flavor mixing in the lepton sector is described. Equations of motion and boundary conditions satisfied by lepton field multiplets are given and the general form of their wave functions are derived. In Sec. III, the equations are solved when brane interactions are diagonal in the generation space. It is shown that neutrino masses naturally become tiny by an inverse seesaw mechanism. In Sec. IV, W couplings of leptons are examined. It will be

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found that the PMNS matrix [51–53] in the normal ordering with $\delta_{\rm CP}=\pi$ is realized consistently with the NuFit-6.0 analysis of neutrino oscillation data [54]. A summary is given in Sec. V. Basis functions for wave functions in the RS space are given in the Appendix.

II. GUT-INSPIRED GHU

The GUT-inspired $SO(5)\times U(1)_X\times SU(3)_C(\equiv\mathcal{G})$ GHU [22] is defined in the RS warped space with the metric [55]

$$ds^2=G_{MN}dx^Mdx^N=e^{-2\sigma(y)}\eta_{\mu\nu}dx^\mu dx^\nu+dy^2, \quad (2.1)$$

where $M,N=0,\ 1,\ 2,\ 3,\ 5,\ \mu,\ \nu=0,\ 1,\ 2,\ 3,\ y=x^5$, $\eta_{\mu\nu}=\mathrm{diag}(-1,+1,+1,+1)$, $\sigma(y)=\sigma(y+2L)=\sigma(-y)$, and $\sigma(y)=ky$ for $0\le y\le L$. In terms of the conformal coordinate $z=e^{ky}$ ($0\le y\le L$, $1\le z\le z_L=e^{kL}$) the metric is expressed as

$$ds^2=\frac{1}{z^2}\left(\eta_{\mu\nu}dx^\mu dx^\nu+\frac{dz^2}{k^2}\right). \quad (2.2)$$

The bulk region $0<y<L$ is anti-de Sitter spacetime with a cosmological constant $\Lambda=-6k^2$. It is sandwiched by the UV brane at $y=0$ and the IR brane at $y=L$. The warp factor z_L is large. The KK mass scale is given by $m_{\rm KK}=\pi k/(z_L-1)\simeq\pi kz_L^{-1}$. Typical values are $z_L\sim10^{11}$ and $m_{\rm KK}\sim13$ TeV.

Gauge fields $A_M^{SO(5)}$, $A_M^{U(1)_X}$, and $A_M^{SU(3)_C}$ of $SO(5)\times U(1)_X\times SU(3)_C$ with gauge couplings g_A, g_B, and g_S satisfy the orbifold boundary conditions (BCs)

$$\begin{pmatrix}A_\mu\\A_y\end{pmatrix}(x,y_j-y)=P_j\begin{pmatrix}A_\mu\\-A_y\end{pmatrix}(x,y_j+y)P_j^{-1}(j=0,1),$$
$$(y_0,y_1)=(0,L), \quad (2.3)$$

where $P_0=P_1=P_{\mathbf{5}}^{SO(5)}=\mathrm{diag}(I_4,-I_1)$ for $A_M^{SO(5)}$ in the vector representation and $P_0=P_1=P_{\mathbf{4}}^{SO(5)}=\mathrm{diag}(I_2,-I_2)$ for $A_M^{SO(5)}$ in the spinorial representation. $P_0=P_1=1$ for $A_M^{U(1)_X}$ and $A_M^{SU(3)_C}$. The orbifold BCs break $SO(5)$ to $SO(4)\simeq SU(2)_L\times SU(2)_R$. In the following, we write $A_M^{SO(5)}=A_M$ and $A_M^{U(1)_X}=B_M$ when no confusion arises.

The 4D Higgs boson doublet $\phi_H(x)$ is the zero mode contained in $A_z=(kz)^{-1}A_y$:

$$A_z^{(j5)}(x,z)=\frac{1}{\sqrt{k}}\phi_j(x)u_H(z)+\cdots,\quad u_H(z)=\sqrt{\frac{2}{z_L^2-1}}z,$$
$$\phi_H(x)=\frac{1}{\sqrt{2}}\begin{pmatrix}\phi_2+i\phi_1\\\phi_4-i\phi_3\end{pmatrix}. \quad (2.4)$$

Without loss of generality we assume $\langle\phi_1\rangle,\langle\phi_2\rangle,\langle\phi_3\rangle=0$, and $\langle\phi_4\rangle\neq0$, which is related to the AB phase θ_H in the fifth dimension by $\langle\phi_4\rangle=\theta_Hf_H$ where

TABLE I. Lepton multiplets are introduced in three generations ($\alpha=1,\ 2,\ 3$). In the second column, $(SU(3)_C,SO(5))_{U(1)_X}$ content is shown. In the third column, $G_{22}=SU(2)_L\times SU(2)_R$ content is shown. In the column "Lefthanded" ("Righthanded"), the parity (P_0,P_1) of the lefthanded (righthanded) components is shown. Brane Majorana fermion fields $\hat\chi^\alpha$ and brane scalar field $\hat\Phi_{(\mathbf{1},\mathbf{4})}$ defined on the UV brane (at $y=0$) are listed in the bottom rows. Note that $\langle\hat\Phi_{[\mathbf{1},\mathbf{2}]}\rangle\neq0$, while $\langle\hat\Phi_{[\mathbf{2},\mathbf{1}]}\rangle=0$.

Field	$\mathcal{G}$	G_{22}	Lefthanded	Righthanded	Name
$\Psi^\alpha_{(\mathbf{1},\mathbf{4})}$	$(\mathbf{1},\mathbf{4})_{-\frac12}$	$[\mathbf{2},\mathbf{1}]$	$(+,+)$	$(-,-)$	$\nu_e\nu_\mu\nu_\tau$
					$e\mu\tau$
		$[\mathbf{1},\mathbf{2}]$	$(-,-)$	$(+,+)$	$\nu'_e\nu'_\mu\nu'_\tau$
					$e'\mu'\tau'$
$\hat\chi^\alpha$	$(\mathbf{1},\mathbf{1})_0$	$[\mathbf{1},\mathbf{1}]$	$\cdots$	$\cdots$	$\eta_e\eta_\mu\eta_\tau$
$\hat\Phi_{(\mathbf{1},\mathbf{4})}$	$(\mathbf{1},\mathbf{4})_{\frac12}$	$[\mathbf{2},\mathbf{1}]$	$\cdots$	$\cdots$	$\hat\Phi_{[\mathbf{2},\mathbf{1}]}$
		$[\mathbf{1},\mathbf{2}]$	$\cdots$	$\cdots$	$\hat\Phi_{[\mathbf{1},\mathbf{2}]}$

$$f_H=\frac{2}{g_w}\sqrt{\frac{k}{L(z_L^2-1)}},\qquad g_w=\frac{g_A}{\sqrt{L}}. \quad (2.5)$$

The matter content relevant in discussing neutrino oscillations and the PMNS matrix is tabulated in Table I. Fields $\Psi^\alpha_{(\mathbf{1},\mathbf{4})}$ ($\alpha=1,2,3$) are in the spinor representation of $SO(5)$, and contain all leptons as zero modes. $\Psi^\alpha_{(\mathbf{1},\mathbf{4})}$ satisfy the orbifold BCs

$$\Psi^\alpha_{(\mathbf{1},\mathbf{4})}(x,y_j-y)=-P_{\mathbf{4}}^{SO(5)}\gamma^5\Psi^\alpha_{(\mathbf{1},\mathbf{4})}(x,y_j+y). \quad (2.6)$$

In addition, there are $SO(5)$ singlet brane fermion fields $\hat\chi^\alpha(x)$ defined on the UV brane. $\hat\chi^\alpha$ satisfies the Majorana condition $(\hat\chi^\alpha)^c=\hat\chi^\alpha$:

$$\hat\chi^\alpha=\begin{pmatrix}\xi_\alpha\\\eta_\alpha\end{pmatrix},\quad(\hat\chi^\alpha)^c=\begin{pmatrix}+(\eta_\alpha)^c\\-(\xi_\alpha)^c\end{pmatrix}=e^{i\delta_C}\begin{pmatrix}+\sigma^2\eta^*_\alpha\\-\sigma^2\xi^*_\alpha\end{pmatrix}. \quad (2.7)$$

The bulk mass parameters of $\Psi^\alpha_{(\mathbf{1},\mathbf{4})}$ are denoted as $(c_{\ell_1},c_{\ell_2},c_{\ell_3})=(c_e,c_\mu,c_\tau)$. The action for $\Psi^\alpha_{(\mathbf{1},\mathbf{4})}$ in the bulk region is given by

$$I^{\rm bulk}_{\rm lepton}=\int d^5x\sqrt{-G}\sum_{\alpha=1}^{3}\bar\Psi^\alpha_{(\mathbf{1},\mathbf{4})}\mathcal{D}(c_{\ell_\alpha})\Psi^\alpha_{(\mathbf{1},\mathbf{4})},$$
$$\mathcal{D}(c)=\Gamma^a e_a{}^M\left(\partial_M+\frac18\omega_{bcM}[\Gamma^b,\Gamma^c]\right.$$
$$\left.-ig_AA_M+\frac{i}{2}g_BB_M\right)-c\sigma'(y), \quad (2.8)$$

where A_M is in the spinorial representation. In terms of $\breve\Psi_\alpha=z^{-2}\Psi^\alpha_{(\mathbf{1},\mathbf{4})}$ the action takes the form

$$I^{\rm bulk}_{\rm lepton}=\int d^4x\int_1^{z_L}\frac{dz}{k}\sum_\alpha\bar{\check\Psi}_\alpha\begin{pmatrix}-k\hat D_-(c_{\ell_\alpha}) & \sigma^\mu D_\mu\\ \bar\sigma^\mu D_\mu & -k\hat D_+(c_{\ell_\alpha})\end{pmatrix}\check\Psi_\alpha,$$

$$\hat D_\pm(c)=\pm D_z+\frac{c}{z},$$

$$D_M=\partial_M-ig_AA_M+\frac{i}{2}g_BB_M. \tag{2.9}$$

The action for $\hat\chi^\alpha(x)$ is given by

$$I^\chi_{\rm brane}=\int d^5x\sqrt{-G}\delta(y)\sum_{\alpha,\beta=1}^{3}\frac{1}{2}\{\delta_{\alpha\beta}\bar{\hat\chi}^\alpha\gamma^\mu\partial_\mu\hat\chi^\beta+M_{\alpha\beta}\bar{\hat\chi}^\alpha\hat\chi^\beta\}. \tag{2.10}$$

The $M_{\alpha\beta}$ term represents Majorana masses. Hermitian nature of the action, $(I^\chi_{\rm brane})^\dagger=I^\chi_{\rm brane}$, implies that $M_{\alpha\beta}=M^*_{\beta\alpha}$. It follows from the Majorana condition $(\hat\chi^\alpha)^c=\hat\chi^\alpha$ that $\bar{\hat\chi}^\alpha\hat\chi^\beta=ie^{-i\delta_C}\eta^t_\alpha\sigma^2\eta_\beta+ie^{i\delta_C}(\eta^t_\alpha\sigma^2\eta_\beta)^*$. For fermion fields $\eta^t_\alpha\sigma^2\eta_\beta=\eta^t_\beta\sigma^2\eta_\alpha$ so that one can set $M_{\alpha\beta}$ to be symmetric $M_{\alpha\beta}=M_{\beta\alpha}$ and real.

There is a brane scalar field $\hat\Phi_{(\mathbf{1},\mathbf{4})}(x)$ on the UV brane at $y=0$, which spontaneously breaks $SU(2)_R\times U(1)_X$ to $U(1)_Y$ with $\langle\hat\Phi_{(\mathbf{1},\mathbf{4})}\rangle\neq0$. It has brane interactions with quark and lepton multiplets, which lead to splitting masses of down-type quarks from those of up-type quarks, and induce an inverse seesaw mechanism for neutrinos as is seen below. We stress that the model is invariant under five-dimensional $SO(5)\times U(1)_X\times SU(3)_C$ gauge transformations which preserve the orbifold boundary conditions, even with brane fields and brane interactions [22]. One may wonder if the condition can be relaxed such that only $SO(4)\times U(1)_X\times SU(3)_C$ gauge invariance is required on the branes. The relaxed condition would allow us to introduce additional $SO(4)$-multiplet fermions and interactions on the branes, which, however, has a danger to spoil anomaly cancellation.

The brane interactions of $\hat\Phi$ with lepton multiplets are given by

$$I^{\rm int}_{\rm brane}=\int d^5x\sqrt{-G}\delta(y)\left\{\sum_{\alpha,\beta}\kappa^\ell_{\alpha\beta}\bar{\hat\chi}^\beta\check\Phi^\dagger_{(\mathbf{1},\mathbf{4})}\Psi^\alpha_{(\mathbf{1},\mathbf{4})}+\text{H.c.}\right\}. \tag{2.11}$$

Here,

$$\hat\Phi_{(\mathbf{1},\mathbf{4})}=\begin{pmatrix}\Phi_{[\mathbf{2},\mathbf{1}]}\\ \Phi_{[\mathbf{1},\mathbf{2}]}\end{pmatrix},\qquad \check\Phi_{(\mathbf{1},\mathbf{4})}=\begin{pmatrix}i\sigma^2\Phi^*_{[\mathbf{2},\mathbf{1}]}\\ -i\sigma^2\Phi^*_{[\mathbf{1},\mathbf{2}]}\end{pmatrix},$$

$$\langle\hat\Phi_{(\mathbf{1},\mathbf{4})}\rangle=\begin{pmatrix}0\\0\\0\\w\end{pmatrix},\qquad \langle\check\Phi_{(\mathbf{1},\mathbf{4})}\rangle=\begin{pmatrix}0\\0\\-w^*\\0\end{pmatrix}. \tag{2.12}$$

Note that both $\hat\Phi_{(\mathbf{1},\mathbf{4})}$ and $\check\Phi_{(\mathbf{1},\mathbf{4})}$ transform as a spinor **4** in $SO(5)$. With $\langle\tilde\Phi_{(\mathbf{1},\mathbf{4})}\rangle\neq0$ there arise brane mass terms

$$I_{\text{brane mass}}=\int d^5x\sqrt{-G}\delta(y)\left\{\frac{m^B_{\alpha\beta}}{\sqrt{k}}\bar{\hat\chi}^\beta\check\nu^\alpha_R+\frac{m^{B*}_{\alpha\beta}}{\sqrt{k}}\bar{\check\nu}^\alpha_R\hat\chi^\beta\right\}, \tag{2.13}$$

where $m^B_{\alpha\beta}=\kappa^\ell_{\alpha\beta}w\sqrt{k}$.

There are two sources for flavor mixing in the lepton sector: Majorana masses $M_{\alpha\beta}$ in Eq. (2.10) and brane masses $m^B_{\alpha\beta}$ in Eq. (2.13). In this paper, we restrict ourselves to the case of diagonal brane masses $m^B_{\alpha\beta}=\delta_{\alpha\beta}m_{B_\alpha}$ for which explicit solutions for neutrino oscillations become available. In this case, the phase of m_{B_α} can be absorbed by redefining $\Psi^\alpha_{(\mathbf{1},\mathbf{4})}$. We take m_{B_α} to be real in the following.

Manipulations are simplified in the twisted gauge [56,57] defined by

$$\tilde A_M(x,z)=\Omega A_M\Omega^{-1}-\frac{i}{g_A}\Omega\partial_M\Omega^{-1},$$

$$\Omega(z)=\exp\{i\theta(z)T^{45}\},\quad \theta(z)=\theta_H\frac{z_L^2-z^2}{z_L^2-1}, \tag{2.14}$$

where T^{jk}'s are $SO(5)$ generators and $A_M=2^{-1/2}\sum_{1\le j<k\le5}A^{(jk)}_MT^{jk}$. In the twisted gauge, the background field vanishes ($\tilde\theta_H=0$). Boundary conditions at the UV brane are modified, whereas boundary conditions at the IR brane remain the same as in the original gauge. Quantities in the twisted gauge are denoted by the tilde sign $\tilde{}$.

Charged leptons are zero modes contained in $\Psi^\alpha_{(\mathbf{1},\mathbf{4})}$. The mass spectrum $m_{\ell^{(n)}}=k\lambda_{\ell^{(n)}}$ ($\ell=e,\mu,\tau$) is determined by

$$S_L(1;\lambda_{\ell^{(n)}},c_\ell)S_R(1;\lambda_{\ell^{(n)}},c_\ell)+\sin^2\frac{1}{2}\theta_H=0, \tag{2.15}$$

where basis functions $S_{L/R}(z,\lambda,c)$ and $C_{L/R}(z,\lambda,c)$ are given by (A3). We follow the notation summarized in Ref. [38]. The lowest modes $e^{(0)}$, $\mu^{(0)}$, and $\tau^{(0)}$ are e, μ, and τ leptons. With m_e, m_μ, and m_τ given, the corresponding bulk mass parameters c_e, c_μ, and c_τ are fixed, and subsequently the mass spectra of their KK towers are determined. For $\theta_H=0.1$ and $m_{\rm KK}=13$ TeV, for instance, $k=1.59\times10^{15}$ GeV, $z_L=3.83\times10^{11}$, and $(|c_e|,|c_\mu|,|c_\tau|)=(1.0068,0.7930,0.6754)$. There is no mixing in charged leptons.

The KK expansion of 5D $e(x,z)$ and $e'(x,z)$ fields in the twisted gauge is given by

$$\begin{pmatrix}\check{\tilde{e}}\\ \check{\tilde{e}}'\end{pmatrix}=\sqrt{k}\sum_{n=0}^{\infty}\left\{e_L^{(n)}(x)\begin{pmatrix}f_L^{e^{(n)}}(z)\\ g_L^{e^{(n)}}(z)\end{pmatrix}+e_R^{(n)}(x)\begin{pmatrix}f_R^{e^{(n)}}(z)\\ g_R^{e^{(n)}}(z)\end{pmatrix}\right\},$$
$$\begin{pmatrix}f_L^{e^{(n)}}(z)\\ g_L^{e^{(n)}}(z)\end{pmatrix}=\frac{1}{\sqrt{r_{e^{(n)}L}}}\begin{pmatrix}\bar{c}_H C_L(z,\lambda_{e^{(n)}},c_e)\\ -i\bar{s}_H\check{S}_L(z,\lambda_{e^{(n)}},c_e)\end{pmatrix},$$
$$\begin{pmatrix}f_R^{e^{(n)}}(z)\\ g_R^{e^{(n)}}(z)\end{pmatrix}=\frac{1}{\sqrt{r_{e^{(n)}R}}}\begin{pmatrix}\bar{c}_H S_R(z,\lambda_{e^{(n)}},c_e)\\ -i\bar{s}_H\check{C}_R(z,\lambda_{e^{(n)}},c_e)\end{pmatrix},\tag{2.16}$$

where $\check{S}_L$ and $\check{C}_R$ are defined in (A3) and $(\bar{c}_H,\bar{s}_H)=(\cos\frac{1}{2}\theta_H,\sin\frac{1}{2}\theta_H)$. The normalization factor for each mode is determined by the condition

$$\int_1^{z_L}dz\{|f(z)|^2+|g(z)|^2\}=1\quad\text{for}\begin{pmatrix}f(z)\\ g(z)\end{pmatrix}.\tag{2.17}$$

One can show that $r_{e^{(n)}L}=r_{e^{(n)}R}$. Similar formulas hold for μ and τ towers.

For neutrinos ν^α, ν'^α, and $\hat{\chi}^\alpha$ intertwine with each other. Equations of motion in the original gauge are given by

$$\sigma^\mu\partial_\mu\begin{pmatrix}\check{\nu}_L^\alpha\\ \check{\nu}_L'^\alpha\end{pmatrix}-k\left(D_-(c_{\ell_\alpha})+\frac{i}{2}\theta'(z)\tau^1\right)\begin{pmatrix}\check{\nu}_R^\alpha\\ \check{\nu}_R'^\alpha\end{pmatrix}=0,$$
$$\bar{\sigma}^\mu\partial_\mu\begin{pmatrix}\check{\nu}_R^\alpha\\ \check{\nu}_R'^\alpha\end{pmatrix}-k\left(D_+(c_{\ell_\alpha})-\frac{i}{2}\theta'(z)\tau^1\right)\begin{pmatrix}\check{\nu}_L^\alpha\\ \check{\nu}_L'^\alpha\end{pmatrix}$$
$$=\frac{m_{B_\alpha}}{\sqrt{k}}\delta(y)\begin{pmatrix}0\\ \eta_\alpha\end{pmatrix},$$
$$\left\{\sigma^\mu\partial_\mu\eta_\alpha-\frac{m_{B_\alpha}}{\sqrt{k}}\check{\nu}_R'^\alpha+M_{\alpha\beta}\eta_\beta^c\right\}\delta(y)=0.\tag{2.18}$$

We use the notation $(\ell_1,\ell_2,\ell_3)=(e,\mu,\tau)$ throughout the paper. The brane interaction (2.13) gives rise to a $\delta(y)$ term on the righthand side of the equations above. ν_R^α and $\nu_L'^\alpha$ are parity odd at $y=0$. Because of the brane interaction, $\nu_L'^\alpha$ becomes discontinuous at $y=0$ as

$$\check{\nu}_L'^\alpha|_{y=\epsilon}=-\check{\nu}_L'^\alpha|_{y=-\epsilon}=-\frac{m_{B_\alpha}}{2\sqrt{k}}\eta_\alpha,$$
$$\check{\nu}_R^\alpha|_{y=0}=0.\tag{2.19}$$

This is seen by integrating the equations in the y coordinate from $-\epsilon$ to ϵ and taking the limit $\epsilon\to0$ with $kD_\pm(c)=e^{-\sigma(y)}[\pm(\partial/\partial y)+c\sigma'(y)]$ in the y coordinate.

Each eigenmode with a mass $m=k\lambda$ has x-dependence given by

$$\check{\nu}_L^\alpha,\check{\nu}_L'^\alpha,\eta_\alpha\propto f_{\pm L}(x,m),\quad \check{\nu}_R^\alpha,\check{\nu}_R'^\alpha,\eta_\alpha^c\propto f_{\pm R}(x,m),$$
$$\sigma^\mu\partial_\mu f_{\pm L}(x,m)=mf_{\pm R}(x,m),$$
$$\bar{\sigma}^\mu\partial_\mu f_{\pm R}(x,m)=mf_{\pm L}(x,m),$$
$$f_{\pm L}(x,m)^c=e^{i\delta_c}\sigma^2 f_{\pm L}(x,m)^*=\pm f_{\pm R}(x,m).\tag{2.20}$$

There are two types of solutions with $f_{\pm L}(x,m)^c=\pm f_{\pm R}(x,m)$. In the twisted gauge, the equations in the bulk region $1<z<z_L$ satisfied by each eigenmode are

$$D_+(c_{\ell_\alpha})\begin{pmatrix}\tilde{\nu}_L^\alpha\\ \tilde{\nu}_L'^\alpha\end{pmatrix}-\lambda\begin{pmatrix}\tilde{\nu}_R^\alpha\\ \tilde{\nu}_R'^\alpha\end{pmatrix}=0,$$
$$D_-(c_{\ell_\alpha})\begin{pmatrix}\tilde{\nu}_R^\alpha\\ \tilde{\nu}_R'^\alpha\end{pmatrix}-\lambda\begin{pmatrix}\tilde{\nu}_L^\alpha\\ \tilde{\nu}_L'^\alpha\end{pmatrix}=0.\tag{2.21}$$

Boundary conditions satisfied at $z=z_L$ are the same as in the original gauge:

$$D_+(c_{\ell_\alpha})\tilde{\nu}_L^\alpha=0,\qquad \tilde{\nu}_L'^\alpha=0,$$
$$\tilde{\nu}_R^\alpha=0,\qquad D_-(c_{\ell_\alpha})\tilde{\nu}_R'^\alpha=0.\tag{2.22}$$

Boundary conditions satisfied at $z=1^+$ are found from (2.19) and the last equation in (2.18).

$$\bar{c}_H\tilde{\nu}_R^\alpha+i\bar{s}_H\tilde{\nu}_R'^\alpha=0,$$
$$i\bar{s}_H\tilde{\nu}_L^\alpha+\bar{c}_H\tilde{\nu}_L'^\alpha+\frac{m_{B_\alpha}}{2\sqrt{k}}\eta_\alpha=0,$$
$$\frac{m_{B_\alpha}}{\sqrt{k}}(i\bar{s}_H\tilde{\nu}_R^\alpha+\bar{c}_H\tilde{\nu}_R'^\alpha)\pm k\lambda\eta_\alpha^c-M_{\alpha\beta}\eta_\beta^c=0.\tag{2.23}$$

Each eigenmode satisfying Eq. (2.21) and boundary conditions (2.22) is expressed as

$$\begin{pmatrix}\tilde{\nu}_L^\alpha(x,z)\\ \tilde{\nu}_L'^\alpha(x,z)\\ \sqrt{k}\eta_\alpha(x)\end{pmatrix}=\begin{pmatrix}a_\alpha C_L(z,\lambda,c_{\ell_\alpha})\\ ib_\alpha S_L(z,\lambda,c_{\ell_\alpha})\\ ih_\alpha\end{pmatrix}f_{\pm L}(x,k\lambda),$$
$$\begin{pmatrix}\tilde{\nu}_R^\alpha(x,z)\\ \tilde{\nu}_R'^\alpha(x,z)\\ \sqrt{k}\eta_\alpha^c(x)\end{pmatrix}=\begin{pmatrix}a_\alpha S_R(z,\lambda,c_{\ell_\alpha})\\ ib_\alpha C_R(z,\lambda,c_{\ell_\alpha})\\ \mp ih_\alpha^*\end{pmatrix}f_{\pm R}(x,k\lambda).\tag{2.24}$$

The eigenvalue λ and corresponding coefficients $(a_\alpha,b_\alpha,h_\alpha)$ are determined by the boundary conditions (2.23). Overall normalization is determined by

$$\sum_{\alpha=1}^{3}\left\{\int_1^{z_L} dz(|a_\alpha|^2\{C_L^\alpha(z)^2+S_R^\alpha(z)^2\}+|b_\alpha|^2\{S_L^\alpha(z)^2+C_R^\alpha(z)^2\})+|h_\alpha|^2\right\}=1 \tag{2.25}$$

where $C_L^\alpha(z)=C_L(z,\lambda,c_{\ell_\alpha})$ etc.

III. SEESAW MECHANISM

We first note that the coefficient b_α in (2.24) is related to a_α by the first relation in (2.23) as

$$b_\alpha=\frac{\bar{c}_H S_R^\alpha}{\bar{s}_H C_R^\alpha}a_\alpha,$$
$$\begin{pmatrix}S_R^\alpha\\ C_R^\alpha\end{pmatrix}=\begin{pmatrix}S_R\\ C_R\end{pmatrix}(1,\lambda,c_{\ell_\alpha}). \tag{3.1}$$

Consistent solutions are found with $a_\alpha^*=a_\alpha$ and $h_\alpha^*=h_\alpha$. It is most convenient to express equations in terms of dimensionless quantities:

$$\tilde{m}_{B_\alpha}=\frac{m_{B_\alpha}}{k},\qquad \tilde{M}_{\alpha\beta}=\frac{M_{\alpha\beta}}{k}. \tag{3.2}$$

The second and third relations in (2.23) are expressed, after eliminating b_α by (3.1), as

$$K\begin{pmatrix}\vec{a}\\ \bar{s}_H\vec{h}\end{pmatrix}=\begin{pmatrix}L & \frac{1}{2}C_B\\ -S_B & \lambda I\mp\tilde{M}\end{pmatrix}\begin{pmatrix}\vec{a}\\ \bar{s}_H\vec{h}\end{pmatrix}=0,$$
$$\vec{a}=\begin{pmatrix}a_1\\ a_2\\ a_3\end{pmatrix},\qquad \vec{h}=\begin{pmatrix}h_1\\ h_2\\ h_3\end{pmatrix}, \tag{3.3}$$

where

$$L=\begin{pmatrix}L_1 & & \\ & L_2 & \\ & & L_3\end{pmatrix},\qquad L_\alpha=S_L^\alpha S_R^\alpha+\bar{s}_H^2,$$
$$S_B=\begin{pmatrix}S_B^1 & & \\ & S_B^2 & \\ & & S_B^3\end{pmatrix},\qquad S_B^\alpha=\tilde{m}_{B_\alpha}S_R^\alpha,$$
$$C_B=\begin{pmatrix}C_B^1 & & \\ & C_B^2 & \\ & & C_B^3\end{pmatrix},\qquad C_B^\alpha=\tilde{m}_{B_\alpha}C_R^\alpha, \tag{3.4}$$

and $(\tilde{M})_{\alpha\beta}=\tilde{M}_{\alpha\beta}$. Note that $L_\alpha=0$ for $\lambda=\lambda_{\ell_\alpha}$, but $L_\alpha\neq 0$ for neutrino states.

The spectrum of neutrino towers is determined by $\det K=0$. It is straightforward to see $\det K=\det L\cdot\det(\lambda I\mp\tilde{M}+\frac{1}{2}S_BC_BL^{-1})$ so that

$$\det\left(\lambda I\mp\tilde{M}+\frac{1}{2}S_BC_BL^{-1}\right)=0. \tag{3.5}$$

The relation (3.5) gives an inverse seesaw mechanism for light neutrino states. To see it explicitly, let us first consider the case of a diagonal Majorana matrix, $\tilde{M}_{\alpha\beta}=\delta_{\alpha\beta}\tilde{M}_\alpha$. The relation (3.5) becomes

$$\lambda\mp\tilde{M}_\alpha+\frac{\tilde{m}_{B_\alpha}^2S_R^\alpha C_R^\alpha}{2L_\alpha}=0. \tag{3.6}$$

Neutrino masses are small and $\lambda\ll\lambda_{\ell_\alpha}$. For $c_{\ell_\alpha}<-\frac{1}{2}$

$$L_\alpha\sim(\lambda_{\ell_\alpha}^2-\lambda^2)\frac{z_L^{2|c_{\ell_\alpha}|+1}}{4c_{\ell_\alpha}^2-1}\sim\lambda_{\ell_\alpha}^2\frac{z_L^{2|c_{\ell_\alpha}|+1}}{4c_{\ell_\alpha}^2-1},$$
$$S_R^\alpha\sim\lambda\frac{z_L^{1+|c_{\ell_\alpha}|}}{1+2|c_{\ell_\alpha}|},\qquad C_R^\alpha\sim z_L^{|c_{\ell_\alpha}|}, \tag{3.7}$$

so that the relation with the upper sign $-$ in (3.6), namely with $f_{+L}(x,k\lambda)$ dependence in (2.24), gives

$$m=k\lambda\sim\frac{m_{\ell_\alpha}^2M_\alpha}{(|c_{\ell_\alpha}|-\frac{1}{2})m_{B_\alpha}^2}. \tag{3.8}$$

With a moderate Majorana mass, say $M_\alpha\sim 10^6$ GeV (10^{10} GeV) small neutrino masses $m_\nu\sim 1$ meV to 10 meV are obtained with appropriate $m_{B_\alpha}=10^6$ GeV $\sim 10^9$ GeV (10^8 GeV $\sim 10^{11}$ GeV). This result was first obtained in the context of gauge-Higgs grand unification in Ref. [27], see also Ref. [58], and was elaborated in the GUT-inspired GHU in Ref. [22]. We note that for $c_{\ell_\alpha}>\frac{1}{2}$, extra KK modes with MeV scale masses appear, which contradicts with the observation. We take, for $\theta_H=0.1$ and $m_{\rm KK}=13$ TeV, $(c_e,c_\mu,c_\tau)=(-1.0068,-0.7930,-0.6754)$.

Equation (3.6) determines the spectrum of neutrino KK towers. For $c_{\ell_\alpha}<0$, the spectrum of the KK modes with $f_{+L/R}(x,k\lambda)$ dependence in (2.24) is given by $\{\lambda_{\nu_\alpha^{+(n)}},n\geq 0\}$, where $\nu_\alpha^{+(0)}$ corresponds to light neutrino states described above. The spectrum of the KK modes with $f_{-L/R}(x,k\lambda)$ dependence in (2.24) is given by $\{\lambda_{\nu_\alpha^{-(n)}},n\geq 1\}$, which contains no light mode. The $\nu_\alpha^{+(n)}$ and $\nu_\alpha^{-(n)}$ modes ($n\geq 1$) are nearly degenerate in masses. The masses of $\nu_\alpha^{+(1)}$ and $\nu_\alpha^{-(1)}$ modes are $O(m_{\rm KK})$.

The formula (3.8) is in the category of the so-called inverse seesaw mechanism. Unlike in the seesaw mechanism of Minkowski-type (type I seesaw) [59–61], at least

three states intertwine with each other in the inverse seesaw mechanism [62,63]. Indeed, the result (3.8) is mimicked by three states ν_L^α, $\nu_R'^{\alpha c}$, and η_α in each generation having mass terms of the form

$$\frac{i}{2}(\nu_L^{\alpha c\dagger}, \nu_R'^{\alpha\dagger}, \eta_\alpha^{c\dagger}) \begin{pmatrix} 0 & m_{\ell_\alpha} & 0 \\ m_{\ell_\alpha} & 0 & m_{B_\alpha} \\ 0 & m_{B_\alpha} & M_\alpha \end{pmatrix} \begin{pmatrix} \nu_L^\alpha \\ \nu_R'^{\alpha c} \\ \eta_\alpha \end{pmatrix} + \text{H.c.}, \tag{3.9}$$

which leads to a tiny neutrino mass $m_{\ell_\alpha}^2 M_\alpha / m_{B_\alpha}^2$. We stress that the formula (3.8) for light neutrino modes has been derived by solving Eq. (3.6) for the entire KK modes, but not by truncating it for three states.

Seesaw mechanisms in RS warped space have been intensively investigated in the literature [64,65], in which SM singlet neutrino fields (denoted as N_α) with independent bulk mass parameters c_{N_α} are introduced in the bulk in addition to regular neutrino fields ν_α with c_{ℓ_α}. It has been shown in Ref. [65] that warped seesaw mechanism is physically inverted with appropriate choice of c_{ℓ_α}, c_{N_α}, and Majorana masses on the UV brane. In the GUT-inspired GHU in RS warped space, the ν'^α fields correspond to the N_α fields in Ref. [65]. In the first generation, for instance, the ν_e, e, ν_e', and e' fields are in one multiplet $\Psi^e_{(\mathbf{1},\mathbf{4})}$ in the spinor representation of $SO(5)$ so that all ν_e, e, ν_e', and e' fields have the same bulk mass parameter c_e. Further, the 4D Higgs field is a part of the fifth dimensional component of $SO(5)$ gauge fields so that the Higgs couplings of lepton fields are regulated by the gauge principle. Neutrino masses are given by Eq. (3.8) in GHU.

The spectrum with general $M_{\alpha\beta}$ can be found as well. We focus on light neutrino states with $\lambda \ll \lambda_{\ell_\alpha}$. By making use of (3.7), the K matrix in Eq. (3.3) with the upper sign in the right-bottom corner, $\lambda I - \tilde{M} \sim -\tilde{M}$, is written as

$$K = \begin{pmatrix} Z_0 & \\ & I \end{pmatrix} \begin{pmatrix} I & \frac{1}{2}\tilde{m}_B \\ -2\lambda \tilde{m}_B^{-1} F^{-1} & -\tilde{M} \end{pmatrix} \begin{pmatrix} \tilde{L} & \\ & I \end{pmatrix},$$
$$Z_0 = \begin{pmatrix} z_L^{|c_e|} & & \\ & z_L^{|c_\mu|} & \\ & & z_L^{|c_\tau|} \end{pmatrix}, \quad \tilde{L} = Z_0^{-1} L,$$
$$\tilde{m}_B = \begin{pmatrix} \tilde{m}_{B_1} & & \\ & \tilde{m}_{B_2} & \\ & & \tilde{m}_{B_3} \end{pmatrix},$$
$$F = \begin{pmatrix} F_1 & & \\ & F_2 & \\ & & F_3 \end{pmatrix}, \quad F_\alpha = \frac{\lambda_{\ell_\alpha}^2}{(|c_{\ell_\alpha}| - \frac{1}{2})\tilde{m}_{B_\alpha}^2}. \tag{3.10}$$

Hence, Eq. (3.3) becomes

$$\tilde{L}\,\vec{a} + \frac{1}{2}\tilde{m}_B \bar{s}_H \vec{h} = 0,$$
$$2\lambda \tilde{m}_B^{-1} F^{-1} \tilde{L}\,\vec{a} + \tilde{M}\bar{s}_H \vec{h} = 0. \tag{3.11}$$

Combining the two equations above, one finds

$$F\tilde{M}\,\vec{h} = \lambda \vec{h}. \tag{3.12}$$

Three neutrino masses $m_{\nu_j} = k\lambda_{\nu_j}$ ($j = 1, 2, 3$) are determined as eigenvalues of FM. For a diagonal Majorana mass matrix $M_{\alpha\beta} = \delta_{\alpha\beta} M_\alpha$, (3.12) reduces to the formula (3.8). The formula (3.12) gives a gauge-Higgs seesaw mechanism for general Majorana masses.

IV. NEUTRINO OSCILLATIONS

The neutrino spectrum $\{\lambda_{\nu_j}\}$ and corresponding eigenvectors are determined by

$$F\tilde{M}\vec{w}_j = \lambda_{\nu_j}\vec{w}_j. \tag{4.1}$$

For each neutrino mass eigenstate, the coefficient vector $\vec{h}_j$ is proportional to $\vec{w}_j$, and the coefficient vector $\vec{a}_j$ is in turn determined by (3.11). Once $\{\vec{a}_j\}$ is fixed, W couplings of leptons are evaluated, the PMNS matrix U_{PMNS} thereby being determined. It follows from (4.1) and $\tilde{M}^\dagger = \tilde{M}$ that

$$(\lambda_{\nu_j}^* - \lambda_{\nu_k})\vec{w}_j{}^\dagger F^{-1}\vec{w}_k = 0. \tag{4.2}$$

It implies that λ_{ν_j} is real and

$$\vec{w}_j{}^\dagger F^{-1}\vec{w}_k = \delta_{jk} n_j, \qquad n_j = \vec{w}_j{}^\dagger F^{-1}\vec{w}_j. \tag{4.3}$$

It follows that $\tilde{M}$ can be expressed as

$$\tilde{M} = F^{-1}\left\{\sum_{j=1}^3 \frac{\lambda_{\nu_j}}{n_j}\vec{w}_j \otimes \vec{w}_j{}^\dagger\right\}F^{-1}. \tag{4.4}$$

As $\tilde{M}_{\alpha\beta} = \tilde{M}_{\beta\alpha}$, $\vec{w}_j$ can be chosen to be real up to an overall phase.

For the ν_j mode, $\vec{h}_j \propto \vec{w}_j$ and $\vec{a}_j = -\frac{1}{2}\bar{s}_H \tilde{L}^{-1}\tilde{m}_B \vec{h}_j$. For charged leptons $\lambda_{\ell_\alpha} \sim (4c_{\ell_\alpha}^2 - 1)^{1/2} z_L^{-|c_{\ell_\alpha}|-0.5}\bar{s}_H$ so that

$$a_{j\alpha} \propto q_\alpha w_{j\alpha}, \qquad q_\alpha = \frac{\tilde{m}_{B_\alpha}}{\lambda_{\ell_\alpha}} \frac{(4c_{\ell_\alpha}^2 - 1)^{1/2}}{z_L}. \tag{4.5}$$

The wave functions of the ν_j mode in Eq. (2.24) are saturated by the $\tilde{\nu}_L^\alpha$ components, that is, contributions of other components to the normalization condition Eq. (2.25) are numerically negligible ($\ll 1$). The KK expansion of the neutrino fields can be expressed as

$$\begin{pmatrix}\tilde{\tilde{\nu}}_L'^e\\ \tilde{\tilde{\nu}}_L'^\mu\\ \tilde{\tilde{\nu}}_L'^\tau\end{pmatrix}=\sqrt{k}\sum_{j=1}^{3}\nu_{jL}(x)\begin{pmatrix}p_{je}f_{eL}^{\nu_j}(z)\\ p_{j\mu}f_{\mu L}^{\nu_j}(z)\\ p_{j\tau}f_{\tau L}^{\nu_j}(z)\end{pmatrix}+\cdots,$$

$$f_{\alpha L}^{\nu_j}(z)=\frac{1}{\sqrt{\hat{r}_{j\alpha}}}C_L(z,\lambda_{\nu_j},c_{\ell_\alpha}),\quad \hat{r}_{j\alpha}=\int_1^{z_L}dzC_L(z,\lambda_{\nu_j},c_{\ell_\alpha})^2,$$

$$p_{j\alpha}=q_\alpha w_{j\alpha}\sqrt{\hat{r}_{j\alpha}/r_j},\quad r_j=\sum_{\alpha=1}^{3}q_\alpha^2|w_{j\alpha}|^2\hat{r}_{j\alpha}. \tag{4.6}$$

Once $\vec{w}_j$ is given, $\vec{p}_j$ is determined. Conversely, with $\vec{p}_j$ specified, $\vec{w}_j$ is determined up to a normalization factor. As neutrino masses are tiny ($m_{\nu_j}<100$ meV), one finds that $\hat{r}_{j\alpha}\sim\hat{r}_\alpha$. For $\theta_H=0.1$ and $m_{\rm KK}=13$ TeV, for instance, $(\hat{r}_1,\hat{r}_2,\hat{r}_3)=(\hat{r}_e,\hat{r}_\mu,\hat{r}_\tau)=(0.3318,0.3867,0.4254)$.

Gauge couplings of leptons are contained in the covariant derivative D_M in Eq. (2.8). Let $\{T^\alpha;\alpha=1\sim10\}=\{T^{a_L},T^{a_R},T^{\hat{b}};a=1\sim3,b=1\sim4\}$ be generators of $SO(5)$. The W boson field, $W_\mu(x)$, is contained in $\sum_{a=1}^{2}(A_\mu^{a_L}T^{a_L}+A_\mu^{a_R}T^{a_R}+A_\mu^{\hat{a}}T^{\hat{a}})$. Its KK expansion in the twisted gauge is given by

$$\frac{1}{\sqrt{2k}}\begin{pmatrix}\tilde{A}_\mu^{1_L}+i\tilde{A}_\mu^{2_L}\\ \tilde{A}_\mu^{1_R}+i\tilde{A}_\mu^{2_R}\\ \tilde{A}_\mu^{\hat{1}}+i\tilde{A}_\mu^{\hat{2}}\end{pmatrix}=\sum_{n=0}^{\infty}W_\mu^{(n)}(x)\begin{pmatrix}h_{W^{(n)}}^L(z)\\ h_{W^{(n)}}^R(z)\\ \hat{h}_{W^{(n)}}(z)\end{pmatrix}+\cdots,$$

$$\begin{pmatrix}h_{W^{(n)}}^L(z)\\ h_{W^{(n)}}^R(z)\\ \hat{h}_{W^{(n)}}(z)\end{pmatrix}=\frac{1}{\sqrt{2r_{W^{(n)}}}}\begin{pmatrix}(1+c_H)C(z;\lambda_{W^{(n)}})\\ (1-c_H)C(z;\lambda_{W^{(n)}})\\ \sqrt{2}s_H\check{S}(z;\lambda_{W^{(n)}})\end{pmatrix},$$

$$c_H=\cos\theta_H,\qquad s_H=\sin\theta_H, \tag{4.7}$$

where the spectrum $m_{W^{(n)}}=k\lambda_{W^{(n)}}$ is determined by $2SC'(1;\lambda_{W^{(n)}})+s_H^2\lambda_{W^{(n)}}=0$. $C(z;\lambda)$, $S(z;\lambda)$, and $\check{S}(z;\lambda)$ are defined in Eq. (A1). Wave functions are normalized by

$$\int_1^{z_L}\frac{dz}{z}\{(|h_{W^{(n)}}^L|^2+|h_{W^{(n)}}^R|^2+|\hat{h}_{W^{(n)}}|^2\}=1. \tag{4.8}$$

The W boson field is $W_\mu(x)=W_\mu^{(0)}(x)$.

The W interaction in the twisted gauge is evaluated from

$$\mathcal{L}_{\rm int}^W=-ig_A\sum_\alpha\int_1^{z_L}\frac{dz}{\sqrt{k}}\bar{\tilde{\Psi}}_{(\mathbf{1},\mathbf{4})}^{\alpha}\gamma^\mu\sum_{a=1}^{2}(\tilde{A}_\mu^{a_L}T^{a_L}+\tilde{A}_\mu^{a_R}T^{a_R}+\tilde{A}_\mu^{\hat{a}}T^{\hat{a}})\tilde{\tilde{\Psi}}_{(\mathbf{1},\mathbf{4})}^{\alpha} \tag{4.9}$$

by inserting the wave functions of leptons and W boson presented above. For leptons, one has

$$\mathcal{L}_{\rm int}^W=-i\frac{g_w}{\sqrt{2}}W_\mu^\dagger(\bar{e}_L,\bar{\mu}_L,\bar{\tau}_L)\gamma^\mu\hat{g}_L^W\begin{pmatrix}\nu_{1L}\\ \nu_{2L}\\ \nu_{3L}\end{pmatrix}+\text{H.c.},$$

$$\hat{g}_L^W=\begin{pmatrix}\hat{g}_L^{We1}&\hat{g}_L^{We2}&\hat{g}_L^{We3}\\ \hat{g}_L^{W\mu1}&\hat{g}_L^{W\mu2}&\hat{g}_L^{W\mu3}\\ \hat{g}_L^{W\tau1}&\hat{g}_L^{W\tau2}&\hat{g}_L^{W\tau3}\end{pmatrix}, \tag{4.10}$$

where

$$(\hat{g}_L^W)_{\alpha j}=G_W[(h_{W^{(0)}}^L,h_{W^{(0)}}^R,\hat{h}_{W^{(0)}});(f_L^{\ell_\alpha},g_L^{\ell_\alpha}),(p_{j\alpha}f_{\alpha L}^{\nu_j},p'_{j\alpha}g_{\alpha L}^{\nu_j})],$$

$$G_W[(h^L,h^R,\hat{h});(f_1,g_1),(f_2,g_2)]=\sqrt{kL}\int_1^{z_L}dz\left\{h^{L*}f_1^*f_2+h^{R*}g_1^*g_2+\frac{i}{\sqrt{2}}\hat{h}^*(f_1^*g_2-g_1^*f_2)\right\}. \tag{4.11}$$

Here, $f_L^{\ell_1}(z)=f_L^{e^{(0)}}(z)$ etc. and $p'_{j\alpha}g_{\alpha L}^{\nu_j}(z)$ is the $\tilde{\tilde{\nu}}_L'^\alpha$ component of the ν_j wave function. Righthanded couplings are $O(10^{-23})$, and have been omitted above. Both $g_L^{\ell_\alpha}(z)$ and $p'_{j\alpha}g_{\alpha L}^{\nu_j}(z)$ are negligibly small so that to high accuracy,

$$(\hat{g}_L^W)_{\alpha j}\simeq p_{j\alpha}\sqrt{kL}\int_1^{z_L}dzh_{W^{(0)}}^{L*}f_L^{\ell_\alpha*}f_{\alpha L}^{\nu_j}\equiv p_{j\alpha}\hat{g}_{\alpha j}^0. \tag{4.12}$$

Furthermore, it turns out that $\hat{g}_{\alpha j}^0\simeq\hat{g}_\alpha^0$ thanks to the smallness of neutrino masses. For $\theta_H=0.1$ and $m_{\rm KK}=13$ TeV, for instance, $(\hat{g}_e^0,\hat{g}_\mu^0,\hat{g}_\tau^0)=(0.997647,0.997644,0.997642)$. The PMNS matrix [51,52] is given by

$$U_{\rm PMNS}=\frac{1}{\hat{g}_e^0}\hat{g}_L^W\simeq\begin{pmatrix}p_{1e}&p_{2e}&p_{3e}\\ p_{1\mu}&p_{2\mu}&p_{3\mu}\\ p_{1\tau}&p_{2\tau}&p_{3\tau}\end{pmatrix}. \tag{4.13}$$

Neutrino oscillations have been observed. The PMNS matrix is parameterized as

$$U_{\rm PMNS} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta_{\rm CP}} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta_{\rm CP}} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \mathcal{P}, \tag{4.14}$$

where $c_{ij} = \cos\theta_{ij}$ and $s_{ij} = \sin\theta_{ij}$ with angles $\theta_{ij} \in [0, \pi/2]$, and phase $\delta_{\rm CP} \in [0, 2\pi]$. $\mathcal{P} = {\rm diag}(e^{i\alpha_1}, e^{i\alpha_2}, 1)$ contains two additional Majorana phases, playing no role in neutrino oscillations. Phases of $\vec{w}_j$ and $\vec{p}_j$ are absorbed in $\mathcal{P}$ and an overall phase of $U_{\rm PMNS}$. In the normal ordering (NO), $m_{\nu_1} < m_{\nu_2} < m_{\nu_3}$, whereas in the inverted ordering (IO) $m_{\nu_3} < m_{\nu_1} < m_{\nu_2}$. In the current scheme, only CP conserving phase $\delta_{\rm CP} = 0$ or π can be realized.

In the NuFit-6.0 global analysis [54], both normal and inverted ordering cases have been analyzed. In the normal ordering case, $\delta_{\rm CP} \sim \pi$ is preferred, whereas in the inverted ordering case, $\delta_{\rm CP} \sim 3\pi/2$ is preferred. We are going to show that the current GHU scenario naturally produces $U_{\rm PMNS}$ in the NO with $\delta_{\rm CP} = \pi$.

According to the NuFit-6.0 global analysis, in the NO

$$m_{\nu_1} = m_0, \quad m_{\nu_2} = \sqrt{m_0^2 + \Delta m_{21}^2}, \quad m_{\nu_3} = \sqrt{m_0^2 + \Delta m_{31}^2},$$
$$\Delta m_{21}^2 \sim 7.49 \times 10^{-5}\,{\rm eV}^2, \quad \Delta m_{31}^2 \sim 2.53 \times 10^{-3}\,{\rm eV}^2, \tag{4.15}$$

and

$$\sin^2\theta_{12} \sim 0.307, \quad \sin^2\theta_{23} \sim 0.561, \quad \sin^2\theta_{13} \sim 0.022, \quad \delta_{\rm CP} \sim \pi,$$
$$U_{\rm PMNS}^{\rm exp} \sim \begin{pmatrix} 0.823 & 0.548 & -0.148 \\ -0.275 & 0.613 & 0.741 \\ 0.497 & -0.569 & 0.655 \end{pmatrix} \equiv \begin{pmatrix} \vec{p}_1^{\,\rm exp} & \vec{p}_2^{\,\rm exp} & \vec{p}_3^{\,\rm exp} \end{pmatrix}. \tag{4.16}$$

As a typical value, we take $m_0 = 1$ meV. Then

$$m_{\nu_1} = 1\,{\rm meV}, \quad m_{\nu_2} = 8.71\,{\rm meV}, \quad m_{\nu_3} = 50.3\,{\rm meV}. \tag{4.17}$$

Now we present Majorana mass matrices which reproduce (4.16) and (4.17). We consider the case with $\theta_H = 0.1$ and $m_{\rm KK} = 13$ TeV, for which $k = 1.59 \times 10^{15}$ GeV, $z_L = 3.83 \times 10^{11}$, and $(c_e, c_\mu, c_\tau) = (-1.0068, -0.7930, -0.6754)$. As seen from (3.11) and (4.4), there remains freedom in choosing the values of F_α in (3.10), or equivalently brane masses m_{B_α} in Eq. (2.13).

Case I: $F_e = F_\mu = F_\tau$

In this case, $F = F_e I$ and the formula (4.4) reduces to

$$\tilde{M} = F_e^{-1} \sum_{j=1}^{3} \frac{\lambda_{\nu_j}}{\vec{w}_j^{\,\dagger} \vec{w}_j} \vec{w}_j \otimes \vec{w}_j{}^{\dagger}. \tag{4.18}$$

Further, it is found that $\vec{w}_j \propto \vec{p}_j$ to high accuracy. It is seen that the set of

$$F_e = F_\mu = F_\tau = 1. \times 10^{-18},$$
$$(m_{B_e}, m_{B_\mu}, m_{B_\tau}) = (6.83 \times 10^5, 1.90 \times 10^8, 4.17 \times 10^9)\,{\rm GeV},$$
$$\vec{w}_j = \vec{p}_j^{\,\rm exp},$$
$$M = k\tilde{M} = \begin{pmatrix} 4.40 & -2.83 & -7.20 \\ -2.83 & 31.0 & 21.2 \\ -7.20 & 21.2 & 24.7 \end{pmatrix} \times 10^6\,{\rm GeV}, \tag{4.19}$$

yields (4.16) and (4.17).

Case II: $F_e \neq F_\mu \neq F_\tau$

In this case, one needs to take more general $\vec{w}_j$. As $\vec{w}_j$ and $\vec{p}_j$ are related to each other by Eq. (4.6), we take

$$w_{je} = p_{je}^{\rm exp},$$
$$w_{j\mu} = \left(\frac{|c_e| + \frac{1}{2}\frac{F_\mu}{F_e}\frac{\hat{r}_{je}}{\hat{r}_{j\mu}}}{|c_\mu| + \frac{1}{2}} \right)^{1/2} p_{j\mu}^{\rm exp} \simeq \sqrt{\frac{F_\mu}{F_e}} p_{j\mu}^{\rm exp},$$
$$w_{j\tau} = \left(\frac{|c_e| + \frac{1}{2}\frac{F_\tau}{F_e}\frac{\hat{r}_{je}}{\hat{r}_{j\tau}}}{|c_\tau| + \frac{1}{2}} \right)^{1/2} p_{j\tau}^{\rm exp} \simeq \sqrt{\frac{F_\tau}{F_e}} p_{j\tau}^{\rm exp}. \tag{4.20}$$

Here we have made use of a fact that $\{(|c_e| + \frac{1}{2})\hat{r}_{je}\}/\{(|c_\mu| + \frac{1}{2})\hat{r}_{j\mu}\} = 1 + O(10^{-9})$, etc. For instance, one finds that the set of

$$(F_e, F_\mu, F_\tau) = (1, 3, 10) \times 10^{-18},$$
$$(m_{B_e}, m_{B_\mu}, m_{B_\tau}) = (6.83 \times 10^5, 1.10 \times 10^8, 1.32 \times 10^9)\ \text{GeV},$$
$$\vec{w}_1 = \begin{pmatrix} 0.823 \\ -0.476 \\ 1.57 \end{pmatrix}, \quad \vec{w}_2 = \begin{pmatrix} 0.548 \\ 1.06 \\ -1.80 \end{pmatrix}, \quad \vec{w}_3 = \begin{pmatrix} -0.148 \\ 1.28 \\ 2.07 \end{pmatrix},$$
$$M = k\tilde{M} = \begin{pmatrix} 4.40 & -1.63 & -2.28 \\ -1.63 & 10.3 & 3.89 \\ -2.28 & 3.89 & 2.47 \end{pmatrix} \times 10^6\ \text{GeV}, \tag{4.21}$$

reproduces (4.16) and (4.17). Similarly, the set of

$$(F_e, F_\mu, F_\tau) = (1, 10, 100) \times 10^{-22},$$
$$(m_{B_e}, m_{B_\mu}, m_{B_\tau}) = (6.83 \times 10^7, 6.00 \times 10^9, 4.17 \times 10^{10})\ \text{GeV},$$
$$\vec{w}_1 = \begin{pmatrix} 0.823 \\ -0.868 \\ 4.97 \end{pmatrix}, \quad \vec{w}_2 = \begin{pmatrix} 0.548 \\ 1.94 \\ -5.69 \end{pmatrix}, \quad \vec{w}_3 = \begin{pmatrix} -0.148 \\ 2.34 \\ 6.55 \end{pmatrix},$$
$$M = k\tilde{M} = \begin{pmatrix} 44.0 & -8.94 & -7.20 \\ -8.94 & 3.10 & 6.72 \\ -7.20 & 6.72 & 2.47 \end{pmatrix} \times 10^9\ \text{GeV}, \tag{4.22}$$

reproduces (4.16) and (4.17).

It is appropriate to comment on the scales of m_{B_α} and M. The brane mass m_{B_α} arises from $\langle \Phi_{(\mathbf{1},\mathbf{4})} \rangle \neq 0$ ($w \neq 0$) in the brane interaction (2.11). With $w \neq 0$ $SO(4) \times U(1)_X \simeq SU(2)_L \times SU(2)_R \times U(1)_X$, gauge symmetry is spontaneously broken to $SU(2)_L \times U(1)_Y$. This breaking scale w is supposed to be much larger than $m_{\rm KK}$. The constraint for m_{B_α} is $m_{\rm KK} \ll m_{B_\alpha} < k$. Further, one can check that the magnitude of Majorana masses in (2.10) must be larger than $m_{\rm KK}$ to have consistent W couplings of leptons [22,38]. The examples presented above, (4.19), (4.21), and (4.22), satisfy these conditions. There remains freedom in GHU in the choice of the scales of m_{B_α} and M. If one assumes that $O(m_{B_\alpha}) = O(M)$, one obtains the same scale of Majorana masses as in the ordinary Minkowski-type (type I) seesaw mechanism as seen from (3.8).

V. SUMMARY

In the GUT-inspired gauge-Higgs unification, the mixing of neutrino mass eigenstates in the W couplings arises from the Majorana mass matrix M. Tiny neutrino masses are explained by the inverse seesaw mechanism. Three neutrino masses $\{m_{\nu_1}, m_{\nu_2}, m_{\nu_3}\}$ are determined by

$$FM\vec{w}_j = m_{\nu_j}\vec{w}_j,$$
$$F_{\alpha\beta} = \delta_{\alpha\beta}F_\alpha, \qquad F_\alpha = \frac{m_{\ell_\alpha}^2}{(|c_{\ell_\alpha}| - \frac{1}{2})m_{B_\alpha}^2}. \tag{5.1}$$

Neutrino masses $\{m_{\nu_j}\}$ are eigenvalues of the matrix FM. With the use of eigenvectors $\vec{w}_j$, the Majorana mass matrix M can be expressed as (4.4), or as

$$M = F^{-1}\left\{\sum_{j=1}^{3} \frac{m_{\nu_j}}{n_j}\vec{w}_j \otimes \vec{w}_j{}^\dagger\right\}F^{-1}. \tag{5.2}$$

The relation (5.1) represents a gauge-Higgs seesaw mechanism for general M.

The PMNS matrix $U_{\rm PMNS}$ is found to be

$$U_{\rm PMNS} = \begin{pmatrix} \vec{p}_1 & \vec{p}_2 & \vec{p}_3 \end{pmatrix},$$
$$\vec{p}_j = \frac{1}{\sqrt{n_j}}F^{-1/2}\vec{w}_j, \qquad \vec{p}_j{}^\dagger\vec{p}_k = \delta_{jk}. \tag{5.3}$$

In terms of $\vec{p}_j$, (5.1) and (5.2) can be written as

$$F^{1/2}MF^{1/2}\vec{p}_j = m_{\nu_j}\vec{p}_j,$$
$$M = F^{-1/2}\left\{\sum_{j=1}^{3} m_{\nu_j}\vec{p}_j \otimes \vec{p}_j{}^\dagger\right\}F^{-1/2}. \tag{5.4}$$

The GHU scenario predicts no CP violation phase in $U_{\rm PMNS}$. Neutrino oscillations with $U_{\rm PMNS}$ in the normal ordering with $\delta_{\rm CP} = \pi$ are reproduced in accordance with the NuFit-6.0 analysis.

To have a nontrivial CP violation phase in $U_{\rm PMNS}$, one needs to consider more general brane interactions and brane masses $m^B_{\alpha\beta}$ in (2.13). In the present paper, we have considered diagonal brane masses $m^B_{\alpha\beta} = \delta_{\alpha\beta} m_{B_\alpha}$. When $m^B_{\alpha\beta}$ is not diagonal, a general CP violation phase $\delta_{\rm CP}$ can arise but it becomes very difficult to solve the equations of motion explicitly. In this regards, we would like to recall that the CKM matrix in the quark sector is naturally reproduced in the GUT-inspired GHU with brane interactions which are diagonal in the generation space [50]. From the viewpoint of grand unification, it is quite natural to imagine that brane interactions in the lepton sector are diagonal as well. It would be interesting to produce KK excited modes of neutrinos at the KK mass scale $m_{\rm KK}$. Effects of lepton number violation could be seen at much lower energy scale than commonly expected, as the scale of Majorana masses can be relatively small. Further investigation is necessary.

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX: BASIS FUNCTIONS

Basis functions used for wave functions of gauge and fermion fields are given in this appendix. For gauge fields, we introduce

$$\begin{aligned} F_{\alpha,\beta}(u,v) &\equiv J_\alpha(u)Y_\beta(v) - Y_\alpha(u)J_\beta(v), \\ C(z;\lambda) &= \frac{\pi}{2}\lambda z z_L F_{1,0}(\lambda z, \lambda z_L), \\ S(z;\lambda) &= -\frac{\pi}{2}\lambda z F_{1,1}(\lambda z, \lambda z_L), \\ C'(z;\lambda) &= \frac{\pi}{2}\lambda^2 z z_L F_{0,0}(\lambda z, \lambda z_L), \\ S'(z;\lambda) &= -\frac{\pi}{2}\lambda^2 z F_{0,1}(\lambda z, \lambda z_L), \\ \check{S}(z;\lambda) &= \frac{C(1;\lambda)}{S(1;\lambda)} S(z;\lambda), \end{aligned} \tag{A1}$$

where $J_\alpha(u)$ and $Y_\alpha(u)$ are Bessel functions of the first and second kind. $C(z;\lambda)$ and $S(z;\lambda)$ satisfy

$$-z\frac{d}{dz}\frac{1}{z}\frac{d}{dz}\begin{pmatrix} C \\ S \end{pmatrix} = \lambda^2 \begin{pmatrix} C \\ S \end{pmatrix}, \tag{A2}$$

with the boundary conditions $C = z_L$, $C' = S = 0$, and $S' = \lambda$ at $z = z_L$, and a relation $CS' - SC' = \lambda z$ holds.

For fermion fields with a bulk mass parameter c, we define

$$\begin{aligned} \begin{pmatrix} C_L \\ S_L \end{pmatrix}(z;\lambda,c) &= \pm\frac{\pi}{2}\lambda\sqrt{z z_L} F_{c+\frac{1}{2},c\mp\frac{1}{2}}(\lambda z, \lambda z_L), \\ \begin{pmatrix} C_R \\ S_R \end{pmatrix}(z;\lambda,c) &= \mp\frac{\pi}{2}\lambda\sqrt{z z_L} F_{c-\frac{1}{2},c\pm\frac{1}{2}}(\lambda z, \lambda z_L), \\ \begin{pmatrix} \check{S}_L \\ \check{C}_R \end{pmatrix}(z;\lambda,c) &= \frac{C_L(1;\lambda,c)}{S_L(1;\lambda,c)} \begin{pmatrix} S_L \\ C_R \end{pmatrix}(z;\lambda,c). \end{aligned} \tag{A3}$$

These functions satisfy

$$\begin{aligned} D_+(c)\begin{pmatrix} C_L \\ S_L \end{pmatrix} &= \lambda \begin{pmatrix} S_R \\ C_R \end{pmatrix}, \\ D_-(c)\begin{pmatrix} S_R \\ C_R \end{pmatrix} &= \lambda \begin{pmatrix} C_L \\ S_L \end{pmatrix}, \\ D_\pm(c) &= \pm\frac{d}{dz} + \frac{c}{z}, \end{aligned} \tag{A4}$$

with the boundary conditions $C_{R/L} = 1$, $D_-(c)C_R = D_+(c)C_L = S_{R/L} = 0$ at $z = z_L$, and $C_L C_R - S_L S_R = 1$.

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