

Cosmological singularity, conformal anomaly and symmetric polynomials

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We consider a spacetime singularity at $t = 0$ arising in a Kasner-type metric that solves the gravitational equations modified by quantum effects of a conformal field theory (CFT). The resulting constraints can be solved efficiently when expressed in terms of symmetric polynomials. Focusing first on the trace part of the modified gravitational equation, we determine the corresponding solution surfaces in the Kasner parameter space. The geometry of these surfaces depends sensitively on the ratio $\eta = A/C$, the quotient of the conformal charges characterizing the underlying CFT. We then fully integrate the conformal anomaly near the singularity for a generic Kasner-type metric and obtain the corresponding stress-energy tensor. Its components are expressed in terms of three symmetric polynomials (of degrees 2, 3, and 4) and depend on seven arbitrary constants, which may be interpreted as parametrizing different choices of the quantum state at the singularity. By imposing a set of constraints, we reduce this parameter space to a single free constant. Subsequently, we solve, at leading order near the singularity, the modified gravitational equations. Among the admissible solutions, we identify, in particular, those that develop a curvature singularity while remaining geodesically complete.

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I. INTRODUCTION

The nature of spacetime singularities remains one of the most important open problems in gravitational physics. In the cosmological context, it is closely linked to the fundamental question of the origin of the observable Universe. Within classical general relativity, substantial progress has been made in identifying the general conditions under which singularities inevitably form. The classical theorems of Penrose and Hawking [1] established that, under broad and physically well-motivated assumptions, the evolution of spacetime inevitably leads to regions in which curvature becomes unbounded and geodesics cannot be extended. In cosmology, such regions typically appear in the past of the Universe's time evolution, making it essential to understand the asymptotic behavior of spacetime as one approaches the singularity.

A major step toward such an understanding was achieved in the seminal works of Belinsky, Khalatnikov, and Lifshitz (BKL) [2], who proposed a detailed asymptotic description of the metric near a spacetime singularity. According to the BKL scenario, time derivatives dominate over spatial ones as the singularity is approached, reducing the Einstein equations to a system of ordinary differential equations. In

this regime, the metric evolves through a sequence of anisotropic contraction phases, the Kasner epochs, each characterized by a set of Kasner exponents and separated by abrupt transitions or bounces. The resulting oscillatory and, in fact, chaotic sequence of epochs is one of the most striking manifestations of nonlinear behavior in general relativity. Further progress in the study of this chaotic dynamics was made in [3].

However, it has long been suspected that the classical behavior near singularities may be radically altered once quantum effects are taken into account. Insight into possible modifications can be gained by studying semiclassical gravity coupled to quantum conformal field theories (CFTs), where the conformal anomaly provides a useful tool for analyzing the backreaction of quantum matter on spacetime. Early work in this direction includes [4–11].

In this paper, we systematically analyze the backreacted semiclassical geometry within the class of Kasner-type metrics. We identify the constraints on the Kasner parameters arising from the conformal anomaly. We then integrate the anomaly explicitly in this class of backgrounds, which allows us to extract the full set of tensorial constraints. These constraints can be solved in closed form in terms of symmetric polynomials of the Kasner parameters.

Our initial motivation was to search for situations in which quantum CFT effects could regularize, or even forbid, the formation of singularities. These expectations turned out to be overly optimistic (except in the isotropic case, already understood in the context of the Starobinsky model [4]). Nevertheless, within the space of admissible

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Kasner parameters, we do find regions in which curvature singularities still form but the spacetime remains geodesically complete. Such spacetimes appear to be free of any known pathologies. The existence of these solutions in semiclassical gravity indicates that quantum effects may significantly modify the classical picture of singularity formation.

II. A CLASS OF KASNER-TYPE METRICS

A. Metric and curvature

In the present paper, we study a general class of cosmological anisotropic metrics that, when extended backward in time, exhibit a cosmological singularity at $t = 0$. Near the singularity, and to leading order, the metrics we consider take the Kasner form,

$$ds^2 = -dt^2 + t^{2a}dx_1^2 + t^{2b}dx_2^2 + t^{2c}dx_3^2, \quad (2.1)$$

where the constant parameters (a, b, c) characterize a particular spacetime within the family. We emphasize that (2.1) captures only the leading behavior of the metric in the limit $t \rightarrow 0$; at later times, subleading corrections, omitted here, will modify this form.

For the metric (2.1), the components of the Einstein tensor $G_\nu^\mu = R_\nu^\mu - \frac{1}{2}\delta_\nu^\mu R$ are

$$\begin{aligned} G_0^0 &= \frac{1}{t^2}(ab + ac + bc), \\ G_1^1 &= -\frac{1}{t^2}(b^2 + c^2 + bc - b - c), \\ G_2^2 &= -\frac{1}{t^2}(a^2 + c^2 + ac - a - c), \\ G_3^3 &= -\frac{1}{t^2}(a^2 + b^2 + ab - a - b), \end{aligned} \quad (2.2)$$

and the Ricci scalar is

$$R = \frac{1}{t^2}(a^2 + b^2 + c^2 + ab + bc + ac - a - b - c). \quad (2.3)$$

These expressions give the leading contributions near the singularity.

B. The regular spacetimes

For generic values of (a, b, c) , the curvature is singular at $t = 0$. To determine which values of the parameters yield a regular spacetime, we examine the quadratic curvature invariant constructed from the components of the Riemann tensor,

$$R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} = \frac{4}{t^4}[a^2(a-1)^2 + b^2(b-1)^2 + c^2(c-1)^2 + a^2b^2 + c^2a^2 + b^2c^2]. \quad (2.4)$$

Since this expression is a sum of non-negative terms, regularity requires that the divergent contribution in (2.4) vanish, which is possible only if each term in the sum vanishes individually. This leads to two types of solutions: either all parameters vanish, $a = b = c = 0$, or one parameter equals 1 while the other two vanish, e.g., $(1, 0, 0)$. In both cases, the spacetime described by (2.1) near $t = 0$ is free of singularities. Although these two forms of the metric appear different, they are in fact locally isometric and represent the same isotropic spacetime in different coordinate systems.

C. Regularity conditions for singular metrics

For all other values of the parameters (a, b, c) , the spacetime described by (2.1) contains a singularity at $(t = 0)$. However, the singularity may be *milder*, or “less singular,” provided that certain additional regularity conditions are satisfied. In what follows, we examine several such conditions.

1. Regular volume density and regular volume

For the metric (2.1), the four-volume density is $\sqrt{-g} = t^{a+b+c}$, which has a finite limit as $t \rightarrow 0$ provided that

$$a + b + c \geq 0. \quad (2.5)$$

The classical Kasner spacetime is an example of such a geometry, with a regular (in fact, vanishing) volume density. In the space of all admissible parameters (a, b, c) , the set of spacetimes with regular volume density occupies a certain region; its complement consists of metrics for which the volume density diverges as $t \rightarrow 0$. Requiring that the integrated four-volume, $V(t) \sim \int dt \sqrt{-g}$, be finite leads to the condition

$$a + b + c > -1. \quad (2.6)$$

If this inequality is not satisfied, then the four-volume between any $t = t_0 > 0$ and $t = 0$ becomes infinite.

2. Integrated curvature is finite

Although the square of the Riemann tensor behaves as $R_{\alpha\beta\mu\nu}^2 \sim t^{-4}$ and is therefore divergent at $t = 0$, its integral $\int dt \sqrt{-g} R_{\alpha\beta\mu\nu}^2$ can nevertheless be finite, provided that

$$a + b + c > 3. \quad (2.7)$$

For $a + b + c = 3$, the integral diverges logarithmically, whereas for $a + b + c < 3$ it diverges by a power law. Similarly, requiring regularity of the integrated Ricci scalar, $\int dt \sqrt{-g} R$, imposes the bound

$$a + b + c > 1. \quad (2.8)$$

At the lower limit of this inequality, $a + b + c = 1$, the integral diverges only logarithmically, which is milder than the power-law divergence encountered when the condition (2.8) is violated.

3. Geodesic completeness

Even if a spacetime possesses a curvature singularity, it may still be geodesically complete, provided that any lightlike geodesic reaches the singularity only at an infinite value of the affine parameter λ . In this sense, the divergent curvature is “infinitely far away” along null geodesics: no observer can physically reach it (or, rather, receive signals from it) within finite proper time or affine parameter.

Consider a light ray propagating along the x^1 direction. The change in the affine parameter between a point at finite t and the singularity at $t = 0$ behaves as $\Delta\lambda \sim \int t^a dt$. This integral diverges when $a \leq -1$. Extending this argument to null rays propagating along any spatial direction x^i , $i = 1, 2, 3$, we obtain the more general condition,

$$a \leq -1, \quad b \leq -1, \quad c \leq -1. \quad (2.9)$$

4. Geodesically complete spacetimes with curvature singularities

This analysis shows that for certain negative values of the parameters, the spacetime near the singularity attains infinite volume and its lightlike geodesics become complete. Since spacetime singularities are typically characterized by geodesic incompleteness, the case in which all parameters (a, b, c) are negative and satisfy (2.9) is particularly appealing, as it appears to evade the usual pathological features.

D. Symmetric polynomials

The metric (2.1) is invariant under permutations of the spatial coordinates (x^1, x^2, x^3) . Consequently, any curvature invariant (with no free indices) must remain symmetric under permutations of the parameters (a, b, c) and can therefore be expressed in terms of symmetric polynomials. It is thus convenient to introduce a basis in the space of symmetric polynomials. In the case of three variables, the independent symmetric polynomials are

$$\begin{aligned} e_1 &= a + b + c, & e_2 &= ab + bc + ac, \\ e_3 &= abc. \end{aligned} \quad (2.10)$$

Other symmetric polynomials of a fixed degree can be constructed from these. In degree 2, there are two independent elementary symmetric polynomials: e_1^2 and e_2 . In degree 3, there are three polynomials: e_1^3 , $e_1 e_2$, and e_3 . In degree 4, there are four: e_1^4 , $e_1^2 e_2$, e_2^2 , and $e_1 e_3$. Below, we provide some useful identities:

$$\begin{aligned} a^2 + b^2 + c^2 &= e_1^2 - 2e_2, \\ a^3 + b^3 + c^3 &= e_1^3 - 3e_1 e_2 + 3e_3, \\ a^2(b + c) + b^2(a + c) + c^2(a + b) &= e_1 e_2 - 3e_3, \\ a^4 + b^4 + c^4 &= e_1^4 - 4e_1^2 e_2 \\ &\quad + 2e_2^2 + 4e_1 e_3, \\ a^3(b + c) + b^3(a + c) + c^3(a + b) &= e_1^2 e_2 - e_1 e_3 - 2e_2^2, \\ a^2 b^2 + a^2 c^2 + b^2 c^2 &= e_2^2 - 2e_1 e_3. \end{aligned} \quad (2.11)$$

It is often convenient, and in some cases particularly useful, to regard the symmetric polynomials (e_1, e_2, e_3) as independent variables instead of the parameters (a, b, c) . The Jacobian of this change of variables is proportional to $(a - b)(b - c)(a - c)$ and is therefore nonvanishing whenever the three parameters (a, b, c) are distinct.

I. Consistency bounds on symmetric polynomials

One should stress that, although *a priori* parameters (a, b, c) may take any real (positive or negative) values, this is not the case for the symmetric polynomials (e_1, e_2, e_3) . These satisfy a number of bounds based on various mathematical inequalities; see [12]. One restriction comes from the relation

$$(a - b)^2 + (b - c)^2 + (c - a)^2 = 2(e_1^2 - 3e_2). \quad (2.12)$$

This implies the bound

$$e_2 \leq \frac{e_1^2}{3}, \quad (2.13)$$

which restricts the possible values of e_1 and e_2 . As follows from (2.12), the bound (2.13) is saturated if and only if all three parameters are equal, $a = b = c$.

On the other hand, the inequality of arithmetic and geometric means (known as the AM-GM inequality) implies that for any non-negative a, b, c the following inequality holds:

$$e_3 \leq \frac{e_1^3}{27}. \quad (2.14)$$

The equality holds if and only if $a = b = c$. The other known inequality for non-negative a, b, c is

$$e_3^2 \leq \frac{e_2^3}{27}, \quad (2.15)$$

and the equality holds if and only if $a = b = c$.

E. Classical Kasner metric

In the classical case [13], the parameters (a, b, c) are those for which the metric (2.1) solves the Einstein

equations in vacuum, $G_\nu^\mu = 0$. The equation $G_0^0 = 0$ implies that the corresponding symmetric polynomial $e_2 = 0$. The equation G_1^1 explicitly separates coordinate x_1 from x_2 and x_3 and, hence, the corresponding equation is symmetric under permutations of b and c but not under permutations that involve the parameter a . Indeed, as is seen from (2.2), the equation $G_1^1 = 0$ implies that (taking into account that $e_2 = 0$)

$$(a - e_1)(e_1 - 1) = 0. \quad (2.16)$$

This equation has two possible solutions. The first is $a = e_1$, which implies that $b + c = 0$ and, from the constraint $e_2 = 0$, one finds that $bc = 0$, i.e., $b = 0$ or $c = 0$. The equations $G_2^2 = G_3^3 = 0$ then take the form

$$a(a - 1) = 0, \quad (2.17)$$

so that $a = 0$ or $a = 1$. Thus, we find two possible solutions: $(0, 0, 0)$ and $(1, 0, 0)$, modulo the permutations. These are regular spacetimes, as we discussed above.

The other solution of (2.16) is $e_1 = 1$. This implies that the equations $G_2^2 = 0$ and $G_3^3 = 0$ are satisfied automatically. Thus, this second-type solution defines (a, b, c) , for which

$$e_1 = 1 \quad \text{and} \quad e_2 = 0. \quad (2.18)$$

Equivalently, using (2.11), these constraints take the usual form

$$a + b + c = 1 \quad \text{and} \quad a^2 + b^2 + c^2 = 1. \quad (2.19)$$

The standard analysis shows that (provided that $a < b < c$) $-\frac{1}{3} \leq a \leq 0$, $0 \leq b \leq \frac{2}{3}$, $\frac{2}{3} \leq c \leq 1$. The equations (2.19) can be solved by introducing a parameter u :

$$a = -\frac{u}{w(u)}, \quad b = \frac{1+u}{w(u)}, \quad c = \frac{u(1+u)}{w(u)},$$

$$w(u) = 1 + u + u^2, \quad u \geq 1. \quad (2.20)$$

Geometrically, in the space of parameters (a, b, c) , it is represented by a circle that is the intersection of a sphere of unit radius ($a^2 + b^2 + c^2 = 1$) and a plane ($a + b + c = 1$). The entire circle corresponds to all possible values of the parameter u .

Alternatively, one can use the symmetric polynomials (e_1, e_2, e_3) as independent variables. As we have seen already, the symmetric polynomials $e_1 = 1$ and $e_2 = 0$. The polynomial e_3 takes negative values in the interval $-\frac{4}{27} \leq e_3 \leq 0$. The lower bound is reached for values $(a = -\frac{1}{3}, b = c = \frac{2}{3})$, so that in the space of variables (e_1, e_2, e_3) , the solution is given by an interval $(e_1 = 1, e_2 = 0, e_3 = v, -\frac{4}{27} \leq v \leq 0)$.

The classical Kasner metric is characterized by a logarithmically divergent integral of the Ricci scalar curvature [see (2.8)]. The classical equations admit two regular spacetimes: $(0, 0, 0)$ and $(1, 0, 0)$; they describe Minkowski spacetime. The other values of (a, b, c) correspond to anisotropic solutions. All of them are singular.

III. CONFORMAL ANOMALY AND BACKREACTION

A. General setup

In the presence of quantum matter, the gravitational equations are modified. In general, the gravitational action is a sum of the classical part and the part due to the quantum matter,

$$W_{\text{gr}} = -\frac{1}{2\kappa} \int \sqrt{-g} R + W_Q, \quad (3.1)$$

where $\kappa = 8\pi G$. For free quantum fields, $W_Q = -\frac{1}{2} \sum_s (-1)^s \det \ln \square_{(s)}$, where s is the spin and $\square_{(s)}$ is the respective field operator. Generally, W_Q is a rather complicated nonlocal functional of the background metric. In the case when the quantum fields are conformal, an important local quantity that characterizes W_Q is available: the conformal anomaly $\mathcal{A} = g^{\mu\nu} \frac{\delta W_Q}{\delta g^{\mu\nu}}$. This is why in the present paper we focus on the case of conformal quantum field theory.

Variation of the gravitational action with respect to the metric gives us the modified Einstein equations,

$$G_\nu^\mu = \kappa (T_Q)_\nu^\mu, \quad (T_Q)_{\mu\nu} = \frac{\delta W_Q}{\delta g^{\mu\nu}}. \quad (3.2)$$

We stress that the background metric is not fixed but appears as a solution to this equation. We call it the *backreacted metric* as it includes the backreaction of the quantum fields on the background metric. The quantum stress-energy tensor $T_{\mu\nu}^Q$ is covariantly conserved,

$$\nabla^\mu T_{\mu\nu}^Q = 0. \quad (3.3)$$

For the conformal quantum matter, the trace of $T_{\mu\nu}^Q$ (henceforth we drop the subscript Q) is known to be a local function of the metric,

$$g^{\mu\nu} T_{\mu\nu} = -\frac{1}{(4\pi)^2} (AE_4 - CW^2), \quad (3.4)$$

where W^2 is the square of the Weyl tensor and E_4 is the Euler density,

$$E_4 = R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} - 4R_{\mu\nu} R^{\mu\nu} + R^2,$$

$$W^2 = R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} - 2R_{\mu\nu} R^{\mu\nu} + \frac{1}{3} R^2. \quad (3.5)$$

A and C are the conformal charges that characterize a given CFT. In a unitary theory, these charges are positive. In this paper, however, we allow A and C to take arbitrary values, thereby effectively including the possibility of nonunitary theories as well as higher-spin theories [14] that may become relevant in the vicinity of the singularity.

B. Our strategy

Our primary goal is to solve the semiclassical gravitational equations (3.2) for the class of Kasner-type metrics (2.1) and to determine the constraints on the parameters (a, b, c) . We achieve this goal in two steps.

Step 1. As already mentioned, the quantum stress tensor $T_{\mu\nu}$ is generally unknown and is defined up to a number of ambiguities. However, its trace is a known local function of the metric. Consequently, the trace of the gravitational equations (3.2) is local and unambiguous,

$$R = \frac{\kappa}{(4\pi)^2} (AE_4 - CW^2). \quad (3.6)$$

In the present section, we analyze this equation for an anisotropic metric whose near-singularity behavior is given by (2.1). To leading order, the left-hand side of Eq. (3.6) behaves as $1/t^2$ in the vicinity of the singularity, whereas the right-hand side behaves as $1/t^4$. Thus, to leading order in $1/t$, Eq. (3.6) for the metric in the general form (2.1) reduces to the requirement that the most divergent term on the right-hand side of (3.6) vanishes. These conditions lead to what we call the Master equation, which will be considered in the next subsection.

Proceeding with Step 1 has the advantage that it does not involve any ambiguities related to the integration of the continuity equation (3.3) for the stress tensor $T_{\mu\nu}^Q$. However, the set of conditions on the parameters (a, b, c) obtained in Step 1 defines a rather broad surface of admissible values in the parameter space. In order to identify a more restricted subset of these conditions, we must consider the remaining tensorial components in Eq. (3.2). This is carried out in Step 2.

Step 2. We first explicitly solve the continuity equation (3.3) for the general form of the Kasner-type metric (2.1), without imposing *a priori* any constraints on the parameters (a, b, c) . This procedure introduces several constants that remain undetermined. Fixing these constants amounts to selecting a particular quantum state of the conformal field. This analysis is carried out in Secs. IV and V, where we focus on the leading divergent terms behaving as $1/t^4$. The resulting stress tensor $T_{\mu\nu}$ is then substituted into Eq. (3.2). As in Step 1, the left-hand side of (3.2) behaves as $1/t^2$, while the right-hand side behaves as $1/t^4$. Therefore, solving Eq. (3.2) requires that the terms in $T_{\mu\nu}^\mu$, that diverge as $1/t^4$, have to vanish. This condition effectively reduces to the vanishing of three symmetric polynomials in the parameters a, b, c , as we show in Sec. V.

The solution of the resulting constraints is considered in Sec. VI.

C. Master equation

With these explanations, we proceed to solve the trace equation (3.6). First, we compute W^2 and E_4 near the singularity. For a Kasner-type metric (2.1), one finds that

$$\begin{aligned} E_4 &= \frac{8}{t^4} abc(a+b+c-3), \\ W^2 &= \frac{4}{3t^4} ((a-1)^2(b-a)(c-a) + (b-1)^2(a-b)(c-b) \\ &\quad + (c-1)^2(a-c)(b-c)). \end{aligned} \quad (3.7)$$

We observe that the left-hand side of Eq. (3.6), which originates from the Ricci scalar, behaves near the singularity as $1/t^2$, while the right-hand side, arising from the conformal anomaly, scales as $1/t^4$. Since the right-hand side diverges more strongly, consistency of the modified gravitational equations requires that it vanishes. This condition imposes a constraint on the parameters (a, b, c) , given by what we refer to as the Master equation:

$$\begin{aligned} \mathcal{M}(a, b, c) &\equiv 6\eta abc(a+b+c-3) \\ &\quad - ((a-1)^2(b-a)(c-a) \\ &\quad + (b-1)^2(a-b)(c-b) \\ &\quad + (c-1)^2(a-c)(b-c)) = 0, \end{aligned} \quad (3.8)$$

where we defined $\eta = A/C$ and we assumed that C is nonvanishing. This equation is symmetric under permutations of the parameters a, b , and c .

The case in which the conformal charge C vanishes is, in a certain sense, degenerate. The Master equation then takes the form

$$abc(a+b+c-3) = 0,$$

which corresponds to four distinct planes in the parameter space (a, b, c) , namely, $a = 0$, $b = 0$, $c = 0$, and $a + b + c = 3$. Each of these planes satisfies the Master equation independently, while leaving the remaining parameters unconstrained.

D. Some particular cases

We now consider some particular cases in which the Master equation simplifies.

1. Isotropic case: $a = b = c$

When all three parameters coincide, the Weyl squared vanishes, $W^2 = 0$, so that the anomaly is determined only by the Euler term:

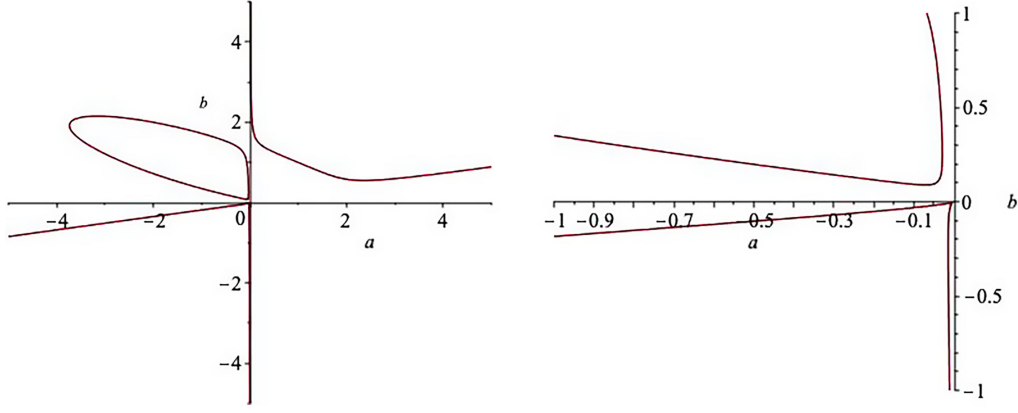


FIG. 1. Plot of the curve $\mathcal{M}(a, b) = 0$ for the value $\eta = 3$ (left) and the zoomed-in view of the region of small negative values of a (right).

$$\mathcal{M}(a, a, a) = 18\eta a^3(a - 1) = 0. \quad (3.9)$$

One solution is $a = 0$. It is the point $(0, 0, 0)$ in the space of parameters. The spacetime in this case is regular at $t = 0$. The other solution is $a = 1$. It is the point $(1, 1, 1)$ in the space of parameters. There is a singularity at $t = 0$ in this case.

2. Semi-isotropic case: $b = c \neq a$

In this case, $b = c \neq a$ and the Master equation becomes a function of two variables, a and b ,

$$\mathcal{M}(a, b) = 6\eta ab^2(a + 2b - 3) - (a - 1)^2(b - a)^2 = 0. \quad (3.10)$$

One obvious solution of this equation, valid for any value of η , is $(a = 1, b = 0)$. This corresponds to a regular spacetime $(1, 0, 0)$. Analysis shows that this solution is

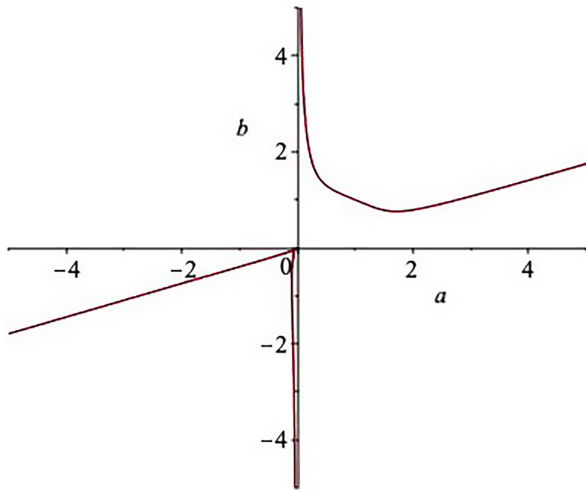


FIG. 2. Plot of the curve $\mathcal{M}(a, b) = 0$ for the value $\eta = \frac{1}{3}$.

isolated, i.e., it cannot be continuously related to any other solution, for any positive value of η .

For negative η , however, it lies on a continuous curve that solves Eq. (3.10). We present in Figs. 1–3 the respective curves that solve Eq. (3.10) for various values of η . One sees that for $0 < \eta < 1$ the curve has two open components: one lies in the domain $a > 0, b > 0$ and the other in the domain $a < 0, b < 0$.

For values $\eta > 1$, a closed curve component appears in the domain $a < 0, b > 0$. This curve approaches quite closely the origin but does not in fact reach it, remaining always in the region $a < 0, b > 0$.

For negative values $\eta < 0$, the curve has three components: one open component in the domain $a < 0, b > 0$, another open component in the domain $a > 0, b < 0$, and a closed curve component in the domain $a > 0, b > 0$ that passes by the points $(a = 0, b = 0)$ and $(a = 1, b = 0)$. These two points correspond to spacetimes without a singularity.

E. General shape of the Master equation surfaces

In a generic case, a solution to the Master equation defines a two-dimensional surface in the space of parameters (a, b, c) . The shape of this surface crucially depends on the parameter $\eta = A/C$ that is determined uniquely by the quantum CFT in question. For a unitary CFT, the parameter η is positive and satisfies certain unitarity bounds that were studied in the literature. As already explained, in the present study we do not restrict ourselves to only unitary theories and thus prefer to explore all possible values of η taking all possible (positive or negative) real numbers. In Figs. 4–6 we present some typical situations with representative values of η .

The analysis shows that as η varies in the interval $0 < \eta < 1$, the surface consists of two open shells. One of them lies in the quadrant of positive values of the parameters (a, b, c) and is bounded by the plane $a + b + c = 3$, so that for all points in this branch, one has that $a + b + c > 3$.

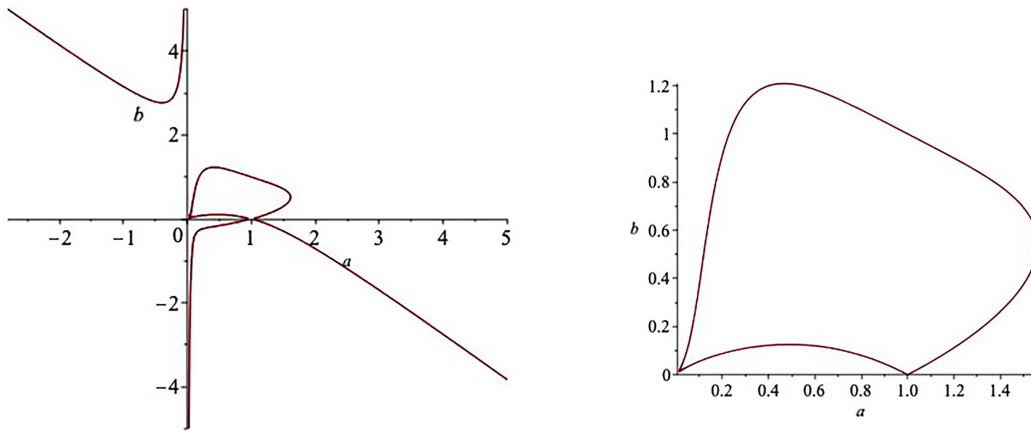


FIG. 3. Plot of the curve $\mathcal{M}(a, b) = 0$ for the value $\eta = -\frac{1}{3}$ (left) and the zoomed-in view of the region of positive values of (a, b) (right).

The respective spacetime has regular integrated Riemann curvature; see (2.7).

The second open shell lies in the region of negative values of all parameters (a, b, c) . It contains a conical point that has coordinates $(0, 0, 0)$. It has three open infinite domains where the conditions (2.9) to have complete geodesics are satisfied. The respective spacetimes have infinite four-volume near the singularity.

The entire surface is presented in Fig. 4 for the value $\eta = 1/2$. Note that the points where all parameters vanish or where one of the parameters is 1 and the others vanish are not shown since they lie outside the continuous surface and represent isolated points in the parameter space.

For values $\eta > 1$, the surface still has two open shells, as in the case above. However, there also appear three closed identical shells, each lying in a region where one of the parameters is negative and the two others are positive. A surface of this type is shown in Fig. 5, where in particular the closed shell is shown in the region $(a < 0, b > 0, c > 0)$. The closed shell then passes by

the points $(a = 0, b = 1, c = 0)$ and $(a = 0, b = 0, c = 1)$ that now are not isolated points and are continuously related to other points in the closed shell component. Geometrically, these three closed shells are identical due to the permutation symmetry in the parameter space (a, b, c) .

For negative values of η , the character of the surface changes again. It contains an open shell in each quadrant where one of the parameters is negative and the two others are positive. Then, in each quadrant where one of the parameters is positive and the two others are negative, there appears an open cone-like shell. Additionally, there appears a closed surface component in the quadrant where all parameters are positive (shown in the right panel of Fig. 5). It passes by the points $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$, all four points representing the spacetime without singularities.

In summary, we find that the configurations that are similar to the classical Kasner solution, where one of the parameters is negative and the other two are positive, occur only for $\eta > 1$ and $\eta < 0$, and are absent for $0 < \eta < 1$. For $\eta > 1$, the corresponding points lie on a compact surface,

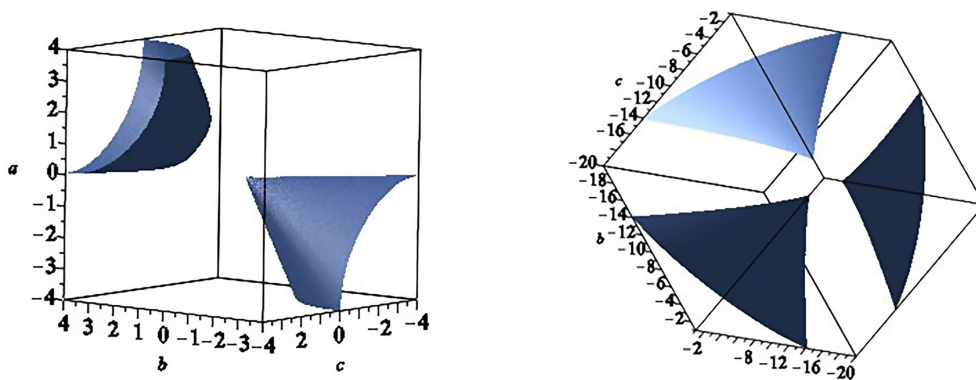


FIG. 4. Surface $\mathcal{M}(a, b, c) = 0$ for $\eta = A/C = 1/2$. Left: two open shells: one for positive values of the parameters (a, b, c) and the other for all negative values. The conical point has coordinates $(0, 0, 0)$ in the parameter space. Right: the three disjoint regions of the surface where each parameter is less than -1 . In the corresponding spacetime, although the Riemann tensor is singular, the geodesics remain complete.

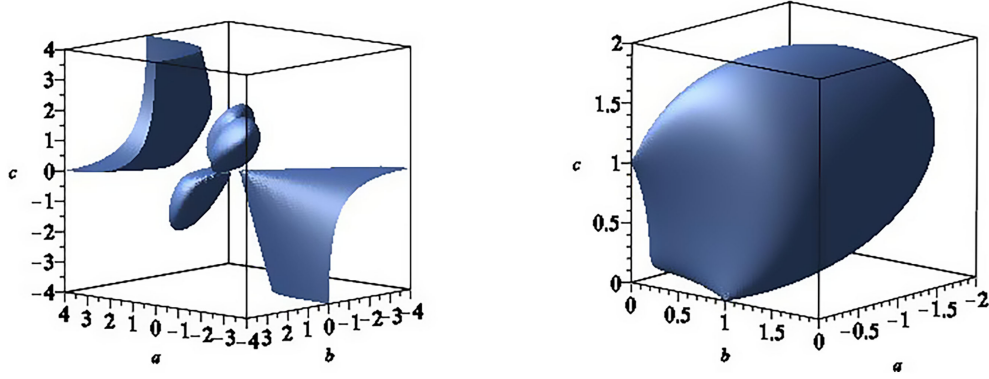


FIG. 5. Left: the surface $\mathcal{M}(a, b, c) = 0$ for $\eta = A/C = 2$. There are three additional closed shells in which one of the parameters is negative and two others are positive. Right: zoomed-in view of such a shell for negative values of a and positive values of b and c .

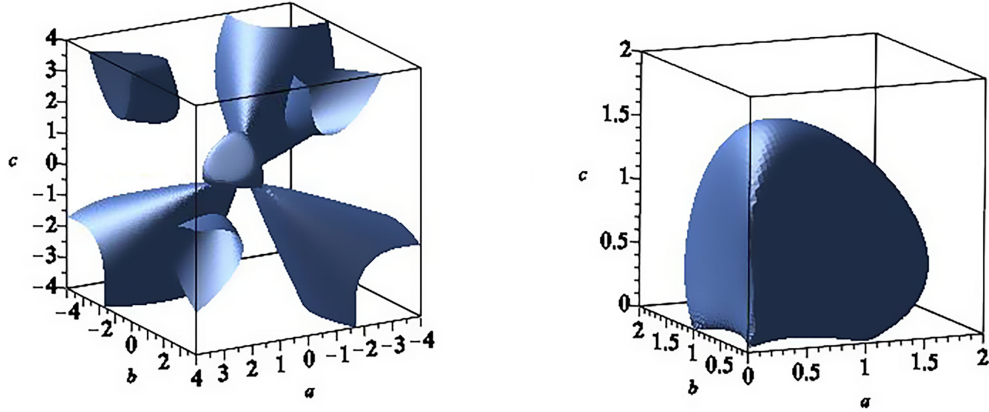


FIG. 6. The surface $\mathcal{M}(a, b, c) = 0$ for $\eta = A/C = -1$. Left: The surface contains an open shell in each region where one of the parameters is negative and two others are positive, an open shell in each region where one of the parameters is positive and two others are negative, and a closed shell in the region where all parameters are positive. Right: zoomed-in view of the region with the closed shell.

so the Kasner parameters remain bounded. In contrast, for $\eta < 0$, the corresponding surface is noncompact, and the absolute values of the parameters are unbounded.

It should be noted that the two-dimensional surfaces that solve the Master equation represent only the trace part of the modified Einstein equations (3.6). The other tensorial components of the Einstein equations will produce further constraints on the parameters (a, b, c) that may restrict the possible solutions for the parameters (a, b, c) to a one-dimensional subset or even to some isolated points. This question will be further addressed later in the paper.

F. Solving the Master equation in terms of symmetric polynomials

The Master equation (3.8) can be rewritten in terms of the symmetric polynomials e_1 , e_2 , and e_3 ,

$$\begin{aligned} \mathcal{M}(e_1, e_2, e_3) &\equiv 6(\eta - 1)e_3(e_1 - 3) - h(e_1, e_2) = 0, \\ h(e_1, e_2) &= e_1^4 - 5e_1^2e_2 + 4e_2^2 - 2e_1^3 + 8e_1e_2 + e_1^2 - 3e_2. \end{aligned} \quad (3.11)$$

1. The case of $\eta = 1$ ($A = C$)

This is the case of a special CFT that plays an important role in the modern developments. The superconformal four-dimensional $SU(N)$ Yang-Mills theory is a CFT of this type. This theory has a dual holographic description in the framework of the AdS/CFT correspondence.

The first term in the Master equation (3.11) in this case vanishes. The symmetric polynomial e_3 then disappears from the Master equation so that it may take any value that is allowed. The Master equation reduces to the equation $h(e_1, e_2) = 0$, which is a quadratic algebraic equation for e_2 , and it can be easily solved,

$$e_2^\pm(e_1) = \frac{1}{8}(e_1 - 1) \left(5e_1 - 3 \pm 3\sqrt{(e_1 - 3)(e_1 - \frac{1}{3})} \right). \quad (3.12)$$

First of all, we notice that there is an isolated solution,

$$e_1 = 1, \quad e_2 = 0. \quad (3.13)$$

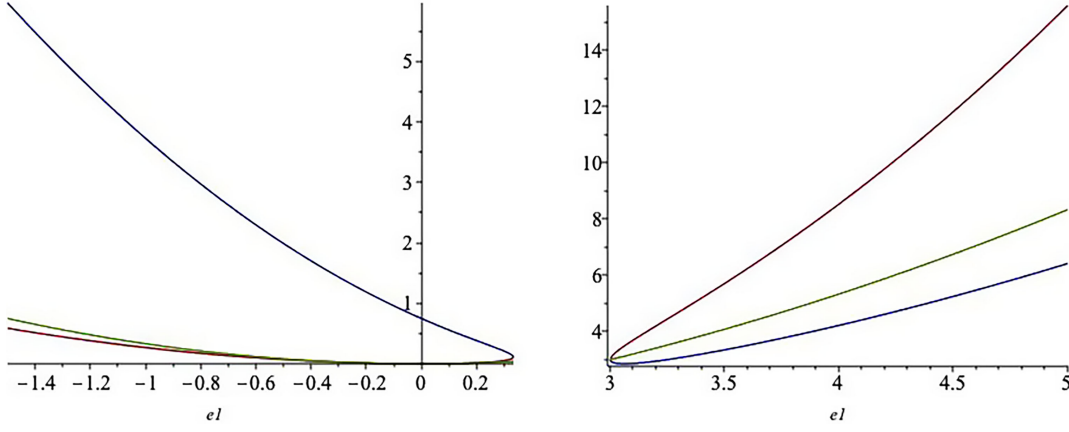


FIG. 7. Plot of the functions $e_2^+(e_1)$ [the lowest curve (left) and the highest curve (right)], $e_2^-(e_1)$ [the highest curve (left) and the lowest curve (right)] and of the bound $e_2^b(e_1) = \frac{e_1^3}{3}$ [the curve in the middle (left and right)] for values $e_1 \leq \frac{1}{3}$ (left) and $e_1 \geq 3$ (right).

This is precisely the classical Kasner solution. It comes out here because for $A = C$, the Riemann tensor disappears in the conformal anomaly that now contains only the Ricci tensor, so that the solutions when the Ricci tensor vanishes are also solutions of Eq. (3.6) with the conformal anomaly. This is the case for the Kasner solution. This solution is isolated from any other possible solution. This is easily seen from the fact that the point $(e_1 = 1, e_2 = 0)$ lies outside of the continuous curves defined by (3.12).

The solution (3.12) has two branches depending on the sign. The expression under the square root is non-negative for two intervals: $e_1 \geq 3$ and $e_1 \leq \frac{1}{3}$. For $e_1 = 0$, in the positive sign branch, $e_2^- = 0$. This is the only point where e_2 vanishes. These values ($e_1 = e_2 = 0$) imply that the parameters (a, b, c) may take only the value $(0, 0, 0)$. This is a spacetime without any singularity.

For all other admissible values of e_1 within the two allowed intervals, $e_2^\pm$ remains positive for both branches. In the interval $e_1 \leq \frac{1}{3}$, the negative branch lies above the positive one, $e_2^- > e_2^+$. Conversely, in the second interval, $e_1 \geq 3$, the ordering is reversed, with the positive branch lying above the negative one, $e_2^+ > e_2^-$.

The bound (2.13) imposes an additional constraint on possible values of the symmetric polynomials. One finds that in the interval $e_1 \geq 3$ the bound selects only the negative branch, e_2^- . In the interval $0 \leq e_1 \leq \frac{1}{3}$, both branches violate the bound, while in the interval $e_1 \leq 0$, the positive-sign branch satisfies the bound. This is shown in Fig. 7.

Thus, we conclude that (i) for values $e_1 \leq 0$, the bound selects the positive-sign branch; (ii) for values $0 < e_1 \leq \frac{1}{3}$, both branches violate the bound and thus there is no solution; and (iii) for values $e_1 \geq 3$, the bound selects the negative branch.

2. The case $\eta \neq 1$ ($A \neq C$)

Consider now a more general situation, when $\eta \neq 1$.

i) First, we consider a special case when $e_1 = 3$. Then, the Master equation (3.11) reduces to the equation $h(e_2, e_1 = 3) = 0$ with a single solution, $e_2 = 3$. Thus, in this case

$$e_1 = 3, \quad e_2 = 3. \quad (3.14)$$

Note that this solution saturates the bound (2.13). It is possible only if all parameters a, b, c are equal, $a = b = c = 1$. We have already mentioned this solution above. It corresponds to a symmetric spacetime with a singularity at $t = 0$.

ii) For $e_1 \neq 3$, the Master equation (3.11) can be solved as follows:

$$e_3 = \frac{1}{6(\eta - 1)} \frac{h(e_1, e_2)}{(e_1 - 3)}, \quad (3.15)$$

where possible values of e_1 and e_2 are restricted by the bound (2.13). If values of e_1 lie in one of the intervals $e_1 \geq 3$ or $e_1 \leq \frac{1}{3}$ function $h(e_1, e_2)$ can be presented as

$$h(e_1, e_2) = (e_2 - e_2^+(e_1))(e_2 - e_2^-(e_1)), \quad (3.16)$$

where $e_2^\pm(e_1)$ are defined in (3.12). For $\frac{1}{3} \leq e_1 \leq 3$, the function $h(e_1, e_2)$ does not vanish and takes positive values.

A particular solution is $e_1 = 1$, $e_2 = 0$, and $e_3 = 0$; this is the point $(1, 0, 0)$ in the parameter space. It corresponds to a regular spacetime.

IV. GENERAL FORM OF ANOMALOUS STRESS TENSOR

In the preceding sections, we analyzed the trace part of the modified gravitational equations that incorporate the conformal anomaly. The full semiclassical gravitational equations (3.2), however, contain additional, tensorial components

that introduce further constraints. Geometrically, these constraints select a subspace of solutions within the surface $\mathcal{M}(a, b, c) = 0$ considered earlier. To investigate their implications, one must first determine the general structure of the anomalous stress-energy tensor. In this section, we examine the complete set of equations for the stress-energy tensor $T_{\mu\nu} = (T_{00}(t), T_{11}(t), T_{22}(t), T_{33}(t))$ of the quantum CFT and provide a full integration of these equations.

The covariant conservation equation $\nabla^\mu T_{\mu\nu} = 0$ in the metric (2.1) for $\nu = 1, 2, 3$ is automatically satisfied (since $T_{0i} = 0, i = 1, 2, 3$), while for $\nu = 0$, one has the equation

$$\partial_t T_0^0 + \frac{(a+b+c)}{t} T_0^0 - \frac{1}{t} (aT_1^1 + bT_2^2 + cT_3^3) = 0. \quad (4.1)$$

On the other hand, the trace anomaly equation (3.6) leads to

$$T_0^0 + T_1^1 + T_2^2 + T_3^3 = -\frac{C}{12\pi^2} \mathcal{M}(a, b, c) \frac{1}{t^4}, \quad (4.2)$$

where $\mathcal{M}(a, b, c)$ is defined in (3.8). Upon solving Eq. (4.1), we represent the components of stress tensor as follows:

$$\begin{aligned} T_0^0(t) &= -\frac{C}{12\pi^2} \frac{G(a, b, c)}{t^4}, \\ T_i^i(t) &= -\frac{C}{12\pi^2} \frac{F_i(a, b, c)}{t^4}, \quad i = 1, 2, 3. \end{aligned} \quad (4.3)$$

Due to the permutation symmetry of the metric (2.1), it is evident that $G(a, b, c)$ is a symmetric function of the parameters a, b , and c . Additionally, the function $F_1(a, b, c)$ is symmetric with respect to the permutation of b and c . The function $F_2(a, b, c)$ is symmetric with respect to a and c , while $F_3(a, b, c)$ is symmetric with respect to a and b .

All three functions can be expressed in terms of a single function $F(a; b, c)$, which satisfies $F(a; b, c) = F(a; c, b)$ as follows:

$$\begin{aligned} F_1(a, b, c) &= F(a; b, c), & F_2(a, b, c) &= F(b; a, c), \\ F_3(a, b, c) &= F(c; a, b). \end{aligned} \quad (4.4)$$

Equations (4.1) and (4.2) become algebraic equations for $G(a, b, c)$ and $F(a; b, c)$:

$$\begin{aligned} G(a, b, c) + F(a; b, c) + F(b; a, c) + F(c; a, b) \\ = \mathcal{M}(a, b, c) \end{aligned} \quad (4.5)$$

and

$$\begin{aligned} F(a; b, c)(2a + b + c - 4) + F(b; a, c)(2b + a + c - 4) \\ + F(c; a, b)(2c + a + b - 4) \\ = (a + b + c - 4)\mathcal{M}(a, b, c). \end{aligned} \quad (4.6)$$

In the semiclassical Einstein equations (3.2) modified by the contribution from a quantum CFT, the left-hand side (due to the Ricci tensor) grows as $1/t^2$ for small t , while the right-hand side, arising from the conformal anomaly, grows as $1/t^4$. The latter therefore dominates and must vanish in any consistent solution. To leading order, the Einstein equations thus reduce to a set of algebraic equations involving two functions of the parameters (a, b, c) :

$$\begin{aligned} G(a, b, c) &= 0, & F(a; b, c) &= 0, \\ F(b; a, c) &= 0, & F(c; a, b) &= 0. \end{aligned} \quad (4.7)$$

One may replace the equation $G(a, b, c) = 0$ with the equation $\mathcal{M}(a, b, c) = 0$, i.e., the anomaly vanishes to this order.

Our strategy now is to solve the algebraic equation (4.6) and then identify the restrictions on the parameters a, b, c due to Eq. (4.7).

A. Homogeneous solution

We pause here for a brief remark. The differential equation (4.1) also admits a homogeneous solution, which can always be added to the general solution of the nonhomogeneous equation. This homogeneous solution corresponds to the case $T_i^i = 0, i = 1, 2, 3$ in Eq. (4.1). It takes the form $T_{0hom}^0 \sim t^{-(a+b+c)}$. The conformal anomaly does not contribute at this order unless $a + b + c = 4$. Therefore, we set $T_{0hom}^0 = 0$. The special case $a + b + c = 4$ requires separate treatment and will not be considered here.

V. KASNER CFT

Both sides of Eq. (4.6) represent a symmetric polynomial of degree 5 of three variables a, b, c . Any such polynomial can be decomposed over the symmetric basis e_1, e_2, e_3 , and there are 16 terms in such a polynomial. Thus, Eq. (4.6) is equivalent to a system of 16 linear equations. On the other hand, $F(a; b, c)$ is a polynomial of degree 4 symmetric in b and c and there may appear 22 terms (and respective coefficients to be determined) in such a polynomial; see Eq. (A2) of Appendix A. The naive counting, thus, suggests that there are $22 - 16 = 6$ coefficients that remain undetermined. The direct analysis (see Appendix A), however, shows that there remain seven undetermined coefficients, so that one of the 22 equations follows from the others. In Appendix A, we solve Eq. (4.6) and find the respective coefficients as functions of these seven coefficients that remain undetermined. (Notice, however, that the choice of these coefficients is rather ambiguous.)

With a fourth-order polynomial $F(a; b, c) = F(a; c, b)$, one can define a third-order polynomial, $E(c; a, b) = E(c; b, a)$, and a second-order polynomial, $K(a; b, c)$, as follows:

$$\begin{aligned} F(a; b, c) - F(b; a, c) &= (a - b)E(c; a, b), \\ E(c; a, b) - E(a; c, b) &= (c - a)K(b; a, c). \end{aligned} \quad (5.1)$$

For a generic polynomial $F(a; b, c)$, the second-order polynomial $K(b; a, c)$ does not have to be totally symmetric with respect to all of its arguments a, b, c . However, for an $F(a; b, c)$ that solves Eq. (4.6), the respective second-order polynomial happens to be totally symmetric, $K(a; b, c) = K(a, b, c)$.

One may define two more symmetric polynomials. The first is a third-degree polynomial,

$$S(a, b, c) = E(a; b, c) + E(b; a, c) + E(c; a, b), \quad (5.2)$$

and the second is a symmetric fourth-degree polynomial,

$$H(a, b, c) = F(a; b, c) + F(b; a, c) + F(c; a, b). \quad (5.3)$$

The important point now is that the polynomial $F(a; b, c) = F(a; c, b)$ can be expressed in term of these three totally symmetric polynomials, $K(a, b, c)$, $S(a, b, c)$, and $H(a, b, c)$,

$$\begin{aligned} F(a; b, c) &= \frac{1}{3}H(a, b, c) + \frac{1}{9}(2a - b - c)S(a, b, c) \\ &\quad + \frac{1}{9}((a - b)(2c - a - b) \\ &\quad + (a - c)(2b - a - c))K(a, b, c). \end{aligned} \quad (5.4)$$

This relation implies that, provided that $(a - b)(b - c) \times (c - a)$ is nonvanishing, the equations $F(a; b, c) = F(b; a, c) = F(c; a, b) = 0$ are equivalent to the vanishing of three symmetric polynomials:

$$H(a, b, c) = 0, \quad S(a, b, c) = 0, \quad K(a, b, c) = 0. \quad (5.5)$$

As is shown in Appendix A, for an $F(a; b, c)$ that solves Eq. (4.6) one finds that,

$$K(a, b, c) = k_1(e_1^2 - 9) + \frac{1}{9}m_3(e_2 - 3) + k_3(e_1 - 3) \quad (5.6)$$

and

$$\begin{aligned} S(a, b, c) &= m_1(e_1^3 - 27) + m_2e_2(e_1 - 3) \\ &\quad + m_3(e_3 - e_2 + 2) + m_4(e_1^2 - 9) \\ &\quad + m_5(e_1 - 3) + 12(3 - e_2). \end{aligned} \quad (5.7)$$

The coefficients $k_1, k_3, m_1, m_2, m_3, m_4, m_5$ can be taken as the seven coefficients that define all other coefficients in the polynomial $F(a; b, c)$.

For the symmetric polynomial $H(a, b, c)$, one finds (see Appendix A) the following form:

$$\begin{aligned} H(a, b, c) &= h_{21}(e_1^2 - 3e_2) + h_{31}(e_1^3 - 3e_1e_2) \\ &\quad + h_{33}\left(e_3 - \frac{1}{9}e_1e_2\right) + h_{41}(e_1^4 - 9e_1^2e_2) \\ &\quad + h_{42}(e_1^2e_2 - 3e_2^2) + h_{44}\left(e_1e_3 - \frac{1}{3}e_2^2\right) \\ &\quad - 2\eta\left(e_1e_2 - \frac{3}{4}e_2^2\right), \end{aligned} \quad (5.8)$$

where $h_{21}, h_{31}, h_{33}, h_{41}, h_{42}, h_{44}$ are expressed in terms of the coefficients $k_1, k_3, m_1, m_2, m_3, m_4, m_5$ according to Eq. (A18) in Appendix A.

The relations (5.4), (5.6)–(5.8) completely solve the problem of finding the stress tensor with the conformal anomaly in a Kasner-type metric (2.1) and define the components of the CFT stress tensor according to Eq. (4.3).

Examining Eqs. (5.6)–(5.8) we can make the following observations. The second-order polynomial $K(a, b, c)$ does not contain any information about the conformal charges. The conformal charge C (associated with the Weyl term in the anomaly) first appears in the third-order symmetric polynomial $S(a, b, c)$, while the conformal charge A (associated with the Euler term) appears in the fourth-order polynomial $H(a, b, c)$. This pattern is reminiscent, albeit only loosely, of the familiar fact that the charge C enters at the level of the two-point correlation function, whereas the charge A enters at the level of the three-point function. We do not pursue this possible analogy further here.

VI. SOLVING THE CONSTRAINTS

Thus, in the previous section we completely integrated the conformal anomaly for a Kasner-type metric (2.1), i.e., we found the corresponding stress-energy tensor that is covariantly conserved and its trace correctly reproduces the anomaly. Now, we want to solve the complete gravitational equations considering the anomalous stress tensor as a source for the gravitational field. These equations are tensorial. They include the vanishing of the anomaly, to leading order in t , as one of the constraints. As we have already explained in the previous sections, the equations of semiclassical gravity, to leading order, reduce to the vanishing of three symmetric functions, $K(a, b, c) = 0$, $S(a, b, c) = 0$, and $H(a, b, c) = 0$, where the functions are given by Eqs. (5.6)–(5.8). They are defined up to seven undetermined parameters: $k_1, k_3, m_1, m_2, m_3, m_4$, and m_5 . If all of these coefficients are nonvanishing, then the equation $K(a, b, c) = 0$ is solved by expressing e_2 as a quadratic function of e_1 , which then should be substituted into the equation $S(a, b, c) = 0$ that is solved by expressing e_3 as a cubic function of e_1 . All of this should be then substituted into the equation $H(a, b, c) = 0$ that becomes a quartic equation for e_1 . This equation may have at most

four real solutions, thus expressing e_1 as a function of the seven coefficients. Thus, in this case, the solution of all three equations in the space of parameters (a, b, c) is represented by four isolated points, the positions of which are determined by the values of the seven coefficients.

This situation, however, is quite restrictive. Indeed, in the case of the classical Kasner metric, the solution is a one-parameter family rather than a set of isolated points. If we look at the classical gravitational equations (2.2), we find that in this case $F(a, b, c) = b^2 + c^2 + bc - (b + c)$, and hence one has that $E(a, b, c) = -(a + b + c - 1)$. This latter function is already symmetric in all three parameters, and hence $K(a, b, c) = 0$ identically. Then, one finds for the symmetric functions $S(a, b, c) = -3(a + b + c - 1)$ and $H(a, b, c) = 2(a + b + c)^2 - 2(a + b + c) - 3(ab + bc + ac)$, so that the equation $S(a, b, c) = 0$ gives us the condition $a + b + c = 1$ and the equation $H(a, b, c) = 0$ gives us the second condition $ab + bc + ac = 0$. Both equations define a one-parameter family of solutions.

Although we do not fully understand the physical motivations for this prescription, let us follow the classical case and assume that the equation $K(a, b, c) = 0$ holds identically and, thus, does not produce any constraints on the parameters a, b, c . This can be achieved by the choice that the three coefficients in (5.6) vanish, i.e., $k_1 = k_3 = m_3 = 0$. In this case, we have only two equations to define the possible values of the parameters a, b, c : $S(a, b, c) = 0$ and $H(a, b, c) = 0$.

We may further restrict the number of nonvanishing coefficients. It is expected that the classical Kasner solution should be also a solution to the semiclassical gravitational equation if $\eta = A/C = 1$. For a general choice of coefficients in (5.7)–(5.8), this is not automatic. Indeed, imposing $e_1 = 1$ and $e_2 = 0$ in these equations, we find that the corresponding solution exists provided a relation between the coefficients holds,

$$m_6 + 13m_1 + 4m_4 - 18 = 0. \quad (6.1)$$

This is a new constraint to be imposed on the coefficients. In order to further restrict the space of coefficients, we further impose conditions that $m_1 = m_4 = 0$ and $m_6 = 18$. With this choice of the coefficients, we arrive at the following expressions for the symmetric functions:

$$S(a, b, c) \equiv e_2(m_2(e_1 - 3) - 12) + 18(e_1 - 1) = 0, \quad (6.2)$$

$$\begin{aligned} H(a, b, c) \equiv & \frac{9}{2}e_3(e_1 - 4)(\eta - 1) - \frac{m_2}{12}e_2(e_1^2 - 3e_2) \\ & - \frac{3}{4}e_1^4 + \frac{9}{4}e_1^3 + \frac{15}{4}e_1^2e_2 - \frac{35}{4}e_1e_2 \\ & - 3e_2^2 - \frac{3}{2}e_1^2 + \frac{9}{2}e_2 = 0, \end{aligned} \quad (6.3)$$

where we used Eq. (A18).

As expected, the system of equations $S(a, b, c) = 0$ and $H(a, b, c) = 0$ is equivalent to the Master equation $\mathcal{M}(a, b, c) = 0$ (3.11). Indeed, the equation $S(a, b, c) = 0$ can be used to express m_2 in terms of e_1 and e_2 that can be substituted into Eq. (6.3); one then obtains

$$4(e_1 - 3)H(a, b, c) = 3(e_1 - 4)\mathcal{M}(a, b, c), \quad (6.4)$$

so that $H(a, b, c) = 0$ implies $\mathcal{M}(a, b, c) = 0$ and vice versa (provided that e_1 is not equal to 3 or 4).

For a given value of the coefficient m_2 , solving Eqs. (6.2) and (6.3) is straightforward. The equation $S(a, b, c) = 0$ can be easily solved to give (assuming that $m_2 \neq -6$)

$$e_2 = -\frac{18(e_1 - 1)}{m_2(e_1 - 3) - 12}. \quad (6.5)$$

The expression (6.5) then is to be substituted into (6.3) to give

$$\begin{aligned} H(a, b, c) &= -\frac{3(e_1 - 4)}{4(e_1 m_2 - 3m_2 - 12)^2} Y(a, b, c) = 0, \\ Y(a, b, c) &= -6e_3(m_2 e_1 - 3m_2 - 12)^2(\eta - 1) \\ &\quad + (e_1 - 1)^2 y(a, b, c), \\ y(a, b, c) &= (m_2^2 e_1^2 (e_1 - 3) - 6m_2(4e_1 - 3)(e_1 - 3) \\ &\quad + 72(2e_1 - 9)). \end{aligned} \quad (6.6)$$

Provided that $e_1 \neq 4$, we arrive at the equation

$$Y(a, b, c) = 0. \quad (6.7)$$

In the case when $\eta \neq 1$, Eq. (6.7) is a linear equation in terms of e_3 and its solution provides us with e_3 as a rational function of e_1 ,

$$e_3 = \frac{(e_1 - 1)^2 y(a, b, c)}{6(\eta - 1)(m_2 e_1 - 3m_2 - 12)^2}. \quad (6.8)$$

Together with (6.5), this gives us (for any given value of m_2) e_2 and e_3 as functions of e_1 , the possible value of which remains arbitrary subject to the consistency bounds that we discussed earlier. We see that one obvious solution for any $\eta \neq 1$ is $e_1 = 1$, $e_2 = 0$, $e_3 = 0$.

In the case when $\eta = 1$, this equation reduces to the equation $(e_1 - 1)^2 y(a, b, c) = 0$. An obvious solution is $e_1 = 1$ (and $e_2 = 0$). This was our choice of the coefficients in Eqs. (5.7) and (5.8) to have this classical Kasner solution of the semiclassical equations when $\eta = 1$. For e_1 different from 1, the equations reduce to $y(a, b, c) = 0$. This is a cubic equation. Thus, it may have at most three real solutions for a given m_2 . For each value of e_1 , the corresponding value of e_2 is then given by Eq. (6.5). The value of e_3 remains arbitrary in this case. This is in full analogy with what we had in the classical case.

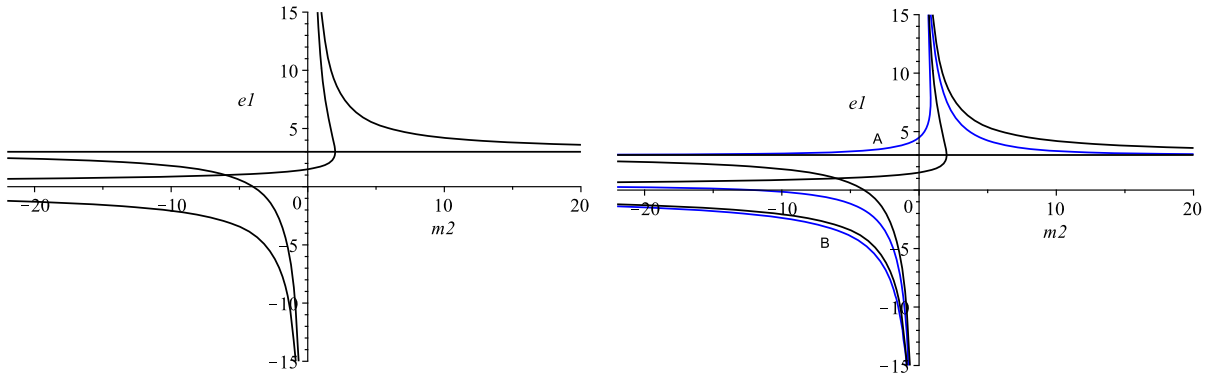


FIG. 8. Left: plot of the zeros of the numerator and denominator of the rational function appearing on the left-hand side of (6.8). The rational function is positive for large positive and large negative values of both e_1 and m_2 , and it changes sign whenever a black contour is crossed. Right: The plot additionally shows the four curves defined by $y(a, b, c) = 0$. Among these, only the two curves labeled **A** and **B** satisfy the consistency bound.

We recall that not any value of e_2 is possible since it should satisfy a consistency bound, $e_2 \leq \frac{e_1^2}{3}$. For e_2 , given by Eq. (6.5), we thus have the following inequality:

$$\frac{(e_1 - 3)(e_1^2 m_2 - 12e_1 + 18)}{(e_1 m_2 - 3m_2 - 12)} \geq 0. \quad (6.9)$$

This inequality should hold for any value of $\eta = A/C$. It imposes important constraints on the possible values of e_1 . The corresponding allowed regions are shown in Fig. 8. The inequality is satisfied for large values of e_1 and m_2 in the upper-right and lower-left corners of the parameter plane, and its sign changes each time one crosses either the black or green line. Figure 8 also displays the four curves that solve the equation $y(a, b, c) = 0$, corresponding to the case $\eta = 1$. Among the four curves, only two of them (labeled **A** and **B** in the figure) satisfy the consistency bound.

A. Singular but geodesically complete metrics as solutions

It is instructive to examine more closely the limit of large negative values of e_1 . This regime is reached for small negative values of m_2 , since $e_1 = \frac{12}{m_2} (1 - \frac{1}{4}m_2 - \frac{1}{12}m_2^2 + O(m_2^3))$, as can be seen from the equation $y(a, b, c) = 0$. This is the blue curve labeled **B** in Fig. 8. From Eq. (6.5), we find that $e_2 = \frac{e_1^2}{4}$ to the leading order, and the consistency bound is clearly satisfied. As e_3 may take any permitted value, we choose $e_3 = s e_1^3$, where $s > 0$ is a parameter. Then, the values of a, b, c are the three roots of the cubic equation $x^3 - e_1 x^2 + e_2 x - e_3 = 0$. This equation has three real roots $x_1 < x_2 < x_3$ provided $s < 1/54$. All roots are negative: $x_1 < e_1/2$, $x_2 < e_1/6$, $x_3 < 0$. For the third root, we have an estimate that $x_3 \simeq 0.045 e_1$ provided $s = 1/108$. Hence, for large negative e_1 (small negative m_2), all three parameters a, b, c that enter the Kasner-type metric (2.1) take negative values that satisfy the condition (2.9).

The corresponding metric therefore possesses a curvature singularity at $t = 0$; however, this singularity lies at an infinite value of the affine parameter for any null geodesic traced backward in time toward it. In other words, any light emitted from the singularity can never reach any observer at a later time. This result demonstrates that within the space of solutions to the semiclassical gravitational equations, there exists a domain of parameters in which the spacetime exhibits a curvature singularity while remaining geodesically complete.

This consideration can be extended to nonvanishing values of η ($A \neq B$). The solution (for a given value of m_2) in this case is fully specified by choosing a value of e_1 that satisfies the condition (6.9). Then, values of e_2 and e_3 are given by Eqs. (6.5) and (6.8), respectively. Provided that m_2 is negative and its value is sufficiently small, we may choose

$$e_1 = \frac{12}{m_2} - \lambda, \quad (6.10)$$

where λ is a free parameter to be further constrained. We find for the respective values of e_2 and e_3 to leading order

$$e_2 = \frac{3}{2(\lambda + 3)} e_1^2 \quad \text{and} \quad e_3 = \frac{(2\lambda + 3)(\lambda - 3)}{12(\lambda + 3)^2(\eta - 1)} e_1^3. \quad (6.11)$$

The conditions $e_2 > 0$ and $e_2 \leq \frac{e_1^2}{3}$ are satisfied for $\lambda > 3/2$. Then, e_3 scales as e_1^3 for large negative values of e_1 , and the coefficient of proportionality is positive if $\lambda > 3$ (for $\eta > 1$) or $3/2 < \lambda < 3$ (for $\eta < 1$). The bound (2.14) imposes an extra constraint on the prefactor in e_3 (6.11). Under these conditions, all three parameters a, b, c take negative values that scale linearly with e_1 and hence satisfy the condition (2.9). Whether all three parameters a, b, c can simultaneously be negative and less than -1 for other values of m_2 requires further investigation.

VII. SEMICLASSICAL SOLUTION IN THE ISOTROPIC CASE: REGULAR SPACETIME

In the previous section, we mostly focused on the case when all three parameters a , b , c take different values. Now we want to discuss the isotropic case when all three parameters have the same value, $a = b = c$. In Sec. III C, we already considered this case when we discussed the condition of the vanishing of the anomaly, $\mathcal{M}(a, a, a) = 18\eta a^3(a - 1)$. There we concluded that this condition selects two possible values: $a = 0$ and $a = 1$. The first is just regular spacetime that looks like Minkowski and the second has a curvature singularity.

In the present section, we would like to extend the discussion to include all tensorial components of the gravitational semiclassical equations.

Let us return to Eqs. (4.5) and (4.6) for the functions $G(a; b, c)$ (related to the component T_0^0 of the anomalous stress tensor) and $F(a; b, c)$ (related to the component T_a^a). As we discussed in Secs. IV–VI, solving these equations involves considerable freedom. In the case when a , b , and c take the same value, this freedom disappears and the result is unique. Indeed, Eqs. (4.5) and (4.6) become

$$\begin{aligned} G(a) + 3F(a) &= 18\eta a^3(a - 1), \\ 12F(a)(a - 1) &= (3a - 4)18\eta a^3(a - 1), \end{aligned} \quad (7.1)$$

where we use the notations $G(a) = G(a; a, a)$ and $F(a) = F(a; a, a)$. This has the unique solution

$$F(a) = \frac{3}{2}\eta(3a - 4)a^3, \quad G(a) = \frac{9}{2}\eta a^4. \quad (7.2)$$

We stress that this solution is unambiguous: no free constants are involved. The gravitational equations then reduce to $F(a) = 0$ and $G(a) = 0$. We find that only the value $a = 0$ remains as a valid solution, while $a = 1$ appears to be spurious. Thus, only the isotropic spacetime, which is free of singularities, satisfies the complete semiclassical equations. This is, of course, a well-known feature of the Starobinsky model [4]. A detailed analysis of the isotropic case is presented in Appendix B, where we show that the absence of the singularity arises from a particular choice of quantum state. The remaining freedom is in the solution of the homogeneous differential equation that we systematically neglected in this paper. In the isotropic case, adding this solution is equivalent to adding an isotropic radiation to the quantum state. This generically leads to the appearance of the singularity. On the other hand, as was explained in [4], the absence of such radiation in the quantum state leads to a completely regular isotropic solution of gravitational equations.

VIII. CONCLUSIONS

In the present paper, we addressed the problem of cosmological singularities in semiclassical gravity modified by a quantum CFT. For Kasner-type metrics, the leading-order behavior in the vicinity of the singularity reduces the problem to a set of algebraic constraints imposed on the Kasner parameters. We determined the complete asymptotic structure of the anomalous stress-energy tensor of the quantum CFT, an uncommon situation, as such analyses can be carried out explicitly only in a few very special cases. Integrating the conformal anomaly for Kasner-type metrics allows us to obtain the full set of constraints on the parameters, which can be solved conveniently when expressed in terms of symmetric polynomials.

The classical Kasner solution appears as a special case of our solutions when $\eta = A/C = 1$ (A and C are the conformal charges), which in particular includes the holographic CFT. In addition to this classical branch, we found a family of solutions that differ substantially from the classical case. Classically, the Kasner parameters are bounded and lie within a finite domain. In contrast, the solutions to the semiclassical equations are generically unbounded. Configurations in which one parameter is negative and the other two are positive, typical of the classical case, are not generic here and, in fact, do not appear when $0 < \eta < 1$. Such configurations arise when $\eta > 1$ and $\eta < 0$.

Moreover, we identified open regions in the parameter space in which all three Kasner parameters are either negative or positive. This suggests that the asymptotic approach to the singularity may differ significantly from the usual BKL scenario. Determining its precise nature requires further investigation based on the results presented here.

Another interesting finding is the existence of solutions in which all three Kasner parameters are negative and less than -1 . We provided explicit examples of such solutions. These solutions are notable because they represent a new type of resolution to the cosmological singularity problem. Although a curvature singularity at $t = 0$ still exists, the geodesics are complete, placing the singularity at infinite affine distance from any physical observer. This scenario deserves a more detailed examination.

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DATA AVAILABILITY

The data are available from the authors upon reasonable request.

APPENDIX A: FUNCTION $F(a; b, c)$ FOR A KASNER-TYPE METRIC

In this appendix, we solve Eq. (4.6) and find the function $F(a; b, c)$. We consider generic variables x, y, z and the function $F(z; x, y) = F(z; y, x)$ symmetric in the last two arguments and assume that it is a polynomial of degree 4 of its arguments. Since it is symmetric in x and y , it can be decomposed in powers of symmetric basic polynomials $(x + y)$ and xy . It can be thus represented as follows:

$$F(z; x, y) = \sum_{n=0}^4 F_{(n)}, \quad F_{(n)} = \sum_{\substack{(l) \\ k+p+q=n}} f_{nl} z^k (y+x)^p (yx)^q, \quad (\text{A1})$$

where f_{nl} are the coefficients to be determined, and the index l changes from 1 to a maximum value that depends on number of independent polynomials at order n . More precisely, we have that

$$\begin{aligned} F_{(0)} &= f_0, & F_{(1)} &= f_{11}(x+y) + f_{12}z, & F_{(2)} &= f_{21}(y+x)^2 + f_{22}xy + f_{23}z^2 + f_{24}(x+y)z, \\ F_{(3)} &= f_{31}(y+x)^3 + f_{32}(x+y)xy + f_{33}xyz + f_{34}z(x+y)^2 + f_{35}z^2(x+y) + f_{36}z^3, \\ F_{(4)} &= f_{41}(x+y)^4 + f_{42}(x+y)^2xy + f_{43}(xy)^2 + f_{44}z(x+y)^3 + f_{45}z(x+y)xy \\ &\quad + f_{46}z^2(x+y)^2 + f_{47}z^2xy + f_{48}z^3(x+y) + f_{49}z^4. \end{aligned} \quad (\text{A2})$$

This representation has to be substituted into Eq. (4.6). This equation is symmetric in a, b , and c and thus can be decomposed over the basis e_1, e_2 , and e_3 defined in (2.10). This equation is a polynomial of degree 5 and thus has 16 independent terms. This means that Eq. (4.6) reduces to 16 different equations for the coefficients f_{nl} . In (A3), there are 22 such coefficients. A naive counting would suggest that there remain $22 - 16 = 6$ undetermined coefficients. The direct calculation, however, shows that there are seven such coefficients. It appears that one of the equations follows from the others. There is an ambiguity in which coefficients are to be taken as these seven coefficients that define all other coefficients. Here is one possible choice to solve the system of linear equations that come from Eq. (4.6):

$$\begin{aligned} f_0 &= 0, & f_{11} &= -9 + 24\eta + 8f_{34} + 2f_{24} + 4f_{33}, \\ f_{12} &= 18 - 48\eta - 16f_{34} - 4f_{24} - 8f_{33}, \\ f_{21} &= -108 + 12\eta - 4f_{34} + 56f_{48} - 184f_{49} - f_{24} - 8f_{45} + 32f_{42}, \\ f_{22} &= 425/2 - 12\eta + f_{24} + 12f_{34} + 2f_{33} - 112f_{48} + 368f_{49} + 16f_{45} - 64f_{42}, \\ f_{23} &= 439/2 - 36\eta + 4f_{34} + f_{24} - 2f_{33} - 112f_{48} + 368f_{49} + 16f_{45} - 64f_{42}, \\ f_{31} &= 57 - 12\eta - 28f_{48} + 94f_{49} + 4f_{45} - 16f_{42} - f_{33}, \\ f_{32} &= -116 + 18\eta - 3f_{34} + f_{33} + 60f_{48} - 196f_{49} - 8f_{45} + 32f_{42}, \\ f_{35} &= -84 + 38f_{48} + 21\eta + f_{34} - 132f_{49} - 6f_{45} + 24f_{42} + 2f_{33}, \\ f_{36} &= -56 + 12\eta + 28f_{48} - 96f_{49} - 4f_{45} + 16f_{42} + f_{33}, \\ f_{41} &= -1/2 - f_{49}, & f_{44} &= 7/2 + 6f_{49} - f_{42} - 2f_{48}, \\ f_{43} &= -59/2 + 9/2\eta - 47f_{49} + 8f_{42} + 15f_{48} - 3f_{45}, \\ f_{46} &= 19/2 - 3/2\eta + 15f_{49} - 3f_{42} - 4f_{48} + f_{45}, \\ f_{47} &= 6\eta - 57/2 - 46f_{49} + 8f_{42} + 15f_{48} - 3f_{45}. \end{aligned} \quad (\text{A3})$$

The seven coefficients $f_{34}, f_{24}, f_{33}, f_{42}, f_{45}, f_{48}, f_{49}$ remain undetermined by the equations.

From the fourth-order polynomial $F(a; b, c)$ one may define a third-order polynomial $E(c; a, b) = E(c; b, a)$ by the relation

$$F(a; b, c) - F(b; a, c) = (a - b)E(c; a, b), \quad (\text{A4})$$

and then also a second-order polynomial $K(b; a, c)$ defined as follows:

$$E(c; a, b) - E(a; c, b) = (c - a)K(b; a, c). \quad (\text{A5})$$

It is an important fact that for $F(a; b, c)$ defined by (A1)–(A4), the polynomial $K(b; a, c)$ is symmetric in (a, b, c) so that $K(b; a, c) = K(a, b, c)$. Additionally, there is a constraint on the coefficients in the decomposition over a symmetric basis, and the exact form then reads

$$K(a, b, c) = k_1(a^2 + b^2 + c^2) + k_2(ab + ac + bc) + k_3(a + b + c) - 3(k_1 + k_2 + k_3). \quad (\text{A6})$$

For the coefficients, one finds

$$\begin{aligned} k_1 &= 2f_{42} + 2f_{48} - 8f_{49} - 5, \\ k_2 &= 14f_{42} - 4f_{45} + 22f_{48} - 72f_{49} + 6\eta - 45, \\ k_3 &= -4f_{34} + 16f_{42} - 4f_{45} + 32f_{48} - 104f_{49} + 6\eta - 58. \end{aligned} \quad (\text{A7})$$

On the other hand, the symmetrization of polynomials $E(a; b, c)$ in all three arguments produces a third-order symmetric polynomial

$$S(a, b, c) = E(a; b, c) + E(b; a, c) + E(c; a, b) \quad (\text{A8})$$

that can be decomposed over the basis e_1 , e_2 , and e_3 . There are in general seven coefficients in this decomposition,

$$S(a, b, c) = e_1^3 m_1 + e_1 e_2 m_2 + e_3 m_3 + e_1^2 m_4 + e_2 m_6 + e_1 m_5 + m_7. \quad (\text{A9})$$

The direct calculation reveals some algebraic relations between the coefficients,

$$\begin{aligned} m_6 + m_3 + 3m_2 + 12 &= 0, \\ m_7 + 27m_1 + 3m_5 + 9m_4 + 2m_3 - 36 &= 0. \end{aligned} \quad (\text{A10})$$

One can use these relations to express m_6 and m_7 in terms of other coefficients. (This is, of course, an ambiguous choice.) This leads to the form

$$\begin{aligned} S(a, b, c) &= m_1(e_1^3 - 27) + m_2 e_2(e_1 - 3) \\ &+ m_3(e_3 - e_2 + 2) + m_4(e_1^2 - 9) \\ &+ m_5(e_1 - 3) + 12(3 - e_2), \end{aligned} \quad (\text{A11})$$

with only five coefficients that are defined as

$$\begin{aligned} m_1 &= -4f_{48} + 28f_{49} - 4f_{42} + 13, \\ m_2 &= 43 - 18f_{48} + 52f_{49} - 14f_{42} + 8f_{45} - 12\eta, \\ m_3 &= 90f_{42} - 36f_{45} + 162f_{48} - 504f_{49} + 54\eta - 315, \\ m_4 &= 272f_{48} - 932f_{49} + 132\eta + 12f_{33} + 8f_{34} + 160f_{42} \\ &\quad - 40f_{45} - 562, \\ m_5 &= -672f_{48} + 2208f_{49} - 216\eta + 12f_{24} - 12f_{33} \\ &\quad + 24f_{34} - 384f_{42} + 96f_{45} + 1317 \end{aligned} \quad (\text{A12})$$

in terms of the independent coefficients f_{nl} .

In fact, it is convenient to choose $k_1, k_3, m_1, m_2, m_3, m_4$, and m_5 as independent variables and express all coefficients f_{nl} in terms of them,

$$\begin{aligned} f_{24} &= \frac{1}{4}m_1 + 2k_1 + \frac{1}{12}m_4 + \frac{1}{18}m_3 + \frac{2}{3}k_3 + \frac{1}{12}m_6, \\ f_{33} &= \frac{1}{12}m_4 + \frac{1}{4}m_1 + \frac{7}{4} - \frac{3}{2}k_1 - \frac{1}{3}m_2 - \frac{5}{27}m_3 - 6\eta + \frac{1}{6}k_3, \\ f_{34} &= \frac{1}{6}m_2 + \frac{1}{4}k_1 + \frac{7}{108}m_3 - \frac{1}{4}k_3, \\ f_{42} &= \frac{1}{12}m_1 + \frac{5}{4} + \frac{11}{12}k_1 - \frac{1}{12}m_2 - \frac{1}{54}m_3, \\ f_{45} &= \frac{1}{6}m_2 + \frac{13}{12}k_1 + \frac{1}{108}m_3 + \frac{1}{6}m_1 - 1 + \frac{3}{2}\eta, \\ f_{48} &= \frac{1}{4}m_1 + \frac{1}{4} + \frac{1}{4}k_1 + \frac{1}{12}m_2 + \frac{1}{54}m_3, \\ f_{49} &= \frac{1}{12}m_1 - \frac{1}{4} + \frac{1}{6}k_1. \end{aligned} \quad (\text{A13})$$

One also finds that

$$k_2 = \frac{1}{9}m_3 + 2k_1. \quad (\text{A14})$$

Then, one has that for the second-order symmetric polynomial

$$K(a, b, c) = k_1(e_1^2 - 9) + \frac{1}{9}m_3(e_2 - 3) + k_3(e_1 - 3) \quad (\text{A15})$$

if it is expressed in terms of the basis of symmetric polynomials.

The next symmetric polynomial, of degree 4, is defined as

$$H(a, b, c) = F(a; b, c) + F(b; a, c) + F(c; a, b). \quad (\text{A16})$$

It can be decomposed over the symmetric basis

$$\begin{aligned}
H(a, b, c) = & h_0 + h_1 e_1 + h_{21} e_1^2 + h_{22} e_2 + h_{31} e_1^3 \\
& + h_{32} e_1 e_2 + h_{33} e_3 + h_{41} e_1^4 \\
& + h_{42} e_1^2 e_2 + h_{43} e_2^2 + h_{44} e_1 e_3.
\end{aligned} \quad (\text{A17})$$

The coefficients h_{nl} then are expressed in terms of $k_1, k_3, m_1, m_2, m_3, m_4$, and m_5 as follows:

$$\begin{aligned}
h_0 &= 0, & h_1 &= 0, \\
h_{21} &= -\frac{3}{4}m_1 - \frac{1}{12}m_5 - \frac{1}{4}m_4 + \frac{1}{18}m_3, \\
h_{22} &= \frac{1}{4}m_5 - \frac{1}{6}m_3 + \frac{9}{4}m_1 + \frac{3}{4}m_4, \\
h_{31} &= -\frac{1}{12}m_4 - \frac{1}{2}k_1 - \frac{1}{6}k_3 - \frac{1}{4}m_1 + \frac{9}{4}, \\
h_{32} &= \frac{9}{4}k_1 + \frac{1}{36}m_3 + \frac{3}{4}k_3 + \frac{3}{4}m_1 + \frac{1}{4}m_4 - \frac{35}{4}, \\
h_{33} &= -18\eta - \frac{27}{4}k_1 - \frac{1}{4}m_3 - \frac{9}{4}k_3 + 18, \\
h_{41} &= -\frac{1}{6}k_1 - \frac{1}{12}m_1 - \frac{3}{4}, \\
h_{42} &= \frac{3}{4}k_1 - \frac{1}{54}m_3 + \frac{1}{4}m_1 - \frac{1}{12}m_2 + \frac{15}{4}, \\
h_{43} &= \frac{1}{12}m_3 + \frac{1}{4}m_2 - 3, \\
h_{44} &= \frac{9}{2}\eta - \frac{9}{4}k_1 - \frac{1}{12}m_3 - \frac{9}{2}.
\end{aligned} \quad (\text{A18})$$

We see that there are nine nonvanishing coefficients. Obviously, not all of them can be independent. Indeed, we find the following relations between these coefficients:

$$\begin{aligned}
h_{22} + 3h_{21} &= 0, \\
h_{33} + 27h_{31} + 9h_{32} + 18\eta &= 0, \\
h_{44} + 9h_{42} + 27h_{41} + 3h_{43} - \frac{9}{2}\eta &= 0.
\end{aligned} \quad (\text{A19})$$

Thus, only six coefficients are independent. For these six independent coefficients, we choose $h_{21}, h_{31}, h_{33}, h_{41}, h_{42}, h_{44}$. The resultant expression for $H(a, b, c)$ is given in the main text.

APPENDIX B: ISOTROPIC COSMOLOGY

Here we present a complete analysis of the isotropic case. The material in this appendix is not entirely original, as the model under consideration is the well-known Starobinsky model [4]. We include this discussion, however, to clarify some points that are not always made explicit.

We consider a general time evolution with a scale factor $a(t)$,

$$ds^2 = -dt^2 + a(t)^2(dx_1^2 + dx_2^2 + dx_3^2). \quad (\text{B1})$$

For this metric, the Weyl tensor vanishes, as is well known. The Euler density, on the other hand takes, the form

$$E_4 = \frac{24\ddot{a}\dot{a}^2}{a^3}. \quad (\text{B2})$$

The components of the quantum stress-energy tensor $T_{\mu\nu}$ then are represented as

$$T_0^0 = \hat{G}(t), \quad T_j^i = \hat{F}(t)\delta_j^i, \quad i, j = 1, 2, 3. \quad (\text{B3})$$

The conservation equation $\nabla_\mu T_\nu^\mu = 0$ produces only one nontrivial constraint,

$$\partial_t \hat{G}(t) + \frac{3\dot{a}}{a} \hat{G} - \frac{3\dot{a}}{a} \hat{F} = 0. \quad (\text{B4})$$

The further constraint comes from the anomaly,

$$\hat{G}(t) + 3\hat{F}(t) = -\mathcal{A}(t), \quad \mathcal{A}(t) = mE_4 = \frac{24m\ddot{a}\dot{a}^2}{a^3}, \quad (\text{B5})$$

where $m = \frac{GA}{2\pi}$ and A is the conformal charge. We assume that $m > 0$. Substituting this into Eq. (B4), one finds that

$$\partial_t \hat{G} + \frac{4\dot{a}}{a} \hat{G} = -\frac{24m\ddot{a}\dot{a}^3}{a^4}. \quad (\text{B6})$$

This differential equation can be solved as follows:

$$\hat{G}(t) = -\frac{6m\dot{a}^4}{a^4} + \frac{C_0}{a^4}, \quad (\text{B7})$$

where the second term is a solution to the homogeneous equation; when the rhs in (B6) vanishes, C_0 is an integration constant. The freedom to choose C_0 is due to the choice of the quantum state. A nonvanishing C_0 can be interpreted as due to the presence of isotropic radiation. One finds that

$$\hat{F}(t) = -\frac{C_0}{3a^4} + \frac{2m\dot{a}^4}{a^4} - \frac{8m\ddot{a}\dot{a}^2}{a^3}. \quad (\text{B8})$$

The semiclassical gravitational equations take the form

$$G_\nu^\mu = T_\nu^\mu, \quad G_\nu^\mu = R_\nu^\mu - \frac{1}{2}\delta_\nu^\mu R. \quad (\text{B9})$$

For the isotropic metric (B1), one finds for the components of the Einstein tensor

$$G_0^0 = -\frac{3}{\dot{a}^2} a^2, \quad G_j^i = -\left(\frac{\dot{a}^2}{a^2} + \frac{2\ddot{a}}{a}\right)\delta_j^i. \quad (\text{B10})$$

The (00) component of Eq. (B9) then leads to the equation

$$6m\left(\frac{\dot{a}}{a}\right)^4 - 3\left(\frac{\dot{a}}{a}\right)^2 - \frac{C_0}{a^4} = 0. \quad (\text{B11})$$

It is solved as follows:

$$\dot{a}^2 = \frac{1}{4m} \left(a^2 \pm \sqrt{a^4 + \frac{8mC_0}{3}} \right). \quad (\text{B12})$$

The behavior of the solution then depends on the value of the integration constant C_0 .

1. $C_0 = 0$

In this case, Eq. (B12) reduces to [excluding the possible solution when $a(t) = \text{const}$]

$$\dot{a}^2 = \frac{a^2}{2m}. \quad (\text{B13})$$

The solution to this equation, $a(t) \sim e^{\pm \frac{t}{\sqrt{2m}}}$, is free of singularities and it describes de Sitter spacetime. This is the solution found in [4].

2. $C_0 > 0$

In this case, only the plus sign can be taken in (B12). In the regime when $a(t)^4 \gg \frac{8mC_0}{3}$, the solution still describes the exponentially growing de Sitter-like Universe. On the other hand, for earlier times, the solution approaches a

singularity, where $a(t)$ vanishes. Assuming that this happens for $t = 0$, one finds that $a(t) \sim t$ in this case.

3. $C_0 < 0$

In this case, both the plus and minus signs in Eq. (B12) are allowed. For the plus sign, the solution in the regime of large $a(t)$ still approaches a de Sitter phase, with $a(t)$ growing exponentially. For the minus sign, by contrast, in the regime of a large scale factor, the solution exhibits a power-law behavior, $a(t) \sim \sqrt{t}$, as in a radiation-dominated universe. In both cases, one has $a(t) \geq a_m$, $a_m = (\frac{-8mC_0}{3})^{1/4}$, so the scale factor never vanishes. However, at the minimal value $a = a_m$, the first derivative $\dot{a}$ remains finite, while the second derivative $\ddot{a}$ diverges. Near this point (assumed to occur at $t = 0$), one finds to leading order in t that $a(t) = a_m(1 + \frac{t}{2m^{1/2}} \pm \frac{\sqrt{2}}{6m^{3/4}} t^{3/2})$. This leads to a curvature singularity at $t = 0$. Note that the corresponding isotropic metric near the singularity is not of the Kasner type.

We conclude that the isotropic case is singularity-free only if the integration constant C_0 vanishes. This corresponds to a particular choice of the quantum state. This choice is, in a sense, distinguished by the fact that the resulting quantum stress-energy tensor can be expressed in terms of local curvature invariants, a representation that is possible when the metric is conformally flat (see [15]) and was employed in [4].

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- [1] R. Penrose, Gravitational collapse and space-time singularities, *Phys. Rev. Lett.* **14**, 57 (1965); S. W. Hawking and R. Penrose, The singularities of gravitational collapse and cosmology, *Proc. R. Soc. A* **314**, 529 (1970); S. Hawking, The occurrence of singularities in cosmology. III. Causality and singularities, *Proc. R. Soc. A* **300**, 187 (1967).
 - [2] V. A. Belinsky, I. M. Khalatnikov, and E. M. Lifshitz, Oscillatory approach to a singular point in the relativistic cosmology, *Adv. Phys.* **19**, 525 (1970); A general solution of the Einstein equations with a time singularity, *Adv. Phys.* **31**, 639 (1982).
 - [3] T. Damour and M. Henneaux, Chaos in superstring cosmology, *Phys. Rev. Lett.* **85**, 920 (2000); T. Damour, M. Henneaux, and H. Nicolai, Cosmological billiards, *Classical Quantum Gravity* **20**, R145 (2003).
 - [4] A. A. Starobinsky, A new type of isotropic cosmological models without singularity, *Phys. Lett.* **91B**, 99 (1980).
 - [5] J. Matyjasek, Quantum fields in Bianchi type I spacetimes: The Kasner metric, *Phys. Rev. D* **98**, 104054 (2018).
 - [6] P. McFadden and K. Skenderis, Holography for cosmology, *Phys. Rev. D* **81**, 021301 (2010).
 - [7] A. Bzowski, P. McFadden, and K. Skenderis, Holographic predictions for cosmological 3-point functions, *J. High Energy Phys.* **03** (2012) 091.
 - [8] E. C. Thomas, F. R. Urban, and A. R. Zhitnitsky, The cosmological constant as a manifestation of the conformal anomaly?, *J. High Energy Phys.* **08** (2009) 043.
 - [9] A. O. Barvinsky, C. Deffayet, and A. Y. Kamenshchik, Anomaly driven cosmology: Big boost scenario and AdS/CFT correspondence, *J. Cosmol. Astropart. Phys.* **05** (2008) 020.
 - [10] O. Trivedi, Recent advances in cosmological singularities, *Symmetry* **16**, 298 (2024).
 - [11] C. Berthiere, D. Sarkar, and S. N. Solodukhin, The fate of black hole horizons in semiclassical gravity, *Phys. Lett. B* **786**, 21 (2018); Y. Potaux, D. Sarkar, and S. N. Solodukhin, Quantum states and their back-reacted geometries in 2D dilaton gravity, *Phys. Rev. D* **105**, 025015 (2022); A. del Río, F. J. Marañón-González, and J. Navarro-Salas, Singularity resolution in spherically reduced 2D semiclassical gravity with negative central charge, *Phys. Rev. D* **111**, 045025 (2025).

-
- [12] J. Michael Steele, *The Cauchy-Schwarz Master Class: An Introduction to the Art of Mathematical Inequalities*, Maa Problem Books Series (Cambridge University Press, 2005).
- [13] E. Kasner, Geometrical theorems on Einstein's cosmological equations, *Am. J. Math.* **43**, 217 (1921).
- [14] A. A. Tseytlin, On partition function and Weyl anomaly of conformal higher spin fields, *Nucl. Phys.* **B877**, 598 (2013).
- [15] N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, England, 1982).