

Renormalization and invariants for two U(1)s

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We revisit the renormalization of models with two U(1) gauge symmetries, in a formulation with noncanonical gauge kinetic terms which is covariant under field reparametrizations among the two gauge bosons. This approach is convenient to study the appearance of kinetic mixing in scale evolution because a coupling matrix is attributed to the gauge kinetic terms. We obtain simple $\overline{\text{MS}}$ renormalization group equations up to two-loop, which can be solved to give effective millicharges at low energy which depend on the running couplings and heavy mass scales of the model. This formulation allows one to construct “invariants” out of running Lagrangian parameters, which are invariant under generic gauge field reparametrizations, including rescalings, and which can be related directly to observables such as the millicharge.

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I. INTRODUCTION

In the presence of more than one U(1) gauge symmetry, the gauge bosons can mix in propagation. This is commonly parametrized as a noncanonical kinetic term

$$\mathcal{L} \supset -\frac{\varepsilon}{2} F_{\mu\nu} V^{\mu\nu}, \quad (1.1)$$

where $F^{\mu\nu}$ and $V^{\mu\nu}$ are the field strength tensors of two U(1)s, and ε can be induced in renormalization group running by vacuum polarization diagrams involving heavy particles charged under both U(1)s (see Fig. 2). As discussed by Okun [1] and Holdom [2] long ago, this “kinetic mixing” can be reparametrized into “millicharges” for the light particles in the model. This setup is a popular example of light new physics scenarios, which is constrained by many experimental searches [3–11], and whose astrophysical, experimental, and cosmological consequences have been reviewed in several studies [12–16]. Extra light U(1)s can impersonate diverse scenarios: they could be the harbinger of a mirror universe [17,18], constitute cold dark matter if massive and produced like axions during inflation [19] or via the misalignment mechanism [20,21], and could modify thunderstorms [22].

This project has two aims: first, to renormalize a toy model with two U(1) gauge symmetries, in order to

understand how kinetic mixing fits into renormalization, and address some naive confusions about the identity of kinetic mixing (Is it a finite model prediction? A running parameter?) and the behavior of the couplings (Do they mix as they run? Do the charges run?). The second aim was to construct invariants (*à la* Jarlskog) that correspond to physical observables, irrespective of the parametrization chosen in the Lagrangian.

The renormalization¹ of models with two U(1)s has been studied carefully [26–28] for a Lagrangian with canonical kinetic terms. Reference [26] obtained renormalization group equations (RGEs) at one loop and two loops for a triangular matrix of U(1) gauge couplings. The subtraction of divergences was discussed in [27,28], where it was observed that there are more divergences than U(1)s. It was nonetheless shown in [28], that if one refrains from introducing off-diagonal kinetic counterterms because the tree-level kinetic terms are diagonal, the S -matrix elements are still renormalisable. Luo and Xiao [29,30] included noncanonical kinetic terms in multi-U(1) models, and generalized the two-loop RGEs of Machacek and Vaughn [31–33] with these terms. This generalization is discussed in more detail in [34], where canonical kinetic terms are preferred, so the kinetic mixing is included in the RGEs via a matrix of couplings as in [26,27]. Poole and Thomsen [35] included the gauge couplings in the kinetic terms for generic gauge groups, and obtained gauge β functions to three-loop order by using formal relations between gauge, Yukawa, and quartic couplings; kinetic

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¹The impact of finite kinetic mixing has also been discussed for two [23] and three U(1)s [24] (see also Ref. [25], and references therein, for an application to supersymmetric grand unified theories).

mixing among $U(1)$ s is discussed in their Appendix A. (The four-loop coefficients in the Poole and Thomsen parametrization were obtained in [36,37].) These results have been included in public codes such as PyR@TE [38,39] and RGBeta [40].

We focus on obtaining and solving RGEs because we are interested in renormalization as scale evolution. Since field reparametrizations are widely used to modify the form of the Lagrangian, we aim for a formulation which we call “covariant,” meaning the expressions remain valid under rotations and rescalings among the two $U(1)$ gauge fields. Such a covariant formulation allows one to obtain observables expressed in terms of “invariants.” The Lagrangian will be constructed with noncanonical kinetic terms, that is, with a coupling matrix containing the (gauge coupling) $^{-2}$ multiplying the kinetic terms. This is appropriate because the physics of interest is the appearance of kinetic mixing during scale evolution, and because this formulation is intrinsically covariant under field reparametrizations. Attributing a gauge coupling matrix to the kinetic terms gives a clean separation between the running gauge couplings, which appear in the gauge kinetic terms, and the conserved charges of matter particles, which appear in the covariant derivative. In this respect, our Lagrangian differs from that of Luo and Xiao [29], where the gauge couplings stay in the covariant derivatives and noncanonical kinetic terms are added.

We obtain an invariant for kinetic mixing, constructed from Lagrangian parameters and invariant under gauge field reparametrizations. It corresponds to the coupling combinations entering S -matrix elements and is expressed in terms of running ($\overline{\text{MS}}$) parameters evaluated at the experimental scale. We give a covariant formulation for the renormalization group equations (RGEs), which are linear at one loop where the solution is two independent running couplings. The third parameter of the kinetic matrix, an angle, is scale independent at one loop, and runs at two loops. We also obtain (in Appendix B) the RGEs for invariants formulated in terms of invariants.

Section II presents various forms of the Lagrangian, discusses the field reparametrizations which make them equivalent, and introduces an “invariant” for kinetic mixing. The $\overline{\text{MS}}$ renormalization, and the RGEs up to two loops are reviewed in Sec. III, in a formulation that is covariant under (scale-independent) field reparametrizations. The RGEs are solved bottom up and top down in Sec. IV, which allows us to check that the invariant of Sec. II corresponds to the effective millicharge. We discuss and summarize in Secs. V and VI.

II. NOTATION AND FIELD REPARAMETRIZATIONS

The Lagrangian of a model with two $U(1)$ gauge symmetries can be formulated in various ways. We prefer Lagrangians with noncanonical kinetic terms because

kinetic mixing is an interesting aspect of such models, and because renormalizing the kinetic matrix is simpler than renormalizing the couplings. This section starts by recalling field reparametrizations and how they motivate invariants in Sec. II A, discuss Lagrangians for two $U(1)$ s in Sec. II B, and presents the kinetic mixing invariant in Sec. II C.

A. Symmetries, reparametrizations, and basis choices

Symmetries and field redefinitions can be defined in the context of the path integral. We restrict our discussion to the Lagrangian, on the glib assumption that modifications to the path integral measure (resulting from field redefinitions) are cancelled out of S -matrix elements by dividing by the vacuum-to-vacuum transition amplitude.

A symmetry is a transformation of the fields and/or coordinates, under which the Lagrangian remains *unchanged*—for instance, a global phase rotation of the scalar field in ϕ^4 theory. Global symmetries have associated conserved currents,² which give quantities that are conserved by the dynamics following from the equations of motion. In perturbative quantum field theory, the symmetries of an interaction term (e.g., phase rotations of charged particles) are transferred to the Feynman rules, and states can be labeled by the eigenvalues of the generators of the symmetries (e.g., the charge).

Reparametrizations are redefinitions of the fields where the Lagrangian parameters change in compensation, but observables are invariant. In the path integral, the fields are integration variables, so can be redefined provided that the integrand remains the same. This is the quantum field theory version of variable changes in usual integration, such as

$$\int dA \exp\{-KA^2 + vA\} = \int \frac{d\tilde{A}}{\sqrt{K}} \exp\left\{-\tilde{A}^2 + \frac{v}{\sqrt{K}}\tilde{A}\right\},$$

$$\tilde{A} = \sqrt{K}A.$$

So in the words of Weinberg (Chapter 7.7 of Ref. [41]), such field redefinitions are associated to redundant Lagrangian parameters. Field reparametrizations have been discussed in the context of EFT [42], and are regularly used to make the kinetic terms canonical in Lagrangians [see e.g., Refs. [23,43] for finite reparametrizations in multi- $U(1)$ models].

For some Lagrangians with canonical kinetic terms, there remain unitary transformations on the fields and parameters, under which the Lagrangian and the

²Obtaining conserved currents is very similar to obtaining Lagrangian equations of motion (EoM), just that for the EoM, a total divergence is dropped because the field variation vanishes at ∞ . Whereas in the case of a symmetry, the Lagrangian is invariant under the field variation, which does not vanish at ∞ , so the total divergence must vanish.

observables are invariant—for instance, rotations in quark flavor space in the Standard Model. In these cases, one must choose a basis to calculate, but it can be convenient to construct “basis-independent” “invariants” out of the Lagrangian parameters. This has been performed for various models [44–46]; a well-known example is the Jarlskog invariant [47,48] (see Ref. [49] for a detailed construction of such CP -sensitive invariants).

The “invariants” obtained here differ from previous constructions, in that they are invariant under arbitrary gauge field redefinitions, rather than just under the residual rotations consistent with canonical kinetic terms. This is a simple generalization because invariants can be associated to diagrams, and propagators have coupling constants when the kinetic terms are noncanonical. It is warranted for this model because kinetic mixing is a noncanonical kinetic term, and the various formulations of the model differ by rescalings as well as rotations among the gauge fields.

B. Lagrangians with two U(1) gauge symmetries

This section discusses different versions of the Lagrangian for our model. They differ by field reparametrizations, so should give equivalent S -matrix elements. Our aim is to study the renormalization of the model, for which we need the bare Lagrangian, but it is convenient to first choose the parametrization of the fields, because it affects the Z -factors.

The Lagrangians describe a toy model containing Dirac fermions with two U(1) gauge symmetries. However, the U(1) of the Standard Model (SM) is chiral hypercharge, so a more realistic model (see e.g., Ref. [50]) would be subject to anomaly cancellation conditions and would have different numerical factors. We envisage that these differences have little impact on our aims.

To fix notation we will consider the gauge bosons of two U(1)s assembled into a vector

$$\vec{A}^\mu = (A^\mu, V^\mu).$$

Both are massless for most of our discussion (masses are discussed for instance in [51]). A Lagrangian can be written for an arbitrary parametrization of the gauge bosons (who therefore wear primes) as

$$\begin{aligned} \mathcal{L}_{nc} = & -\frac{1}{4}(F^{\mu\nu'}V^{\mu\nu'}) \begin{bmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{bmatrix} \begin{pmatrix} F'_{\mu\nu} \\ V'_{\mu\nu} \end{pmatrix} \\ & + \sum_f \bar{f}(i\not{D}' - m_f)f + \mathcal{L}_{gf} + \mathcal{L}_{neglect}, \end{aligned} \quad (2.1)$$

where the gauge bosons interact with fermions f via the covariant derivative

$$D^{\mu'} = \partial^\mu + i\vec{v}' \cdot \vec{A}^{\mu'}, \quad (2.2)$$

with $\vec{v}' \cdot \vec{A}' = Q'_f A^{\mu'} + Y'_f V^{\mu'}$, so there are no gauge couplings in the covariant derivative.³ The “noncanonical” (nc) gauge kinetic terms can be written more compactly as $-\frac{1}{4}\vec{F}^{\mu\nu'}[K]\vec{F}'_{\mu\nu}$, where Hermiticity and positivity imply that $[K]$ is a real symmetric positive definite matrix.

The fermions $\{f\}$ will later be divided into η_j s of mass m_j which are charged under both U(1)s, SM ψ s which interact principally with A^μ , and shadow χ s which interact principally with the shadow photon V^μ . The terms in $\mathcal{L}_{neglect}$ which we neglect are sketched in Appendix A.

For the gauge-fixing term we choose $\mathcal{L}_{gf} = -\frac{1}{2}\partial_\mu \vec{A}^\mu[\Xi]\partial_\nu \vec{A}^\nu$, where $[\Xi]$ is real and symmetric. In the following we will choose $[\Xi] = [K]$ for the renormalized Lagrangian, which corresponds to the Feynman gauge for multiple U(1)s and our choice of the kinetic term. This is peculiar because $[K]$ renormalizes and $[\Xi]$ does not; however, the same peculiarity arises in Feynman gauge with canonical kinetic terms, where $\xi = 1$ is not preserved by the renormalization group equations. Yet this is inconsequential for all on-shell matrix elements and for the running of $[K]$ and m_f , as those are all independent of $[\Xi]$.

This gauge-fixing choice is symmetric in U(1) space, so suitable for massless bosons. This can be compared to Ref. [28], which gives a detailed discussion of gauge-fixing two U(1)s (in the canonical Lagrangian formalism) where the second U(1) is massive, with distinct gauge choices for the two U(1)s. We envisage that a gauge boson mass m_V can be neglected in the scale evolution of the model at scales above m_V , so our results should apply, but that at the scale m_V , the massless model should be matched onto a massive one, where a different gauge-fixing scheme could be chosen.

We find the above normalization of the gauge kinetic term advantageous when discussing the renormalization of the theory: The U(1) Ward identities guarantee that $\vec{v}'_f$, $\vec{A}$, and $[\Xi]$ do not renormalize, so only $[K]$ and the fermion masses m_f run.

The form of the Lagrangian can be changed by gauge fields reparametrizations, without affecting observables. Since renormalization will be discussed in Sec. III, it is useful to distinguish whether these gauge field transformations depend, or not, on the renormalization scale μ .

Consider first the case of μ -independent changes to $\mathcal{L}_{nc}$, which will later serve to simplify the initial conditions of the RGEs of this Lagrangian. At any fixed scale, the kinetic matrix $[K]$ can be diagonalized with an orthogonal rotation O_K :

$$O_K[K]O_K^T = [D_K],$$

³This differs from the Lagrangian of [29], where there are gauge couplings in the covariant derivatives and noncanonical kinetic terms.

where the eigenvalues $1/e^2$ and $1/g^2$ of $[K]$ correspond to the gauge couplings when the charges are defined as $(Q_f, \tilde{Y}_f) = (Q'_f, Y'_f) O_K^T$. A further reparametrization of the gauge fields allows one to equate the eigenvalues of the kinetic terms (at tree level, or at some fixed scale in the renormalized theory): we redefine $V^\mu \rightarrow \frac{e}{g} V^\mu$, so that $\tilde{Y}_f \rightarrow Y_f = \frac{g}{e} \tilde{Y}_f$ and the diagonal kinetic matrix becomes

$$[D_K] = \begin{bmatrix} \frac{1}{e^2} & 0 \\ 0 & \frac{1}{g^2} \end{bmatrix}, \quad g^2 = e^2 \quad (2.3)$$

which is then invariant under $O(2)$ rotations in the space of the two $U(1)$ s. So imposing diagonal kinetic terms is not a basis choice for massless gauge fields. The charge vectors corresponding to this kinetic matrix are

$$\vec{v}_f = \begin{pmatrix} Q_f \\ Y_f \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{g}{e} \end{bmatrix} [O_K] \begin{pmatrix} Q'_f \\ Y'_f \end{pmatrix}. \quad (2.4)$$

Consider now renormalization-scale-dependent gauge field redefinitions, which allow to change the form of the Lagrangian. An interesting transformation is to diagonalize $[K]$ at all scales, and absorb the eigenvalues of D_K into the gauge fields,

$$\vec{A}_{\text{can}}^\nu(\mu) = [D_K^{1/2}](\mu) O_K(\mu) \vec{A}^\nu, \quad (2.5)$$

where, exceptionally, the μ dependence is given explicitly. This field reparametrization induces a canonical kinetic term in the Lagrangian at all scales,

$$\mathcal{L}_{\text{can}} = -\frac{1}{4} \vec{F}_{\text{can}}^{\mu\nu} \vec{F}_{\text{can},\mu\nu} + \sum_f \bar{f}(i\vec{D}_{\text{can}} - m_f)f, \quad (2.6)$$

where the covariant derivatives now contain the couplings, $D_{\text{can}}^\mu = \partial^\mu + i\vec{u}_f \cdot \vec{A}_{\text{can}}^\mu$, with $\vec{u}_f = (Q_f e, \tilde{Y}_f g)$.

Another common formulation for a Lagrangian with two $U(1)$ s, is to add Eq. (1.1) to a Lagrangian with canonical kinetic terms, where the matter particles $\{\psi\}$ interact with A^μ , and the $\{\chi\}$ interact with V^μ :

$$\begin{aligned} \mathcal{L}_{\text{other}} = & -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{1}{4} V^{\mu\nu} V_{\mu\nu} - \frac{\varepsilon}{2} V^{\mu\nu} F_{\mu\nu} \\ & + \sum_\psi \bar{\psi}(i\partial - eQ_\psi A)\psi + \sum_\chi \bar{\chi}(i\partial - eY_\chi V)\chi. \end{aligned} \quad (2.7)$$

By making a field reparametrization, such a Lagrangian can be transformed into either of the more general Lagrangians given in Eqs. (2.1) and (2.6). The parameters of the various Lagrangians can be compared using the invariants discussed in Sec. II C.

Finally, we give our conventions for the relative normalization of couplings and charges. The gauge couplings parametrize the relative importance of interactions vs propagation for the gauge fields, whereas charges are required for ≥ 2 matter particles, to modulate the relative strength of their gauge interactions. These two roles are clearly separated in the noncanonical Lagrangian of Eq. (2.1), but the relative normalization can be changed ($Q \rightarrow xQ$, $e \rightarrow e/x$) by a field definition $A^\mu \rightarrow A^\mu/x$. We often address this redundancy by choosing $e = g$ and $Q_\tau = -1$ at some scale.

C. A kinetic mixing invariant

In this section, we construct an invariant for kinetic mixing, which corresponds to fermion scattering, and is unchanged under generic gauge field redefinitions including rescalings. An invariant for a process with external gauge bosons is sketched in Appendix C. Invariants for nonrenormalizable interactions such as the Lamb shift could also be considered.

To construct an invariant associated to Fig. 1, requires the noncanonical gauge boson propagator. After subtracting total derivatives from the quadratic gauge boson terms in $\mathcal{L}_{nc}$ [Eq. (2.1)] and inverting, the propagator is

$$-i[K]^{-1} \left\{ \frac{g^{\mu\nu}}{q^2} - \frac{q^\mu q^\nu}{q^4} \right\} - i[\Xi]^{-1} \frac{q^\mu q^\nu}{q^4} \rightarrow -i[K]^{-1} \frac{g^{\mu\nu}}{q^2}, \quad (2.8)$$

where the Feynman gauge choice of $[\Xi] = [K]$ is implemented after the arrow. So the invariant corresponding to tree scattering amplitudes between particles f_1 and f_2 (see Fig. 1) is

$$I_{f_1 f_2} = \vec{v}_{f_1} \cdot [K^{-1}] \cdot \vec{v}_{f_2} = \vec{u}_{f_1} \cdot \vec{u}_{f_2}, \quad (2.9)$$

where $\vec{v}_f$ are the charge vectors of Eq. (2.4) and the (coupling constant)² are contained in $[K]^{-1}$. The second expression for the invariant is constructed from the Lagrangian with canonical kinetic terms; it is less useful than the first expression because it is not explicitly invariant under the field redefinitions that transform between the Lagrangians of Eqs. (2.1), (2.6), and (2.7).

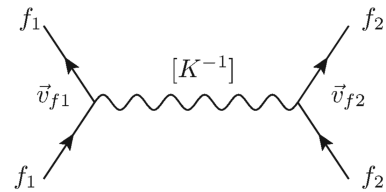


FIG. 1. Tree-level scattering between fermions f_1 and f_2 in a 2 $U(1)$ model parametrized by the Lagrangian with noncanonical kinetic terms of Eq. (2.1). This diagram corresponds to the invariant of Eq. (2.9).

Focusing on a single shadow fermion χ , the inner products of Eq. (2.9) can be combined to obtain a ratio of invariants corresponding to the kinetic mixing or “millicharge,”

$$I_\varepsilon \equiv \frac{\vec{v}_\chi \cdot [K^{-1}] \cdot \vec{v}_\psi}{\vec{v}_\psi \cdot [K^{-1}] \cdot \vec{v}_\psi}, \quad (2.10)$$

where ψ is a SM fermion. I_ε is invariant under generic gauge field reparametrizations, so can be used to compare the parameters of Lagrangians written in different bases and with different field normalizations. In a Lagrangian with canonical kinetic terms, this invariant corresponds to the “millicharge” of shadow particle χ normalized to the SM charge of ψ

$$I_\varepsilon = \frac{Q_\chi}{Q_\psi} \Big|_{\text{can}},$$

where the SM fermions ψ interact exclusively with A^μ , and the shadow χ s interact principally with the second U(1). In the Lagrangian with noncanonical kinetic terms of Eq. (2.7), $I_\varepsilon = -\varepsilon \frac{Y_{\chi g}}{Q_{\psi e}} \Big|_{\text{other}}$.

III. RENORMALIZATION

We are interested in renormalization as scale evolution, so this section gives the RGEs for our model up to two loops. We use dimensional regularization, and renormalize in $\overline{\text{MS}}$, so will later implement our RGEs in effective field theory in order to resum the correct logarithms [52]. The RGEs for the kinetic matrix $[K]$ are covariant in U(1) space and linear at one loop, so we will renormalize the noncanonical Lagrangian of Eqs. (2.1) and (2.3). The results given here agree with [29,35,53].

A. The bare Lagrangian

In dimensional regularization, the bare Lagrangian in $4 - 2\varepsilon$ spacetime dimensions is

$$\begin{aligned} \mathcal{L}_0 = & -\frac{1}{4} \vec{F}^{\mu\nu} [K_0] \vec{F}_{\mu\nu} + \sum_f \bar{f}_0 (i\vec{D} - m_{f,0}) f_0 \\ & - \frac{1}{2} \partial_\mu \vec{A}^\mu [\Xi_0] \partial_\nu \vec{A}^\nu, \end{aligned} \quad (3.1)$$

where the bare fields and parameters can depend on the regulator ε but should not depend on the scale μ . This Lagrangian can be reexpressed in terms of Z factors and running parameters as

$$\begin{aligned} \mathcal{L}_0 = & -\frac{1}{4} \mu^{-2\varepsilon} \vec{F}^{\mu\nu} [Z_K]^T [K] [Z_K] \vec{F}_{\mu\nu} - \frac{1}{2} \partial_\mu \vec{A}^\mu [\Xi] \partial_\nu \vec{A}^\nu \\ & + \sum_f Z_f \bar{f} (i\vec{D} - Z_{m,f} m_f) f, \end{aligned} \quad (3.2)$$

where $D^\mu = \partial^\mu + i\vec{v} \cdot \vec{A}$, and the running parameters have the mass dimension expected in four dimensions and can be related to observations. The Z s contain the counterterms, which in $\overline{\text{MS}}$ can be expanded as a Laurent series in $1/\varepsilon$:

$$Z = 1 + \frac{1}{\varepsilon} Z^{(1)} + \frac{1}{\varepsilon^2} Z^{(2)} + \dots, \quad (3.3)$$

where

$$\frac{1}{\varepsilon} \equiv \left(\frac{1}{\varepsilon} - \gamma + \ln 4\pi \right). \quad (3.4)$$

The $[Z_K]$ matrices will be given in Eqs. (3.6) and (3.7). Several comments are in order about the Lagrangian (3.2).

Firstly, the gauge fields do not renormalize [54] and have mass-dimension one. This occurs because the vacuum polarization diagrams renormalize the kinetic matrix $[K]$ rather than the gauge fields. The usual Ward identities, such as $Z_1 = Z_2$ in QED, are therefore more transparent. In addition, since the Ward identities imply that the gauge boson counterterms are transverse, this implies that $[\Xi]$ is not renormalized, $Z_\Xi = 1$. As discussed after Eq. (2.1), we take $[K] = [\Xi]$ which corresponds to Feynman gauge.

Secondly, the renormalization is multiplicative and Z_K is a matrix, in order to obtain Z_K factors which are explicitly covariant in U(1) space,

$$\begin{aligned} [Z_K]^T [K] [Z_K] = & \left(\mathbb{1} + \frac{1}{\varepsilon} \begin{bmatrix} Z_{K,11}^{(1)} & Z_{K,21}^{(1)} \\ Z_{K,12}^{(1)} & Z_{K,22}^{(1)} \end{bmatrix} + \dots \right) \begin{bmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{bmatrix} \\ & \times \left(\mathbb{1} + \frac{1}{\varepsilon} \begin{bmatrix} Z_{K,11}^{(1)} & Z_{K,12}^{(1)} \\ Z_{K,21}^{(1)} & Z_{K,22}^{(1)} \end{bmatrix} + \dots \right) \\ \approx & [K] + \frac{1}{\varepsilon} [Z_K^{(1)}]^T [K] + \frac{1}{\varepsilon} [K] [Z_K^{(1)}], \end{aligned} \quad (3.5)$$

where $[K]$ is symmetric and real, $[Z_K]$ is not required to be symmetric, and the $+\dots$ are terms higher order in $1/\varepsilon$ which are neglected.

This covariant formalism for $[Z_K]$ differs from the common choice of writing a Z -factor for each element of a matrix. A first difference is that $[Z_K]$ has four matrix elements to implement the three counterterms that could be required for the symmetric $[K]$. We do not consider this to be a problem because we later obtain a relation between elements of $[Z_K]$. A second difference is that our formalism allows a counterterm for K_{12} , even if $[K]$ is diagonal. We consider this acceptable because the rotation to diagonalize $[K]$ is μ dependent at ≥ 2 loops, meaning that the RGEs regenerate an initially vanishing K_{12} (in conventions where the charges are μ independent). However, Ref. [27] attached importance to the number of counterterms, and Ref. [28] showed that, in a model with a fermion, a massless and a massive gauge boson, observables could

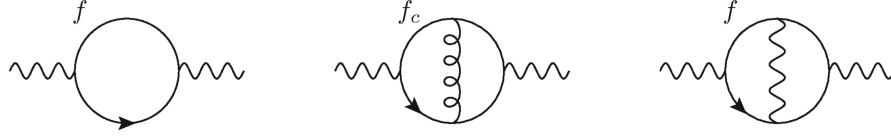


FIG. 2. The one-loop diagram and representative two-loop diagrams contributing to vacuum polarization in a 2 U(1) model containing various fermions $\{f\}$. There are additional two-loop diagrams with the gauge boson vertices differently ordered on the fermion loop; only colored fermions f_c contribute in the QCD diagram.

be renormalized without introducing a counterterm for K_{12} , but with mixing between the gauge couplings.

The $1/\hat{\epsilon}$ term of the Z_K matrix will be required to obtain the RGEs. It arises from vacuum polarization diagrams as illustrated in Fig. 2. For canonical kinetic terms (and with the gauge choice $\xi = 1$), these diagrams are written as

$$\frac{-ig_{\mu\rho}}{q^2} i[\Pi^{\rho\sigma}](q^2) \frac{-ig_{\sigma\nu}}{q^2},$$

where the amputated bubble $i[\Pi^{\rho\sigma}]$ is a matrix in the space of the two U(1)s, commonly written as $i(q^2 g^{\mu\nu} - q^\mu q^\nu)[\Pi(q^2)]$, and presented in many textbooks. For the noncanonical Lagrangian, the coupling constants e^2 and g^2 are in the amputated external propagators [see Eq. (2.8)], rather than in $[\Pi(q^2)]$. So for Dirac fermions at one loop, using Eq. (3.5)

$$Z_K^{(1)}|_{1\text{ loop}} = -\frac{1}{24\pi^2} [K^{-1}] \left[\sum_f \vec{v}_f \vec{v}_f^T \right],$$

$$\left[\sum_f \vec{v}_f \vec{v}_f^T \right] = \begin{bmatrix} \sum_f Q_f^2 & \sum_f Q_f Y_f \\ \sum_f Q_f Y_f & \sum_f Y_f^2 \end{bmatrix}, \quad (3.6)$$

where there is an additional factor of 1/2 in the one-loop coefficient for chiral fermions.

The two-loop contribution to the matrix $Z_K^{(1)}$ is expected to be numerically small compared to one loop (in our model where kinetic mixing arises at one loop; models where kinetic mixing first arises at higher order have been considered [55]). However, it is formally interesting because it has a different functional dependence on the couplings. Two-loop effects have been calculated for QED [56] (see also the summary in the three-loop calculation [57]), and we will include two-loop contributions to the RGEs by generalizing the QED β function given in [53]. For this, we only need the “covariants” [matrices in the 2 U(1) space] corresponding to the two-loop diagrams of Fig. 2. Since the amputated one-loop diagram gave the matrix $\sum_f \vec{v}_f \vec{v}_f^T$, the two-loop QCD and U(1) diagrams respectively give

$$Z_K^{(1)}|_{2\text{ loop}} = b_c g_s^2 [K^{-1}] \left[\sum_c \vec{v}_c \vec{v}_c^T \right] + b_{U1} [K^{-1}] \left[\sum_f (\vec{v}_f \cdot [K^{-1}] \cdot \vec{v}_f) \vec{v}_f \vec{v}_f^T \right], \quad (3.7)$$

where b_c and b_{U1} are numerical coefficients given at Eq. (3.9), g_s is the running coupling of QCD, and c indexes colored fermions.

B. Renormalization group equations

This section derives the RGEs for the kinetic matrix $[K]$ (following the review [58]), from which RGEs for the gauge couplings can be obtained.

The $\overline{\text{MS}}$ RGEs for $[K]$ can be obtained by imposing that the bare $[K_0]$ of Eq. (3.1) is independent of the renormalization scale μ :

$$0 = \mu \frac{d}{d\mu} [K_0] = \mu \frac{d}{d\mu} (\mu^{-2\hat{\epsilon}} [Z_K]^T [K] [Z_K]),$$

where Z_K is given in Eqs. (3.3) and (3.5). Below we use $\mu \frac{d}{d\mu} = \frac{d}{dt}$. Approximating $Z_K^{-1} \approx I$, this gives

$$2\hat{\epsilon}[K] = \left(\frac{d}{dt} [Z_K^T] \right) [K] + \frac{d}{dt} [K] + [K] \left(\frac{d}{dt} [Z_K] \right),$$

where $\frac{d}{dt} [Z_K]$ is an expansion of terms that diverge as $1/\hat{\epsilon}^n$ for $n \geq 1$, and recall that the only scale dependence in $[Z_K]$ is via the couplings. Expanding the β_K -matrix $\frac{d}{dt} [K]$ in positive powers of $\hat{\epsilon}$, as $[\beta^{(0)}] + \hat{\epsilon}[\beta^{(1)}] + \dots$, it is straightforward to show that the $[\beta^{(n)}]$ vanish for $n \geq 2$, that $[\beta^{(1)}] = 2[K]$, and therefore that only $\frac{d}{dt} [Z_K^{(1)}]$ is relevant to determine the four-dimensional $[\beta_K^{(0)}]$:

$$[\beta_K^{(0)}] = - \left(\frac{d}{dt} [Z_K^{(1)}]^T \right) [K] - [K] \left(\frac{d}{dt} [Z_K^{(1)}] \right). \quad (3.8)$$

Up to two loops, from Eqs. (3.6) and (3.7),

$$\begin{aligned} \frac{d}{dt}[Z_K^{(1)}] &= \left(\frac{d}{dt}[K^{-1}] \right) \left(-\frac{1}{24\pi^2} \left[\sum_f \vec{v}_f \vec{v}_f^T \right] + b_c g_s^2 \left[\sum_c \vec{v}_c \vec{v}_c^T \right] + b_{U1} \left[\sum_f (\vec{v}_f \cdot [K^{-1}] \cdot \vec{v}_f) \vec{v}_f \vec{v}_f^T \right] \right) \\ &+ b_c \left(\frac{d}{dt} g_s^2 \right) [K^{-1}] \left[\sum_c \vec{v}_c \vec{v}_c^T \right] + b_{U1} [K^{-1}] \left[\sum_f \left(\vec{v}_f \cdot \left(\frac{d}{dt} [K^{-1}] \right) \cdot \vec{v}_f \right) \vec{v}_f \vec{v}_f^T \right] \end{aligned}$$

so using $\frac{d}{dt}[K^{-1}] = -[K^{-1}](\frac{d}{dt}[K])[K^{-1}]$ and the substitution $\frac{d}{dt}[K] \rightarrow [\beta^{(1)}] = 2[K]$ inside $\frac{d}{dt}[Z_K^{(1)}]$ gives

$$[\beta_K^{(0)}] = - \left(\frac{1}{6\pi^2} \left[\sum_f \vec{v}_f \vec{v}_f^T \right] - 8b_c g_s^2 \left[\sum_c \vec{v}_c \vec{v}_c^T \right] - 8b_{U1} \left[\sum_f (\vec{v}_f \cdot [K^{-1}] \cdot \vec{v}_f) \vec{v}_f \vec{v}_f^T \right] \right).$$

For a single U(1), the one-loop term reduces to the QED result, so we extract b_c and b_{U1} comparing to the two-loop QED RGEs of [53]. This gives the β -matrix for two U(1)s, up to two loops, which should be applicable in any basis:

$$\begin{aligned} \frac{d}{dt}[K] &= 2\hat{e}[K] - \frac{1}{6\pi^2} \begin{bmatrix} \sum_f Q_f^2 & \sum_f Q_f Y_f \\ \sum_f Q_f Y_f & \sum_f Y_f^2 \end{bmatrix} \\ &- \frac{\alpha_s(t)}{6\pi^3} \begin{bmatrix} \sum_c Q_c^2 & \sum_c Q_c Y_c \\ \sum_c Q_c Y_c & \sum_c Y_c^2 \end{bmatrix} - \sum_f \frac{\vec{v}_f \cdot [K^{-1}] \cdot \vec{v}_f}{32\pi^4} \begin{bmatrix} Q_f^2 & Q_f Y_f \\ Q_f Y_f & Y_f^2 \end{bmatrix}, \end{aligned} \quad (3.9)$$

where the last two terms arise at two loops, f indexes all the fermions with masses below the scale of interest, α_s is the strong coupling, and c are colored particles.

The one-loop contribution to the β -matrix, which is the second term on the right-hand side of Eq. (3.9), can be diagonalized by a scale-independent rotation, implying that at one loop, there is always a basis choice with two running couplings and constant charges. At two loops, so generically to all orders, the rotation to diagonalize the β -matrix will depend on the renormalization scale μ .⁴ This μ dependence will be discussed in the next section; since these RGEs are in a massless renormalization scheme, the impact of mass scales must be included by hand, and will be more important than the running of the diagonalization angle.

IV. SOLVING THE RGEs

In this section, we study the scale evolution of the millicharge invariant, by solving the RGEs for the kinetic matrix after making a convenient choice of fields. In Appendix B, we obtain and solve the one-loop RGEs for invariants in terms of invariants.

It is notorious that dimensional regularization, combined with massless renormalization schemes like $\overline{\text{MS}}$, gives RGEs that resum the “wrong” logarithms. However, implementing these RGEs in effective field theory allows

⁴This agrees with [27], where the coupling matrix was renormalized in the presence of canonical kinetic terms, and three running parameters were expected. However [27] expected three running parameters already at one loop.

one to recover a more Wilsonian scaling of parameters [52], relevant to observables. So we transform our original model into a tower of submodels, each defined in a scale slice between two mass thresholds of the original model, to contain the particles which are light compared to the scale of the slice. Then we sequentially solve the RGE of Eq. (3.9) passing through this tower of submodels. Section IV A runs the RGEs bottom-up in a submodel, in order to show that the eigenbasis of the running couplings can in principle be observed. Section IV B solves the one-loop RGEs top-down, from $\Lambda_{\text{mod}} \rightarrow m_b$, to show that the invariant $\vec{v}_\psi [K^{-1}] \vec{v}_\chi$ is proportional to the millicharge of a shadow fermion χ . Section IV C includes two-loop QCD contributions to the β -matrix of the U(1) couplings, in order to explore the consequences of the β -matrix being scale dependent.

A. Observing the basis where the couplings are diagonal

We start at a low energy $\gtrsim m_b$, in an EFT containing the photon, the paraphoton, the gluon, the SM fermions (except the t) and some shadow fermions $\{\chi\}$ lighter than the b . This EFT is defined between m_b and m_W ; at m_W it matches to an EFT with more SM particles, and at m_b , it matches to an EFT without the b . The aim is to explore what can be learned about this model from observations of τ s and one of the χ s. In particular, is it in principle possible to observe the angle θ_K diagonalizing the RGEs, and to observe whether this angle is scale dependent? Associated to this question of scale dependence of θ_K is a notion distinguishing model parameters that run from parameters that are μ independent. We come back to this question in Sec. V A.

At the scale m_b , consider scatterings among τ s and a particular χ . In a noncanonical gauge field parametrization such that the kinetic terms are diagonal and the photon is the gauge boson coupling to the τ , one has

$$\vec{v}_\tau = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad \vec{v}_\chi = \begin{pmatrix} Q_\chi \\ Y_\chi \end{pmatrix},$$

$$D_K^{-1}(m_b) = \begin{bmatrix} e^2 & 0 \\ 0 & g^2 \end{bmatrix} \quad \text{with} \quad e^2(m_b) = g^2(m_b). \quad (4.1)$$

Scattering the χ s and τ s with themselves and each other

gives three observables evaluated at m_b^2 , which determine $e^2(m_b)$, Q_χ , and Y_χ :

$$\begin{aligned} \vec{v}_\tau[K^{-1}]\vec{v}_\tau(m_b) &= e^2(m_b), \\ \vec{v}_\tau[K^{-1}]\vec{v}_\chi(m_b) &= -Q_\chi e^2(m_b), \\ \vec{v}_\chi[K^{-1}]\vec{v}_\chi(m_b) &= Q_\chi^2 e^2(m_b) + Y_\chi^2 g^2(m_b). \end{aligned} \quad (4.2)$$

The three observables can be translated to the scale m_W , by running the RGEs for $[K]$ [see Eq. (3.9)] upward from $m_b \rightarrow m_W$. This also gives

$$\begin{aligned} K^{-1}(m_W) &= \frac{1}{\det\{K\}} \begin{bmatrix} K_{22}(m_W) & -K_{12}(m_W) \\ -K_{12}(m_W) & K_{11}(m_W) \end{bmatrix} \\ &\approx \begin{bmatrix} e^2(m_W) & -e^2(m_W)g^2(m_W)K_{12}(m_W) \\ -e^2(m_W)g^2(m_W)K_{12}(m_W) & g^2(m_W) \end{bmatrix}, \end{aligned} \quad (4.3)$$

where, in the basis of Eq. (4.1), we allowed that the RGEs could induce off-diagonal kinetic terms for the gauge bosons, and, at one loop

$$K_{12}(m_W) = -\frac{1}{6\pi^2} \sum_\chi Q_\chi Y_\chi \ln \frac{m_W}{m_b}.$$

The approximation for K^{-1} relies on $e^2(m_W)g^2(m_W)K_{12}(m_W) \ll 1$. If the three observables are measured at m_W with arbitrary experimental precision, they allow one to determine $K_{11}^{-1}(m_W) \approx e^2(m_W)$, $K_{22}^{-1}(m_W) \approx g^2(m_W)$, and $K_{12}^{-1}(m_W) \approx -e^2 g^2 K_{12}(m_W)$. Measuring these three observables at some intermediate scale $q^2 \sim (40 \text{ GeV})^2$ would also allow one to obtain $e^2(q)$, $g^2(q)$, and $K_{12}(q)$.

It is in principle possible to determine whether the eigenvalues of $[K]$ run with scale, and whether the angle to diagonalize it runs with scale, by measuring matrix elements of $[K]$ with infinite precision at three scales. To see this, start at some given scale, for instance m_b , where D_K can be chosen $\propto I$. The basis where the β -fn is diagonal still exists, but it cannot be identified from observations. So at m_b , we take $e^2 = g^2 = |\vec{u}_\tau|^2$, and measure Q_χ and Y_χ as above. Then, as above, measuring $K_{ij}^{-1}(m_W)$ allows one to solve for the eigenvalues and diagonalization angle at m_W :

$$\lambda_{\pm K} = \frac{1}{2} \left\{ \text{Tr}[K^{-1}] \pm \sqrt{\text{Tr}[K^{-1}]^2 - 4 \det[K^{-1}]} \right\},$$

$$\tan 2\theta_K = \frac{2K_{12}^{-1}(m_W)}{K_{22}^{-1}(m_W) - K_{11}^{-1}(m_W)}. \quad (4.4)$$

Measuring the three observables at some other scale allows one to verify whether the angle is scale dependent, and thereby whether there are two or three running parameters in $[K]$. This applies provided one does not cross particle thresholds.

B. An expression for the millicharge in terms of running couplings

This section starts at a high scale Λ_{mod} where the model is defined, and runs the kinetic matrix $[K]$ at one loop down to the low scale m_b . The RGEs are solved by running through a series of submodels, which contain only the particles which are lighter than the scales where the submodel is defined. This allows one to evaluate, at m_b , the millicharge invariant of Eq. (2.10).

The particle content of the model at Λ_{mod} is motivated by Holdom's study [2]: there are two massless gauge bosons and three groups of fermions (previously all called f): the heavy $\{\eta_j\}$, with $m_1 > \dots > m_N > m_t$, are charged under both U(1)s, $\{\psi\}$ are the Standard Model fermions interacting principally with one of the gauge bosons, and the $\{\chi\}$ are some number of "shadow" fermions with masses $< m_b$ who interact principally with the orthogonal vector boson. All the fermions are taken Dirac, although $\{\psi\}$ and $\{\chi\}$ could be chiral.

In massless renormalization schemes, one must implement by hand the mass thresholds when solving the RGEs, in order that the running of the couplings resums the leading-log corrections to S -matrix elements. This is a familiar recipe in EFT: when running down in scale, every time a mass threshold is crossed, the massive particle is removed, and the matrix elements of the models above and

below the threshold are equated. We follow this recipe “at one loop,” meaning that the parameters run according to the one-loop RGEs [see Eq. (3.9)], the tree matrix elements of $[K]$ are continuous when crossing a threshold $\mu = m$, and particles of mass m are removed from the β function at $\mu < m$.

At the high scale Λ_{mod} , we suppose that the ψ s and χ s do not interact at tree level:

$$\vec{v}_\chi \cdot [K^{-1}] \cdot \vec{v}_\psi|_{\Lambda_{\text{mod}}} = 0.$$

In order to solve RGEs, we need initial conditions for $[K]$ at Λ_{mod} . A convenient choice is to take $[K] \propto I$, as described in Eq. (2.3). With this form of $[K]$, which is then $O(2)$ invariant, the A^μ can be chosen to interact with the ψ s, and V^μ interacts with the χ s. This basis choice, which implies

$$\vec{v}_\psi = \begin{pmatrix} Q_\psi \\ 0 \end{pmatrix}, \quad \vec{v}_\chi = \begin{pmatrix} 0 \\ Y_\chi \end{pmatrix}, \quad (4.5)$$

is convenient because only the η s contribute to the off-diagonals of $[K]$.

Running $[K]$ down to the scale m_1 with the one-loop RGEs gives

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{bmatrix}(m_1) = \begin{bmatrix} \frac{1}{e^2} & 0 \\ 0 & \frac{1}{g^2} \end{bmatrix}(\Lambda_{\text{mod}}) + \frac{1}{6\pi^2} \begin{bmatrix} \sum_f Q_f^2 & \sum_\eta Q_\eta Y_\eta \\ \sum_\eta Q_\eta Y_\eta & \sum_f Y_f^2 \end{bmatrix} \ln \frac{\Lambda_{\text{mod}}}{m_1},$$

where f runs over all the fermions in the model, η runs over all the N η s, and $g(\Lambda_{\text{mod}}) = e(\Lambda_{\text{mod}})$.

At m_1 , the full model is matched onto the submodel constructed by removing η_1 from the full model: the K_{ij} are taken continuous across the threshold, and η_1 is removed from the β function at lower scales. Running down to m_2 , this gives

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{bmatrix}(m_2) = \begin{bmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{bmatrix}(m_1) + \frac{1}{6\pi^2} \begin{bmatrix} \sum_{f'} Q_f^2 & \sum_{\eta=2}^N Y_\eta Q_\eta \\ \sum_{\eta=2}^N Y_\eta Q_\eta & \sum_{f'} Y_f^2 \end{bmatrix} \ln \frac{m_1}{m_2},$$

where the primed sums over run over all the fermions f except η_1 .

Repeating this matching process across all the thresholds down to the mass m_b gives

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{bmatrix}(m_b) = \begin{bmatrix} \frac{1}{e^2} & 0 \\ 0 & \frac{1}{g^2} \end{bmatrix}(\Lambda_{\text{mod}}) + \frac{1}{6\pi^2} \begin{bmatrix} \sum_f Q_f^2 \ln \frac{\Lambda_{\text{mod}}}{\tilde{m}_f} & \sum_\eta Y_\eta Q_\eta \ln \frac{\Lambda_{\text{mod}}}{m_\eta} \\ \sum_\eta Y_\eta Q_\eta \ln \frac{\Lambda_{\text{mod}}}{m_\eta} & \sum_f Y_f^2 \ln \frac{\Lambda_{\text{mod}}}{\tilde{m}_f} \end{bmatrix}, \quad (4.6)$$

where in the logarithms, $\tilde{m}_f$ is m_n for $f = \eta_n$, m_t for $f = t$, and m_b for f any SM or shadow particle of $m < m_b$.

The overlap $\vec{v}_\chi \cdot K^{-1} \cdot \vec{v}_\psi$ parametrizes the scattering between SM and shadow fermions. To evaluate this overlap at the scale m_b , we need to invert $[K](m_b)$, which can be performed analogously to Eq. (4.3).

$$\vec{v}_\chi \cdot K^{-1} \cdot \vec{v}_\psi(m_b) \approx -e^2(m_b)g^2(m_b)Q_\psi Y_\chi K_{12}(m_b). \quad (4.7)$$

We calculated this invariant by remaining in the same basis [given in Eq. (4.5)] through the whole sequence of submodels between the full model and low energy. So the kinetic terms were not diagonal in the submodels, but this should not pose a problem because we are calculating an invariant. The millicharge invariant of Eq. (2.10), which gives the millicharge of χ normalized to the charge of ψ , is therefore

$$I_e(m_b) \approx -\frac{g^2(m_b)}{6\pi^2} Y_\chi \sum_\eta Y_\eta Q_\eta \ln \frac{\Lambda_{\text{mod}}}{m_\eta}, \quad (4.8)$$

where the charges on the right-hand side are in the high-scale model basis of Eq. (4.5). As anticipated, the effect of RG running is to give a millicharge to χ . In this one-loop expression, the millicharge is proportional to the running shadow coupling² (evaluated at the scale where the millicharge is probed).

C. QCD at two loops

Two-loop contributions to the running of $[K]$ are illustrated in Fig. 2 and correspond to the last two terms on the right side of Eq. (3.9). They are suppressed relative to the one-loop term by respectively $\sim \alpha_s/\pi$ and $\sim \alpha_e/\pi$, so are unlikely to be of numerical importance. The QCD corrections are discussed here because they make the β -matrix scale dependent; therefore the rotation angle to diagonalize $[K]$ could be μ dependent. So at two loops and beyond, $[K]$ will be characterized by its two eigenvalues (the inverse coupling²s) which already were μ dependent at one loop, and by a running angle. This section aims to see the effects of having a third running parameter in $[K]$;

however, we do not perform a full two-loop analysis which would require one-loop matching calculations.

The two-loop QCD contribution can be included by implementing the one-loop solution to the RGE of α_s :

$$\alpha_s(t) = \frac{\alpha_s(t_i)}{1 + \alpha_s(t_i) \frac{\beta_{s0}}{2\pi} \ln \frac{\mu}{\mu_i}} \quad (4.9)$$

with $\beta_{s0} = (11 - 2N_{fc}/3)$ and N_{fc} the number of color triplet Dirac fermions. Integrating the right side of the RGE equation (3.9), where only the one-loop U(1) and two-loop U(1)-QCD corrections are included, and using $\int d\alpha_s = \int d\alpha_s/\alpha_s$, gives

$$\begin{aligned} [K](\mu_f) = [K](\mu_i) - \frac{1}{6\pi^2} \begin{bmatrix} \sum_f Q_f^2 & \sum_f Q_f Y_f \\ \sum_f Q_f Y_f & \sum_f Y_f^2 \end{bmatrix} \ln \frac{\mu_f}{\mu_i} \\ + \frac{1}{3\pi^2 \beta_{s0}} \begin{bmatrix} \sum_c Q_c^2 & \sum_c Q_c Y_c \\ \sum_c Q_c Y_c & \sum_c Y_c^2 \end{bmatrix} \ln \frac{\alpha_s(\mu_f)}{\alpha_s(\mu_i)}, \end{aligned} \quad (4.10)$$

where f runs over all the Dirac fermions in the model, and c runs over the color components of the colored triplets.

The angle to diagonalize $[K](\mu_f)$ is simple to obtain. For a model where only SM fermions ψ are colored, in a basis where $\vec{v}_\psi = (Q_\psi, 0)$ and the parametrization where $[K](\mu_i) \propto I$:

$$\tan 2\theta = \frac{\sum_f Q_f Y_f}{\sum_f (Y_f^2 - Q_f^2) + \frac{2}{\beta_{s0}} \sum_c Q_c^2 \frac{\ln[\alpha_{sf}/\alpha_{si}]}{\ln[\mu_f/\mu_i]}}, \quad (4.11)$$

where $\alpha_{sf} = \alpha_s(\mu_f)$. One sees that the angle is scale independent if QCD is neglected, and that the angle vanishes if $\sum Q_f Y_f = 0$, which corresponds to no particles charged under both U(1)s, and SM particles interacting via QCD and QED [with a disjoint shadow U(1)]. In the case where SM QCD corrections are included to a model with kinetic mixing, then $[\sum_f \vec{v}_f \vec{v}_f^T]$ and $[\sum_c \vec{v}_c \vec{v}_c^T]$ are not simultaneously diagonalizable, so the eigenbasis of $[K]$ rotates with scale, and $[K]$ is described by two μ -dependent couplings and a μ -dependent angle.

The impact of a running angle in $[K]$ on S -matrix elements seems minimal. Our aim in running RGEs is to resum leading-log corrections into the couplings, such that the calculation of a tree-level matrix element gives a leading-log improved result. This resummation was obtained in the previous section at one loop, and gave an effective millicharge depending on the running couplings at the experimental scale, and logarithmically on the mass scales of the full model. So the effective charges at low energy are already functions of μ at one loop. The form of the equation for the millicharge, Eq. (4.8), is unchanged when two-loop QCD is included for SM particles in the

evolution from $\Lambda_{\text{mod}} \rightarrow m_b$, but the expression for $e^2(m_b)$ is augmented by the two-loop QCD running from Λ_{mod} .

V. DISCUSSION

A. Parameters, predictions, and which is kinetic mixing?

One of the puzzles motivating this study was the identity of kinetic mixing. This section argues that, as expected, it is a prediction of the model.

The model is defined by its symmetries, particle content, and parameters such as masses, coupling constants, and charges. To define, at a given scale, our model of N_f fermions coupled to two massless U(1)s, all nonredundant (or equivalently “physical”) parameters must be input: N_f masses, and $2N_f - 1$ couplings and charges [27]. It is straightforward to check that the number of couplings and charges corresponds to the number of independent invariants $I_{f_1 f_2}$ [see Eq. (2.9)].

At one loop, the expression for the millicharge invariant is given in Eq. (4.8). It is a function of the running coupling $g^2(\mu)$, of charges, and of (running) mass parameters of the theory, such as the masses m_i of the η_i , or the scale Λ_{mod} where the model is defined. So modulo Λ_{mod} which we discuss next, the invariant at low energy is clearly a prediction of the model, in the case where the model’s parameters were input at some higher scale.

There has been some discussion in the literature (see e.g., Ref. [27]) about whether $\ln \Lambda_{\text{mod}}$ should appear in the formula for low-energy kinetic mixing because it suggests an undesirable dependence on a cutoff. However, we are undisturbed by Λ_{mod} because it either cancels out in the sum over η_j in the one-loop result for the low-energy millicharge given in Eq. (4.8) or it is the physical scale of the theory where $\vec{v}_\psi [K]^{-1} \vec{v}_\chi(\mu) = 0$. [Recall that the η s are the heavy fermions charged under both U(1)s, which generate the kinetic mixing.]

An input assumption of our model is that the SM and shadow fermions do not scatter off each other at the high scale Λ_{mod} : $\vec{v}_\psi [K]^{-1} \vec{v}_\chi(\Lambda_{\text{mod}}) = 0$. This assumption is equivalent to imposing that there is a basis where $[K](\Lambda_{\text{mod}}) \propto I$, the SM particles have charges Q_ψ but $Y_\psi = 0$, and the shadow particles have charges Y_χ but $Q_\chi = 0$ [see Eq. (4.5)]. If the β -matrix is diagonal in this basis, then $\vec{v}_\psi [K]^{-1} \vec{v}_\chi = 0$ from $\Lambda_{\text{mod}} \rightarrow m_1$, and at one loop the off-diagonals of the β -matrix vanish: $0 = \sum_{\eta_j} Q_j Y_j$. So in this case, Λ_{mod} cancels out of the low energy millicharge [Eq. (4.8)].

If the β -matrix is *not* diagonal in the $Y_\psi = Q_\chi = 0$ basis at Λ_{mod} , then the vanishing of $\vec{v}_\psi [K]^{-1} \vec{v}_\chi$ at Λ_{mod} is an accidental cancellation that occurs only at Λ_{mod} , which allows to calculate Λ_{mod} from the charges, masses, and couplings of the model.

So in summary the model predicts the millicharge invariants at low energies in terms of the N_f masses and the $2N_f - 1$ coupling and charges.

B. If millicharges run, what happened to current conservation and anomalies?

Current conservation, in classical electrodynamics, is a consequence of the rephasing invariance of the charged particles, and applies in conjunction with the equations of motion. In quantum field theory, where the equations of motion may not be satisfied, they can be exchanged for Ward-Takahashi identities, which follow from a residual local symmetry that survives gauge fixing. These identities for our model can be obtained by making a BRS transformation on the noncanonical Lagrangian of Eq. (2.1) augmented by gauge-fixing terms as in Eq. (3.2) and by ghosts. These identities confirm that the charges do not run, as anticipated in Sec. III A. Anomaly cancellation works as usual in this formalism, where the kinetic mixing in the gauge boson propagators is external to the triangle diagram. So one sees that, in the noncanonical formalism, anomalies cancel and charge conservation identities are satisfied as usual.

The situation is less clear with the canonical Lagrangian of Eq. (2.6), where the millicharge of χ is proportional to the invariant I_e of Eq. (4.8). So already at one loop, there is a charge which depends on scale via the running couplings. However, the path integral is invariant under field reparametrizations, and S -matrix elements are functions of reparametrization-invariant combinations of Lagrangian parameters. So the physical consequences of Ward-Takahashi identities and anomaly cancellation should remain in the canonical parametrization. But it is reassuring to check that the effective charges in the canonical formalism are conserved at vertices, due to the vanishing of the sum of the charges of the fields participating in each term of the Lagrangian. Similarly, the hydrogen atom is neutral in the canonical formulation because anomaly cancellation relates the charges of the electrons and quarks in the noncanonical Lagrangian, and the proton and electron charge vectors transform the same way in going from the noncanonical to canonical formulation.

C. What about a covariant canonical formalism?

The charge vectors $\vec{u}_f$ of the canonical Lagrangian (2.6) are physically attractive because their inner products appear in S -matrix elements, and they are invariant under rescaling of the charges into couplings. However, it is difficult to formulate and solve covariant-in-U(1)-space RGEs for $\vec{u}$. We encountered two difficulties.

The first is that the one-loop RGE for e in QED is nonlinear: $\mu \frac{d}{d\mu} e = \frac{1}{12\pi^2} e^3$. So the generalization of this differential equation to the vector of couplings $\vec{u}$ is not obvious. This is to be compared with the one-loop RGE for

the kinetic term coefficient $1/e^2$, which is linear and trivial to generalize to 2 U(1)s.

The second issue, is that rotations [in two-U(1) space] of the charge vector $\vec{u}$ are unrelated to the rotation which diagonalizes the kinetic terms. This can be seen from Eq. (2.5), which implies that $\vec{u} = D_K^{-1} O_K \vec{v}$. If the rotation to diagonalize $[K]$ is scale dependent, which is not the case at one loop but appears true to all orders, then that rotation, given as O_K in Eq. (2.5), is inconveniently intricated inside $\vec{u}$. Furthermore, $\vec{u}$ appears to contain two (running) parameters, so its unclear what happened to the angle.

A more promising approach that has been carefully explored [26,27,34], is to define a coupling matrix $[E]$, such that $\vec{u}_f = [E] \vec{v}_f$. However, this requires defining $[E]$ as the square root of $[K^{-1}]$, and the commutator of $[E]$ with its derivative seems required to obtain its RGEs. Therefore it appears more convenient to work with $[K]$.

D. Invariants

The invariants of Eqs. (2.9) and (2.10) have many desirable features: they are invariant under rotations and rescaling of the U(1) gauge fields, they are simple, can be constructed diagrammatically, and they correspond to the coupling combinations that appears in S -matrix elements.

This is a rare combination. Most such constructions, including the familiar Jarlskog invariant, are only invariant under field rotations in a Lagrangian with canonical kinetic terms. In addition, invariants constructed from the Lagrangian are often only remotely related to S -matrix elements. For instance, in the SM quark sector, the Jarlskog invariant is a trace (so corresponds to a closed diagram), and is constructed from tree-level parameters. Whereas S -matrix elements measuring quark CP violation do not correspond to closed diagrams, and are proportional to simpler CP invariants and running parameters [59]. It has been suggested [60] that such S -matrix combinations could be useful in reconstructing new physics models; however, in our two U(1) model, the Lagrangian and S -matrix invariants are the same.

An additional remarkable feature of the invariants $I_{f_i f_j}$ of Eq. (2.9) is that their RGEs can be expressed in terms of invariants (see Appendix B). This opens tantalizing perspectives: S -matrix elements are functions of invariants; if in addition, RGEs can be written in invariant fashion, then perhaps there is no need for the Lagrangian parameters deemed redundant by Weinberg [41].

VI. SUMMARY

The path integral and Green's functions are invariant under field redefinitions—but renormalization-scale-dependent redefinitions can modify which parameters run. In a model with kinetic mixing between two U(1) gauge bosons, we obtain and solve the $\overline{\text{MS}}$ RGEs [Eq. (3.9)] in a

parametrization where the kinetic terms have a running coupling matrix $[K]$ but there are no $U(1)$ gauge couplings in the covariant derivatives. We rediscovered that in this parametrization [$\sim -\frac{1}{4e^2}FF$ for a single $U(1)$], the gauge fields are not renormalized, the Ward identities are transparent, and the RGEs are linear at one loop so trivial to solve. $[K]$ at one loop contains two running couplings and can be diagonalized by a finite angle. This parametrization is particularly convenient for multi- $U(1)$ models because kinetic mixing is generated automatically in RG evolution, and because the formalism is covariant under the μ -independent field reparametrizations which are commonly used in such models. To have a measure of kinetic mixing that is independent of the field parametrization, we constructed a generalized invariant for kinetic mixing [Eq. (4.8)]. Recall that usual invariants (e.g. Jarlskog) are not modified by unitary transformations which leave the kinetic terms invariant in the Lagrangian; we generalized this notion to obtain an invariant which is unchanged under generic gauge field reparametrizations including rescalings.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: NEGLECTED LAGRANGIAN TERMS

This appendix discusses the various terms not considered in the Lagrangian of Eq. (2.1).

Non-Abelian gauge interactions were not written because they are independent, at the Lagrangian level, of the $U(1)$ s. We also neglected terms such as

$$\begin{aligned} \mathcal{L}_{\text{neglect}} = & -\frac{\epsilon_{\mu\nu\alpha\beta}}{2}(F^{\mu\nu}V^{\alpha\beta})\begin{bmatrix} T_{11} & T_{12} \\ T_{12} & T_{22} \end{bmatrix}\begin{pmatrix} F^{\alpha\beta} \\ V^{\alpha\beta} \end{pmatrix} \\ & + \sum_j D^{\mu'}\phi_j^\dagger D'_\mu\phi_j - V(\phi_j) + (\lambda^{jkn}\phi_j\bar{f}_kf_n + \text{H.c.}), \end{aligned} \quad (\text{A1})$$

where the Yukawa interactions are gauge invariant if the sum of the participating charge vectors vanishes: $\vec{v}'_\phi + \vec{v}'_{fk} + \vec{v}'_{fn} = 0$. [This condition remains satisfied under rotations and rescaling in the two- $U(1)$ -space.⁵] Multiple scalars, non-Abelian gauge sectors, and Yukawa interactions are included in [29]. Scalar fields are neglected here (despite that they could induce gauge boson masses) because we are interested in the running of parameters at scales above possible gauge boson masses. Finally, in QED, the CP -odd θ term can be written as a total divergence, and this remains true for T_{12} terms in the presence of 2 $U(1)$ s,

$$\epsilon^{\mu\nu\alpha\beta}F'_{\mu\nu}V'_{\alpha\beta} = 4\partial_\mu(\epsilon^{\mu\nu\alpha\beta}A'_\nu\partial_\alpha V'_\beta).$$

So we also neglect these $\epsilon\vec{F}'[T]\vec{F}'$ terms because they should be irrelevant for S -matrix elements, although they could be interesting in certain topologies or for space-time dependent T_{ij} [54].

APPENDIX B: INVARIANT FORM OF THE RGEs

In the main body of the text we solved the renormalization group equations for the coefficient of the kinetic term $[K]$, which then determined the running of the invariants. Given the simple solution for the RGEs of $[K]$ this is a numerically efficient and practical approach, even though K transforms under field redefinitions. Hence, it is interesting to note that the RGE and their solution can be derived solely in terms of the invariants. Writing $t = \ln \mu$ and sandwiching the RGE for K inverse with the charge vectors of fermion f_1 and f_2 ,

$$\begin{aligned} \frac{d}{dt}v_{f_1}^T[K]^{-1}v_{f_2} = & -\frac{1}{6\pi^2}v_{f_1}^T[K]^{-1}\sum_f v_f v_f^T[K]^{-1}v_{f_2} \\ & -\frac{\alpha_s(t)}{6\pi^3}v_{f_1}^T[K]^{-1}\sum_c v_c v_c^T[K]^{-1}v_{f_2}, \end{aligned} \quad (\text{B1})$$

we find a RGE equation that is expressed only in terms of the invariant matrix defined in Eq. (2.9). (Note that we sum over all fermions f , treating each colored state distinct, and only over the colored triplets c , when we include mixed QCD effects.) Using this equation, we find for the inverse of the invariant matrix

$$\frac{d}{dt}[I]^{-1} = [I]^{-1}[\dot{I}][I]^{-1} = -\frac{1}{6\pi^2}\mathbb{1} - \frac{\alpha_s(t)}{6\pi^3}\delta_c, \quad (\text{B2})$$

⁵In a particular field parametrization, the sum of the charges only needs to be an integer in order for the Yukawa interaction to be invariant. But the integer should be zero to allow field reparametrizations which rescale the couplings $e \rightarrow e/x$ and charges $Q \rightarrow xQ$.

where $\mathbb{1}$ denotes the unit matrix in fermion space, while δ_c is 1 for a fermion that is a colored triplet component and 0 otherwise.

Integration and expressing the scale dependence of α_s in terms of the QCD β function results in a solution for $[I]^{-1}(t)$, whose inverse reads

$$I_{f_1 f_2}(\mu) = I_{f_1 f}(\mu_0) \left[\mathbb{1} - \frac{1}{6\pi^2} \ln\left(\frac{\mu}{\mu_0}\right) [I(\mu_0)] + \frac{1}{3\pi^2 \beta_{s0}} \ln\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right) [\delta_c I(\mu_0)] \right]_{ff_2}^{-1}. \quad (\text{B3})$$

The above solution involves the inversion of larger matrices, that can be still performed efficiently with numerical methods. In the following we will perform this inversion through an expansion in $I_{ff'} \ln / (6\pi^2)$ in order to match the above result to the millicharged invariant of Eq. (2.10). Considering fermion fields ψ , η , and χ , defined as in Sec. IV B we find

$$I_\epsilon(\mu) \approx \frac{1}{6\pi^2 I_{\psi\psi}} \sum_{h \in \eta} I_{\psi h} I_{h\chi} \ln\left(\frac{m_h}{\Lambda_{\text{mod}}}\right). \quad (\text{B4})$$

APPENDIX C: AN INVARIANT FOR $\tau^+ \tau^- \rightarrow \gamma + E_{\text{miss}}$

This appendix discusses an invariant for a process with external gauge bosons: $\tau^+ \tau^- \rightarrow A + V$. Physically, this would give a final state of $\gamma +$ missing energy. In the case of two massless gauge bosons, the physical photon is the gauge boson that interacts with the SM, so this process should not occur. However, in the case where V^μ is massive, the photon A^μ is defined as the massless gauge boson, and $\tau^+ \tau^- \rightarrow \gamma + E_{\text{miss}}$ becomes possible. In the case where V^μ gets a mass from the vacuum expectation value of a scalar ϕ , this appears in the noncanonical Lagrangian (2.1) as

$$\mathcal{L}_{nc} \supset \langle \phi \rangle^2 \vec{A}^\mu [\vec{v}_\phi \vec{v}_\phi^T] \vec{A}^\mu.$$

The $U(1) \times U(1)$ structure of the two-point function for V^μ can be written in an arbitrary basis as

$$D_V^{\mu\nu} \propto [K^{-1}] \frac{[\vec{v}_\phi \vec{v}_\phi^T]}{\vec{v}_\phi \cdot [K^{-1}] \cdot \vec{v}_\phi} [K^{-1}], \quad [\vec{v}_\phi \vec{v}_\phi^T] = \begin{bmatrix} Q_\phi^2 & Q_\phi Y_\phi \\ Q_\phi Y_\phi & Y_\phi^2 \end{bmatrix}, \quad (\text{C1})$$

where the simplest field parametrization in which to implement this matrix is to take $[K] \propto I$, and choose the basis such that $Q_\phi = 0$. The equivalent coupling structure for the orthogonal gauge boson A is orthogonal in the $U(1)$ space, so

$$D_A^{\mu\nu} \propto [K^{-1}] \epsilon \frac{[\vec{v}_\phi \vec{v}_\phi^T]}{\vec{v}_\phi \cdot \epsilon [K^{-1}] \epsilon \cdot \vec{v}_\phi} \epsilon [K^{-1}], \quad (\text{C2})$$

where ϵ is the antisymmetric 2×2 matrix with unit entries. The matrix element squared for $|\mathcal{M}(\tau\tau \rightarrow A + V)|^2$ is then proportional to

$$\frac{(\vec{v}_\tau \cdot [K^{-1}] \cdot \vec{v}_\phi)^2 (\vec{v}_\tau \cdot [K^{-1}] [\epsilon] \cdot \vec{v}_\phi)^2}{|(\vec{v}_\phi \cdot \epsilon [K^{-1}] \epsilon \cdot \vec{v}_\phi) (\vec{v}_\phi \cdot [K^{-1}] \cdot \vec{v}_\phi)|}. \quad (\text{C3})$$

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