

Double heavy quarkonia production with color-octet channels at a Z factory and at the CEPC and FCC-ee

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(Received 31 January 2025; accepted 12 May 2025; published 11 June 2025)

Within the nonrelativistic QCD framework, we calculate the double heavy quarkonium (double charmonium and double bottomonium) production at a future super Z factory and at the Circular Electron Positron Collider (CEPC) and Electron-Positron Future Circular Collider (FCC-ee). The complete color-octet (CO) channels in the γ^*/Z^* -propagated process are considered along with the color-singlet (CS) channels at leading order of α_s . We found that the contributions of CO states to the total cross section are significant or dominant for many processes within the energy region at a Z factory and at the CEPC and FCC-ee. The experimental measurements will help us to verify the CO mechanism. Among these CO channels, the gluon fragmentation process (into $^3S_1^8$ states at leading order) is most important. The comparison between the theoretical results and future data may give constraints to the CO matrix elements. Additionally, we consider the relativistic corrections to both the CS and CO channels which decrease the cross sections significantly. Specially, the K factors are about 0.5 for most charmonium channels. We get estimates of the events for double heavy quarkonium production. The final events of $J/\psi + \eta_c$, $J/\psi + J/\psi$, and $\eta_c + \eta_c$ production, with J/ψ reconstructed by lepton pair and η_c reconstructed by hadronic decays, would be (1, 8, 4) and (11, 77, 37) at the CEPC (2-yr) and at the FCC-ee (4-yr) for the Z factory mode, respectively.

DOI: [10.1103/c7jj-y61n](https://doi.org/10.1103/c7jj-y61n)

I. INTRODUCTION

Heavy quarkonium is a bound state of a heavy quark (Q) and its antiquark ($\bar{Q}$) with a nonrelativistic nature. Nonrelativistic QCD (NRQCD) [1] has been widely accepted to study heavy quarkonium production and decay. Under this framework, the heavy quarkonium production can be factorized into short-distance coefficients (SDCs) and long-distance matrix elements (LDMEs). The SDCs, which represent the production of intermediate $Q\bar{Q}$ pairs, are perturbatively calculable. The LDMEs, which are nonperturbative universal parameters, describe the hadronization of the $Q\bar{Q}$ pair into a physical quarkonium state. These LDMEs can be obtained from potential models, lattice QCD calculations, or extracted from experimental data.

The color-octet mechanism (COM) is a key component of the NRQCD approach, introduced by systematically accounting for the higher-Fock components of a quarkonium

state. Unlike the color-singlet model (CSM), the intermediate $Q\bar{Q}$ pair in COM can be in a color-octet (CO) configuration with different quantum numbers from the final quarkonium state. With the COM, infrared divergences encountered in the CSM can be eliminated [2], and the ψ' anomaly observed at hadron colliders can be explained [3–5], which is considered a significant achievement of NRQCD. However, there are still some challenges that question the NRQCD approach. One of these is the universality problem of the CO LDMEs. LDMEs extracted from hadronic scattering measurements by different theoretical groups have shown inconsistencies [6–9]. Additionally, for inclusive and exclusive charmonium production at B factories, higher-order corrections to the color-singlet (CS) channel often saturate the data, leaving little room for the COM [10–24]. We refer the reader to the reviews in Refs. [25–29] for more information on the current status of heavy quarkonia physics.

Several future e^+e^- collider projects are proposed, including the Circular Electron Positron Collider (CEPC) [30], the Future Circular Collider (FCC) [31,32], and the GigaZ program of the International Linear Collider [33–35]. Additionally, a super Z factory with a high luminosity of $\mathcal{L} \simeq 10^{34-36} \text{ cm}^{-2} \text{ s}^{-1}$ [36] is also proposed. These facilities provide excellent opportunities to revisit both exclusive and inclusive heavy quarkonium production, thereby allowing for a more thorough verification of the COM. Within the

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CSM, the exclusive production of double heavy quarkonia in e^+e^- annihilation at Z factories has been extensively studied at leading order (LO) [14,37,38]. Next-to-leading-order (NLO) QCD corrections have also been considered [39–41]. Specifically, NLO QCD corrections for double S -wave B_c meson production [42] and associated S -wave charmonium-bottomonium production [43] have been obtained. Recently, Liao *et al.* [44–46] have investigated the production of double charmonia, double bottomonia, and double B_c mesons at LO. Pure QED processes also contribute observably to these production mechanisms [41]. On the other hand, it has been found that the contributions of the COM are significant in semiexclusive processes [47]. Therefore, it is also necessary to study the effects of COM in both exclusive and inclusive heavy quarkonia production processes at future e^+e^- colliders.

In the present paper, we will study the production of double heavy quarkonia (double charmonia or double bottomonia) within the COM framework. The final states for these processes include $J/\psi + \eta_c$, $J/\psi + J/\psi$, $J/\psi + \chi_{cJ}$, $J/\psi + h_c$, $\eta_c + \eta_c$, $\eta_c + h_c$, $\eta_c + \chi_{cJ}$, $\Upsilon + \eta_b$, $\Upsilon + \Upsilon$, $\Upsilon + \chi_{bJ}$, $\Upsilon + h_b$, $\eta_b + \eta_b$, $\eta_b + h_b$, and $\eta_b + \chi_{bJ}$.

The rest of this paper is organized as follows. In Sec. II, we present the formula and method used in our study. In Sec. III, we provide the input parameters and LDMEs utilized in this work. In Sec. IV, we analyze all possible production channels and present the results of the cross sections and differential cross sections. We also discuss the generation of events and the associated uncertainties. Finally, a summary is given in Sec. V.

II. FRAMEWORK

A. Diagrams and amplitudes

Within the NRQCD framework, the production cross sections can be divided into two parts, SDCs and LDMEs [1]:

$$\hat{\sigma}(e^+ + e^- \rightarrow H_1 + H_2) = \sum_{mn} \frac{F_{mn}}{m_{Q1}^{d_m-4} m_{Q2}^{d_n-4}} \langle 0 | \mathcal{O}_m^{H_1} | 0 \rangle \langle 0 | \mathcal{O}_n^{H_2} | 0 \rangle. \quad (1)$$

SDCs F_{mn} are labeled by the subscripts m and n for different production channels. The factor $m_Q^{d_{m,n}-4}$ is introduced to make the SDCs F_{mn} dimensionless. The Fock state expansion formalism for heavy quarkonium up to the order of $\mathcal{O}(v^2)$ can be expressed as below, where v is the relative velocity between quark and antiquark in the heavy quarkonium rest frame:

$$\begin{aligned} |H\rangle &= \mathcal{O}(1) |Q\bar{Q}^{(2S+1)L_J^{[1]}}\rangle + \mathcal{O}(v) |Q\bar{Q}^{(2S+1)(L\pm 1)_J^{[8]}}g\rangle \\ &+ \mathcal{O}(v^2) |Q\bar{Q}^{(2(S\pm 1)+1)L_J^{[8]}}g\rangle \\ &+ \mathcal{O}(v^2) |Q\bar{Q}^{(2S+1)L_J^{[1,8]}}gg\rangle + \dots \end{aligned} \quad (2)$$

For specific heavy quarkonium mesons, the formula are written as

$$\begin{aligned} |J/\psi\rangle &= \mathcal{O}(1) |c\bar{c}({}^3S_1^{[1]})\rangle + \mathcal{O}(v) |c\bar{c}({}^3P_J^{[8]})g\rangle \\ &+ \mathcal{O}(v^2) |c\bar{c}({}^1S_0^{[8]})g\rangle + \mathcal{O}(v^2) |c\bar{c}({}^3S_1^{[8]})gg\rangle + \dots, \end{aligned} \quad (3)$$

$$\begin{aligned} |\eta_c\rangle &= \mathcal{O}(1) |c\bar{c}({}^1S_0^{[1]})\rangle + \mathcal{O}(v) |c\bar{c}({}^1P_1^{[8]})g\rangle \\ &+ \mathcal{O}(v^2) |c\bar{c}({}^3S_1^{[8]})g\rangle + \mathcal{O}(v^2) |c\bar{c}({}^1S_0^{[8]})gg\rangle + \dots, \end{aligned} \quad (4)$$

$$|\chi_{cJ}\rangle = \mathcal{O}(1) |c\bar{c}({}^3P_J^{[1]})\rangle + \mathcal{O}(v) |c\bar{c}({}^3S_1^{[8]})g\rangle + \dots, \quad (5)$$

$$|h_c\rangle = \mathcal{O}(1) |c\bar{c}({}^1P_1^{[1]})\rangle + \mathcal{O}(v) |c\bar{c}({}^1S_0^{[8]})g\rangle + \dots, \quad (6)$$

$$\begin{aligned} |\Upsilon\rangle &= \mathcal{O}(1) |b\bar{b}({}^3S_1^{[1]})\rangle + \mathcal{O}(v) |b\bar{b}({}^3P_J^{[8]})g\rangle \\ &+ \mathcal{O}(v^2) |b\bar{b}({}^1S_0^{[8]})g\rangle + \mathcal{O}(v^2) |b\bar{b}({}^3S_1^{[8]})gg\rangle + \dots, \end{aligned} \quad (7)$$

$$\begin{aligned} |\eta_b\rangle &= \mathcal{O}(1) |b\bar{b}({}^1S_0^{[1]})\rangle + \mathcal{O}(v) |b\bar{b}({}^1P_1^{[8]})g\rangle \\ &+ \mathcal{O}(v^2) |b\bar{b}({}^3S_1^{[8]})g\rangle + \mathcal{O}(v^2) |b\bar{b}({}^1S_0^{[8]})gg\rangle + \dots, \end{aligned} \quad (8)$$

$$|\chi_{bJ}\rangle = \mathcal{O}(1) |b\bar{b}({}^3P_J^{[1]})\rangle + \mathcal{O}(v) |b\bar{b}({}^3S_1^{[8]})g\rangle + \dots, \quad (9)$$

$$|h_b\rangle = \mathcal{O}(1) |b\bar{b}({}^1P_1^{[1]})\rangle + \mathcal{O}(v) |b\bar{b}({}^1S_0^{[8]})g\rangle + \dots. \quad (10)$$

We will consider both the CS and CO intermediate states in the following calculations. Our calculations include the QCD contributions [at $\mathcal{O}(\alpha_s^2\alpha^2)$] and the electric-weak (EW) contributions [at $\mathcal{O}(\alpha^4)$] at tree level. At this order, it includes $H_1(CS) + H_2(CS)$ (CS pair channels) and $H_1(CO) + H_2(CO)$ (CO pair channels) for the final states. We need to point out that the CO channels actually represent the inclusive processes for the soft gluon emission or absorption. The complete calculations of LO inclusive production invoking $CS + CO + g$ and $CO + CO + g$ which are of the NLO of α_s compared with CS and CO pair channels. In our calculations, this implies that other final particles besides the measured double quarkonium are integrated over. Besides, for CO pair channels, the two final states, with an energy close to half of the center-of-mass (c.m.) energy, recoil without the emission of additional light particles and, thus, contribute to the exclusive double quarkonia production in the experimental measurements. This specific configuration leads to the breakdown of factorization due to non-perturbative effects which, fortunately, are suppressed by the inverse powers of the collision energy and are usually negligible. We employed the FeynArts [48] package to generate the Feynman diagrams and amplitudes and utilized the

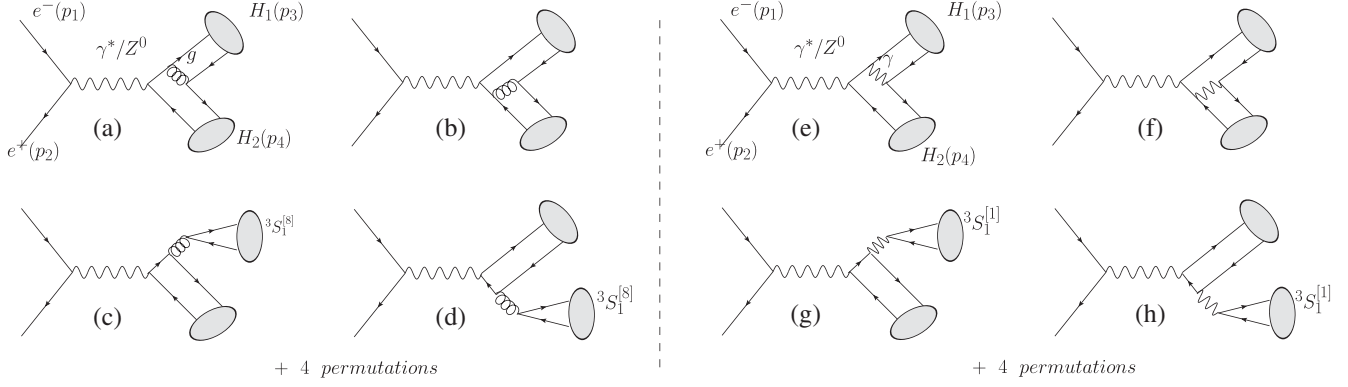


FIG. 1. CS and CO Feynman diagrams for $e^-(p_1) + e^+(p_2) \rightarrow \gamma^*/Z^0 \rightarrow H_1(p_3) + H_2(p_4)$ at the tree level. The permutation diagrams can be obtained by reversing the quark line. The QCD diagrams are shown on the left side (a)–(d), and the EW diagrams are shown on the right side (e)–(h). (c) and (d) pertain exclusively to the CO channels. (g) and (h) pertain exclusively to the CS channels.

FeynCalc [49] package to manipulate the amplitudes. The typical diagrams for CS and CO channels are presented in Fig. 1 through the single boson exchange process $e^-(p_1) + e^-(p_2) \rightarrow \gamma^*/Z^0(p) \rightarrow H_1(p_3) + H_2(p_4)$. The t -channel EW diagrams (two-boson exchange process) are not presented which give small contributions except for J/ψ pair and Υ pair production. The corresponding amplitudes in perturbative QCD and EW are expressed as

$$i\mathcal{M} = \sum_n \bar{v}(p_2) \mathcal{L}_\mu u(p_1) D^{\mu\nu} \mathcal{A}_\nu^{(n)}, \quad (11)$$

where the vertex $\mathcal{L}^\mu$ and propagator $D_{\mu\nu}$ are defined, respectively, as

$$\begin{aligned} \mathcal{L}^\mu &= -ie\gamma^\mu, & \frac{-ie}{4\cos\theta_W \sin\theta_W} \gamma^\mu (1 - 4\sin^2\theta_W - \gamma^5), \\ D_{\mu\nu} &= \frac{-ig_{\mu\nu}}{p^2}, & \frac{-ig_{\mu\nu}}{p^2 - m_Z^2 + im_Z\Gamma_Z}. \end{aligned} \quad (12)$$

In the above definitions, the first (second) term corresponds to the process propagated by γ^* (Z^0). $e = \sqrt{4\pi\alpha}$, θ_W stands for the Weinberg angle, and m_Z and Γ_Z denote the mass and width of the Z^0 boson, respectively. In one certain channel, the hadron part of the amplitude is computed through the summation of all the diagrams, with each diagram being marked by a script, namely, $\mathcal{A}^{(n)}$. For the P -wave state, the LO amplitude is expressed as the contraction of the orbital polarization vector $\epsilon(L_z)$ and the derivative of the original amplitude with respect to the relative momentum (denoted by q) between the $Q\bar{Q}$ pair:

$$\mathcal{A} = \epsilon^\rho(L_z) \frac{\partial}{\partial q^\rho} \tilde{\mathcal{A}}|_{q \rightarrow 0}. \quad (13)$$

Through the projection operator method, $Q\bar{Q}$ is projected onto specific spin and color quantum numbers. For instance, the original amplitudes of the diagrams labeled (a) and (c) in Fig. 1 are written as below:

$$\begin{aligned} \tilde{\mathcal{A}}_\nu^{(a)} &= ig^2 \text{Tr} \left[\Pi_1 \gamma_\rho \frac{(\not{p}_{3Q} + \not{p}_{3\bar{Q}} + \not{p}_{4Q}) + m_Q}{[(p_{3Q} + p_{3\bar{Q}} + p_{4Q})^2 - m_Q^2](p_{3\bar{Q}} + p_{4Q})^2} \mathcal{V}_\nu \Pi_2 \gamma^\rho \right], \\ \tilde{\mathcal{A}}_\nu^{(c)} &= -g^2 \text{Tr}[\gamma^\rho \Pi_1] \text{Tr} \left[\Pi_2 \gamma_\rho \frac{(\not{p}_{3Q} + \not{p}_{3\bar{Q}} + \not{p}_{4Q}) + m_Q}{[(p_{3Q} + p_{3\bar{Q}} + p_{4Q})^2 - m_Q^2](p_{3Q} + p_{3\bar{Q}})^2} \mathcal{V}_\nu \right]. \end{aligned} \quad (14)$$

The phase difference between the two amplitudes with a sign change arises because one diagram can be obtained by exchanging the identical final-state fermions in the other. The vertex $\mathcal{V}_\nu$ for γ/Z propagation is defined as follows:

$$\mathcal{V}_\nu = ie e_Q \gamma_\nu, \quad \frac{ie}{4\cos\theta_W \sin\theta_W} \gamma_\nu (1 - 4|e_Q|\sin^2\theta_W - \gamma^5), \quad (15)$$

where e_Q is the charge of heavy quark. The projection operator Π_i , with the subscript i indicating the specific states, is constituted as the multiplicative combination of the spin projection and the color projection. The expressions of spin-singlet and spin-triplet projection operators are, respectively,

$$\Pi(00) \equiv \sum_{s_Q s_{\bar{Q}}} \langle s_Q s_{\bar{Q}}; 00 | v(p_{\bar{Q}}) \bar{u}(p_Q) = \frac{1}{2\sqrt{2}(E_q + m_Q)} (-\not{p}_{\bar{Q}} + m_Q) \gamma_5 \frac{\not{p}_Q + \not{p}_{\bar{Q}} + 2E_q}{2E_q} (\not{p}_Q + m_Q), \quad (16)$$

$$\Pi(1s_z) \equiv \sum_{s_Q s_{\bar{Q}}} \langle s_Q s_{\bar{Q}}; 1s_z | v(p_{\bar{Q}}) \bar{u}(p_Q) = \frac{1}{2\sqrt{2}(E_q + m_Q)} (-\not{p}_{\bar{Q}} + m_Q) \not{\epsilon}(s_z) \frac{\not{p}_Q + \not{p}_{\bar{Q}} + 2E_q}{2E_q} (\not{p}_Q + m_Q), \quad (17)$$

where $\epsilon(L_z)$ is the spin polarization vector and E_q is the total energy of quark or antiquark in the rest frame of the meson. Color projection operators assume the form of $\sum_{ij} \frac{\delta_{ij}}{\sqrt{N_c}}$ in the case of CS and $\sqrt{2}T_{ij}^a$, $a = 1, 2, \dots, 8$, for CO.

The requirement for the overall colorless nature of the final state dictates that both particles must be either in the CS state or in the CO state. For the CS channel, the color factors of diagrams (a) and (b) are $\frac{N_c^2-1}{2N_c}$, while for the CO channel, they are $-\frac{\delta_{ab}}{2N_c}$, where a and b are the color indexes of final states. In contrast, diagrams (c) and (d) can be only in the CO state, and the gluon fragment into the $^3S_1^{[8]}$ state. The total color factors are $\frac{\delta_{ab}}{2}$. For the CS channel, the color factors of diagrams (e) and (f) are 1, while for the CO channel, they are δ_{ab} . Diagrams (g) and (h) can be only in the CS state, and the final γ^* fragment into the $^3S_1^{[1]}$ state. The total color factors are N_c .

B. Kinematics and relativistic corrections (RCs)

We compute the cross sections of each channel up to the NLO of $\mathcal{O}(v^2)$, and the conventional approach is employed

to derive the RCs, as referenced in [13,18–20,50,51]. For the S -wave state, the amplitudes are expanded to the second order of $|\vec{q}|$, whereas for the P -wave state, the expansion is carried out to the third order. Here, $|\vec{q}|$ which is a Lorentz scalar represents the magnitude of the three-momentum of either Q or $\bar{Q}$ within the rest frame of the meson.

Consequently, we have $E_q = \sqrt{m_Q^2 + \vec{q}^2}$ and the meson mass can be approximated as $2E_q$. The momentum of $Q(\bar{Q})$ in arbitrary frame is expressed as $p_Q = \frac{1}{2}p + q$ ($p_{\bar{Q}} = \frac{1}{2}p - q$), where q is obtained via Lorentz boost from $(0, \vec{q})$. Therefore, after integrating out the angles of q , the corresponding relationship between the polynomial term of q and $|\vec{q}|^2$ can be ascertained. $q^\alpha q^\beta \rightarrow \frac{|\vec{q}|^2}{3} \pi^{\alpha\beta}$ for the S -wave state, $q^\alpha q^\beta q^\sigma \rightarrow \frac{|\vec{q}|^3}{5} (\pi^{\alpha\beta} \epsilon_{L_z}^\sigma + \pi^{\alpha\sigma} \epsilon_{L_z}^\beta + \pi^{\beta\sigma} \epsilon_{L_z}^\alpha)$ for the P -wave state, where $\pi^{\alpha\beta} = -g^{\alpha\beta} + \frac{p^\alpha p^\beta}{p^2}$. Furthermore, the normalization factor of $\sqrt{\frac{m_Q}{E_Q}}$ is incorporated into the amplitudes.

The kinematics variables should be also expanded. For two-body final states, the Lorentz invariant Mandelstam variables are defined as

$$s = (p_1 + p_2)^2 = (p_3 + p_4)^2, \quad t = (p_3 - p_1)^2 = (p_4 - p_2)^2, \quad u = (p_4 - p_1)^2 = (p_3 - p_2)^2 \quad (18)$$

with the relationship $s + t + u = M_{H_1}^2 + M_{H_2}^2$. The variable s is independent of $|\vec{q}|^2$, while t and u are dependent on $|\vec{q}|^2$. We define t_0 as the value of t when $|\vec{q}_1| = |\vec{q}_2| = 0$, and u_0 as the value of u when $|\vec{q}_1| = |\vec{q}_2| = 0$. Consequently, the following expansions are derived, as presented in [50]:

$$t = t_0 - \frac{4[t_0 + 4m_Q^2]}{s - 16m_Q^2} (|\vec{q}_1|^2 + |\vec{q}_2|^2) + \mathcal{O}(v^4), \quad (19)$$

$$u = u_0 - \frac{4[u_0 + 4m_Q^2]}{s - 16m_Q^2} (|\vec{q}_1|^2 + |\vec{q}_2|^2) + \mathcal{O}(v^4). \quad (20)$$

The matrix elements of $\langle v^2 \rangle$ are defined by the ratios of the NLO LDMEs in v^2 to the LO LDMEs:

$$\langle v^2 \rangle \equiv \frac{\langle 0 | \mathcal{P}^H(2s+1 L_J) | 0 \rangle}{m_Q^2 \langle 0 | \mathcal{O}^H(2s+1 L_J) | 0 \rangle}, \quad (21)$$

which adhere to the velocity power scaling rules and are of the order of v^2 . Moreover,

$$v^2 = \langle v^2 \rangle [1 + \mathcal{O}(v^4)]. \quad (22)$$

The four-fermion operators $\mathcal{O}^H(2s+1 L_J)$, which are of dimension 6 for the LO, and the dimension-8 four-fermion operators $\mathcal{P}^H(2s+1 L_J)$ utilized for the RCs, are defined in accordance with Ref. [1].

III. PARAMETERS

We will take the following input parameters [52] and LDMEs [53] in the present work:

$$\begin{aligned} \alpha_s(2m_c) &= 0.26, & \alpha_s(2m_b) &= 0.18, & \alpha &= 1/137, \\ m_c &= 1.5 \text{ GeV}, & m_b &= 4.7 \text{ GeV}, & m_Z &= 91.1876 \text{ GeV}, \\ \Gamma_Z &= 2.4952 \text{ GeV}, & \sin^2 \theta_w &= 0.2312, & v_{c\bar{c}}^2 &= 0.23, & v_{b\bar{b}}^2 &= 0.1. \end{aligned} \quad (23)$$

Here, the strong running coupling constant α_s is calculated by the two-loop formula

$$\frac{\alpha_s(\mu)}{4\pi} = \frac{1}{\beta_0 L} - \frac{\beta_1 \ln L}{\beta_0^3 L^2}, \quad (24)$$

where $L = \ln(\mu^2/\Lambda_{\text{QCD}}^2)$ with $\Lambda_{\text{QCD}} \simeq 338$ MeV and $\beta_0 = (11/3)C_A - (4/3)T_f n_f$ and $\beta_1 = (34/3)C_A^2 - 4C_F T_f n_f - (20/3)C_A T_f$ are the one-loop and two-loop coefficients of the QCD beta function, respectively. n_f is the active quark flavors which is set to 3 for heavy quarkonia. The values of v^2 for CS and CO are estimated via the Gremm-Kapustin relation [54,55]

$$v^2 = v_1^2 = v_8^2 = \frac{M_{Q\bar{Q}} - 2m_Q^{\text{pole}}}{m_Q^{\text{QCD}}}, \quad (25)$$

where m_Q^{pole} is quark pole mass, m_Q^{QCD} is the quark mass in NRQCD, and $M_{Q\bar{Q}}$ is the heavy quarkonium mass.

For the LDMEs, the CS ones are derived from potential models, while the CO ones are extracted from the LO calculations that fit to the hadronic colliders with the feeddown contributions. Specifically, the LDMEs of charmonium [56–64] are

$$\begin{aligned} \langle \mathcal{O}^{J/\psi}[^3S_1^{[1]}] \rangle &= 1.2 \text{ GeV}^3, \\ \langle \mathcal{O}^{J/\psi}[^1S_0^{[8]}] \rangle &= 0.0180 \pm 0.0087 \text{ GeV}^3, \\ \langle \mathcal{O}^{J/\psi}[^3S_1^{[8]}] \rangle &= 0.0013 \pm 0.0013 \text{ GeV}^3, \\ \langle \mathcal{O}^{J/\psi}[^3P_0^{[8]}] \rangle &= (0.0180 \pm 0.0087)m_c^2 \text{ GeV}^3, \\ \langle \mathcal{O}^{\eta_c}[^1S_0^{[1]}] \rangle &= \frac{1}{3} \times 1.2 \text{ GeV}^3, \\ \langle \mathcal{O}^{\eta_c}[^1S_0^{[8]}] \rangle &= \frac{1}{3} \times (0.0013 \pm 0.0013) \text{ GeV}^3, \\ \langle \mathcal{O}^{\eta_c}[^3S_1^{[8]}] \rangle &= 0.0180 \pm 0.0087 \text{ GeV}^3, \\ \langle \mathcal{O}^{\eta_c}[^1P_1^{[8]}] \rangle &= 3 \times (0.0180 \pm 0.0087)m_c^2 \text{ GeV}^3, \\ \langle \mathcal{O}^{h_c}[^1P_1^{[1]}] \rangle &= 3 \times 0.054m_c^2 \text{ GeV}^3, \\ \langle \mathcal{O}^{h_c}[^1S_0^{[8]}] \rangle &= 3 \times (0.00187 \pm 0.00025) \text{ GeV}^3, \\ \langle \mathcal{O}^{\chi_{c0}}[^3P_0^{[1]}] \rangle &= 0.054m_c^2 \text{ GeV}^3, \\ \langle \mathcal{O}^{\chi_{c0}}[^3S_1^{[8]}] \rangle &= 0.00187 \pm 0.00025 \text{ GeV}^3, \\ \langle \mathcal{O}^{\chi_{c1}}[^3P_1^{[1]}] \rangle &= 3 \times 0.054m_c^2 \text{ GeV}^3, \\ \langle \mathcal{O}^{\chi_{c1}}[^3S_1^{[8]}] \rangle &= 3 \times (0.00187 \pm 0.00025) \text{ GeV}^3, \\ \langle \mathcal{O}^{\chi_{c2}}[^3P_2^{[1]}] \rangle &= 5 \times 0.054m_c^2 \text{ GeV}^3, \\ \langle \mathcal{O}^{\chi_{c2}}[^3S_1^{[8]}] \rangle &= 5 \times (0.00187 \pm 0.00025) \text{ GeV}^3 \end{aligned} \quad (26)$$

and those of bottomonium [56,57,63–65] are

$$\begin{aligned} \langle \mathcal{O}^{\Upsilon}[^3S_1^{[1]}] \rangle &= 10.9 \text{ GeV}^3, \\ \langle \mathcal{O}^{\Upsilon}[^1S_0^{[8]}] \rangle &= (0.0121 \pm 0.0400) \text{ GeV}^3, \\ \langle \mathcal{O}^{\Upsilon}[^3S_1^{[8]}] \rangle &= (0.0477 \pm 0.0334) \text{ GeV}^3, \\ \langle \mathcal{O}^{\Upsilon}[^3P_0^{[8]}] \rangle &= 5 \times (0.0121 \pm 0.0400)m_b^2 \text{ GeV}^3, \\ \langle \mathcal{O}^{\chi_{b0}}[^3P_0^{[1]}] \rangle &= 0.1m_b^2 \text{ GeV}^3, \\ \langle \mathcal{O}^{\chi_{b0}}[^3S_1^{[8]}] \rangle &= 0.1008 \text{ GeV}^3, \\ \langle \mathcal{O}^{\chi_{b1}}[^3P_1^{[1]}] \rangle &= 3 \times 0.1m_b^2 \text{ GeV}^3, \\ \langle \mathcal{O}^{\chi_{b1}}[^3S_1^{[8]}] \rangle &= 3 \times 0.1008 \text{ GeV}^3, \\ \langle \mathcal{O}^{\chi_{b2}}[^3P_2^{[1]}] \rangle &= 5 \times 0.1m_b^2 \text{ GeV}^3, \\ \langle \mathcal{O}^{\chi_{b2}}[^3S_1^{[8]}] \rangle &= 5 \times 0.1008 \text{ GeV}^3, \\ \langle \mathcal{O}^{\eta_b}[^1S_0^{[1]}] \rangle &= \frac{1}{3} \langle \mathcal{O}^{\Upsilon}[^3S_1^{[1]}] \rangle = 3.633 \text{ GeV}^3, \\ \langle \mathcal{O}^{\eta_b}[^1S_0^{[8]}] \rangle &= \frac{1}{3} \langle \mathcal{O}^{\Upsilon}[^3S_1^{[8]}] \rangle = (0.0159 \pm 0.0111) \text{ GeV}^3, \\ \langle \mathcal{O}^{\eta_b}[^3S_1^{[8]}] \rangle &= \langle \mathcal{O}^{\Upsilon}[^1S_0^{[8]}] \rangle = (0.0121 \pm 0.0400) \text{ GeV}^3, \\ \langle \mathcal{O}^{\eta_b}[^1P_1^{[8]}] \rangle &= 3 \times \langle \mathcal{O}^{\Upsilon}[^3P_0^{[8]}] \rangle \\ &= 3 \times 5 \times (0.0121 \pm 0.0400)m_b^2 \text{ GeV}^3, \\ \langle \mathcal{O}^{h_b}[^1P_1^{[1]}] \rangle &= 3 \times \langle \mathcal{O}^{\chi_{b0}}[^3P_0^{[1]}] \rangle = 3 \times 0.1m_b^2 \text{ GeV}^3, \\ \langle \mathcal{O}^{h_b}[^1S_0^{[8]}] \rangle &= 3 \times \langle \mathcal{O}^{\chi_{b0}}[^3S_1^{[8]}] \rangle = 3 \times 0.1008 \text{ GeV}^3. \end{aligned}$$

The heavy quark spin symmetry (HQSS) is exhibited by them [1]:

$$\begin{aligned} \langle \mathcal{O}^{\chi_{cJ}}[^3P_J^{[1]}] \rangle &= (2J+1) \langle \mathcal{O}^{\chi_{c0}}[^3P_0^{[1]}] \rangle, \\ \langle \mathcal{O}^{\chi_{cJ}}[^3S_1^{[8]}] \rangle &= (2J+1) \langle \mathcal{O}^{\chi_{c0}}[^3S_1^{[8]}] \rangle, \\ \langle \mathcal{O}^{\eta_c}[^1S_0^{[1]}/^1S_0^{[8]}] \rangle &= \frac{1}{3} \langle \mathcal{O}^{J/\psi}[^3S_1^{[1]}/^3S_1^{[8]}] \rangle, \\ \langle \mathcal{O}^{\eta_c}[^3S_1^{[8]}] \rangle &= \langle \mathcal{O}^{J/\psi}[^1S_0^{[8]}] \rangle, \\ \langle \mathcal{O}^{\eta_c}[^1P_1^{[8]}] \rangle &= 3 \langle \mathcal{O}^{J/\psi}[^3P_0^{[8]}] \rangle, \\ \langle \mathcal{O}^{h_c}[^1P_1^{[1]}/^1S_0^{[8]}] \rangle &= 3 \langle \mathcal{O}^{\chi_{c0}}[^3P_0^{[1]}/^3S_1^{[8]}] \rangle. \end{aligned} \quad (27)$$

We also see that these parameters approximately satisfy the velocity scaling rule of NRQCD [1]:

$$\begin{aligned} \langle \mathcal{O}^{J/\psi}[^3S_1^{[1]}] \rangle &\sim m_c^3 v_c^3, & \langle \mathcal{O}^{\chi_{cJ}}[^3P_J^{[1]}] \rangle &\sim m_c^5 v_c^5, \\ \langle \mathcal{O}^{J/\psi}[^3S_1^{[8]}] \rangle &\sim m_c^3 v_c^7, & \langle \mathcal{O}^{\chi_{cJ}}[^3S_1^{[8]}] \rangle &\sim m_c^3 v_c^5, \\ &\dots, & & \end{aligned} \quad (28)$$

and the CS LDMEs satisfy the quark potential model [1]:

$$\begin{aligned}
\frac{\langle \mathcal{O}^{J/\psi}[^3S_1^{[1]}] \rangle}{2N_c \times 3} &\simeq \frac{|R_s(0)|^2}{4\pi}, & \frac{\langle \mathcal{O}^{\eta_c}[^1S_0^{[1]}] \rangle}{2N_c} &\simeq \frac{|R_s(0)|^2}{4\pi}, \\
\frac{\langle \mathcal{O}^{\chi_{cJ}}[^3P_J^{[1]}] \rangle}{2N_c(2J+1)} (J=0,1,2) &\simeq \frac{3|R'_p(0)|^2}{4\pi}, & \frac{\langle \mathcal{O}^{h_c}[^1P_1^{[1]}] \rangle}{2N_c \times 3} &\simeq \frac{3|R'_p(0)|^2}{4\pi}.
\end{aligned} \tag{29}$$

The CS LDMEs of charmonia used in this work, which correspond to the values of $|R_s(0)|^2 = 0.838$ and $|R'_p(0)|^2 = 0.085$, closely approximate the value $[|R_s(0)|^2 = 0.810, |R'_p(0)|^2 = 0.075]$ in Ref. [66], compared to maximal $[|R_s(0)|^2 = 2.458, |R'_p(0)|^2 = 0.322]$ [67] and minimal $[|R_s(0)|^2 = 0.565, |R'_p(0)|^2 = 0.053]$ [68] set in Refs. [44,69], and those of bottomonia $[|R_s(0)|^2 = 7.61, |R'_p(0)|^2 = 1.54]$ closely approximate the value $[|R_s(0)|^2 = 6.477, |R'_p(0)|^2 = 1.417]$ in Ref. [66], compared to maximal $[|R_s(0)|^2 = 16.12, |R'_p(0)|^2 = 5.874]$ [67] and minimal $[|R_s(0)|^2 = 5.298, |R'_p(0)|^2 = 1.111]$ [70] set in Refs. [44,69].

Analogous correlations among the LDMEs are also valid for bottomonia.

IV. RESULTS AND DISCUSSION

A. Cross sections and CO contributions

In the energy region of a Z factory, the pure EW contributions are also significant [41]. For the J/ψ pair and Υ pair production, we consider the t -channel EW contributions which are significant either at a B factory or

at the Z factory [71–73]. For other processes, the t -channel EW contributions could be neglected compared with the s -channel contributions. The γ^* -propagated processes contribute predominantly at the B factory but are negligible compared with the Z^0 -propagated processes in the energy region of the Z factory. And for the γ^* -propagated processes, due to the additional C -parity conservation constraint imposed by the γ^* propagator, these processes exhibit fewer production channels. Therefore, in our calculations, the total cross sections will include the complete contributions from QCD and EW processes, as well as γ^*/Z^0 -propagated processes. The interferences were also taken into account.

In Table I, we present the contributions stemming from all combinations of the CS and CO states that comply with the conservation laws of color, C parity, and combined CP parity. Upon examining the table, it becomes evident that the CO channels contribute substantially more than the CS channels in several processes, particularly for the combinations involving the $^3S_1^{[8]}$ state. This indicates that the processes of gluon fragmentation into the intermediate $^3S_1^{[8]}$ state might play a crucial role. The significance of the $^3S_1^{[8]}$

TABLE I. Production channels and corresponding cross sections (units $\times 10^{-4}$ fb) at the Z^0 pole in $\mathcal{O}(v^0)$. The percentage inside the brackets is the proportion of CO. The negligible results ($< 0.1 \times 10^{-4}$ fb) are not shown.

H_1				H_2						
				Charmonia						
η_c	J/ψ	32.1 (66.1%)	χ_{cJ}	11.9($J=0$) (42.7%)	16.1($J=1$) (95.0%)	39.5($J=2$) (64.3%)	h_c	4.4 (31.0%)	η_c	39.2 (100.0%)
$^1S_0^{[1]}$	$^3S_1^{[1]}$	10.9	$^3P_J^{[1]}$	6.8	0.8	14.1	$^1P_1^{[1]}$	3.1	$^1S_0^{[1]}$	
$^3S_1^{[8]}$	$^1S_0^{[8]}$	4.4	$^3S_1^{[8]}$	2.0	6.1	10.2	$^1S_0^{[8]}$	1.4	$^3S_1^{[8]}$	9.8
	$^3S_1^{[8]}$	1.4								
	$^3P_J^{[8]}$	13.2								
$^1S_0^{[8]}$	$^3S_1^{[8]}$		$^3S_1^{[8]}$			0.1			$^3S_1^{[8]}$	0.1
	$^3P_J^{[8]}$								$^1P_1^{[8]}$	
$^1P_1^{[8]}$	$^1S_0^{[8]}$		$^3S_1^{[8]}$	3.0	9.1	15.2	$^1S_0^{[8]}$		$^3S_1^{[8]}$	29.3
	$^3S_1^{[8]}$	2.1							$^1P_1^{[8]}$	
	$^3P_J^{[8]}$									
J/ψ	J/ψ	2737.4 (0.1%)	χ_{cJ}	4.1($J=0$) (48.4%)	8.2($J=1$) (72.5%)	13.1($J=2$) (75.7%)	h_c	33.3 (0.3%)		
$^3S_1^{[1]}$	$^3S_1^{[1]}$	2736.0	$^3P_J^{[1]}$	2.1	2.3	3.2	$^1P_1^{[1]}$	33.2		
$^3S_1^{[8]}$	$^3S_1^{[8]}$	0.1	$^3S_1^{[8]}$	0.1	0.4	0.7	$^1S_0^{[8]}$	0.1		

(Table continued)

TABLE I. (Continued)

H_1				H_2						
Charmonia										
$1S_0^{[8]}$	$3P_J^{[8]}$		$3S_1^{[8]}$	0.5	1.4	2.3				
	$3S_1^{[8]}$	0.3								
$3P_J^{[8]}$	$3P_J^{[8]}$		$3S_1^{[8]}$	1.4	4.1	6.9	$1S_0^{[8]}$			
	$3S_1^{[8]}$	0.9								
Bottomonia										
η_b	Υ	37.6 (5.2%)	χ_{bJ}	8.1($J=0$) (37.7%)	15.0($J=1$) (60.7%)	28.5($J=2$) (53.3%)	h_b	7.8 (9.6%)	η_b	0.3 (100.0%)
$1S_0^{[1]}$	$3S_1^{[1]}$	35.7	$3P_J^{[1]}$	5.0	5.9	13.3	$1P_1^{[1]}$	7.1	$1S_0^{[1]}$	
$3S_1^{[8]}$	$1S_0^{[8]}$		$3S_1^{[8]}$	0.3	1.0	1.6	$1S_0^{[8]}$	0.7	$3S_1^{[8]}$	
	$3S_1^{[8]}$	0.2								
	$3P_J^{[8]}$	0.4								
$1S_0^{[8]}$	$3S_1^{[8]}$	0.2	$3S_1^{[8]}$	0.3	1.0	1.6			$3S_1^{[8]}$	
	$3P_J^{[8]}$								$1P_1^{[8]}$	
$1P_1^{[8]}$	$1S_0^{[8]}$		$3S_1^{[8]}$	2.4	7.2	12.0	$1S_0^{[8]}$		$3S_1^{[8]}$	0.3
	$3S_1^{[8]}$	1.1							$1P_1^{[8]}$	
	$3P_J^{[8]}$									
Υ	Υ	36.7 (6.1%)	χ_{bJ}	13.3($J=0$) (39.6%)	17.1($J=1$) (92.4%)	32.9($J=2$) (80.0%)	h_b	25.2 (11.7%)		
$3S_1^{[1]}$	$3S_1^{[1]}$	34.4	$3P_J^{[1]}$	8.0	1.3	6.6	$1P_1^{[1]}$	22.2		
$3S_1^{[8]}$	$3S_1^{[8]}$	0.3	$3S_1^{[8]}$	1.3	3.9	6.5	$1S_0^{[8]}$	2.9		
$1S_0^{[8]}$	$3P_J^{[8]}$		$3S_1^{[8]}$	0.2	0.7	1.2				
	$3S_1^{[8]}$	0.1								
$3P_J^{[8]}$	$3P_J^{[8]}$	0.1	$3S_1^{[8]}$	3.7	11.2	18.7	$1S_0^{[8]}$	0.1		
	$3S_1^{[8]}$	1.8								

state has also been emphasized in Refs. [47,74], where the semiexclusive heavy quarkonia production and the inclusive η_Q production at the Z factory were investigated.

One could check these results with the fragmentation functions to the $3S_1^{[8]}$ states at $\mathcal{O}(\alpha_s)$ in the high-energy limit ($M_H \rightarrow 0$):

$$\begin{aligned} \sigma(e^+e^- \rightarrow H_1 + g \rightarrow H_1 + H_2 + X) \\ = \sigma(e^+e^- \rightarrow H_1 + g) \int_0^1 dz D_{g \rightarrow H_2}(z), \end{aligned} \quad (30)$$

where $z = 2E_2/\sqrt{s}$. At LO of $\mathcal{O}(\alpha_s)$, the fragmentation functions are given as $D_{g \rightarrow H_2}(3S_1^{[8]})(z) = \frac{\pi\alpha_s\delta(1-z)}{3(N_c^2-1)} \frac{\langle \mathcal{O}^{H_2}[3S_1^{[8]}] \rangle}{m_c^2}$.

It should be noted that the importance of the $3S_1^{[8]}$ state may be reduced when considering the NLO corrections. As for the fragmentation diagrams, NLO fragmentation functions have been widely studied [75–83], and other Fock states also contribute to the cross sections in

addition to the $3S_1^{[8]}$ state. In Table II, the values of the integrated fragmentation functions for $J/\psi/\eta_c$ are presented (the factorization scale μ set to $2m_c$). The contributions from $3S_1^{[8]}$ would be enhanced by a factor

TABLE II. The integrated fragmentation function at next-to-leading order. The analytical expressions are taken from Ref. [75], and the factorization scale μ set to $2m_c$.

State(n)	$\frac{m_Q^{2L+3}}{m_c^2} \frac{\int_0^1 dz D_{g \rightarrow H(n)}(z)}{\langle \mathcal{O}^H[n] \rangle}$
$1S_0^{[1]}$	$0.06\alpha_s^2$
$1S_0^{[8]}$	$0.02\alpha_s^2$
$3S_1^{[8]}$	$0.13\alpha_s + 0.42\alpha_s^2$
$3P_J^{[8]}$	$-0.28\alpha_s^2$
$1P_1^{[8]}$	$0.01\alpha_s^2$

of 1.84. And the contributions from the $^3S_1^{[8]}$ state may also be enhanced by the NLO corrections to the part $e^+e^- \rightarrow H_1 + g$ of the diagrams. It is found that relatively large negative contributions from the $^3P_J^{[8]}$ state cancel the contributions from the $^3S_1^{[8]}$ state when adopting the values of LDMEs in Eq. (26). Note that, here, we adopted a relatively large $\langle^3P_0^{[8]}\rangle$ and relatively small $\langle^3S_1^{[8]}\rangle$ in Eq. (26) compared with the values extracted from the hadron colliders with complete NLO calculations. And a lower constraint of $\langle^3P_0^{[8]}\rangle$ obtained from inclusive J/ψ production in e^+e^- colliders [84] is also smaller than the value we adopted.

It is interesting to compare this work with the double quarkonium hadroproduction. At hadron colliders, the double J/ψ production serves as an excellent platform for understanding the CO mechanism and double parton scattering, sparking a significant amount of research [50,85–93]. Based on the calculations at LO, apart from the double-fragmentation octet channels, other octet channels (with single-fragmentation-like processes and no-fragmentation processes) also make considerable contributions [88]. Moreover, this work on lepton colliders also highlights the importance of the octet contributions (mainly from single-fragmentation processes) at the LO level. It can be expected that uncalculated NLO corrections especially for the octet channels will also have

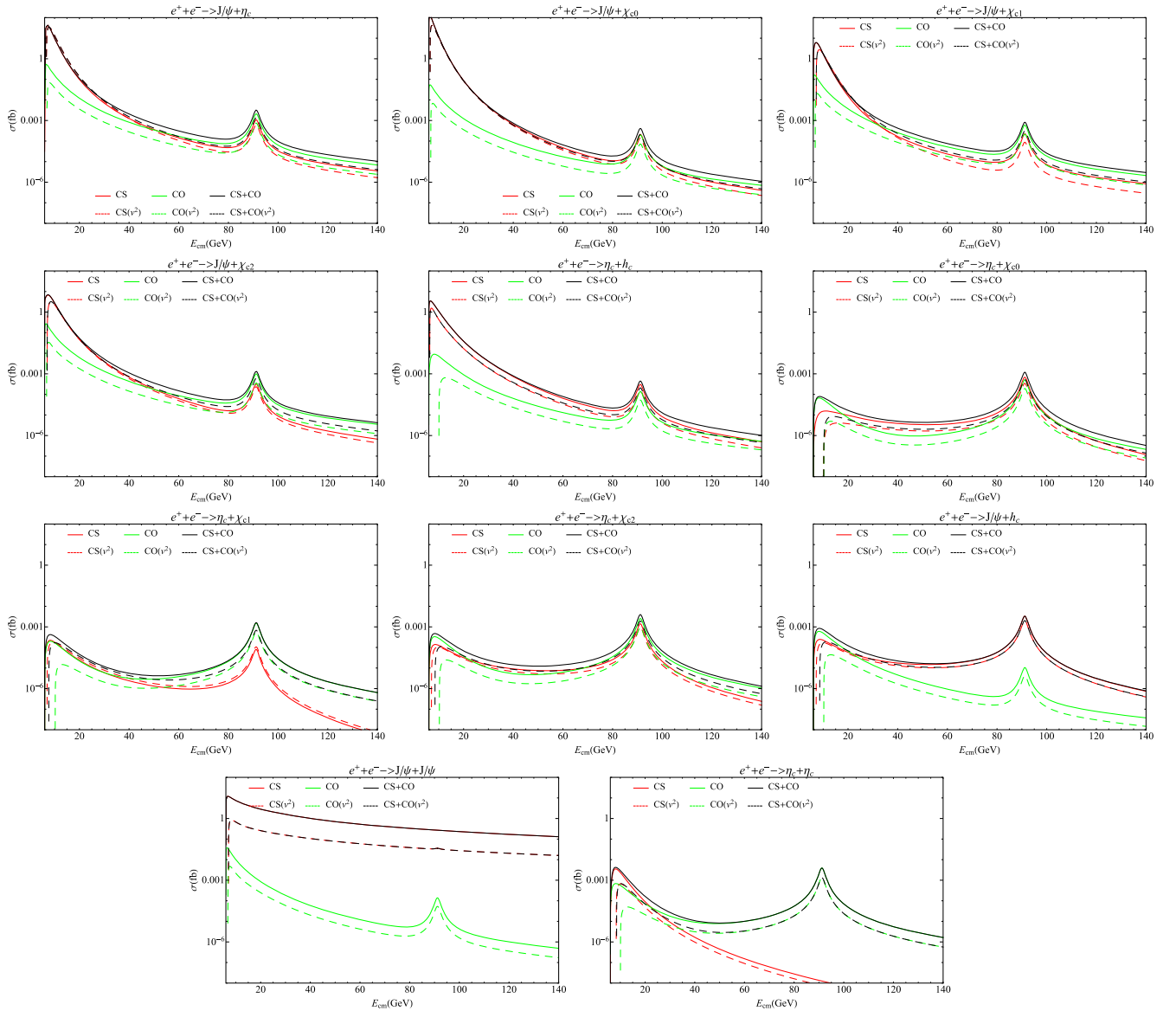


FIG. 2. Cross sections (σ) versus c.m. energy ($E_{\text{cm}} = \sqrt{s}$) for double charmonium production. The solid line represents LO and the dashed line represents next-to-leading order in v^2 (NLO) results. The red line represents the CS channel, the green line represents the total CO channels, and the black line represents the sum of CS and CO.

visible contributions. In particular, for the pair production of J/ψ , the predominant contribution to LO cross section comes from the t -channel two-photon fragmentation process which is in CS mode. When considering the NLO effects, there would be the CO contributions from the single-fragmentation processes and nonfragmentation processes in the two-photon channels.

The cross sections vary with c.m. energy $\sqrt{s}$ are presented in Figs. 2 and 3. In the energy region of the B factory, the γ^* propagation process dominates. Near the energy of the Z^0 pole, the Z^0 propagation process dominates due to the resonance effects. Differing from the case in the B factory where the production of double charmonium is mainly due to the CSM with the CO contribution

being negligible, at the Z factories, the CO contribution prevails in a number of production processes. In Figs. 4 and 5, the ratios $\sigma_{\text{CO}}/\sigma_{\text{Total}}$ versus $\sqrt{s}$ have been illustrated, enabling a more direct and intuitive perception of the CO contributions to the total cross sections. It should be noted that, in the production of double η_c or double η_b , the contributions stemming from the CS channels vanish when the t channel is left out of consideration. Even so, the CS contributions are still several orders of magnitude smaller than the CO contribution, resulting in pure CO cross sections.

Relativistic corrections (RCs) exert a substantial impact on double charmonia production. At the Z^0 pole and for higher collision energies, the K factors associated with the

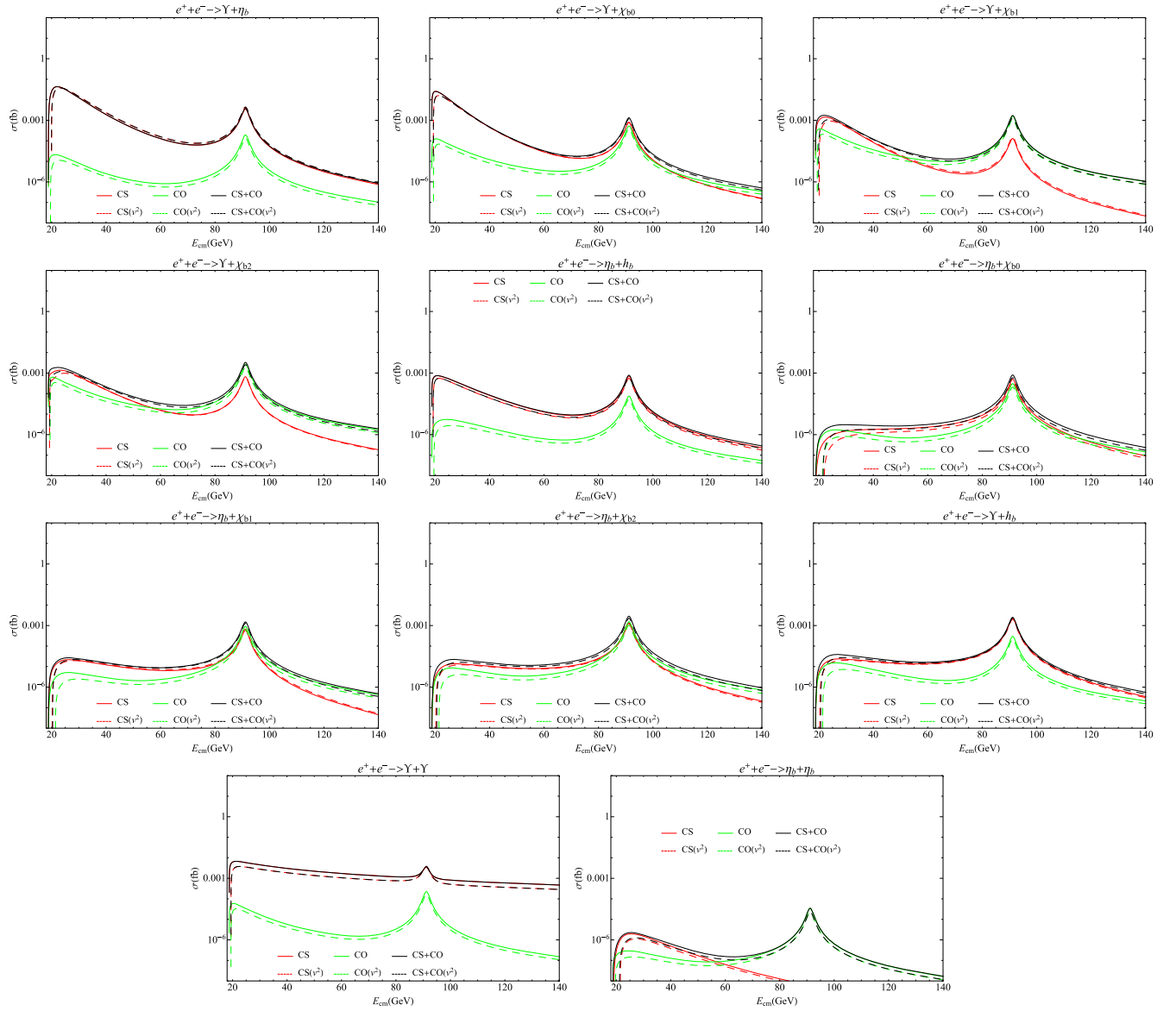


FIG. 3. Cross sections (σ) versus c.m. energy ($E_{\text{cm}} = \sqrt{s}$) for double bottomonium production. The solid line represents LO and the dashed line represents next-to-leading order in v^2 (NLO) results. The red line represents the CS channel, the green line represents the total CO channels, and the black line represents the sum of CS and CO.

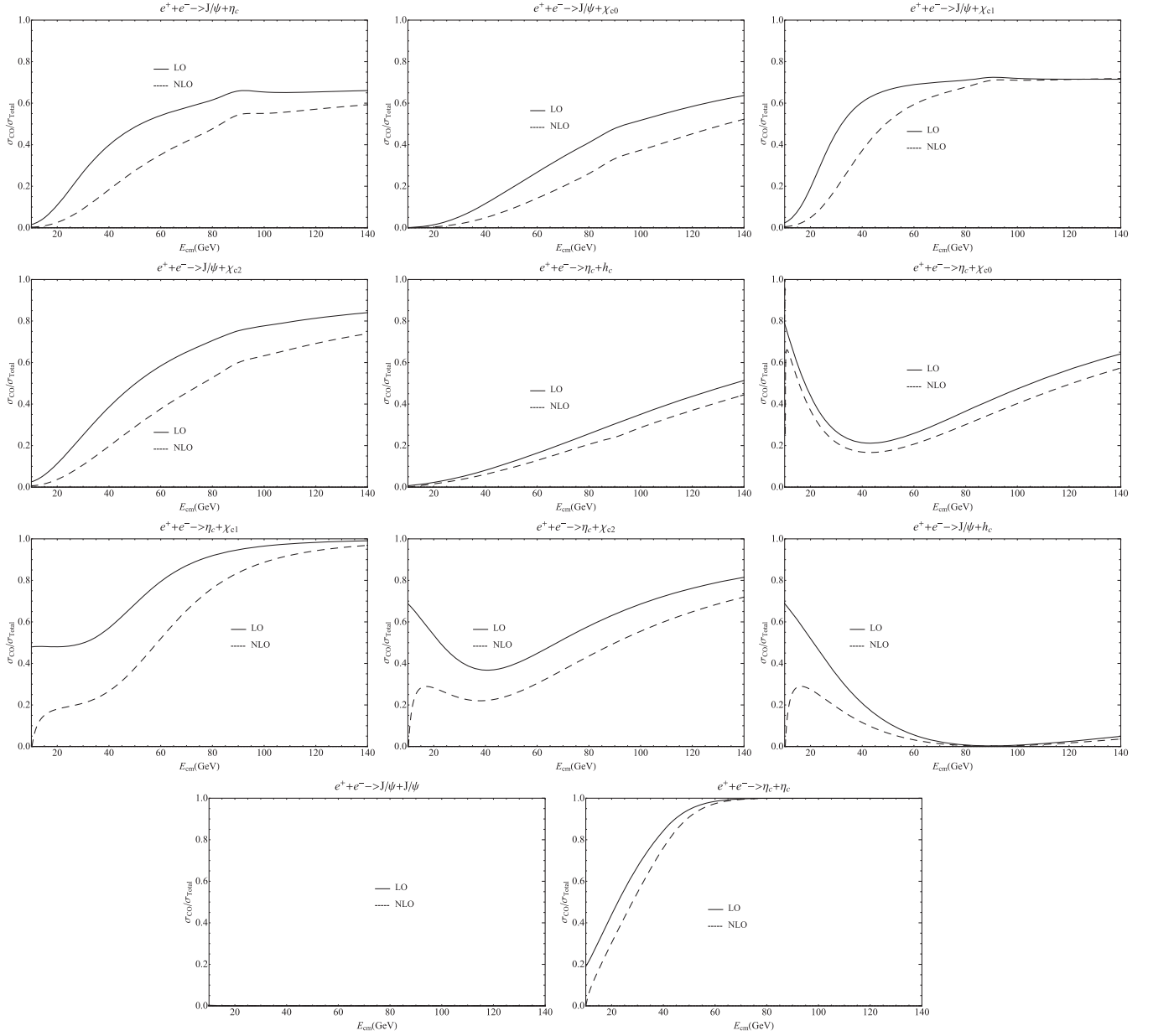


FIG. 4. The ratios $(\sigma_{CO}/\sigma_{\text{Total}})$ versus c.m. energy ($E_{cm} = \sqrt{s}$) for double charmonium production. The solid line represents LO and the dashed line represents next-to-leading order in v^2 (NLO) results.

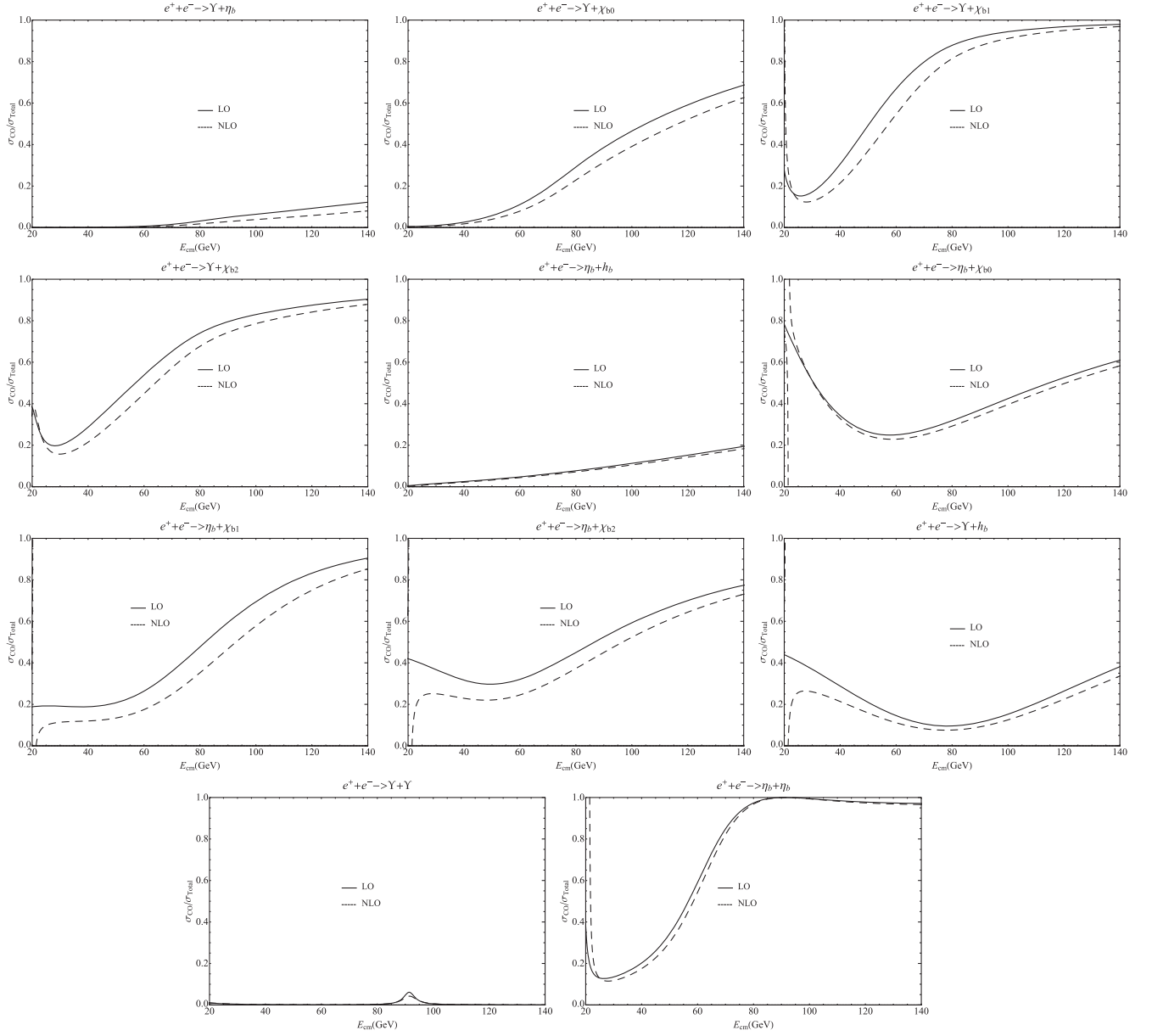


FIG. 5. The ratios $(\sigma_{CO}/\sigma_{\text{Total}})$ versus c.m. energy ($E_{\text{cm}} = \sqrt{s}$) for double bottomonium production. The solid line represents LO and the dashed line represents next-to-leading order in v^2 (NLO) results.

TABLE III. Production cross sections (units $\times 10^{-4}$ fb) of the double heavy quarkonia at $\sqrt{s} = 91.1876$ GeV. LO and NLO(v^2) mean leading-order and next-to-leading-order results in the v^2 expansions, respectively. CS means the cross section of the color-singlet channel, and CO means cross sections of all the color-octet channels.

	Charmonia				Bottomonia				
	CS		CS + CO		CS		CS + CO		
	LO	NLO(v^2)	LO	NLO(v^2)	LO	NLO(v^2)	LO	NLO(v^2)	
$J/\psi + J/\psi$	2736.0	355.7	2737.4	356.2	$\Upsilon + \Upsilon$	34.4	36.8	36.7	38.4
$J/\psi + \eta_c$	10.9	6.2	32.1	13.7	$\Upsilon + \eta_b$	35.7	43.0	37.6	44.3
$J/\psi + h_c$	33.2	20.2	33.3	20.3	$\Upsilon + h_b$	22.2	20.4	25.2	22.5
$J/\psi + \chi_{c0}$	2.1	1.4	4.1	2.1	$\Upsilon + \chi_{b0}$	8.0	7.9	13.3	11.7
$J/\psi + \chi_{c1}$	2.3	0.9	8.2	3.0	$\Upsilon + \chi_{b1}$	1.3	1.5	17.1	12.9
$J/\psi + \chi_{c2}$	3.2	2.3	13.1	5.9	$\Upsilon + \chi_{b2}$	6.6	6.4	32.9	25.4
$\eta_c + \eta_c$	0.0	0.0	39.2	13.3	$\eta_b + \eta_b$	0.0	0.0	0.3	0.2
$\eta_c + h_c$	3.1	1.5	4.4	2.1	$\eta_b + h_b$	7.1	5.5	7.8	6.0
$\eta_c + \chi_{c0}$	6.8	3.4	11.9	5.4	$\eta_b + \chi_{b0}$	5.0	3.8	8.1	5.9
$\eta_c + \chi_{c1}$	0.8	1.1	16.1	6.8	$\eta_b + \chi_{b1}$	5.9	6.7	15.0	12.8
$\eta_c + \chi_{c2}$	14.1	9.4	39.5	19.0	$\eta_b + \chi_{b2}$	13.3	12.1	28.5	22.4

NLO(v^2) cross sections falls within the range of 0.3–0.5. And for bottomonia production, the influence of RCs is relatively minor. Here, the K factor is in the range of 0.7–0.8. In Appendix A, we depict the ratios of $\sigma_{NLO(v^2)}/\sigma_{LO}$ (i.e., the K factors) as a function of $\sqrt{s}$ in Figs. 15 and 16. Correspondingly, in Appendix B, we elucidate the ratios of the SDCs corresponding to the RCs and those at LO in the large energy limit. Notably, the NLO QCD corrections can also be remarkably significant, with the K factor reaching as high as 2–5, as reported in [39,41].

The numerical results of the total cross section at $\sqrt{s} = m_Z$ are illustrated in Table III. For comparison, we present the total cross sections of CS as well as CS plus CO, at the order of v^0 and v^2 , respectively. Our LO CS results are in agreement with Refs. [12,37,41,44,94] when using the same input parameters.¹ Our NLO(v^2) CS result of $\gamma^* \rightarrow J/\psi + \eta_c$ is consistent with Ref. [18].

In Figs. 6 and 7, we depict the differential cross section $d\sigma/d\cos\theta$, where θ denotes the scattering angle of H_1 . It is

evident that within the CS channel, a concave distribution pattern emerges for $J/\psi + \eta_c/\chi_{c1}/J/\psi, \eta_c + \chi_{c1}$ and $\Upsilon + \eta_b/\chi_{b1}/\Upsilon, \eta_b + \chi_{b1}$. In contrast, the remaining processes within this channel exhibit a bulge distribution. As for the total CO channel, a concave distribution is characteristic of all its processes.

The differential cross section $\frac{d\sigma}{dp_t}$ can be written as

$$\frac{d\sigma}{dp_t} = \left| \frac{d\cos\theta}{dp_t} \right| \left(\frac{d\sigma}{d\cos\theta} \right) = \frac{p_t}{|\vec{p}_3| \sqrt{p_3^2 - p_t^2}} \left(\frac{d\sigma}{d\cos\theta} \right) \quad (31)$$

and the momentum of the H_1 heavy quarkonia:

$$\vec{p}_3 = \frac{\sqrt{\lambda[s, (2E_{q3})^2, (2E_{q4})^2]}}{2\sqrt{s}}. \quad (32)$$

Combining Eqs. (31) and (32), we can also get an $\mathcal{O}(v^2)$ expression:

$$\frac{d\sigma}{dp_t} = \frac{4p_t s^{3/2} [s(s - 4p_t^2) - 16m_Q^2(2p_t^2 - s)(v^2 - 2) - 256m_Q^4(v^2 - 1)]}{[s(s - 16m_Q^2)]^{3/2}(s - 16m_Q^2 - 4p_t^2)^{3/2}} \left(\frac{d\sigma}{d\cos\theta} \right) + \mathcal{O}(v^4). \quad (33)$$

¹We find that a few processes are in contradiction with Ref. [44], our results of $\sigma(J/\psi + J/\psi)_{Z^0}(\sigma(\Upsilon + \Upsilon)_{Z^0})$ are half of those in Ref. [44], and the difference might be due to the particle identities. Moreover, in our results, $\sigma(J/\psi + h_c)_{\gamma^*}$ and $\sigma(\Upsilon + h_b)_{\gamma^*}$ are zero because of the C -parity conservation, which are also different from the values in Ref. [44].

The results are shown in Figs. 8 and 9, it is seen that the p_t distribution change is more stable for the CO channel than for the CS channel.

B. Events

CEPC has two interaction points. And for its Z factory mode, the designed integrated luminosity within two

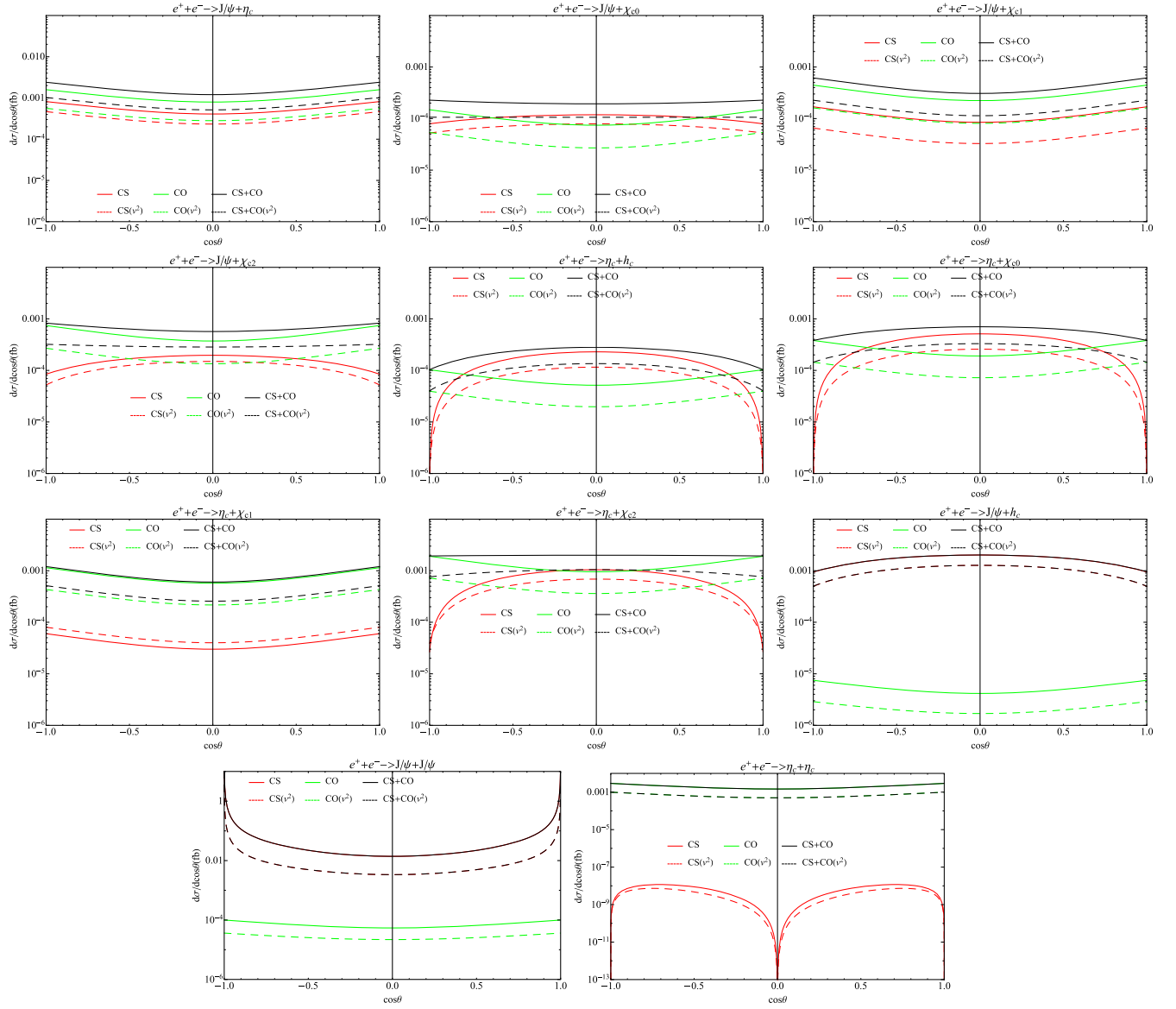


FIG. 6. The differential cross sections $d\sigma/d\cos\theta$ for double charmonium production at $\sqrt{s} = m_Z$. The solid line represents LO and the dashed line represents next-to-leading order in v^2 (NLO) results. The red line represents the CS channel, the green line represents the total CO channels, and the black line represents the sum of CS and CO.

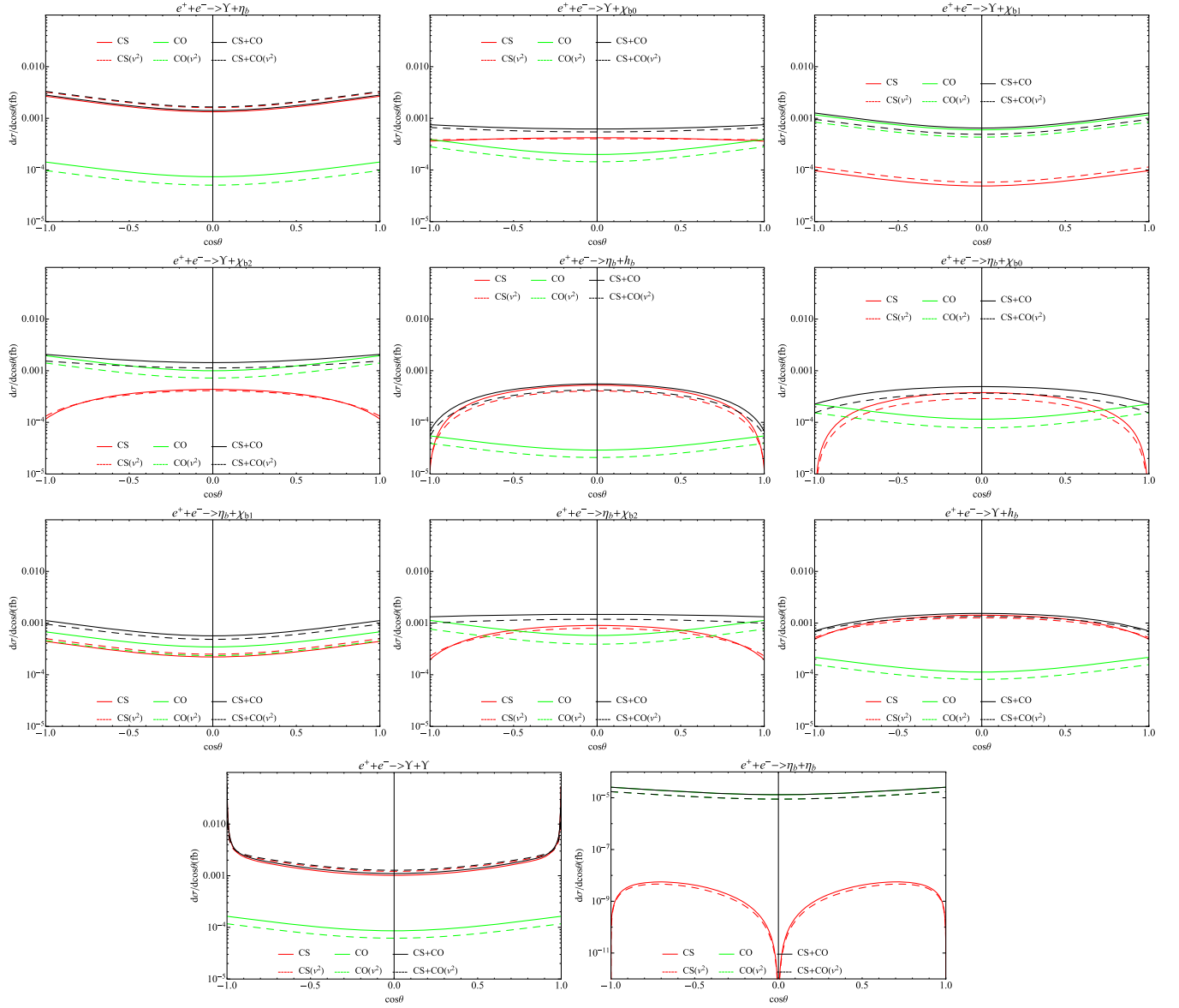


FIG. 7. The differential cross sections $d\sigma/d\cos\theta$ for double bottomonium production at $\sqrt{s} = m_Z$. The solid line represents LO and the dashed line represents next-to-leading order in v^2 (NLO) results. The red line represents the CS channel, the green line represents the total CO channels, and the black line represents the sum of CS and CO.

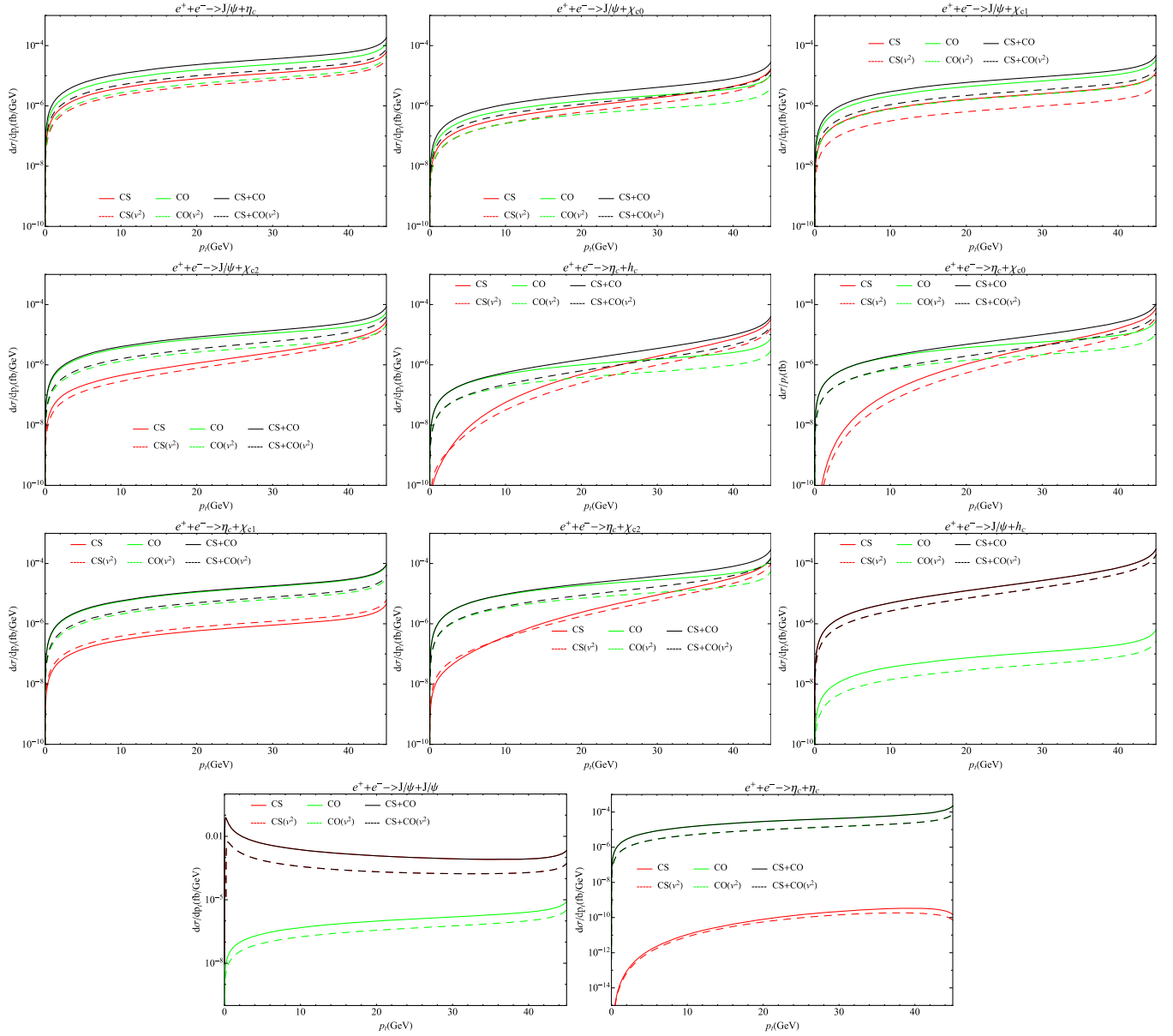


FIG. 8. The differential cross sections ($d\sigma/dp_T$) for double charmonium production at $\sqrt{s} = m_Z$. The solid line represents LO and the dashed line represents next-to-leading order in v^2 (NLO) results. The red line represents the CS channel, the green line represents the total CO channels, and the black line represents the sum of CS and CO.

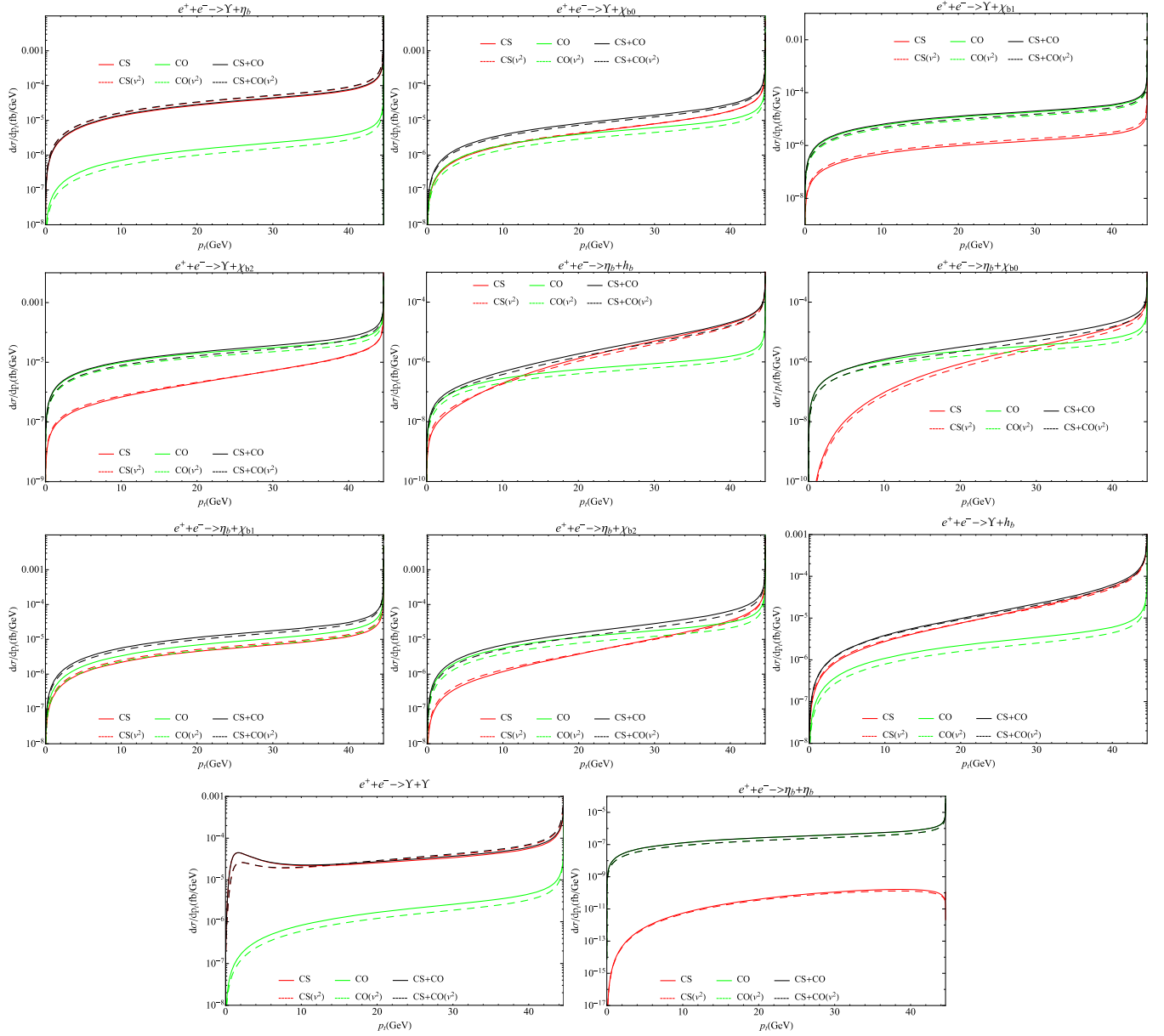


FIG. 9. The differential cross section ($d\sigma/dp_T$) for double bottomonium production at $\sqrt{s} = m_Z$. The solid line represents LO and the dashed line represents next-to-leading order in v^2 (NLO) results. The red line represents the CS channel, the green line represents the total CO channels, and the black line represents the sum of CS and CO.

TABLE IV. The events of double heavy quarkonia production at $\sqrt{s} = 91.1876$ GeV for the CEPC and FCC-ee. In each cell, the values outside and inside the brackets are for leading-order and next-to-leading-order results in the v^2 expansions, respectively.

CEPC (16 ab ⁻¹)					
	CS	CS + CO		CS	CS + CO
$J/\psi + J/\psi$	4378(569)	4380(570)	$\Upsilon + \Upsilon$	55(59)	59(61)
$J/\psi + \eta_c$	17(10)	51(22)	$\Upsilon + \eta_b$	57(69)	60(71)
$J/\psi + h_c$	53(32)	53(32)	$\Upsilon + h_b$	36(33)	40(36)
$J/\psi + \chi_{c0}$	3(2)	7(3)	$\Upsilon + \chi_{b0}$	13(13)	21(19)
$J/\psi + \chi_{c1}$	4(1)	13(5)	$\Upsilon + \chi_{b1}$	2(2)	27(21)
$J/\psi + \chi_{c2}$	5(4)	21(9)	$\Upsilon + \chi_{b2}$	11(10)	53(41)
$\eta_c + \eta_c$	0(0)	63(21)	$\eta_b + \eta_b$	0(0)	1(0)
$\eta_c + h_c$	5(2)	7(3)	$\eta_b + h_b$	11(9)	13(10)
$\eta_c + \chi_{c0}$	11(6)	19(9)	$\eta_b + \chi_{b0}$	8(6)	13(9)
$\eta_c + \chi_{c1}$	1(2)	26(11)	$\eta_b + \chi_{b1}$	9(11)	24(21)
$\eta_c + \chi_{c2}$	23(15)	63(30)	$\eta_b + \chi_{b2}$	21(19)	46(36)
FCC-ee (150 ab ⁻¹)					
$J/\psi + J/\psi$	41041(5335)	41061(5343)	$\Upsilon + \Upsilon$	516(552)	550(576)
$J/\psi + \eta_c$	163(93)	481(206)	$\Upsilon + \eta_b$	535(645)	564(665)
$J/\psi + h_c$	498(304)	499(304)	$\Upsilon + h_b$	333(305)	378(337)
$J/\psi + \chi_{c0}$	32(21)	61(32)	$\Upsilon + \chi_{b0}$	120(119)	199(176)
$J/\psi + \chi_{c1}$	34(13)	123(45)	$\Upsilon + \chi_{b1}$	20(23)	257(194)
$J/\psi + \chi_{c2}$	48(35)	196(89)	$\Upsilon + \chi_{b2}$	99(95)	494(380)
$\eta_c + \eta_c$	0(0)	587(199)	$\eta_b + \eta_b$	0(0)	5(3)
$\eta_c + h_c$	46(23)	67(31)	$\eta_b + h_b$	106(82)	118(91)
$\eta_c + \chi_{c0}$	102(52)	179(80)	$\eta_b + \chi_{b0}$	75(58)	121(89)
$\eta_c + \chi_{c1}$	12(16)	241(103)	$\eta_b + \chi_{b1}$	88(100)	225(193)
$\eta_c + \chi_{c2}$	211(141)	593(285)	$\eta_b + \chi_{b2}$	200(182)	428(336)

years is 16 ab⁻¹ [30]. The FCC-ee, with a designed luminosity of 150 ab⁻¹ [31], would operate as a super Z factory in its first four years. The designed luminosity of FCC-ee is 9 times that of CEPC. The estimated events for CEPC and FCC-ee are shown in Table IV.

We use the K factors of the NLO cross sections for QCD and EW channels in α_s from Ref. [41] [$K_{\text{QCD}}^{(\alpha_s)} \simeq 3.75, 3.9, 2.55, 2.5$ and $K_{\text{EW}}^{(\alpha_s)} \simeq 1.08, 1.01, 0.775, 0.908$ for $J/\psi + \eta_c$, $J/\psi + J/\psi$, $\Upsilon + \eta_b$, and $\Upsilon + \Upsilon$]. Respectively, the NLO(α_s) cross sections would be obtained as $\sigma_{\text{CS}(\text{QCD})}^{(\alpha_s)} = (0.7; 1.3; 58.9; 28.9) \times 10^{-4}$ fb, $\sigma_{\text{CS}(\text{EW})}^{(\alpha_s)} = (8.7; 54.8; 1.0; 2.4) \times 10^{-4}$ fb,² and with the interference contributions between QCD and EW channels as $(2.5; 8.4; 11.2; 11.1) \times 10^{-4}$ fb. The CO cross sections are $(20.9; 1.4; 1.9; 2.2) \times 10^{-4}$ fb. The t -channel double-photon-fragmentation QED contributions are $(0; 2673.0; 0; 9.1) \times 10^{-4}$ fb. Therefore, the total cross section at $\mathcal{O}(v^0)$ would be $(32.7; 2738.9; 73.1; 53.8) \times 10^{-4}$ fb, and the final events would be (52, 4382, 117, 86) and (491, 41083, 1096, 806) for the two colliders.

²The cross sections obtained by summing the QCD and EW contributions for $\Upsilon + \eta_b$ and $\Upsilon + \Upsilon$ processes are different from those in Ref. [41]. This is due to the adoption of different parameters, especially α_s and LDMEs.

The reconstructed events from their decay products can be estimated as

$$N = N[H_1 + H_2] \times Br_1 \times Br_2 \times \epsilon_1 \times \epsilon_2, \quad (34)$$

where Br_i , $i = 1, 2$, are branching fractions of heavy quarkonium for H_1 and H_2 , respectively. And ϵ_i is the corresponding detection efficiency. At CEPC and FCC-ee, precise Higgs boson and electroweak measurements require the detector to efficiently and precisely identify leptons, photons, τ leptons, jets, and flavors [30,31]. Therefore, here, we assume the detection efficiency is 100%. J/ψ and Υ are reconstructed from leptonic decays. η_c is reconstructed from hadronic decays. h_c is reconstructed via the decay chain $h_c \rightarrow \eta_c + \gamma \rightarrow (\text{hadron decay}) + \gamma$. The final results are presented in Table V. Here, for $(J/\psi + \eta_c, J/\psi$ pair, and Υ pair) production, we have introduced NLO(α_s) corrections mentioned above to the CS cross sections.

C. Uncertainties

The uncertainties in cross sections arise from the heavy quark mass, LDMEs, renormalization scale, and the deviation of collision energy from m_Z . Among them, the LDMEs may bring large uncertainties. Unlike CS LDMEs, which are usually obtained from the potential model, all the CO LDMEs are extracted from the experimental data.

TABLE V. The reconstructed events of double heavy quarkonia from their decay products at $\sqrt{s} = 91.1876$ GeV for the CEPC and FCC-ee. In each cell, the values outside and inside the brackets are for leading-order and next-to-leading-order results in the v^2 expansions, respectively. The efficiency is selected as 100%.

$H_1 + H_2$	H_1		H_2		CEPC (16 ab ⁻¹)		FCC (150 ab ⁻¹)	
	Decay	Br	Decay	Br	CS	CS + CO	CS	CS + CO
$J/\psi + J/\psi$	$l^+ l^- (l = e, \mu)$	12%	$l^+ l^- (l = e, \mu)$	12%	63(8)	63(8)	591(77)	592(77)
$J/\psi + \eta_c$	...	...	Hadron decay	43%	1(1)	3(1)	8(5)	25(11)
$J/\psi + h_c$	...	...	$\eta_c \gamma \rightarrow (\text{hadron decay}) + \gamma$	21.5%	1(1)	1(1)	13(8)	13(8)
$\eta_c + \eta_c$	Hadron decay	43%	Hadron decay	43%	0(0)	12(4)	0(0)	109(37)
$\eta_c + \chi_{c2}$	...	...	$\gamma l^+ l^- (\gamma J\psi)$	2.3%	0(0)	1(0)	2(1)	6(3)
$\Upsilon + \Upsilon$	$l^+ l^- (l = e, \mu)$	4.86%	$l^+ l^- (l = e, \mu)$	4.86%	0(0)	0(0)	1(1)	2(1)

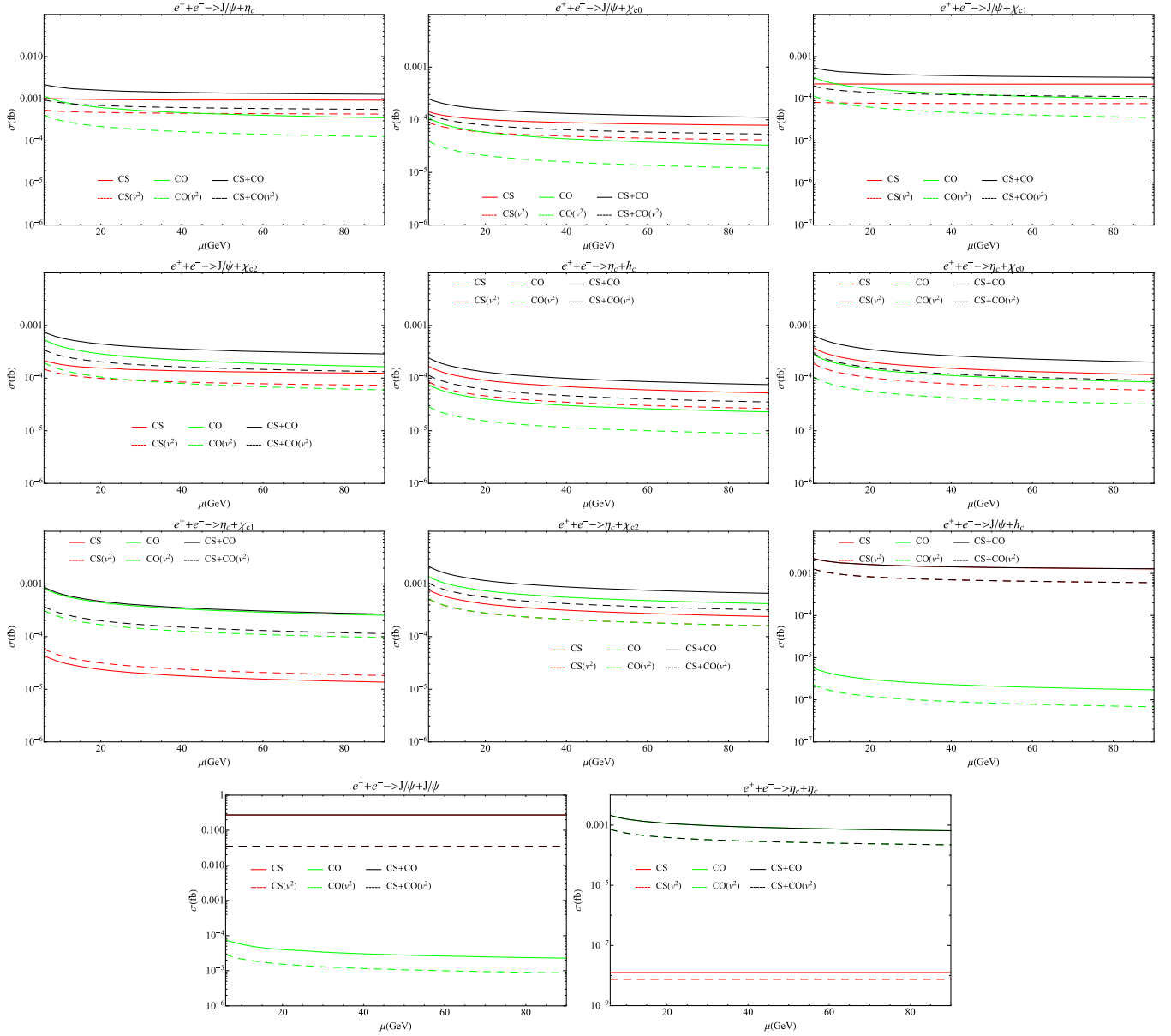


FIG. 10. Cross sections (σ) versus renormalization scale (μ) for double charmonium production at $\sqrt{s} = m_Z$. The solid line represents LO and the dashed line represents next-to-leading order in v^2 (NLO) results. The red line represents the CS channel, the green line represents the total CO channels, and the black line represents the sum of CS and CO.

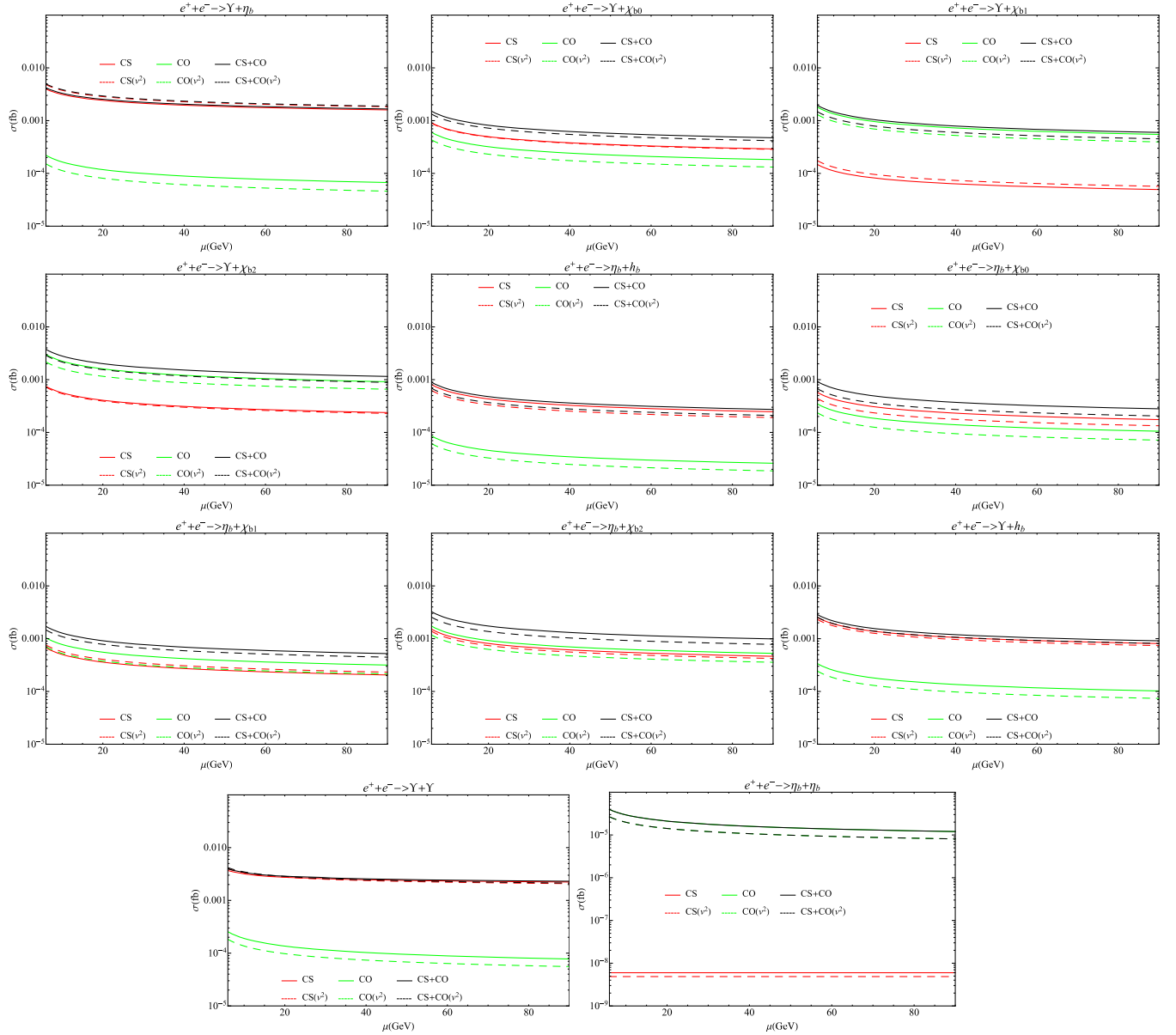


FIG. 11. Cross section (σ) versus renormalization scale (μ) for double bottomonium production at $\sqrt{s} = m_Z$. The solid line represents LO and the dashed line represents next-to-leading order in v^2 (NLO) results. The red line represents the CS channel, the green line represents the total CO channels, and the black line represents the sum of CS and CO.

Unfortunately, the values of CO LDMEs fitted by different works differ greatly [2,5–9,95–103]. As an example of discussing the effects of the LDMEs on the cross sections, we will discuss the $J/\psi + \eta_c$ production in the next subsection and not discuss other channels.

The uncertainties caused by the renormalization scale (μ) are shown in Figs. 10 and 11.

To show the sensitivity of the total cross sections to the collision energy around the Z^0 peak, we calculate the total cross sections using $E_{cm} = (1 \pm 3\%)m_Z$. The results are given in Tables VI and VII, where we use the same ratios as in Ref. [47].

$$R_- = \frac{\sigma(E_{cm} = 97\%m_Z)}{\sigma(E_{cm} = m_Z)},$$

$$R_+ = \frac{\sigma(E_{cm} = 103\%m_Z)}{\sigma(E_{cm} = m_Z)}. \quad (35)$$

The total cross section decreases to 15%–20% of its peak values, consistent with the results in Tables VIII and IX of Ref. [47], which studied the semiexclusive processes.

Next, to see the uncertainty caused by the heavy quark mass, we take $m_c = (1.5 \pm 0.15)\text{GeV}$, $m_b = (4.7 \pm 0.15)\text{GeV}$. The results are shown in Tables VIII and IX. For the

TABLE VI. Total cross sections (units $\times 10^{-4}$ fb) up to $\mathcal{O}(v^2)$ for the double charmonia production, with varying values of E_{cm} . The ratio $R_{\pm}$ shows how the cross sections are changed with varying E_{cm} .

	CS			CO			Total		
	97% m_Z	103% m_Z	$\begin{pmatrix} R_- \\ R_+ \end{pmatrix}$	97% m_Z	103% m_Z	$\begin{pmatrix} R_- \\ R_+ \end{pmatrix}$	97% m_Z	103% m_Z	$\begin{pmatrix} R_- \\ R_+ \end{pmatrix}$
$J/\psi + \eta_c$	1.2	1.1	$\begin{pmatrix} 19\% \\ 18\% \end{pmatrix}$	1.4	1.4	$\begin{pmatrix} 18\% \\ 18\% \end{pmatrix}$	2.6	2.5	$\begin{pmatrix} 19\% \\ 18\% \end{pmatrix}$
$J/\psi + h_c$	3.8	3.2	$\begin{pmatrix} 19\% \\ 16\% \end{pmatrix}$	0.0	0.0	$\begin{pmatrix} 19\% \\ 19\% \end{pmatrix}$	3.8	3.2	$\begin{pmatrix} 19\% \\ 16\% \end{pmatrix}$
$J/\psi + \chi_{c0}$	0.3	0.2	$\begin{pmatrix} 20\% \\ 18\% \end{pmatrix}$	0.1	0.1	$\begin{pmatrix} 18\% \\ 19\% \end{pmatrix}$	0.4	0.4	$\begin{pmatrix} 19\% \\ 18\% \end{pmatrix}$
$J/\psi + \chi_{c1}$	0.2	0.2	$\begin{pmatrix} 19\% \\ 19\% \end{pmatrix}$	0.4	0.4	$\begin{pmatrix} 18\% \\ 19\% \end{pmatrix}$	0.6	0.6	$\begin{pmatrix} 19\% \\ 19\% \end{pmatrix}$
$J/\psi + \chi_{c2}$	0.5	0.4	$\begin{pmatrix} 20\% \\ 18\% \end{pmatrix}$	0.7	0.7	$\begin{pmatrix} 18\% \\ 19\% \end{pmatrix}$	1.1	1.1	$\begin{pmatrix} 19\% \\ 18\% \end{pmatrix}$
$\eta_c + h_c$	0.3	0.3	$\begin{pmatrix} 20\% \\ 18\% \end{pmatrix}$	0.1	0.1	$\begin{pmatrix} 19\% \\ 19\% \end{pmatrix}$	0.4	0.4	$\begin{pmatrix} 19\% \\ 18\% \end{pmatrix}$
$\eta_c + \chi_{c0}$	0.6	0.5	$\begin{pmatrix} 19\% \\ 16\% \end{pmatrix}$	0.3	0.3	$\begin{pmatrix} 18\% \\ 17\% \end{pmatrix}$	1.0	0.9	$\begin{pmatrix} 18\% \\ 16\% \end{pmatrix}$
$\eta_c + \chi_{c1}$	0.2	0.2	$\begin{pmatrix} 20\% \\ 15\% \end{pmatrix}$	1.0	1.0	$\begin{pmatrix} 18\% \\ 17\% \end{pmatrix}$	1.2	1.1	$\begin{pmatrix} 18\% \\ 17\% \end{pmatrix}$
$\eta_c + \chi_{c2}$	1.8	1.5	$\begin{pmatrix} 19\% \\ 16\% \end{pmatrix}$	1.7	1.6	$\begin{pmatrix} 18\% \\ 17\% \end{pmatrix}$	3.5	3.1	$\begin{pmatrix} 18\% \\ 16\% \end{pmatrix}$
$J/\psi + J/\psi$	669.4	620.0	$\begin{pmatrix} 94\% \\ 87\% \end{pmatrix}$	0.2	0.2	$\begin{pmatrix} 19\% \\ 18\% \end{pmatrix}$	670.0	620.0	$\begin{pmatrix} 94\% \\ 87\% \end{pmatrix}$
$\eta_c + \eta_c$	0.0	0.0	$\begin{pmatrix} 120\% \\ 84\% \end{pmatrix}$	4.7	4.5	$\begin{pmatrix} 18\% \\ 17\% \end{pmatrix}$	4.7	4.5	$\begin{pmatrix} 18\% \\ 17\% \end{pmatrix}$

TABLE VII. Total cross sections (units $\times 10^{-4}$ fb) up to $\mathcal{O}(v^2)$ for the double bottomonia production, with varying values of E_{cm} . The ratio $R_{\pm}$ shows how the cross sections are changed with varying E_{cm} .

	CS			CO			Total		
	97% m_Z	103% m_Z	$\begin{pmatrix} R_- \\ R_+ \end{pmatrix}$	97% m_Z	103% m_Z	$\begin{pmatrix} R_- \\ R_+ \end{pmatrix}$	97% m_Z	103% m_Z	$\begin{pmatrix} R_- \\ R_+ \end{pmatrix}$
$\Upsilon + \eta_b$	8.3	6.7	$\begin{pmatrix} 19\% \\ 16\% \end{pmatrix}$	0.2	0.2	$\begin{pmatrix} 17\% \\ 17\% \end{pmatrix}$	8.6	6.9	$\begin{pmatrix} 19\% \\ 16\% \end{pmatrix}$
$\Upsilon + h_b$	3.9	3.2	$\begin{pmatrix} 19\% \\ 16\% \end{pmatrix}$	0.4	0.4	$\begin{pmatrix} 17\% \\ 17\% \end{pmatrix}$	4.2	3.6	$\begin{pmatrix} 19\% \\ 16\% \end{pmatrix}$
$\Upsilon + \chi_{b0}$	1.5	1.2	$\begin{pmatrix} 19\% \\ 16\% \end{pmatrix}$	0.7	0.7	$\begin{pmatrix} 17\% \\ 17\% \end{pmatrix}$	2.2	1.9	$\begin{pmatrix} 19\% \\ 16\% \end{pmatrix}$
$\Upsilon + \chi_{b1}$	0.3	0.2	$\begin{pmatrix} 20\% \\ 15\% \end{pmatrix}$	2.0	2.0	$\begin{pmatrix} 17\% \\ 17\% \end{pmatrix}$	2.3	2.2	$\begin{pmatrix} 18\% \\ 17\% \end{pmatrix}$
$\Upsilon + \chi_{b2}$	1.2	1.0	$\begin{pmatrix} 19\% \\ 16\% \end{pmatrix}$	3.3	3.3	$\begin{pmatrix} 17\% \\ 17\% \end{pmatrix}$	4.5	4.3	$\begin{pmatrix} 18\% \\ 17\% \end{pmatrix}$
$\eta_b + h_b$	1.0	0.9	$\begin{pmatrix} 18\% \\ 16\% \end{pmatrix}$	0.1	0.1	$\begin{pmatrix} 17\% \\ 17\% \end{pmatrix}$	1.1	1.0	$\begin{pmatrix} 18\% \\ 16\% \end{pmatrix}$
$\eta_b + \chi_{b0}$	0.7	0.6	$\begin{pmatrix} 19\% \\ 16\% \end{pmatrix}$	0.4	0.4	$\begin{pmatrix} 17\% \\ 17\% \end{pmatrix}$	1.1	1.0	$\begin{pmatrix} 18\% \\ 16\% \end{pmatrix}$
$\eta_b + \chi_{b1}$	1.3	1.0	$\begin{pmatrix} 20\% \\ 15\% \end{pmatrix}$	1.1	1.1	$\begin{pmatrix} 17\% \\ 17\% \end{pmatrix}$	2.4	2.1	$\begin{pmatrix} 19\% \\ 16\% \end{pmatrix}$
$\eta_b + \chi_{b2}$	2.3	1.9	$\begin{pmatrix} 19\% \\ 16\% \end{pmatrix}$	1.8	1.8	$\begin{pmatrix} 17\% \\ 17\% \end{pmatrix}$	4.1	3.7	$\begin{pmatrix} 18\% \\ 16\% \end{pmatrix}$
$\Upsilon + \Upsilon$	12.5	9.6	$\begin{pmatrix} 34\% \\ 26\% \end{pmatrix}$	0.3	0.3	$\begin{pmatrix} 18\% \\ 17\% \end{pmatrix}$	12.8	9.9	$\begin{pmatrix} 18\% \\ 17\% \end{pmatrix}$
$\eta_b + \eta_b$	0.0	0.0	$\begin{pmatrix} 119\% \\ 84\% \end{pmatrix}$	0.0	0.0	$\begin{pmatrix} 17\% \\ 17\% \end{pmatrix}$	0.0	0.0	$\begin{pmatrix} 17\% \\ 17\% \end{pmatrix}$

TABLE VIII. Total cross sections (units $\times 10^{-4}$ fb) up to $\mathcal{O}(v^2)$ for the double charmonia production at $\sqrt{s} = m_Z$, with varying values of m_c . The uncertainties in the third column of each are the deviations from the central values corresponding to $m_c = 1.5$ GeV.

m_c	CS			CO			Total		
	1.35 GeV	1.65 GeV	Uncertainty	1.35 GeV	1.65 GeV	Uncertainty	1.35 GeV	1.65 GeV	Uncertainty
$J/\psi + \eta_c$	8.510	4.732	+2.300 – 1.477	11.47	5.102	+3.971 – 2.397	19.98	9.835	+6.271 – 3.874
$J/\psi + h_c$	25.41	16.57	+5.167 – 3.677	0.063	0.029	+0.021 – 0.013	25.48	16.60	+5.188 – 3.690
$J/\psi + \chi_{c0}$	1.707	1.184	+0.305 – 0.218	1.099	0.489	+0.380 – 0.230	2.806	1.673	+0.685 – 0.448
$J/\psi + \chi_{c1}$	1.271	0.631	+0.397 – 0.243	3.296	1.467	+1.141 – 0.689	4.566	2.098	+1.537 – 0.931
$J/\psi + \chi_{c2}$	2.967	1.902	+0.624 – 0.440	5.493	2.444	+1.901 – 1.148	8.460	4.347	+2.525 – 1.588
$\eta_c + h_c$	1.916	1.275	+0.368 – 0.273	0.805	0.358	+0.279 – 0.168	2.721	1.633	+0.647 – 0.441
$\eta_c + \chi_{c0}$	4.268	2.833	+0.825 – 0.610	2.947	1.306	+1.024 – 0.617	7.215	4.139	+1.848 – 1.227
$\eta_c + \chi_{c1}$	1.069	1.064	+0.002 – 0.003	8.840	3.918	+3.071 – 1.852	9.909	4.982	+3.073 – 1.855
$\eta_c + \chi_{c2}$	11.51	7.860	+2.097 – 1.552	14.73	6.529	+5.118 – 3.087	26.24	14.39	+7.216 – 4.639
$J/\psi + J/\psi$	679.0	199.8	+323.3 – 155.9	0.796	0.375	+0.262 – 0.160	679.8	200.2	+323.6 – 156.0
$\eta_c + \eta_c$	0.000	0.000	+0.000 – 0.000	20.35	9.005	+7.076 – 4.266	20.35	9.006	+7.076 – 4.266

TABLE IX. Total cross sections (units $\times 10^{-4}$ fb) up to $\mathcal{O}(v^2)$ for the double bottomonia production at $\sqrt{s} = m_Z$, with varying values of m_b . The uncertainties in the third column of each are the deviations from the central values corresponding to $m_b = 4.7$ GeV.

m_b	CS			CO			Total		
	4.55 GeV	4.85 GeV	Uncertainty	4.55 GeV	4.85 GeV	Uncertainty	4.55 GeV	4.85 GeV	Uncertainty
$\Upsilon + \eta_b$	44.12	42.00	+1.105 – 1.014	1.526	1.162	+0.197 – 0.167	45.64	43.16	+1.302 – 1.181
$\Upsilon + h_b$	21.46	19.37	+1.093 – 0.998	2.436	1.870	+0.306 – 0.260	23.89	21.24	+1.399 – 1.257
$\Upsilon + \chi_{b0}$	8.112	7.721	+0.204 – 0.187	4.354	3.329	+0.554 – 0.471	12.47	11.05	+0.758 – 0.658
$\Upsilon + \chi_{b1}$	1.554	1.516	+0.020 – 0.019	13.06	9.987	+1.662 – 1.412	14.61	11.50	+1.681 – 1.430
$\Upsilon + \chi_{b2}$	6.716	6.021	+0.364 – 0.332	21.77	16.65	+2.770 – 2.353	28.48	22.67	+3.134 – 2.684
$\eta_b + h_b$	5.915	5.112	+0.421 – 0.382	0.618	0.473	+0.078 – 0.066	6.532	5.586	+0.499 – 0.448
$\eta_b + \chi_{b0}$	4.138	3.577	+0.294 – 0.267	2.372	1.795	+0.312 – 0.265	6.510	5.373	+0.606 – 0.531
$\eta_b + \chi_{b1}$	6.707	6.620	+0.043 – 0.044	7.115	5.385	+0.937 – 0.794	13.82	12.01	+0.980 – 0.838
$\eta_b + \chi_{b2}$	12.78	11.53	+0.658 – 0.601	11.86	8.975	+1.561 – 1.323	24.64	20.50	+2.219 – 1.923
$\Upsilon + \Upsilon$	39.49	34.55	+2.671 – 2.266	1.835	1.413	+0.228 – 0.194	41.32	35.96	+2.899 – 2.459
$\eta_b + \eta_b$	0.000	0.000	+0.000 – 0.000	0.268	0.203	+0.035 – 0.030	0.268	0.203	+0.035 – 0.030

charmonium production, the uncertainties associated with the $m_c = 1.5 \pm 0.15$ GeV variation are as follows.

- (i) For the CS channel, the uncertainties are about 0.2% for $\eta_c + \chi_{c1}$ production, about 16%–27% for $J/\psi + h_c/\chi_{c0}/\chi_{c2}$, $\eta_c + h_c/\chi_{c0}/\chi_{c2}/\eta_c$ production, about 24%–37% for $J/\psi + \eta_c$ production, about 28%–45% for $J/\psi + \chi_{c1}$ production, and about 44%–91% for double J/ψ production.
- (ii) For total CO channels, about 30%–50% for $J/\psi + h_c/J/\psi$ production and about 32%–53% for all other production processes.
- (iii) For total cross sections, it is about 18%–26% for $J/\psi + h_c$ production, about 21%–32% for $J/\psi + \chi_{c0}, \eta_c + h_c$ production, about 23%–38% for $\eta_c + \chi_{c0}/\chi_{c2}$ production, about 27%–46% for $J/\psi + \eta_c/\chi_{c2}, \eta_c + \chi_{c1}$ production, about 31%–53% for $J/\psi + \chi_{c1}, \eta_c + \eta_c$ production, and about 44%–91% for double J/ψ production.

As for the bottomonium production, the uncertainties associated with the variation $m_b = 4.7 \pm 0.15$ GeV are as follows:

- (i) For the CS channel, the uncertainties are 1% for $\Upsilon/\eta_b + \chi_{b1}$ production, about 2%–3% for $\Upsilon + \eta_b/\chi_{b0}$ production, about 5%–6% for $\Upsilon + \chi_{b2}/h_b, \eta_b + \chi_{b2}$ production, and about 6%–8% for $\eta_b + h_b/\chi_{b0}/\eta_b, \Upsilon + \Upsilon$ production.
- (ii) For total CO channels, it is about 12%–15% for all production processes.
- (iii) For total cross sections, it is about 3% for $\Upsilon + \eta_b$ production, about 6% for $\Upsilon + h_b/\chi_{b0}$ production, about 6%–8% for $\eta_b + h_b/\chi_{b1}, \Upsilon + \Upsilon$ production, about 9%–10% for $\eta_b + \chi_{b0}/\chi_{b2}$ production, about 11%–13% for $\Upsilon + \chi_{b1}/\chi_{b2}$ production, and about 13%–15% for double η_b production.

D. $J/\psi + \eta_c$ production

In this subsection, we more specifically present the CS and CO cross sections of $J/\psi + \eta_c$ production with uncertainties. It is known that the $J/\psi + \eta_c$ production at B factories has been thoroughly studied. The large gap between LO calculation [12,13] and data [10,11] was eliminated

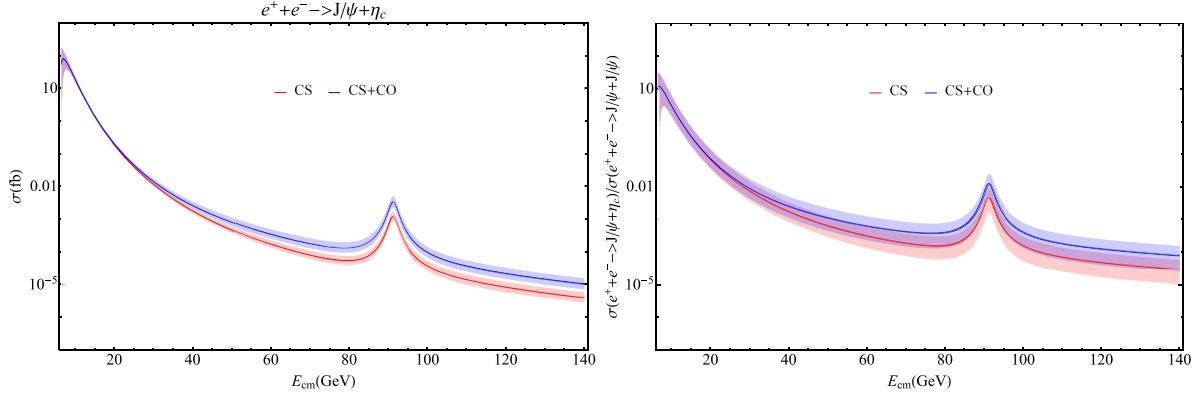


FIG. 12. The cross sections with uncertainties of $J/\psi + \eta_c$ production (left panel). The ratios between the cross sections of $J/\psi + \eta_c$ production and that of double J/ψ production (right panel).

by NLO [15,16,22] QCD corrections and RCs [18,21]. Recently, next-to-next-to-leading-order QCD corrections were also computed and exhibit reasonable perturbative convergence behavior for the cross sections [23,24]. For these works, due to the suppression of Z^0 propagator and the small CO proportion, considering γ^* propagating processes within CSM is enough. While in the Z factory energy region, the γ^* propagated processes can be neglected, and one must consider CS and CO channels via Z^0 propagator.

The CS and CO cross sections, which are obtained by combining QCD and EW processes and have uncertainties, are presented in the left panel in Fig. 12. The uncertainties are induced by the quark mass. The central lines correspond to $m_c = 1.5$ GeV. The upper and lower bounds of the bands correspond to $m_c = 1.35, 1.65$ GeV, respectively. Here, besides the RCs of CS and CO channels, we also include the NLO QCD corrections to the CS channels. The K factors of NLO cross sections are estimated from the calculations in Ref. [41], i.e., $K_{\text{QCD}} = 1.21 - 0.37 \ln(4m_c^2/s)$ and $K_{\text{QED}} = 0.67 - 0.06 \ln(4m_c^2/s)$. Furthermore, in an attempt to eliminate the uncertainties associated with CS LDMEs, we plot the ratios of the production rates of $J/\psi + \eta_c$ to those of double J/ψ in the right panel.

Compared with the CS LDMEs, the CO LDMEs have greater uncertainties. Next, we discuss the effects of CO LDMEs on the cross sections. As described in the foregoing content, the gluon fragmentation processes largely dominate the LO cross section. By comparing the short-distance cross sections (denoted by $\hat{\sigma}$) of $e^+e^- \rightarrow \gamma^*/Z^0 \rightarrow Q\bar{Q}(^{2s+1}L_J^8) + g$ as seen in Fig. 13, we have

$$\begin{aligned} \frac{\hat{\sigma}[Q\bar{Q}(^3P_J^8) + g]m_c^2}{\hat{\sigma}[Q\bar{Q}(^1S_0^8) + g]} \Big|_{m_c \ll E_{cm}} &= 3, \\ \frac{\hat{\sigma}[Q\bar{Q}(^1P_1^8) + g]m_c^2}{\hat{\sigma}[Q\bar{Q}(^3S_1^8) + g]} \Big|_{m_c \ll E_{cm}} &= 1. \end{aligned} \quad (36)$$

The relations also hold for $H + \gamma$ production [37,104] at Z factories, which could be used to check the correctness.

Therefore, it is convenient to define several matrix elements as below:

$$\begin{aligned} M_1 &= \langle \mathcal{O}^{J/\psi}(^1S_0^{[8]}) \rangle = \langle \mathcal{O}^{\eta_c}(^3S_1^{[8]}) \rangle, \\ M_2 &= \langle \mathcal{O}^{J/\psi}(^3S_1^{[8]}) \rangle = 3 \langle \mathcal{O}^{\eta_c}(^1S_0^{[8]}) \rangle, \\ M_3 &= \langle \mathcal{O}^{J/\psi}(^1S_0^{[8]}) \rangle + 3 \langle \mathcal{O}^{J/\psi}(^3P_0^{[8]}) \rangle / m_c^2 \\ &= \langle \mathcal{O}^{\eta_c}(^3S_1^{[8]}) \rangle + \langle \mathcal{O}^{\eta_c}(^1P_1^{[8]}) \rangle / m_c^2, \\ M_s &= M_1 M_3 + \frac{M_2^2}{3}, \\ M_t &= \frac{M_2}{3} (M_1 + M_3). \end{aligned} \quad (37)$$

Here, HQSS has been applied in the above expressions. Finally, utilizing Eq. (30), the cross section is expressed as the sum of two parts with two independent matrix elements M_s and M_t :

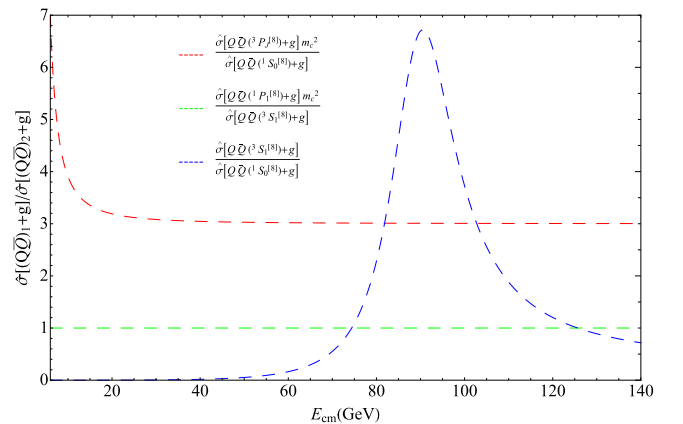


FIG. 13. The ratios of the short-distance cross sections of different production channels.

TABLE X. Color-octet long-distance matrix elements of J/ψ extracted from hadronic production through six works [6–9,60,64]. The definitions of some symbols in this table are $M_3 = \langle \mathcal{O}^{J/\psi}({}^1S_0^8) \rangle + 3\langle \mathcal{O}^{J/\psi}({}^3P_0^8) \rangle/m_c^2$, $M_s = M_1 M_3 + \frac{M_2^2}{3}$, and $M_t = \frac{M_2}{3}(M_1 + M_3)$.

	$M_1: \langle \mathcal{O}^{J/\psi}({}^1S_0^8) \rangle$ (GeV ³)	$\langle \mathcal{O}^{J/\psi}({}^3P_0^8) \rangle$ (GeV ⁵)	M_3 (GeV ³)	$M_2: \langle \mathcal{O}^{J/\psi}({}^3S_1^8) \rangle$ (GeV ³)	$M_s(10^{-4} \text{ GeV}^6)$	$M_t(10^{-4} \text{ GeV}^6)$
Butenschön	0.0497	−0.0161	0.0282	0.00224	14.03	0.58
Chao	0.089	0.0126	0.1058	0.0030	94.19	1.95
	0	0.054	0.072	0.014	0.65	3.36
	0.11	0	0.11	0	121	0
Gong	0.097	−0.02138	0.0685	−0.0046	66.52	−2.54
Bodwin	0.099	0.011	0.1137	0.011	112.97	7.80
Zhang	0.0078	0.0383	0.0589	0.01	4.93	2.22
Sharma ^a	0.018	0.0405	0.072	0.0013	12.97	0.39

^aLDMEs adopted in this paper.

$$\begin{aligned}
\sigma(J/\psi + \eta_c) = & \left\{ \hat{\sigma}[Q\bar{Q}({}^1S_0^8)] + g \frac{M_3}{N_c^2 - 1} + \hat{\sigma}[Q\bar{Q}({}^3S_1^8)] + g \frac{M_2}{3(N_c^2 - 1)} \right\} \times 0.13\alpha_s \frac{M_1}{m_c^3} \\
& + \left\{ \hat{\sigma}[Q\bar{Q}({}^3S_1^8)] + g \frac{M_3}{3(N_c^2 - 1)} + \hat{\sigma}[Q\bar{Q}({}^1S_0^8)] + g \frac{M_2}{3(N_c^2 - 1)} \right\} \\
& \times 0.13\alpha_s \frac{M_2}{m_c^3} = \frac{0.13\alpha_s}{m_c^3(N_c^2 - 1)} \{ \hat{\sigma}[Q\bar{Q}({}^1S_0^8)] + g M_s + \hat{\sigma}[Q\bar{Q}({}^3S_1^8)] + g M_t \}. \quad (38)
\end{aligned}$$

In Table X, the values of the LDMEs which were extracted from the hadronic production data of J/ψ by six works are listed. Actually, a wide range of values can satisfy the yields and polarizations of the J/ψ production at hadron colliders [7], which results in the difficulty of determining the exact values of the CO LDMEs. Therefore, the uncertainties of the LDMEs may affect the numerical value of the cross section greatly. As reflected in the table, it can be seen that the values of M_s and M_t also have great uncertainties. The total (CS + CO) cross sections utilizing these LDMEs are compared in Fig. 14. Near the Z-pole area, the uncertainties are about one order of magnitude, and the results of our calculations in this paper are at the

lower edge of the range of the cross-section values obtained using all these LDMEs. This indicates that the CO channels may make more significant contributions. Compared with the $J/\psi + \eta_c$ production, the uncertainties of η_c pair production would be larger due to the wide range of values of M_1 [corresponding to the matrix element of fragmentation into $\eta_c({}^3S_1^8)$], with the lowest being the zero value.

V. SUMMARY

In this paper, we have studied the production of double heavy quarkonium in e^+e^- annihilation at the Z factories, considering the CO contributions along with CS channels up to $\mathcal{O}(v^2)$.

We considered CS channels including QCD and EW processes and all the possible pairings of CO channels at the level of tree diagrams in the calculations. The CO channels contribute substantially more than the CS channels in several processes, particularly for the combinations involving the ${}^3S_1^8$ state. This indicates that the processes of gluon fragmentation into the intermediate ${}^3S_1^8$ state might play a crucial role at leading order of α_s . Meanwhile, the relativistic corrections are significant and the LO cross sections would be suppressed by the factors of at least 50% for charmonia and 20% for bottomonia, respectively. With the results of cross sections, we estimate the events of double heavy quarkonium production for future experimental measurements (CEPC and FCC-ee).

The COM is indispensable in hadron collision processes, while in the double charmonium production via e^+e^- annihilation at B factories the COM is negligible. In this

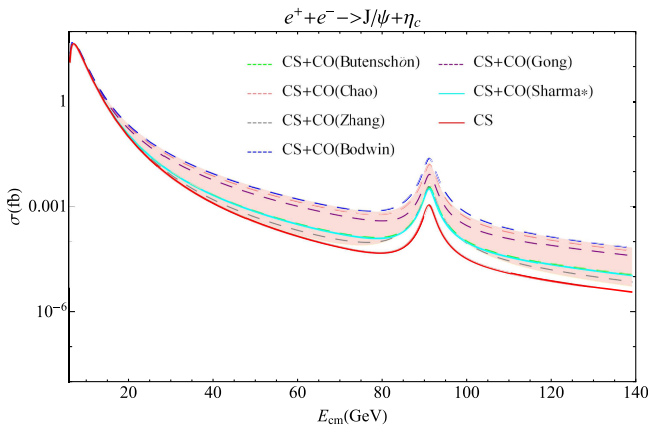


FIG. 14. The cross sections of $J/\psi + \eta_c$ production with different sets of the LDMEs, where the uncertainty band is obtained by adopting the extreme cases of LDMEs in Ref. [7].

work, we show that the COM is significant or dominant in e^+e^- annihilation at a Z factory and higher collision energy. It is expected that the COM is also significant for some other heavy quarkonia production processes (e.g., inclusive production processes or D -wave heavy quarkonia production) at the Z factory and higher energy region. However, the COM may be negligible for B_c meson production, since the gluon cannot be fragmented into a $^3S_1^{[8]} c\bar{b}$ quark pair. And it will be important to continue to calculate QCD loop corrections to these processes to complement the study of heavy quarkonia production at future e^+e^- colliders.

ACKNOWLEDGMENTS

The authors thank Professor Xiao-Shen Kang for valuable and helpful discussions and thank

Dr. Guang-Yu Wang for providing the code of NLO radiative corrections for J/ψ pair production. This work was supported by the National Natural Science Foundation of China (No. 11705078).

DATA AVAILABILITY

The data supporting this study's findings are available within the article.

APPENDIX A

The ratios of $\sigma_{NLO}(v^2)/\sigma_{LO}$ (i.e., the K factors) as a function of $\sqrt{s}$ are shown as follows.

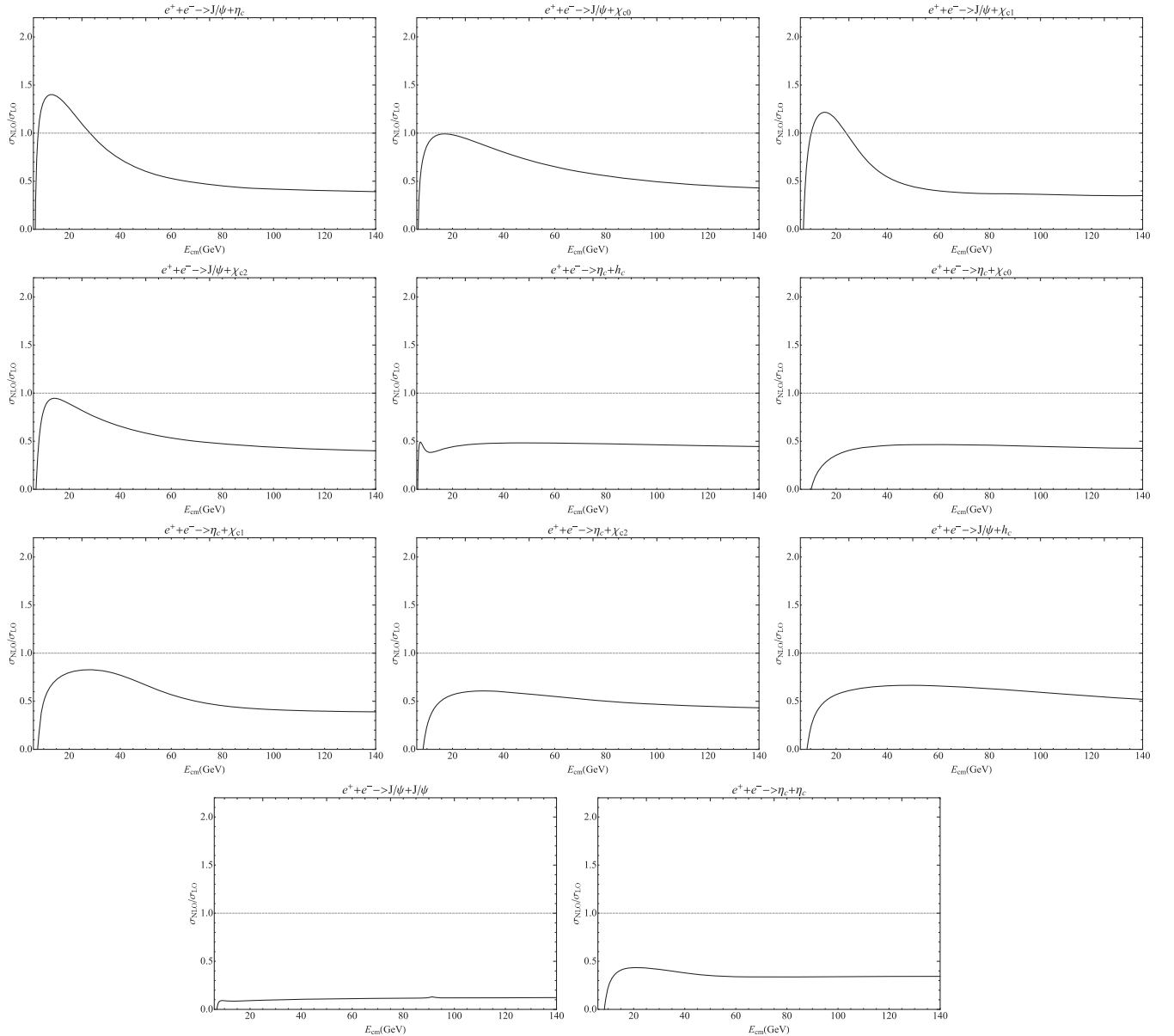


FIG. 15. The K factors for next-to-leading-order cross sections in v^2 as a function of c.m. energy for the production of double charmonium.

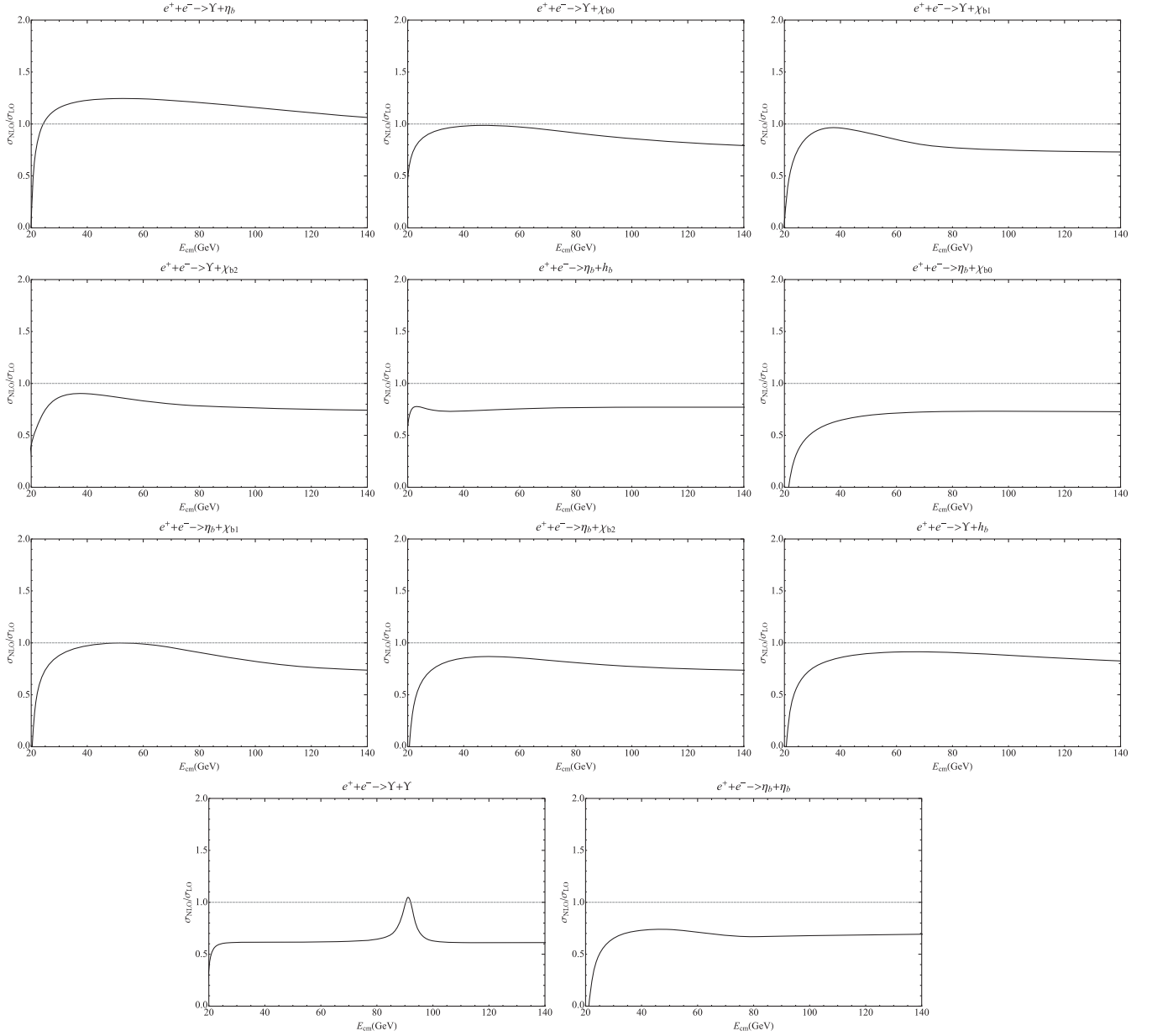


FIG. 16. The K factors for next-to-leading-order cross sections in v^2 as a function of c.m. energy for the production of double bottomonium.

APPENDIX B

Rather than presenting the analytical expression of the SDCs, we provide in Table XI the ratios of relativistic correction SDCs (denoted as G) to LO SDCs (denoted as F) in the high c.m. energy limit ($m_Q^2 \ll s$). For specific states, the ratios are determined by the topologies of the Feynman

diagrams. Consequently, each ratio is applicable to both charmonia and bottomonia in the channels with consistent states. The gluon fragmentation processes into $^3S_1^{[8]}$ are dominated for the $^3S_1^{[8]}$ channels associated with $^1S_0^{[8]}$, $^3P_J^{[8]}$, and $^1P_1^{[8]}$ states. Their ratios can be verified against Refs. [50,105]. And the photon fragmentation processes

TABLE XI. Ratios of relativistic correction SDCs (denoted as G) to LO SDCs (denoted as F) in the high c.m. energy limit ($m_Q^2 \ll s$). In each cell, we define $R_i \equiv G[n_i]/F[n_1 + n_2]$, $c = 1, 8$ for CS or CO states and (nf/f) as the nonfragmentation and fragmentation processes corresponding to (a),(b),(e),(f) and (c),(d),(g),(h) diagrams in Fig. 1. The first four rows correspond to negative total C parity of final states and are applicable to γ^*/Z^0 -propagated processes, while the second group of three rows correspond to positive C parity and are applicable only to Z^0 -propagated processes. The last row corresponds to the t -channel processes.

$(\text{nf})^3S_1^{[1,8]} + {}^1S_0^{[1,8]}$	$R_1 = \frac{3}{2}$	$R_2 = \frac{11}{6}$	$(\text{nf})^3S_1^{[1,8]} + {}^3P_0^{[1,8]}$	$R_1 = -\frac{1}{6}$	$R_2 = -\frac{13}{10}$	$(\text{nf})^3S_1^{[1,8]} + {}^3P_1^{[1,8]}$	$R_1 = \frac{3}{2}$	$R_2 = \frac{1}{2}$
$(\text{nf})^3S_1^{[1,8]} + {}^3P_2^{[1,8]}$	$R_1 = -\frac{1}{6}$	$R_2 = -\frac{7}{10}$	$(\text{nf})^1S_0^{[1,8]} + {}^1P_1^{[1,8]}$	$R_1 = -\frac{5}{6}$	$R_2 = -\frac{13}{10}$	$(f)^3S_1^{[1,8]} + {}^1S_0^{[1,8]}$	$R_1 = -\frac{11}{6}$	$R_2 = -\frac{5}{6}$
$(f)^3S_1^{[1,8]} + {}^3P_0^{[1,8]}$	$R_1 = -\frac{11}{6}$	$R_2 = -\frac{13}{10}$	$(f)^3S_1^{[1,8]} + {}^3P_1^{[1,8]}$	$R_1 = -\frac{11}{6}$	$R_2 = -\frac{11}{10}$	$(f)^3S_1^{[1,8]} + {}^3P_2^{[1,8]}$	$R_1 = -\frac{11}{6}$	$R_2 = -\frac{7}{10}$
$(f)^3S_1^{[8]} + {}^3P_J^{[8]}$	$R_1 = -\frac{11}{6}$	$R_2 = -\frac{31}{30}$	$(\text{nf})^3P_J^{[8]} + {}^1P_1^{[8]}$	$R_1 = -\frac{11}{10}$	$R_2 = -\frac{13}{10}$			
$(\text{nf})^3S_1^{[1,8]} + {}^1P_1^{[1,8]}$	$R_1 = -\frac{1}{6}$	$R_2 = -\frac{13}{10}$	$(\text{nf})^1S_0^{[1]} + {}^3P_0^{[1]}$	$R_1 = -\frac{5}{6}$	$R_2 = -\frac{13}{10}$	$(\text{nf})^1S_0^{[1]} + {}^3P_1^{[1]}$	$R_1 = \frac{7}{6}$	$R_2 = \frac{3}{10}$
$(\text{nf})^1S_0^{[1]} + {}^3P_2^{[1]}$	$R_1 = -\frac{5}{6}$	$R_2 = -\frac{7}{10}$	$(\text{nf})^1S_0^{[8]} + {}^3P_J^{[8]}$	$R_1 = -\frac{5}{6}$	$R_2 = -\frac{9}{10}$	$(f)^3S_1^{[1,8]} + {}^1P_1^{[1,8]}$	$R_1 = -\frac{11}{6}$	$R_2 = -\frac{13}{10}$
$(f)^3S_1^{[8]} + {}^3S_1^{[8]}$	$R_1 = R_2 = -1$		$(\text{nf})^3P_J^{[8]} + {}^3P_J^{[8]}$	$R_1 = R_2 = -1$		$(\text{nf})^1P_1^{[8]} + {}^1P_1^{[8]}$	$R_1 = R_2 = \frac{6}{5}$	
${}^3S_1^{[1]} + {}^3S_1^{[1]}$	$R_1 = R_2 = -\frac{11}{6}$		${}^1S_0^{[1]} + {}^1S_0^{[1]}$	$R_1 = R_2 = -\frac{1}{2}$				

into ${}^3S_1^{[1]}$ are dominated for the CS channels of J/ψ production associated with η_c , $\chi_{c0,1,2}$, and h_c . Their ratios can be verified against Ref. [106]. In the high c.m. energy limit, the K factors for $NLO(v^2)$ can be obtained from the

ratios. In the energy region near the Z^0 pole, the K factors would be jointly determined by the ratios of the non-fragmentation and fragmentation processes for J/ψ production.

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