

Positivity and cluster structures in Landau analysis

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Landau analysis in momentum twistor space can be formulated as the study of varieties of lines in three-dimensional projective space, together with their projections and discriminants. Within this framework, we define enumerative invariants (leading singularity (LS) degrees) that count leading singularities. Leading Landau singularities (LS discriminants) arise as discriminants detecting the collision of leading singularities. We uncover a recursive mechanism underlying Landau singularities, governed by substitution maps between Grassmannians. Applying this framework, we prove positivity and factorization into cluster variables for the LS discriminant of a large class of Landau diagrams at arbitrary loop order. This provides a first-principles explanation for the emergence of positivity and cluster algebra structures in scattering amplitudes of planar $\mathcal{N} = 4$ super Yang-Mills theory.

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I. INTRODUCTION

Modern progress in perturbative quantum field theory relies increasingly on computing scattering amplitudes and understanding their singularity structure. Scattering amplitudes are built from integrals of the form

$$\mathcal{I}(\mathbf{M}) = \int_{\Gamma} \frac{N(\mathbf{L}; \mathbf{M})}{D(\mathbf{L}; \mathbf{M})} d\mu(\mathbf{L}). \quad (1)$$

The ingredients are the loop variables $\mathbf{L}$, external kinematics $\mathbf{M}$, integration measure μ , and contour Γ . The explicit form of (1) depends on the choice of kinematic variables [1]. In the case of a Feynman integral in momentum representation, $D(\mathbf{L}; \mathbf{M})$ is the product of propagators associated with the internal edges of the corresponding Feynman diagram. The integral $\mathcal{I}(\mathbf{M})$ is a transcendental function involving, e.g., multiple polylogarithms, elliptic integrals, and Calabi-Yau periods. A first step toward computing $\mathcal{I}(\mathbf{M})$ is determining its singularities, namely, the locus in the external kinematics where $\mathcal{I}(\mathbf{M})$ develops poles or branch cuts. This locus is the *Landau variety*.

The core idea of *Landau analysis* [2–13] is the determination of the Landau variety from the singular locus $\{D(\mathbf{L}; \mathbf{M}) = 0\}$ of the integrand in (1), and its interplay with the cycle Γ [14,15]. In this letter we use *momentum*

twistor variables [16] to exploit the rich algebro-geometric structure of the Grassmannian. In these variables, vanishing propagators give incidence conditions for pairs of lines in projective 3-space. Hence, on-shell spaces are subvarieties in a product of Grassmannians of lines [17,18].

The Landau variety is the branch locus of the *Landau map*, the projection from the on-shell space to the space of external kinematics [12]. When the Landau map has finite fibers, we call its branch locus *leading Landau singularity*. The points in the fiber correspond to maximal residues of the integrand in (1). These *leading singularities* (LS) appear as maximal iterated discontinuities of (1) and as algebraic prefactors in analytic evaluations [19].

Momentum twistor variables are well suited for describing planar dual conformal invariant integrals [20]. Our results have important implications for scattering amplitudes in planar $\mathcal{N} = 4$ super Yang-Mills (SYM) theory. Here, additional geometric structures arise, such as positive geometries and cluster algebras [21,22]. At fixed multiplicity and loop order, the full color-ordered amplitude can be written as a single integral (1), whose integrand is the *canonical form* of the loop amplituhedron [21].

The connection between scattering amplitudes and cluster algebras was first observed by Golden *et al.* [22].

This was followed by the discovery of *cluster adjacencies* [23] and related patterns [24], culminating in the cluster bootstrap program [25]. Despite these advances, a first-principles explanation for the emergence of cluster structures in particle physics has remained elusive. At tree level, where amplitudes are rational functions, recent progress [26] has explained the appearance of cluster structures in terms of the positive geometry of the amplituhedron. A key

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ingredient is the introduction of *BCFW promotion maps*. These *cluster quasi-homomorphisms* mirror the Britto-Cachazo-Feng-Witten (BCFW) recursive construction of amplitudes [27]. At loop level, cluster structures are known to appear in finite, dual-conformally invariant quantities, such as remainder functions and the nontrivial part of ratio functions, beginning at six points [22].

A. Main contributions

We develop a framework for *Landau analysis on the Grassmannian*, in which (leading and superleading) Landau singularities are expressed as discriminants and resultants. We uncover recursive structures governing these objects, whose key ingredient is a class of *substitution maps* closely related to *promotion maps* for cluster algebras [28]. This provides a pathway toward proving two central conjectures about singularities of planar $\mathcal{N} = 4$ SYM amplitudes: *positivity*, namely, regularity in the positive kinematic region, and the emergence of cluster algebras, which we formulate as *cluster factorization* of Landau singularities into cluster variables in the rational regime. The broader geometric and algebraic constructions are developed in [18].

We prove both positivity and cluster factorization at all loop orders for planar Landau diagrams whose *dual loop graph* G is a tree, i.e., the subgraph of the dual diagram whose vertices correspond to loop variables. In addition, we perform extensive numerical checks for planar Landau diagrams up to four loops. Our approach yields a first-principles explanation for the emergence of positivity and cluster structures in planar $\mathcal{N} = 4$ SYM amplitudes, while also providing effective tools for the computation of Landau singularities.

II. ON-SHELL SPACES AND THE LANDAU MAP

The Grassmannian $\text{Gr}(2,4)$ is the space of lines in projective 3-space $\mathbb{P}^3$. In momentum twistors, propagators become incidences of lines ([18], Sec. II), and on-shell spaces [29] are subvarieties in $\text{Gr}(2,4)^\ell$. Fix $\ell \geq 1$ and consider a connected undirected simple graph G on ℓ vertices. We define the *line incidence variety* $V_G \subseteq \text{Gr}(2,4)^\ell$ of G as the configurations of ℓ lines $\mathbf{L} = (L_1, \dots, L_\ell)$, satisfying the incidence $\langle L_i, L_j \rangle = 0$ for every edge ij of G .

Questions about the irreducibility, dimension, and complete intersection of V_G were resolved in [17] (Sec. V). Both answers rely on the rigidity theory of graphs in the plane [30]. For example, the variety of the triangle V_{K_3} is nine-dimensional and has two irreducible components $V_{[3]}$ and $V_{[3]}^*$, of concurrent and coplanar lines. This geometric understanding is crucial for Landau analysis, as in [11].

To the ℓ loop lines $\mathbf{L}$ we adjoin d external lines $\mathbf{M} = (M_1, \dots, M_d)$. Let $u \in \mathbb{N}^\ell$ with $|u| = \sum_{i=1}^\ell u_i = d$ and form a graph G_u by attaching u_i pendant edges to

vertex i . Each pendant edge corresponds to an external line incident to L_i . We call $\mathcal{L} = G_u$ a *Landau diagram*—this is actually the planar dual of a usual Landau diagram [5]. The line incidence variety $V_{\mathcal{L}} \subseteq \text{Gr}(2,4)^{\ell+d}$ is the on-shell space of $\mathcal{L}$: it contains the lines (kinematics) satisfying all incidences (cut or on-shell conditions) prescribed by $\mathcal{L}$.

Given a Landau diagram $\mathcal{L} = G_u$, consider the map

$$\psi_{\mathcal{L}}: V_{\mathcal{L}} \rightarrow \text{Gr}(2,4)^d, \quad (\mathbf{L}; \mathbf{M}) \mapsto \mathbf{M}. \quad (2)$$

The key idea from Landau analysis is that the Landau variety is recovered as the union of the branch loci of $\psi_{\mathcal{L}}$, where G and u vary within a family determined by the structure of (1) [14,15]. For fixed external lines $\mathbf{M}$, the fiber $\psi_{\mathcal{L}}^{-1}(\mathbf{M})$ consists of all loop lines satisfying the on-shell conditions associated to edges of $\mathcal{L}$. The Landau variety detects the locus in the external kinematics where the fiber is nongeneric [14,15]. We focus on *leading Landau diagrams* $\mathcal{L}$, when the generic fiber of $\psi_{\mathcal{L}}$ consists of finitely many points. We address the following questions:

- (1) Size: what is the cardinality of the generic fiber?
- (2) Discriminants: which $\mathbf{M}$ have nongeneric fiber?
- (3) Reality: for which real $\mathbf{M}$ is the whole fiber real?
- (4) Rationality: for which constraints on the external data are all points in the fiber rational in $\mathbf{M}$?

III. LEADING SINGULARITY DEGREE

Fix a graph G and $u \in \mathbb{N}^\ell$ with $|u| = d = \dim(V_G)$. We form a Landau diagram $\mathcal{L} = G_u$ with Landau map $\psi_u = \psi_{\mathcal{L}}$. We expect the fiber $\psi_u^{-1}(\mathbf{M})$ to be empty or finite. Its points are configurations of ℓ lines, fully localized by imposing the on-shell conditions. The points are the *leading singularities*. We define the *LS degree* by

$$\gamma_u := |\psi_u^{-1}(\mathbf{M})| \quad \text{for generic } \mathbf{M}. \quad (3)$$

The LS degrees count solutions to Schubert problems, i.e., incidence problems for lines in projective space. If $G = K_1$ is the one-vertex graph and $u = (4)$, then $\gamma_{(4)} = 2$: four generic lines in $\mathbb{P}^3$ have two transversals.

From an algebraic viewpoint, γ_u is a coefficient in the multidegree $[V_G]$, see [17] (Sec. III). This polynomial represents a cohomology class and packages, for the fixed dual loop graph G , all leading Landau diagrams G_u , and their LS degrees γ_u . When V_G is a complete intersection ([17], Thm 5.4), meaning $\dim(V_G) = 4\ell - |G|$, then

$$[V_G] = 2^\ell \cdot \prod_{i=1}^\ell t_i \cdot \prod_{ij \in G} (t_i + t_j) = \sum_{u \in \mathbb{N}^\ell} \gamma_u t_1^{5-u_1} \cdots t_\ell^{5-u_\ell}.$$

For example, $[V_G] = 16 \cdot t_1^2 t_2^2 t_3^2 + 8 \cdot t_1^3 t_2^2 t_3 + \text{perm.}$ for $G = K_3$. The coefficient 16 is the LS degree of triple pentagon G_u with $u = (3, 3, 3)$ [31]. For all graphs up to eight vertices, i.e., leading Landau diagrams with up to

eight loops (planar and nonplanar), we determine the dimension, irreducible decomposition, and multidegree in [17]. The data are presented at [32].

IV. LANDAU DISCRIMINANTS AND RESULTANTS

We focus on *leading* and *superleading* Landau singularities, where $d - \dim(V_G)$ equals 0 or 1, respectively. If $\mathcal{L}$ is a leading Landau diagram ($|u| = d$) then the branch locus of $\psi_{\mathcal{L}}$ detects when leading singularities collide. This property is encoded by the *LS discriminant* $\Delta_{\mathcal{L}}(\mathbf{M})$,

$$\Delta_{\mathcal{L}}(\mathbf{M}) = 0 \quad \text{whenever } |\psi_{\mathcal{L}}^{-1}(\mathbf{M})| < \text{generic value}. \quad (4)$$

This is an analog of the univariate discriminant. For instance, the LS discriminant of the *four-mass box* with $G = K_1$ and $u = (4)$ is the 4×4 Gram determinant $\langle M_i M_j \rangle$.

Let $\mathcal{L}$ be a superleading Landau diagram. We have $\psi_{\mathcal{L}}^{-1}(\mathbf{M}) = \emptyset$ for generic $\mathbf{M}$. The *SLS (superleading singularity) resultant* satisfies

$$R_{\mathcal{L}}(\mathbf{M}) = 0 \quad \text{whenever } \psi_{\mathcal{L}}^{-1}(\mathbf{M}) \neq \emptyset. \quad (5)$$

This captures when an overdetermined polynomial system admits a solution. As an example, the SLS resultant of the *pentagon* with $G = K_1$ and $u = (5)$ is the determinant of the 5×5 Gram matrix $\langle M_i M_j \rangle$, as above.

Our full paper [18] reports on computations of Landau discriminants and resultants. We give insight into leading and superleading singularities of amplitudes and Feynman integrals. We also study the degrees of discriminants and resultants ([18], Thm 4.4, Thm 5.1, Prop 5.3). Lastly, we look into *next-to-leading* singularities, where the generic fibers of the Landau map are curves: we give a formula for their genera ([18], Sec. VI) and compute the Landau singularity of the double box with all external momenta off-shell.

V. RECURSIVE LANDAU ANALYSIS

One of our key discoveries is that many Landau singularities have a *recursive* structure via graph decompositions. Our approach is more algebro-geometric than Ref. [2].

Consider a Landau diagram that decomposes as $\mathcal{L} = G_u \cup H_u$, $\mathcal{L}_1 = G_{1,u_1} \cup H_{1,u_1}$, and $\mathcal{L}_2 = G_{2,u_2} \cup H_{2,u_2} \cup E$ such that $u = (u_1, u_2)$, $G = G_1 \cup G_2 \cup E$, with E the edges between G_1 and G_2 . Assume that $\mathcal{L}_1$ is *leading*, which means that $|u_1| = \dim(V_{G_1})$. We say that $\mathcal{L}$ is *reducible by* $\mathcal{L}_1$. Write $\mathbf{M} = (\mathbf{M}^{(1)}, \mathbf{M}^{(2)})$ for the $|u_1| + |u_2|$ external lines. Solving the on-shell conditions for $\mathcal{L}_1$ for fixed $\mathbf{M}^{(1)}$ already fully localizes the loop lines in G_1 to γ_1 configurations $\mathbf{L}_r^{(1)} = \mathbf{L}_r^{(1)}(\mathbf{M}^{(1)})$, where γ_1 is the LS degree of $\mathcal{L}_1$ and $r = 1, \dots, \gamma_1$. This structure yields a

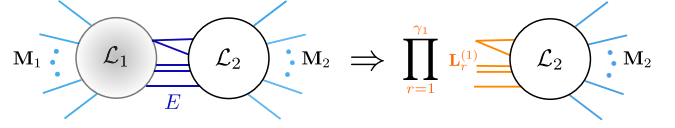


FIG. 1. The recursion for Landau singularities of a Landau diagram $\mathcal{L}$ that is reducible by a leading Landau diagram $\mathcal{L}_1$.

decomposition of the on-shell space, and hence of the fiber of the Landau map ψ . As a result, the branch locus of ψ can be characterized in terms of that of the Landau maps ψ_i on $\mathcal{L}_i$. A branch point $\mathbf{M} = (\mathbf{M}^{(1)}, \mathbf{M}^{(2)})$ of ψ is such that either $\mathbf{M}^{(1)}$ is a branch point of ψ_1 , or $(\mathbf{L}_r^{(1)}, \mathbf{M}^{(2)})$ is a branch point for ψ_2 for some r . This also implies an algebraic factorization: if $\text{LS}_{\mathcal{L}}$ and $\text{LS}_{\mathcal{L}_i}$ are the Landau singularities of $\mathcal{L}$ and $\mathcal{L}_i$, then

$$\text{LS}_{\mathcal{L}}(\mathbf{M}) = \Delta_{\mathcal{L}_1}(\mathbf{M}^{(1)}) \cdot \mathcal{E}(\mathbf{M}^{(1)}) \cdot \prod_{r=1}^{\gamma_1} \varphi_r^* \text{LS}_{\mathcal{L}_2}(\mathbf{M}). \quad (6)$$

Each *substitution map* φ_r^* inserts the lines $\mathbf{L}_r^{(1)}(\mathbf{M}^{(1)})$ in the polynomial $\text{LS}_{\mathcal{L}_2}$ according to $\mathcal{L}$, see Fig. 1. The *extraneous factor* $\mathcal{E}(\mathbf{M}^{(1)})$ is a rational function in $\mathbf{M}^{(1)}$ reflecting the (projective) ambiguity in how we define the substitution maps—by possibly redefining φ_r , we can set $\mathcal{E}(\mathbf{M}^{(1)}) = 1$. Despite each factor $\varphi_r^* \text{LS}_{\mathcal{L}_2}$ in (6) being algebraic, the full product is rational ([18], Thm 7.4).

We now focus on *leading* and *superleading* Landau diagrams $\mathcal{L}$. Here, $\text{LS}_{\mathcal{L}}$ is replaced by $\Delta_{\mathcal{L}}$ and $R_{\mathcal{L}}$, respectively. This recursion is the main engine behind our positivity and cluster structure results, discussed below.

For the four-mass box $\mathcal{L}_1$ with $G_1 = K_1$ and $u_1 = (4)$, we have $\gamma_1 = 2$ points $\mathbf{L}_{\pm}^{(1)} = (L_{\pm})$ in the Landau fiber over $\mathbf{M}^{(1)} = (A, B, C, D)$, see [18] [Eq. (26)]. Namely, $L_{\pm} = \varepsilon \cdot (p(x^{(\pm)})B) \star (p(x^{(\pm)})C)$, where $p(x) = d_1 + xd_2$ parametrizes the line $D = d_1d_2$ and $\star$ is the geometric intersection. The roots $x^{(\pm)} = (-a_1 \pm \sqrt{a_1^2 - 4a_0a_2})/(2a_2)$ are written in terms of *chain polynomials* $\langle abc|de|fgh \rangle := \langle abcd \rangle \langle efgh \rangle - \langle abce \rangle \langle dfgh \rangle$. Namely, $a_2 = \langle d_2A|B|d_2C \rangle$, $a_1 = \langle d_1A|B|d_2C \rangle + \langle d_2A|B|d_1C \rangle$, and $a_0 = \langle d_1A|B|d_1C \rangle$. The normalization $\varepsilon = 2a_2/\sqrt{\langle BC \rangle \langle BD \rangle \langle CD \rangle}$ is chosen such that $L_{\pm}$ have degree $(1, 1, 1, 1)$ in (A, B, C, D) .

Example ($\ell = 2$). Let $\mathcal{L} = G_u$ be the pentabox with $G = K_2$ and $u = (4, 3)$. This is reducible by the four-mass box $\mathcal{L}_1$, and (6) yields (omitting the prefactor $\Delta_{\mathcal{L}_1}$)

$$\Delta_{\mathcal{L}}(\mathbf{M}) = \prod_{\sigma \in \{\pm\}} \Delta_{K_{1,(4)}}(\varphi_{\pm}(\mathbf{M})). \quad (7)$$

The *four-mass box substitution maps* $\varphi_{\pm}$ are defined by $\varphi_{\pm}(\mathbf{M}) = (L_{\pm}(\mathbf{M}^{(1)}), \mathbf{M}^{(2)})$. This is depicted in Fig. 2.

The same recursion holds for the SLS resultant (indicating superleading Landau singularities). Here, $\mathcal{L}$ is the

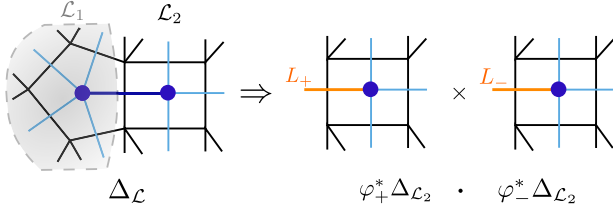


FIG. 2. Recursion for the LS discriminant of the pentabox.

double box, $G = K_2$, and $u = (4, 4)$. Then, Eq. (7) holds by replacing $\Delta_{K_2, (4,3)}$ by $R_{K_2, (4,4)}$, and $\Delta_{K_1, (4)}$ by $R_{K_1, (5)}$.

In general, if $\mathcal{L}$ is reducible by the four-mass box $\mathcal{L}_1$,

$$\Delta_{\mathcal{L}}(\mathbf{M}) = \prod_{\sigma \in \{\pm\}} \Delta_{\mathcal{L}_2}(\varphi_{\pm}(\mathbf{M})). \quad (8)$$

A similar formula holds for SLS resultants if $\mathcal{L}$ is super-leading. This works well for all trees G . We stress that the absence of the extraneous factor crucially depends on our choice of $L_{\pm}$, and hence of $\varphi_{\pm}$, as in [18] [Eq. (26)].

VI. REALITY AND POSITIVITY

We consider a notion of positivity on the external data. Assume that $\mathcal{L}$ is planar and embedded in a disk, so that the d external lines $\mathbf{M}$ have a circular order. Write $M_i = z_{p_i} z_{p_{i+1}}$ and collect the points z_p in a $4 \times n$ matrix $\mathbf{z} = [z_1 | \dots | z_n]$ with $n \leq 2d$. We remove equal columns if some external lines intersect. We call $\mathbf{M}$ *positive* if $\mathbf{z} \in \text{Gr}_{>0}(4, n)$, the positive Grassmannian, i.e., the subset of $\text{Gr}(4, n)$ represented by matrices with positive ordered maximal minors. This notion of positivity is expected to yield a vast family of real Schubert problems.

Reality conjecture: If $\mathcal{L}$ is planar leading and $\mathbf{M}$ is positive, then all points in the fiber $\psi_{\mathcal{L}}^{-1}(\mathbf{M})$ are real.

Reality is tightly related to positivity. A rational function on the Grassmannian $\text{Gr}(k, n)$ is called *Grassmann copositive* if it is positive on $\text{Gr}_{>0}(k, n)$. Nontrivial examples are the LS discriminant $\Delta_{K_1, (4)}$ of the box and the SLS resultant of the pentagon $R_{K_1, (5)}$, see [18] (Prop 9.2). All amplitudes in planar $\mathcal{N} = 4$ SYM are expected to be regular on the positive Grassmannian $\text{Gr}_{>0}(4, n)$.

Positivity conjecture: If $\mathcal{L}$ is planar leading (or super-leading), then $\Delta_{\mathcal{L}}$ (or $R_{\mathcal{L}}$) is Grassmann copositive.

Our result in [18] (Thm 9.4, Cor 9.7, Prop 9.8) states the following:

Theorem (reality and positivity for trees): Both the reality and positivity conjectures are true if G is a tree.

The key to the proof is that the substitution maps $\varphi_{\pm}$ in (7) and (8) are *copositive*, i.e., they map positive lines $\mathbf{M}$ to positive lines ([18], Lemma 9.3). Together with the recursive structure seen above, one derives both conjectures inductively on the number of vertices of G .

The positivity conjecture for LS discriminants and resultants are meant for each irreducible component of

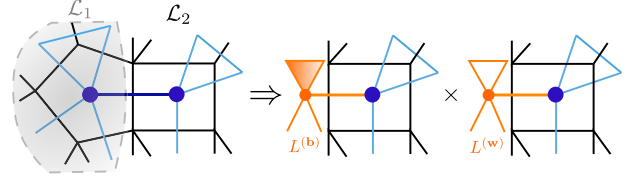


FIG. 3. LS discriminant recursion of the rational pentabox.

V_G separately, see [18] (Sec. X) for details. We verified the reality conjecture computationally for all planar graphs G on $\ell = 3, 4$ vertices with HomotopyContinuation.jl [33]. For each G and u such that $\gamma_u \neq 0$, we checked for 10^5 random $\mathbf{M}$ that $\psi_{\mathcal{L}}^{-1}(\mathbf{M})$ has only real points. Similarly, we checked copositivity of the substitution maps. Hence, G is a “seed” for the recursion in (6) that yields positivity for other infinite families of Landau diagrams.

VII. RATIONALITY AND CLUSTER STRUCTURE

We now degenerate the external data $\mathbf{M}$ so that all points in the Landau fiber are rational functions of $\mathbf{M}$. We set some of the external lines to be incident by fixing a graph H_u on d vertices, and restricting $\mathbf{M}$ to lie in $V_{H_u} \subseteq \text{Gr}(2, 4)^d$. We denote the resulting Landau diagram by $\mathcal{L} = G_u \cup H_u$ and its Landau map by $\psi_{\mathcal{L}}: V_{\mathcal{L}} \rightarrow V_{H_u}$. We call $\mathcal{L}$ *rational* if it is a leading Landau diagram and each point in the fiber of $\psi_{\mathcal{L}}$ is a rational function of $\mathbf{M}$.

Physically, when two external lines intersect, the corresponding external momentum at that vertex of the dual graph is on-shell, otherwise it is off-shell, while the propagators remain massless, see Fig. 3. Thus, one starts from fully off-shell kinematics and imposes on-shell conditions until $\mathcal{L}$ is rational. In [18] (Sec. VIII), we study rationality for special graphs G .

Example ($\ell = 1$). The four-mass box is made rational by setting $\langle M_1 M_2 \rangle = 0$. This is the *three-mass box*: $G = K_1$, $u = (4)$, and H_u being one edge. The two transversals to $M_1 = (12)$, $M_2 = (23)$, $M_1 = (45)$, and $M_1 = (67)$ are $L^{(b)} = (245) \star (267)$, $L^{(w)} = ((123) \star (45), (123) \star (67))$, and $\Delta_{K_1, (4)}$ becomes $\Delta_{\mathcal{L}} = \langle L^{(b)} L^{(w)} \rangle = \langle 245 | 13 | 672 \rangle^2$. This is the square of a cluster variable in $\text{Gr}(4, 7)$.

Cluster algebras famously govern the symbol alphabet of planar $\mathcal{N} = 4$ SYM amplitudes. We formulate this phenomenon in the present setting as follows.

Cluster factorization conjecture: Let $\mathcal{L}$ be a planar leading rational Landau diagram. Then, $\Delta_{\mathcal{L}}$ factors into a product of cluster variables for $\text{Gr}(4, n)$.

The origin of cluster algebras in this context was until now mysterious. We suggest an explanation: cluster variables arise as discriminants encoding collisions of rational leading singularities. Moreover, Eq. (6) reveals a recursive mechanism behind cluster factorization: assume $\Delta_{\mathcal{L}_1}$ and $\text{LS}_{\mathcal{L}_2}$ are products of cluster variables. If each substitution map φ_r^* is a *cluster quasi-homomorphism* [34], then in particular it sends cluster variables to cluster variables, and

hence $\text{LS}_{\mathcal{L}}$ factors into cluster variables. We now show this mechanism for trees G using the three-mass box as seed.

Let $\mathcal{L}$ be the rational pentabox of Fig. 3: $G = K_2$, $u = (4, 3)$, and H_u with two edges. Vertex 1 has the external lines $a_1 a_2, a_2 a_3, a_4 a_5, a_6 a_7$, and vertex 2 has the external lines $b_1 b_2, b_2 b_3, b_4 b_5$. These are ordered. By the recursion, the LS discriminant $\Delta_{\mathcal{L}}$ factors as follows:

$$\langle b_2 b_4 b_5 | b_1 b_3 | b_2, (a_4 a_1 a_2) \star (a_4 a_6 a_7) \rangle^2, \cdot \langle b_2 b_4 b_5 | b_1 b_3 | b_2, (a_3 a_4 a_5) \star (a_1 a_2), (a_3 a_4 a_5) \star (a_6 a_7) \rangle^2. \quad (9)$$

More generally, let $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2$ be a Landau diagram reducible by the three-mass box $\mathcal{L}_1$. By [18] (Prop 12.3),

$$\Delta_{\mathcal{L}} = \varphi_{\mathbf{b}}^* \Delta_{\mathcal{L}_1} \cdot \varphi_{\mathbf{w}}^* \Delta_{\mathcal{L}_2}. \quad (10)$$

The maps φ_{σ}^* , for $\sigma \in \{\mathbf{b}, \mathbf{w}\}$, substitute the rational solutions $L^{(\sigma)}$ of the three-mass box from the previous example into $\Delta_{\mathcal{L}_2}$. These maps are cluster quasi-homomorphisms ([18], Lemma 12.5), for example, $\varphi_{\mathbf{b}}^*$ is closely related to *BCFW promotion* ([35], Sec. IV). Hence, they send cluster variables to cluster variables, so the two factors in (9) are cluster variables in $\text{Gr}(4, 12)$. More generally, recursions of the form (10) preserve factorization into cluster variables.

Theorem ([18], Thm 12.6): the cluster factorization conjecture is true if the underlying graph G is a tree.

The first Landau diagram whose graph G is not a tree is $\mathcal{L} = G_u \cup H_u$ with $G = K_3$, $u = (3, 3, 3)$, and one pendant triangle per vertex. We restrict to the irreducible component of V_G where the three lines are concurrent. The Landau map has eight rational points in its fiber, one for each coloring (black $\mathbf{b}$ for concurrent or white $\mathbf{w}$ for coplanar) of the three pendant triangles. The LS discriminant $\Delta_{\mathcal{L}}$ factors into 12 irreducible factors, one for each edge of the 3-cube labeled by $\{\mathbf{b}, \mathbf{w}\}^3$. Each factor is the square of a cluster variable. Let $a_1 a_2, a_2 a_3$, and $a_4 a_5$ denote the external lines at vertex 1, and similarly with b_i and c_i at vertices 2 and 3. The 15 vectors defining these lines form a totally positive 4×15 matrix $\mathbf{z}$, whose columns are ordered as $a_1, a_2, \dots, a_5, b_1, b_2, \dots, b_5, c_1, c_2, \dots, c_5$. One of the factors is $\langle a_2 a_4 a_5 | a_1 a_3 | a_2, (b_2 b_4 b_5) \star (c_2 c_4 c_5) \rangle^2$, which is a cluster variable for $\text{Gr}(4, 15)$. Each factor can be written as the pairing $\langle L_i^{(\sigma_1)} L_i^{(\sigma_2)} \rangle$, where the colorings σ_1 and σ_2 differ in exactly one position i . Similarly, the coplanar component has 12 analogous factors.

This suggests the following general picture. For planar rational Landau diagrams, the rational points in the fibers correspond to colorings of the ℓ pendant triangles, according to whether they form triples of coplanar or concurrent lines. The LS discriminant is expected to factor into cluster variables, each associated with a pair of colorings σ_1, σ_2 that differ at a single position i , see [18] (Conj 8.12). The

corresponding factor is given by the pairing $\langle L_i^{(\sigma_1)} L_i^{(\sigma_2)} \rangle$, for suitable representatives of $L_i^{(\sigma_1)}$ and $L_i^{(\sigma_2)}$.

The mechanism underlying both positivity and cluster structure of LS discriminants is the recursion (6), together with the interplay with the substitution maps. By relating line incidence varieties to positroid varieties and amplituhedron maps, we show in [18] (Secs. 10–12) that these substitution maps coincide with *promotion maps* [28] between Grassmannians defined from plabic graphs. General promotion maps are conjectured to preserve positivity and, in the rational regime, to define cluster quasi-homomorphisms, referred to as *cluster promotion maps* ([28], Conjectures 4.16 and 10.13). This leads to the following:

Substitution maps conjecture: Let $\mathcal{L}$ be a planar Landau diagram reducible by $\mathcal{L}_1$, as in (6). Then, the substitution maps φ_r are copositive, i.e., they map positive lines to positive lines. Moreover, in the rational regime, the pull-backs φ_r^* are cluster quasi-homomorphisms.

VIII. OUTLOOK

This letter, together with the papers [17,18], develops a rigorous framework for Landau analysis on the Grassmannian. We uncover a recursive mechanism governing a large class of Landau singularities, bridging their algebro-geometric description as discriminants and resultants with positivity and cluster structures seen in planar $\mathcal{N} = 4$ SYM. Directions for future research are outlined in [18] (Sec. 13). These include the extension of our framework to subleading singularities, the relation between the Landau analysis in the Grassmannian and the loop amplituhedron, and cluster adjacencies from discriminants across different strata of Landau singularities. Another relevant direction is to consider more degenerate external kinematics configurations, which would enable further comparison with existing computations for certain all-loop families of graphs [36–39].

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

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