

Lubkin-Page typicality bounds for type II von Neumann factors

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Typicality arguments for emergent spacetime rely on the Lubkin-Page bounds, which show that generic quantum states have vanishing correlations between subsystems. These bounds assume a tensor-product Hilbert space (a type I von Neumann algebra), but the observable algebras in quantum field theory and quantum gravity are generically type II or type III, raising the question of whether the bounds survive. We prove that they do for all type II von Neumann factors. For the hyperfinite type II₁ factor with a tripartite decomposition $R \cong A \otimes B \otimes E$, the mutual information between subsystems A and B vanishes as $O((d_A d_B / d_E)^2)$ in finite-dimensional approximations, provided $d_A d_B \leq d_E$ (Theorem 1). For Type II_∞ factors, including the gravitational algebras constructed via the crossed-product method by Witten and by Chandrasekaran, Longo, Penington, and Witten, the bound acquires an additional exponential suppression controlled by the Bekenstein-Hawking entropy (Theorem 2). We identify the obstructions to extending the result to type III factors and discuss the open question of whether the commutant of the observable algebra can serve as a natural thermal bath that tightens the bound further.

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Introduction. Arguments that spacetime geometry emerges from quantum entanglement [1–3] rest on a quantitative foundation: the Lubkin-Page theorem [4,5], which establishes that generic quantum states have near-maximal subsystem entropy and vanishing intersubsystem correlations. Combined with the conjecture that entanglement is necessary for connected geometry [1], this implies that geometry-supporting states occupy an exponentially thin sliver of the Hilbert space [6]. The argument has concrete observational consequences: a pregeometric initial phase produces a cutoff in the primordial power spectrum that may account for the anomalous suppression of the CMB quadrupole observed by COBE, WMAP, and Planck [6].

A central concern is that these typicality bounds assume a tensor-product Hilbert space (a type I von Neumann algebra). In quantum field theory, the local observable algebras are generically type III factors [7], for which the von Neumann entropy is not directly defined. Recent work by Witten [8] and Chandrasekaran, Longo, Penington, and Witten [9] has shown that gravitational effects modify these to type II algebras, which admit a well-defined entropy (up to an additive constant) and a semifinite trace. Whether the

Lubkin-Page bounds extend to these algebras is therefore a question of direct physical relevance: if they do not, the typicality-based arguments for emergent spacetime would be undermined precisely in the setting where they are most needed.

In a companion paper [10], we proved that the Lubkin-Page bounds survive and are tightened in finite-dimensional Hilbert spaces with direct-sum-of-tensor-product structure, the generic form arising from gauge invariance and superselection rules. Here we take the next step: we formulate and prove Lubkin-Page bounds for type II von Neumann factors (both II₁ and II_∞), the simplest infinite-dimensional settings in which a trace and a well-defined entropy exist. For type II_∞ factors, which include the gravitational algebras of Refs. [8,9], the Bekenstein-Hawking entropy provides an exponential suppression of intersubsystem correlations for typical states.

Background.

Type II₁ factors: A type II₁ factor M is an infinite-dimensional von Neumann algebra with trivial center and a unique faithful normal tracial state $\tau: M \rightarrow \mathbb{C}$ satisfying $\tau(\mathbf{1}) = 1$ and $\tau(xy) = \tau(yx)$ for all $x, y \in M$. The trace plays the role of the normalized matrix trace $\text{Tr}(\cdot)/d$ in finite dimensions.

The GNS construction associated with τ produces a Hilbert space $L^2(M, \tau)$ on which M acts by left multiplication, with a cyclic and separating vector Ω such that $\tau(x) = \langle \Omega, x\Omega \rangle$.

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The prototypical example is the hyperfinite type II_1 factor R , which is (up to isomorphism) the unique approximately finite-dimensional (AFD) type II_1 factor [11]. It can be realized as the weak closure of an increasing chain of matrix algebras

$$M_{d_1}(\mathbb{C}) \subset M_{d_2}(\mathbb{C}) \subset M_{d_3}(\mathbb{C}) \subset \cdots \subset R, \quad (1)$$

where $d_1|d_2|d_3|\cdots$ and $R = \overline{\bigcup_n M_{d_n}(\mathbb{C})}^{\text{weak}}$. The trace of R restricts to the normalized matrix trace $\text{Tr}(\cdot)/d_n$ on each M_{d_n} .

Subfactors and the Jones index: A subfactor $A \subset M$ of type II_1 factors has an associated Jones index $[M:A]$, which measures the “relative size” of A within M [12,13]. When M factors as a tensor product $M \cong A \otimes B$ of type II_1 factors (with B the relative commutant $A' \cap M$), the Jones index satisfies

$$[M:A] = [\tau_{M_1}(e_A)]^{-1}, \quad (2)$$

where $M_1 = \langle M, e_A \rangle$ is the Jones basic construction and e_A is the Jones projection implementing the trace-preserving conditional expectation $E_A: M \rightarrow A$. In the finite-dimensional analog $M = M_d(\mathbb{C})$, $A = M_{d_A}(\mathbb{C}) \otimes \mathbf{1}_{d_B}$, the Jones index is $[M:A] = d_B^2$, the square of the complement dimension. A large Jones index corresponds to a “small” subsystem A inside a “large” algebra M , the regime in which the Lubkin bound gives strong suppression of intersubsystem correlations.

Entropy for type II_1 factors: For a normal state φ on a type II_1 factor M with density operator ρ_φ (defined by $\varphi(x) = \tau(\rho_\varphi x)$ for all $x \in M$), the entropy relative to the trace is

$$S(\varphi|\tau) = -\tau(\rho_\varphi \log \rho_\varphi) = S(\varphi), \quad (3)$$

where $S(\varphi) = -\tau(\rho_\varphi \log \rho_\varphi)$ is the Segal entropy and we have used $\log \tau(\mathbf{1}) = \log 1 = 0$. For the tracial state itself, $\rho_\tau = \mathbf{1}$, with Segal entropy $S(\tau) = -\tau(\mathbf{1} \log \mathbf{1}) = 0$.

In the finite-dimensional case $M = M_d(\mathbb{C})$ with $\tau = \text{Tr}(\cdot)/d$, a pure state φ has $\rho_\varphi = d|\psi\rangle\langle\psi|$ (since $\tau(\rho_\varphi x) = \langle\psi|x|\psi\rangle$ requires $\rho_\varphi = d|\psi\rangle\langle\psi|$), giving $S(\varphi) = -\log d$: a pure state has the minimum Segal entropy, reflecting that it is maximally “ordered” relative to the tracial state.

The restriction of φ to a subfactor $A \subset M$ has density $\rho_{\varphi|_A}$ defined by $\varphi(x) = \tau_A(\rho_{\varphi|_A} x)$ for $x \in A$, and entropy $S(\varphi|_A) = -\tau_A(\rho_{\varphi|_A} \log \rho_{\varphi|_A})$.

The quantum mutual information (MI) between A and B (assuming $M \cong A \otimes B$) is

$$I(A:B)_\varphi = S(\varphi|_A) + S(\varphi|_B) - S(\varphi). \quad (4)$$

For the tracial state, $\varphi|_A = \tau_A$ and $\varphi|_B = \tau_B$, so $I(A:B)_\tau = 0$. The tracial state is the “maximally uncorrelated” state, the analog of the maximally mixed state in finite dimensions.

Typicality in finite-dimensional approximations.

The tripartite structure: The Lubkin-Page bound requires that the subsystems of interest are small relative to the total system: for a Haar-random pure state on $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_E$, the mutual information $I(A:B)$ is small when the environment E is much larger than A and B combined. Without the environment, a Haar-random pure state on $\mathcal{H}_A \otimes \mathcal{H}_B$ has $I(A:B) \approx 2 \log \min(d_A, d_B)$, which is *maximal*, not small.

To apply the Lubkin bound in the type II_1 setting, we therefore need a tripartite decomposition $R \cong A \otimes B \otimes E$, where A and B are the subsystems of interest and E is an environment factor. All three are subfactors of the hyperfinite type II_1 factor R , and all three are hyperfinite. (By Connes’s theorem [11], a II_1 factor is hyperfinite if and only if it is injective. Injectivity passes to von Neumann subalgebras [14], so A , B , and E , as subfactors of the injective factor R , are themselves injective and hence hyperfinite.)

The compatible approximating tower: Since A , B , and E are each hyperfinite, they admit increasing chains of matrix algebras

$$A_n \cong M_{d_n^{(A)}}(\mathbb{C}), \quad B_n \cong M_{d_n^{(B)}}(\mathbb{C}), \quad E_n \cong M_{d_n^{(E)}}(\mathbb{C}). \quad (5)$$

We construct the approximating tower for R by taking tensor products

$$M_n = A_n \otimes B_n \otimes E_n \cong M_{d_n}(\mathbb{C}), \quad d_n = d_n^{(A)} \cdot d_n^{(B)} \cdot d_n^{(E)}. \quad (6)$$

By construction, $M_n \subset M_{n+1} \subset \cdots \subset R$, and $R = \overline{\bigcup_n M_n}^{\text{weak}}$. The tensor-product structure is built in at every level.

Lubkin bound at each level: At level n , the GNS Hilbert space of M_n with the normalized trace τ is $L^2(M_n, \tau) \cong \mathbb{C}^{d_n^2}$. The tripartite tensor-product structure (6) induces a decomposition of the GNS space

$$\mathbb{C}^{d_n^2} \cong \mathbb{C}^{(d_n^{(A)})^2} \otimes \mathbb{C}^{(d_n^{(B)})^2} \otimes \mathbb{C}^{(d_n^{(E)})^2}. \quad (7)$$

A Haar-random unit vector $|\psi_n\rangle \in \mathbb{C}^{d_n^2}$ defines a pure state on the full space. Tracing over the environment E_n gives a mixed state on $A_n \otimes B_n$. The standard Lubkin bound [4,5] on the mutual information between A_n and B_n gives

$$\langle I(A_n : B_n)_{\varphi_n} \rangle \leq \frac{(d_n^{(A)})^2 \cdot (d_n^{(B)})^2}{2 \ln 2 \cdot (d_n^{(E)})^2}, \quad (8)$$

where we have used $d_n^{(A)} \cdot d_n^{(B)} \leq d_n^{(E)}$ (small subsystems relative to environment) and retained only the dominant term. Here the mutual information is computed between the GNS-level subsystems $\mathbb{C}^{(d_n^{(A)})^2}$ and $\mathbb{C}^{(d_n^{(B)})^2}$; since tracing out degrees of freedom cannot increase mutual information, this bounds the physical (left-action) mutual information $I(A_n : B_n)$ from above. The bound vanishes as $d_n^{(E)} \rightarrow \infty$, i.e., as the environment grows.

The limit.

Convergence of states: As $n \rightarrow \infty$, the Haar-random vector states on M_n define a sequence of random states on R . The inclusion $M_n \hookrightarrow M_{n+1} \hookrightarrow \dots \hookrightarrow R$ induces, for each $|\psi_n\rangle \in \mathbb{C}^{d_n^2}$, a state φ_n on R defined by $\varphi_n(x) = \langle \psi_n | E_n(x) | \psi_n \rangle$, where $E_n : R \rightarrow M_n$ is the trace-preserving conditional expectation.

Lower semicontinuity of the mutual information: The mutual information can be expressed as a single relative entropy

$$I(A : B)_\varphi = S(\varphi|_{A \vee B} \| \varphi|_A \otimes \varphi|_B), \quad (9)$$

where $A \vee B$ is the algebra generated by A and B (which equals $A \otimes B$ in our setting, a proper subalgebra of R since $R \cong A \otimes B \otimes E$). The Araki relative entropy is jointly lower semicontinuous in both arguments with respect to the weak-* topology [15,16]. Applied to (9), this gives: for any sequence of states φ_n converging weak-* to φ ,

$$I(A : B)_\varphi \leq \liminf_{n \rightarrow \infty} I(A_n : B_n)_{\varphi_n}. \quad (10)$$

The direction of the inequality is the one we need: if the finite-level mutual information is small, the limit mutual information is also small.

The typicality bound: Combining (8) and (10)

Theorem 1. Let R be the hyperfinite type II_1 factor with a tripartite tensor-product decomposition $R \cong A \otimes B \otimes E$. Let $\{M_n = A_n \otimes B_n \otimes E_n\}$ be the compatible approximating tower constructed in Sec. III A, and let φ_n be the state on $A_n \otimes B_n$ obtained by tracing out the environment E_n from a Haar-random vector in $L^2(M_n, \tau)$. Assume that $d_n^{(A)} \cdot d_n^{(B)} \leq d_n^{(E)}$ for all sufficiently large n . Then the Haar-averaged mutual information satisfies

$$\langle I(A_n : B_n)_{\varphi_n} \rangle \leq \frac{(d_n^{(A)})^2 \cdot (d_n^{(B)})^2}{2 \ln 2 \cdot (d_n^{(E)})^2} \xrightarrow{n \rightarrow \infty} 0, \quad (11)$$

provided $d_n^{(E)}$ grows faster than $d_n^{(A)} \cdot d_n^{(B)}$. For any weak-* limit state $\varphi = \lim \varphi_n$,

$$I(A : B)_\varphi = 0. \quad (12)$$

Proof. At each level n , the GNS space is tripartite: $\mathbb{C}^{(d_n^{(A)})^2} \otimes \mathbb{C}^{(d_n^{(B)})^2} \otimes \mathbb{C}^{(d_n^{(E)})^2}$. A Haar-random pure state on this space gives, after tracing out E_n , a mixed state on $A_n \otimes B_n$ whose MI is bounded by (8). Since $(d_n^{(E)})^2 \rightarrow \infty$ faster than $(d_n^{(A)})^2 \cdot (d_n^{(B)})^2$, the bound vanishes.

For the limit: since $I(A_n : B_n)_{\varphi_n} \geq 0$ and $\langle I(A_n : B_n)_{\varphi_n} \rangle \rightarrow 0$, Markov's inequality implies $I(A_n : B_n)_{\varphi_n} \rightarrow 0$ in probability. For any weak-* limit state φ , lower semicontinuity (10) then yields the result $I(A : B)_\varphi = 0$. ■

Remark on the role of the environment. The environment factor E is essential. For a bipartite decomposition $R \cong A \otimes B$ with no environment, a Haar-random pure state on $L^2(A_n \otimes B_n, \tau)$ has MI of order $2 \log[\min(d_n^{(A)}, d_n^{(B)})]^2$, which is *maximal*, not small: the Lubkin-Page theorem says the subsystem entropy is near maximal for a pure state, so the two factors are nearly maximally entangled. The environment provides the “bath” into which the correlations are diluted. Physically, E represents all degrees of freedom outside the two subsystems of interest, and the Lubkin bound says that when E is large, the reduced state on $A \otimes B$ is nearly uncorrelated.

Remark on the GNS left-right decomposition. The GNS Hilbert space $L^2(M_n, \tau)$ admits a left-right decomposition $\mathcal{H}_L \otimes \mathcal{H}_R$ with $\dim \mathcal{H}_L = \dim \mathcal{H}_R = d_n$, where the observable algebra acts on $\mathcal{H}_L$ and its commutant acts on $\mathcal{H}_R$. If the Haar measure on the full GNS space $\mathbb{C}^{d_n^2}$ is the appropriate typicality ensemble, then the commutant acts as a thermal bath that automatically places the observable subsystems in the small-subsystem regime, and the dimensional condition $d_n^{(A)} \cdot d_n^{(B)} \leq d_n^{(E)}$ in Theorem 1 can be dropped, yielding a tighter bound $O(1/(d_n^{(E)})^2)$ that depends only on the environment dimension. However, whether this ensemble is physically appropriate depends on whether the commutant represents independent physical degrees of freedom or is merely a representation-theoretic auxiliary of the GNS construction. In algebraic quantum field theory and holography, for instance, Tomita-Takesaki theory identifies the commutant with the physical degrees of freedom of the causal complement, such as a black hole interior, naturally motivating this tighter ensemble [7]. We leave this question open and present the tripartite result with the explicit dimensional condition, which is rigorous and does not depend on the interpretation of the commutant.

Remark on finite Jones index. Jones's celebrated theorem [12] shows that the index of a subfactor can take finite values in the set $\{4 \cos^2(\pi/m) : m \geq 3\} \cup [4, \infty)$, but these

arise from subfactors that do *not* decompose R as a tensor product. For an irreducible subfactor with $A' \cap R = \mathbb{C}\mathbf{1}$ (i.e., the only operators in R that commute with every element of A are scalar multiples of the identity), no tensor-product complement exists, and the mutual information between A and a “complement” is not defined in the tensor-product sense. Extending the typicality bound to finite-index subfactors would require a reformulation in terms of the conditional expectation $E_A: R \rightarrow A$ and the associated relative entropy $S(\varphi\|\varphi \circ E_A)$, which we leave to future work.

Interpretation: The theorem establishes that for the hyperfinite II_1 factor with a tripartite tensor-product decomposition, Haar-random vector states in finite-dimensional approximations have mutual information between any two factors that vanishes as the environment factor grows. The rate of vanishing is controlled by the ratio $(d_n^{(A)} \cdot d_n^{(B)} / d_n^{(E)})^2$.

This is the direct analog of the finite-dimensional Lubkin-Page result: in a tripartite system $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_E$, the MI between A and B for a Haar-random pure state is small when $d_E \gg d_A \cdot d_B$. In the type II_1 case, the environment is always infinite-dimensional (in the limit), and the MI vanishes exactly.

The tracial state τ (which has $I = 0$) is the “maximally uncorrelated” state. The theorem confirms that Haar-random vector states converge to this uncorrelated behavior as the approximation level increases, exactly as in the finite-dimensional Lubkin-Page theory.

Extension to type II_∞ and discussion of type III.

Type II_∞ factors: A type II_∞ factor has the form $M = R \otimes \mathcal{B}(\mathcal{H})$, where R is a type II_1 factor and $\mathcal{H}$ is a separable infinite-dimensional Hilbert space. It admits a semifinite trace $\tau = \tau_R \otimes \text{Tr}$ (faithful and normal, but $\tau(\mathbf{1}) = \infty$).

Tripartite subfactor structure. Consider two tensor-product subfactors $\mathcal{A} = A_0 \otimes \mathcal{B}(\mathcal{H}_A)$ and $\mathcal{B} = B_0 \otimes \mathcal{B}(\mathcal{H}_B)$, where A_0, B_0, E_0 are type II_1 subfactors with $R \cong A_0 \otimes B_0 \otimes E_0$, and $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_E$. The environment is $\mathcal{E} = E_0 \otimes \mathcal{B}(\mathcal{H}_E)$, and the full algebra decomposes as $M \cong \mathcal{A} \otimes \mathcal{B} \otimes \mathcal{E}$.

Doubly indexed approximation. We introduce two approximation parameters: n for the II_1 component and D for the type I_∞ component. For the II_1 part, construct the compatible tower $\{A_{0,n} \otimes B_{0,n} \otimes E_{0,n}\}$ as in Sec. III A. For the type I_∞ part, truncate to d_A , d_B , and d_E -dimensional subspaces of $\mathcal{H}_A$, $\mathcal{H}_B$, and $\mathcal{H}_E$, respectively, with $D = d_A \cdot d_B \cdot d_E$.

The truncated algebra at level (n, D) is

$$M_{n,D} = \underbrace{(A_{0,n} \otimes M_{d_A})}_{\mathcal{A}_{n,D}} \otimes \underbrace{(B_{0,n} \otimes M_{d_B})}_{\mathcal{B}_{n,D}} \otimes \underbrace{(E_{0,n} \otimes M_{d_E})}_{\mathcal{E}_{n,D}}. \quad (13)$$

GNS space. The GNS Hilbert space of $M_{n,D}$ with the normalized trace decomposes as

$$L^2(M_{n,D}, \tau) \cong \underbrace{\mathbb{C}^{(d_n^{(A)})^2 d_A^2}}_{\text{sub } \mathcal{A}} \otimes \underbrace{\mathbb{C}^{(d_n^{(B)})^2 d_B^2}}_{\text{sub } \mathcal{B}} \otimes \underbrace{\mathbb{C}^{(d_n^{(E)})^2 d_E^2}}_{\text{env}}. \quad (14)$$

Theorem 2. Let $M = R \otimes \mathcal{B}(\mathcal{H})$ be a type II_∞ factor with the tripartite decomposition above. Let $\varphi_{n,D}$ be the state on $\mathcal{A}_{n,D} \otimes \mathcal{B}_{n,D}$ obtained by tracing out the environment from a Haar-random vector in $L^2(M_{n,D}, \tau)$. Assume that $d_n^{(A)} \cdot d_A \cdot d_n^{(B)} \cdot d_B \leq d_n^{(E)} \cdot d_E$. Then

$$\langle I(\mathcal{A}_{n,D} : \mathcal{B}_{n,D})_{\varphi_{n,D}} \rangle \leq \frac{(d_n^{(A)})^2 d_A^2 \cdot (d_n^{(B)})^2 d_B^2}{2 \ln 2 \cdot (d_n^{(E)})^2 d_E^2}. \quad (15)$$

The bound vanishes provided $d_n^{(E)} \cdot d_E / (d_n^{(A)} \cdot d_A \cdot d_n^{(B)} \cdot d_B) \rightarrow \infty$, giving

$$I(\mathcal{A} : \mathcal{B})_\varphi = 0 \quad (16)$$

for any weak-* limit state φ .

Proof. The GNS decomposition (14) is a tripartite tensor product. A Haar-random pure state, after tracing out the environment, gives a mixed state on $\mathcal{A}_{n,D} \otimes \mathcal{B}_{n,D}$. The Lubkin bound on the MI gives (15). Lower semicontinuity of the Araki relative entropy gives the limit statement (16) by the same argument as Theorem 1. ■

Connection to gravitational algebras. The crossed-product construction of Witten [8] promotes a type III_1 algebra (the local observable algebra of a QFT) to a type II_∞ factor $M = \mathcal{A} \rtimes_\sigma \mathbb{R}$, where σ is the modular automorphism group. In the gravitational setting [9], the observer’s Hamiltonian provides a natural energy cutoff, and the Bekenstein-Hawking entropy $S_{\text{BH}} = A/(4G\hbar)$ bounds the effective dimension of the truncated Hilbert space: $D_{\text{eff}} \sim e^{S_{\text{BH}}}$. If the environment (the exterior of a black hole, or the complement of the observed region) has Bekenstein-Hawking entropy $S_{\text{BH}}^{(E)}$, then $d_E \sim e^{S_{\text{BH}}^{(E)}}$ and the bound (15) is exponentially small

$$\langle I \rangle \lesssim \frac{d_A^2 \cdot d_B^2}{2 \ln 2} \cdot e^{-2S_{\text{BH}}^{(E)}}. \quad (17)$$

For macroscopic black holes, $S_{\text{BH}}^{(E)} \gg 1$ and the bound is exponentially suppressed: typical states of the gravitational algebra have negligible correlations between two local subsystems when the environment is large.

Remark. The individual entropies $S(\varphi|_{\mathcal{A}})$ and $S(\varphi|_{\mathcal{B}})$ diverge as $D \rightarrow \infty$ (because the trace is infinite), but the mutual information remains finite and bounded because the divergent contributions cancel in $I = S(\mathcal{A}) + S(\mathcal{B}) - S(\mathcal{AB})$. In the crossed-product setting, this corresponds to

the well-known fact that the entropy is defined only up to an additive constant [8], but the MI is unambiguous.

Type III factors: Type III factors have no trace, no density matrices, and no von Neumann entropy. The local algebras of quantum field theory in Minkowski space are type III₁ [7].

For type III, the only well-defined entropic quantity is the Araki relative entropy [15]

$$S(\varphi|\omega) = -\langle \Omega_\varphi | \log \Delta_{\omega,\varphi} | \Omega_\varphi \rangle, \quad (18)$$

where Ω_φ is the vector representative of φ in the natural positive cone and $\Delta_{\omega,\varphi}$ is the relative modular operator.

The mutual information can be defined as

$$I(\varphi; \mathcal{M}_A, \mathcal{M}_B) = S(\varphi|_{\mathcal{M}_A \vee \mathcal{M}_B} \| \varphi|_{\mathcal{M}_A} \otimes \varphi|_{\mathcal{M}_B}), \quad (19)$$

provided the algebras $\mathcal{M}_A$ and $\mathcal{M}_B$ are independent (their generated algebra is isomorphic to the tensor product). This reduces to the standard MI for type I.

A Lubkin-type bound for type III faces three obstructions:

No trace. There is no analog of the “maximally mixed state” to serve as the reference state in the Lubkin bound. The vacuum state or KMS state could serve as a substitute, but the bound would then depend on the choice of reference state, unlike the type I case where the maximally mixed state is canonical.

No Haar measure. The state space of a type III factor is infinite-dimensional and does not carry a natural uniform measure. The notion of “typical state” must be replaced by an ensemble-dependent notion: typical relative to a given energy constraint, temperature, or dynamical evolution. The microcanonical ensemble (uniform measure on an energy shell) or the eigenstate thermalization hypothesis (ETH) could provide the necessary substitute.

No dimension. The Jones index, which controls the rate of concentration for type II₁, has an analog for type III subfactors (the Kosaki index or the Longo index [17]), but its relationship to concentration of measure is unexplored.

Despite these obstructions, there is reason for cautious optimism. The Bisognano-Wichmann theorem [18] relates the modular flow of the vacuum state to geometric (boost) transformations, providing a connection between the algebraic structure and spacetime geometry that could anchor a typicality argument. The ETH, if it holds for the relevant algebras, would provide a dynamical notion of typicality that does not require a Haar measure. And the recent constructions of type II algebras from type III via crossed products [8,9] suggest that the type III obstruction may be an artefact of the continuum limit that is resolved by gravitational effects.

A complete Lubkin-Page theorem for type III algebras would constitute a major advance, placing typicality-based

arguments for emergent spacetime on a fully rigorous algebraic footing. The type II results proved here is a step in that direction.

Conclusion. We have shown that the Lubkin-Page typicality bounds extend to both type II₁ and type II_∞ von Neumann factors, establishing that generic states of these algebras have vanishing inter-subsystem correlations. For type II_∞ factors, the bound applies directly to the gravitational algebras constructed via the crossed-product method [8,9], where the Bekenstein-Hawking entropy provides an exponential suppression: typical states of these algebras cannot support connected spacetime geometry, just as in the finite-dimensional case.

Together with the companion result for direct-sum Hilbert spaces [10], this closes the most technically substantive algebraic objection to typicality-based arguments for emergent spacetime: the physical Hilbert space of quantum gravity, whether it has superselection sectors (type I with direct-sum structure), a semifinite trace (type II), or both, does not rescue geometry-supporting states from their exponential rarity.

The extension to type III factors remains open and requires new ideas to replace the trace, the Haar measure, and the notion of dimension. The most promising avenues are the eigenstate thermalization hypothesis as a dynamical substitute for typicality, the modular theory of Tomita-Takesaki as a substitute for the partial trace, and the Kosaki-Longo index as a substitute for the Jones index. There is also the open question of whether the commutant of the observable algebra, which in holographic settings corresponds to the physical degrees of freedom of the causal complement, can serve as a natural thermal bath that tightens the bounds further.

It is also worth noting a different line of research in which general relativity emerges as a hydrodynamic approximation to underlying quantum systems [19–24]. As synthesized in Ref. [24], this framework explicitly breaks a quantum gravitational system into finite-dimensional subsystems, replacing the continuous type III algebras of algebraic quantum field theory (AQFT) with type I_N algebras. It prescribes the modular Hamiltonian of each causal diamond according to rules conjectured by Jacobson, Carlip, and Solodukhin [19–22] (with a cutoff on the conformal field theory) and generalized by Banks and Zurek [23]. The quantum dynamics is described by an infinite number of independent time evolution operators in a Hilbert bundle over the space of timelike geodesics on the hydrodynamic background, where entanglement plays the role of the connection: the entanglement spectrum on overlapping diamonds must be the same, independent of which geodesic is used to compute time evolution. This allows one to compute time evolution outside the instantaneous causal diamond of any given geodesic. Furthermore, the “empty diamond state” of the background geometry is

posited to be a maximal entropy state allowed by quantum gravity, contrary to AQFT, whereas states with localized excitations in a diamond reduce the entropy. All of this applies for non-negative cosmological constants or diamonds parametrically smaller than the anti-de Sitter (AdS) radius. For larger diamonds in AdS space, one builds tensor networks with these small diamonds as nodes, where time evolution follows the (inverse) tensor network renormalization group of Evenbly and Vidal [25]. Because this holographic space-time approach relies fundamentally on finite-dimensional subsystems from the outset, the typicality bounds of Lubkin and Page directly apply, bypassing the need for infinite-dimensional algebraic generalizations.

While such finite-dimensional frameworks elegantly circumvent the issue by explicitly converting continuous

type III algebras to discrete type I_N systems, whether a fully algebraic Lubkin-Page theorem can be formulated natively for type III factors in the continuum limit is nevertheless one of the central open problems at the interface of algebraic quantum field theory and quantum gravity. If the Bekenstein-Hawking bound renders the physical Hilbert space finite-dimensional, however, or as explicitly realized in the hydrodynamic approach discussed above, the type III structure is an artefact of the continuum approximation, and the results proved here and in [10] apply directly.

Data availability. There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

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