

Entropy bounds from quantum thermodynamics

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Within an inherently classical perspective, there is always an unavoidable energy cost associated with the information deletion and this common lore is at the heart of the Landauer conjecture that does not impose, *per se*, any relevant limit on the information acquisition. Although such a mindset should generally apply to systems of any size, its quantum mechanical implications are particularly intriguing and, for this reason, we examine here a minimal physical structure where the system and the environment are described, respectively, by a pair of quantum oscillators coupled by an appropriate Hermitian interaction able to amplify the entropy of the initial state. Since at the onset of the dynamical evolution the system is originally in a pure state, its entropy variation is always positive semidefinite and the Landauer conjecture should not impose any constraint. Nonetheless, provided the quantum amplification is effective, it turns out that the entropy variation of the system always undershoots the heat transferred to the environment. When the initial thermal state of the environment is characterized by a chemical potential, the entropy growth is bounded both by the particles and by the heat flowing to the environment. The limits deduced in the quantum thermodynamical framework are also scrutinized from a field theory standpoint where species of different spins are copiously produced (especially in a cosmological context) thanks to the rapid variation of the space-time curvature.

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I. INTRODUCTION

According to the current wisdom, a fundamental energy (or heat) cost must always be associated with the information erasure and this perspective supports the view that information is unavoidably linked to an empirical representation [1]. There also exist complementary viewpoints suggesting that mathematical and geometrical structures should not depend upon the specific devices employed for their computational assessment [2]. Between these two interesting perspectives the former is more common than the latter when applying thermodynamics to information theory, or vice versa. In particular, the so-called Landauer conjecture [3–7] would suggest that the irreversible deletion of one bit of information demands an energy cost larger than (or equal to) $\kappa_B T_e \ln 2$ where κ_B is the Boltzmann constant and T_e is the temperature of the environment. This formulation implicitly propounds that there should not be any restriction connected to the acquisition of information; in other words the energy cost

associated with the information erasure is unavoidable but the information acquisition is not constrained, at least within the purely classical perspective where, among other things, the Landauer conjecture provides a possible set of solutions to the Maxwell paradox [3,4] (see also [8] for a review). In more quantitative terms, if $\Delta Q_e \geq 0$ is the amount of heat flowing to the environment and ΔS_s is the variation of the entropy of the system the Landauer conjecture stipulates that

$$\Delta Q_e \geq -\kappa_B T_e \Delta S_s. \quad (1.1)$$

When information is erased from the system this means that $\Delta S_s < 0$ and Eq. (1.1) demands a specific energy cost; for instance, to erase one bit of information $\Delta S_s = -\ln 2$ and Eq. (1.1) would imply that the heat flowing to the environment should be, at least, comparable with $\kappa_B T_e \ln 2$.

The bound of Eq. (1.1) holds in spite of the size of the underlying physical structures and it should then apply, in particular, to *any* quantum system. Furthermore, Eq. (1.1) is effectively constraining only when $\Delta S_s < 0$ but it should not imply any limit when $\Delta S_s > 0$: in this case Eq. (1.1) is obviously verified since a positive semidefinite increment must always exceed a negative contribution. In this paper, following a recent suggestion [9], we feel that it is both necessary and interesting to discuss at length one of the simplest quantum mechanical playgrounds where the

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potential bounds on the entropy growth could be quantitatively scrutinized. The system is initially in a pure state given by a quantum oscillator in the vacuum while the environment is instead constituted by a further oscillator in a mixed state whose density operator is characterized by Bose-Einstein weights. The Hermitian interactions between the system and the environment follow from the requirement that the von Neumann entropy of the initial state is amplified and this may occur, as we shall demonstrate, in the quantum mechanical description of parametric amplification [10–12] (see also [13,14]). After revisiting the standard tenets of parametric amplification within a quantum thermodynamical perspective, it will be shown that the variation of the entropy of the system is always positive (i.e. $\Delta S_s \geq 0$) but it is nonetheless constrained by the heat transfer according to the following bound:

$$\kappa_B T_e \Delta S_s \leq \Delta Q_e. \quad (1.2)$$

From the quantum mechanical viewpoint both Eqs. (1.1) and (1.2) hold under the assumption that the environment is initially in a thermal state with temperature T_e while the system is in a pure state (e.g. the vacuum). We shall additionally argue that Eq. (1.2) can be generalized to the field theoretical situation where particles are produced because of the variation of the space-time curvature [15,16] in the cases of curvature phonons [17,18] and of gravitons [19–22] (see also [23–27]).

Before plunging into the details of the discussion we stress that, although in Eqs. (1.1) and (1.2) the Boltzmann constant κ_B has been included, we are going to adopt hereunder the natural system of units where $\hbar = c = \kappa_B = 1$. In this system Newton’s constant G , the Planck length ℓ_P and the Planck mass M_P are then related as $\ell_P = \sqrt{8\pi}/M_P = 8\pi G$. In the remaining three subsections of this Introduction we shall present, step by step, the classical viewpoint leading to Eq. (1.1), the basic tenets of the quantum thermodynamical approach and the overall logic of this investigation.

A. The classical perspective

In classical information theory the Shannon entropy measures the lack of specific knowledge one has about a given system [28–30]. If we suppose the system is given by a particle that can be in two different energy levels (e.g. + and –) with equal probability [i.e. $p_+^{(\text{in})} = p_-^{(\text{in})} = 1/2$] the Shannon entropy associated with this initial situation is

$$S_s^{(\text{in})} = -p_+^{(\text{in})} \ln p_+^{(\text{in})} - p_-^{(\text{in})} \ln p_-^{(\text{in})} = \ln 2. \quad (1.3)$$

Let us now imagine that the final state of the system always excludes the presence of the particle in the lower state so that $S_s^{(\text{fin})} = 0$ since $p_+^{(\text{fin})} = 1$ and $p_-^{(\text{fin})} = 0$. In this process the information of the initial state has been erased

since any potential knowledge about the initial state of the particle is forever obliterated. Moreover, as expected, the variation of the Shannon entropy is negative

$$\Delta S_s = S_s^{(\text{fin})} - S_s^{(\text{in})} = -\ln 2 < 0, \quad (1.4)$$

and the Landauer conjecture of Eq. (1.1) would then demand that for the deletion of this one bit of information the minimal energy cost is given by¹ $\Delta Q_e \geq T \ln 2$. Generally speaking, if a certain state p_n has probability 1 this means that the “surprise” tends to zero where the surprise, in classical information theory, is simply the natural logarithm of $1/p_n$ [30], i.e. $-\ln p_n$. This also means that, after erasing the information, the surprise and the entropy decrease since the Shannon entropy is, in practice, the average surprise of a given state. Conversely, more surprise implies instead an increase of the information and of the entropy. Since the removal of information requires a decrease of the entropy of the system, when $\Delta S_s > 0$ the condition imposed by Eq. (1.1) does not seem restrictive: a physical quantity which is positive semi-definite (i.e. $\Delta Q_e \geq 0$) always exceeds a negative contribution (i.e. $-T\Delta S_e$). Once more the deletion of information costs energy whereas its acquisition remains unconstrained, at least within the classical perspective. Even if the potential saturation of the bound given by Eq. (1.1) is under debate [31], the classical logic behind the Landauer conjecture has been experimentally verified in a number of different frameworks [32,33] and will not be questioned hereunder.

B. The quantum mechanical perspective

Since Eq. (1.1) applies in spite of the physical dimensions of the underlying structures, it should be particularly relevant for any quantum system [8] where the Landauer conjecture is customarily formulated by introducing the Hilbert spaces of the system and of the environment [34]. Both the “system” and the “environment” are described by the corresponding Hamiltonian operators that we shall denote, respectively, by $\hat{H}_s$ and $\hat{H}_e$. The interaction Hamiltonian (denoted by $\hat{H}_{se}$) is usually expected to be Hermitian so that the final and the initial density matrices are in fact related by the unitary operator $\hat{\mathcal{U}}(t_{\text{in}}, t_{\text{fin}})$ accounting for the global time evolution of the system and of the environment,

$$\hat{\rho}(t_{\text{fin}}) = \hat{\mathcal{U}}(t_{\text{in}}, t_{\text{fin}}) \hat{\rho}(t_{\text{in}}) \hat{\mathcal{U}}^\dagger(t_{\text{in}}, t_{\text{fin}}), \quad (1.5)$$

where $\hat{\rho}(t)$ denotes the total density matrix encompassing the system and the environment. Occasionally $\hat{\rho}(t)$ is called

¹To simplify the notation and to avoid the proliferation of indices we shall always denote by T the temperature of the environment (i.e. $T_e = T$); since there is no other temperature appearing in the discussion this notation is unambiguous.

the proper density operator [35] as opposed to the improper one (obtained by tracing the total density matrix either over the degrees of freedom of the system or over the ones of the environment). Bearing in mind this standard terminology, the quantum mechanical formulation of the Landauer conjecture stipulates that the system and the environment are (i) initially uncorrelated and that (ii) the environment is in a thermal state [34]. The absence of correlations at t_{in} demands that $\hat{\rho}(t_{\text{in}}) = \hat{\rho}_s(t_{\text{in}}) \otimes \hat{\rho}_e(t_{\text{in}})$; thus Eq. (1.5) can also be written as

$$\hat{\rho}(t_{\text{fin}}) = \hat{\mathcal{U}}(t_{\text{in}}, t_{\text{fin}}) \hat{\rho}_s(t_{\text{in}}) \otimes \hat{\rho}_e(t_{\text{in}}) \hat{\mathcal{U}}^\dagger(t_{\text{in}}, t_{\text{fin}}). \quad (1.6)$$

The lack of initial correlations between the system and the environment does not imply that the two shall remain uncorrelated later on. On the contrary we are specifically interested in the situation where the entropy of the system increases between the initial and the final stages of the dynamical evolution. At any generic time t the reduced (or improper) density matrices are obtained by tracing $\hat{\rho}(t)$ with respect to the degrees of freedom either of the system or of the environment. This means, in particular, that at t_{fin}

$$\hat{\rho}_s(t_{\text{fin}}) = \text{Tr}_e[\hat{\rho}(t_{\text{fin}})], \quad \hat{\rho}_e(t_{\text{fin}}) = \text{Tr}_s[\hat{\rho}(t_{\text{fin}})], \quad (1.7)$$

where Tr_e and Tr_s indicate, respectively, the traces over the degrees of freedom of the environment and of the system.

The corresponding von Neumann entropies and the associated heat transfer follow from the reduced density matrices of Eq. (1.7); for instance the von Neumann entropy of the system is given by

$$S_s(t_{\text{fin}}) = -\text{Tr}[\hat{\rho}_s(t_{\text{fin}}) \ln \hat{\rho}_s(t_{\text{fin}})], \quad (1.8)$$

where the trace is now performed directly over the dynamical degrees of freedom of the system. Similarly the expectation value of the heat transferred to the environment will be

$$Q_e(t_{\text{fin}}) = \text{Tr}[\hat{H}_e(t_{\text{fin}}) \hat{\rho}_e(t_{\text{fin}})]. \quad (1.9)$$

In terms of Eqs. (1.8) and (1.9) the variation of the von Neumann entropy of the system and the heat flowing to the environment are given, respectively, by

$$\Delta S_s = S_s[\hat{\rho}_s(t_{\text{fin}})] - S_s[\hat{\rho}_s(t_{\text{in}})], \quad (1.10)$$

$$\Delta Q_e = Q_e[\hat{\rho}_e(t_{\text{fin}})] - Q_e[\hat{\rho}_e(t_{\text{in}})]. \quad (1.11)$$

The heat transferred to the environment following from Eq. (1.11) must coincide with the expectation values of the Hamiltonian of the environment (evaluated in the

Schrödinger description) between the final and the initial states of the evolution,

$$\Delta Q_e = \langle \text{fin} | \hat{H}_e(t_{\text{in}}) | \text{fin} \rangle - \langle \text{in} | \hat{H}_e(t_{\text{in}}) | \text{in} \rangle, \quad (1.12)$$

where now the Hamiltonian is evaluated at t_{in} . It can finally happen, in the present context, that the initial state allows for the presence of a chemical potential. In this situation we may expect that also the particles flow to the environment and the corresponding increment, in analogy with Eq. (1.12), is

$$\Delta N_e = \langle \text{fin} | \hat{N}_e(t_{\text{in}}) | \text{fin} \rangle - \langle \text{in} | \hat{N}_e(t_{\text{in}}) | \text{in} \rangle, \quad (1.13)$$

where $\hat{N}_e$ is now the averaged multiplicity of the environment. The inclusion of ΔN_e is not usually considered in the quantum formulation of the Landauer conjecture but it might be relevant in our present discussion, as we are going to argue when discussing the nature of the entropy bounds.

C. Quantum parametric amplification

From the quantum mechanical viewpoint the Landauer conjecture becomes then a restriction on the mutual relation between the energy increment of the environment and the variation of the von Neumann entropy of the system. The explicit form of this restriction is exactly given by Eq. (1.1) where however ΔQ_e and ΔS_s are now computed from the expectation values of the Hamiltonian and from the von Neumann entropy given, respectively, by Eqs. (1.10) and (1.11).

We point out in this investigation that quantum mechanics does not necessarily dictate that the growth of the entropy of the system (i.e. $\Delta S_s > 0$) must remain completely unconstrained. On the contrary the purpose of this analysis is to show that the variation of the entropy of the system may well be bounded by the heat transfer as suggested by Eq. (1.2). For the actual derivation of the bound expressed by Eq. (1.2) we shall be examining here the minimal situation where the system and the environment correspond, respectively, to a pair of quantum oscillators with frequencies ω_s and ω_e

$$\hat{H}_s = \omega_s(\hat{s}^\dagger \hat{s} + 1/2) = \omega_s(\hat{N}_s + 1/2), \quad (1.14)$$

$$\hat{H}_e = \omega_e(\hat{e}^\dagger \hat{e} + 1/2) = \omega_e(\hat{N}_e + 1/2), \quad (1.15)$$

where $[\hat{s}, \hat{e}] = 0$ and similarly for the associated number operators $\hat{N}_s$ and $\hat{N}_e$, i.e. $[\hat{N}_s, \hat{N}_e] = 0$. The presence of the zero-point energy in the Hamiltonians is not essential but we shall anyway stick to the expressions of Eqs. (1.14) and (1.15).

To avoid spurious effects that might influence the conclusions, the interplay between the system and the

environment should satisfy the following three plausible requirements: (i) the mutual interactions must be Hermitian to avoid violations of the unitary evolution; (ii) they must lead to a positive heat transfer to the environment (i.e. $\Delta Q_e \geq 0$); and (iii) they must also yield a positive entropy variation of the system (i.e. $\Delta S_s \geq 0$). In case these three conditions are verified it remains to be understood if ΔS_s is limited [as suggested by Eq. (1.2)] or if it is unbounded, as implied by the classical situation and by the conventional form of the Landauer conjecture. As we shall see, the conditions spelled out above are potentially verified in the context of the quantum theory of parametric amplification [10–14]. However, although the creation and the annihilation operators of Eqs. (1.14) and (1.15) may be connected to the signal and idler modes, the quantum thermodynamical analysis pursued here suggests a different interpretation that is dictated by the correlation properties of the initial density operators.

After these general premises the layout of this investigation is, in short, the following. In Sec. II, the coupled quantum evolutions of system and of the environment are discussed in the Heisenberg description when the free Hamiltonians are represented by Eqs. (1.14) and (1.15) and with particular attention to the symmetries of the problem. In Sec. III, we focus on the specific form of the reduced density operators and connect their properties with the irreducible representations of the $SU(1, 1)$ group. The heat flow and the entropy are directly computed and analyzed in Sec. IV, while Sec. V is devoted to the scrutiny of the physical bounds connecting the variation of the entropy with the heat flow. At the end of Sec. V, we also comment on the possible presence of a chemical potential. In Sec. VI, motivated by the cosmological evolution, we examine the quantum mechanical considerations leading to the bound of Eq. (1.2) from the standpoint of field theory, with particular attention to the production of spin 0 and spin 2 particles in curved background geometries. Section VII contains our concluding remarks.

II. THE SYSTEM AND THE ENVIRONMENT

A. The interaction Hamiltonian

The requirements formulated at the end of Sec. I imply that the interaction Hamiltonian must be Hermitian to avoid nonunitary effects coming from the coupled evolution of the system and of the environment; moreover the explicit quantum mechanical forms of ΔS_s and ΔQ_e [see Eqs. (1.10)–(1.12)] is positive semidefinite provided $\hat{H}_{se}$ does not commute with the sum of the number operators of the system and of the environment. A general form of $\hat{H}_{se}$ satisfying these three independent physical conditions turns out to be [10–12] (see also [13,14])

$$\hat{H}_{se} = g(t) \hat{s}^\dagger \hat{e}^\dagger e^{-i\omega t} + g^*(t) \hat{e} \hat{s} e^{i\omega t}, \quad (2.1)$$

where $g(t) = q(t)e^{i\theta_n}$ and $\omega = \omega_s + \omega_e$. The evolution of $\hat{s}$ and $\hat{e}$ follows from the total Hamiltonian obtained from the sum of Eqs. (1.14), (1.15), and (2.1), i.e.

$$\hat{H} = \hat{H}_s + \hat{H}_e + \hat{H}_{se}. \quad (2.2)$$

In the Heisenberg description the evolution of the operators is given by

$$\frac{d\hat{s}}{dt} = i[\hat{H}, \hat{s}] = -i\omega_s \hat{s} - i g e^{-i\omega t} \hat{e}^\dagger, \quad (2.3)$$

$$\frac{d\hat{e}}{dt} = i[\hat{H}, \hat{e}] = -i\omega_e \hat{e} - i g e^{-i\omega t} \hat{s}^\dagger. \quad (2.4)$$

The solution of Eqs. (2.3) and (2.4) can be expressed in terms of four complex functions, namely,

$$\hat{s}(t) = u_s(t) \hat{s} - v_s(t) \hat{e}^\dagger, \quad (2.5)$$

$$\hat{e}(t) = u_e(t) \hat{e} - v_e(t) \hat{s}^\dagger, \quad (2.6)$$

where, to avoid possible confusions, the following notations have been adopted:

$$\hat{s} = \hat{s}(t_{\text{in}}), \quad \hat{e} = \hat{e}(t_{\text{in}}), \quad (2.7)$$

and will be enforced throughout the whole discussion; the calligraphic operators $\hat{s}$ and $\hat{e}$ actually appear in the Schrödinger description that plays a relevant role in the determination of the asymptotic states (see, in this respect, Sec. III). We should also stress that, because of the unitarity of the evolution, the functions $[u_e(t), v_e(t)]$ and $[u_s(t), v_s(t)]$ must satisfy, at any stage of the dynamical evolution, the conditions $|u_s(t)|^2 - |v_s(t)|^2 = 1$ and $|u_e(t)|^2 - |v_e(t)|^2 = 1$.

B. Solutions of the coupled evolution

After inserting Eqs. (2.5) and (2.6) into Eqs. (2.3) and (2.4) the evolution of the four unknown functions becomes²

$$\begin{aligned} \frac{du_s}{dt} &= -i\omega_s u_s + i g(t) e^{-i\omega t} v_e^*, \\ \frac{dv_s}{dt} &= -i\omega_s v_s + i g(t) e^{-i\omega t} u_e^*, \end{aligned} \quad (2.8)$$

²In principle the four complex functions subjected to the conditions $|u_s(t)|^2 - |v_s(t)|^2 = 1$ and $|u_e(t)|^2 - |v_e(t)|^2 = 1$ should be equivalent to six real functions. However as a result of the dynamical evolution the system (2.8) and (2.9) depends on one real function and five real phases that can be reduced to 3 by fixing two of them.

$$\begin{aligned}\frac{du_e}{dt} &= -i\omega_e u_e + ig(t)e^{-i\omega t} v_s^*, \\ \frac{dv_e}{dt} &= -i\omega_e v_e + ig(t)e^{-i\omega t} u_s^*,\end{aligned}\quad (2.9)$$

and the solutions of Eqs. (2.8) and (2.9) are

$$\begin{aligned}u_s(t) &= e^{i(\vartheta_s - \omega_s t)} \cosh r, & v_s(t) &= e^{i(\bar{\vartheta}_s - \omega_s t)} \sinh r, \\ u_e(t) &= e^{i(\vartheta_e - \omega_e t)} \cosh r, & v_e(t) &= e^{i(\bar{\vartheta}_e - \omega_e t)} \sinh r,\end{aligned}$$

where $dr/dt = q(t)$ while

$$\bar{\vartheta}_e = \theta - \vartheta_s, \quad \bar{\vartheta}_s = \theta - \vartheta_e. \quad (2.10)$$

In Eq. (2.10) we introduced the new variable $\theta = (\theta_{\text{in}} + \pi/2)$ which is going to control the phases of the final solution; indeed, without loss of generality, we can always set $\vartheta_s = \vartheta_e = 0$ so that the evolution of the operators $\hat{s}(t)$ and $\hat{e}(t)$ can be ultimately expressed as

$$\hat{s}(t) = e^{-i\omega_s t} [\cosh r \hat{s} - \sinh r e^{i\theta} \hat{e}^\dagger], \quad (2.11)$$

$$\hat{e}(t) = e^{-i\omega_e t} [\cosh r \hat{e} - \sinh r e^{i\theta} \hat{s}^\dagger]. \quad (2.12)$$

When $\vartheta_s \neq 0$ and $\vartheta_e \neq 0$ the further contributions can always be absorbed in a redefinition of the overall phases appearing outside the square brackets in Eqs. (2.11) and (2.12). It is now convenient to introduce a pair of unitary operators [36–38] denoted hereunder by $\hat{\mathcal{H}}(\delta)$ and $\hat{\Sigma}(z)$:

$$\hat{\mathcal{H}}(\delta) = e^{-i\delta_s \hat{s}^\dagger \hat{s} - i\delta_e \hat{e}^\dagger \hat{e}}, \quad (2.13)$$

$$\hat{\Sigma}(z) = e^{z^* \hat{s} \hat{e} - z \hat{s}^\dagger \hat{e}^\dagger}, \quad (2.14)$$

where $z = r(t)e^{i\theta}$ while $\delta_e = \omega_e t$ and $\delta_s = \omega_s t$. In terms of Eqs. (2.13) and (2.14) the results of Eqs. (2.11) and (2.12) can be written as

$$\hat{s}(t) = \hat{\Sigma}^\dagger(z) \hat{\mathcal{H}}^\dagger(\delta) \hat{s} \hat{\mathcal{H}}(\delta) \hat{\Sigma}(z), \quad (2.15)$$

$$\hat{e}(t) = \hat{\Sigma}^\dagger(z) \hat{\mathcal{H}}^\dagger(\delta) \hat{e} \hat{\mathcal{H}}(\delta) \hat{\Sigma}(z). \quad (2.16)$$

The late-time density operator can then be expressed in terms of $\hat{\mathcal{H}}(\delta)$ and $\hat{\Sigma}(z)$,

$$\hat{\rho}(t, t_{\text{in}}) = \hat{\mathcal{H}}(\delta) \hat{\Sigma}(z) \hat{\rho}_{se}(t_{\text{in}}) \hat{\Sigma}^\dagger(z) \hat{\mathcal{H}}^\dagger(\delta). \quad (2.17)$$

C. Symmetries of the Hamiltonian

As anticipated at the beginning of this discussion, the *total* Hamiltonian of Eq. (2.2) commutes with the

difference between the number operators of the system and of the environment

$$\hat{\mathcal{Q}} = \hat{N}_e - \hat{N}_s, \quad [\hat{H}, \hat{\mathcal{Q}}] = 0, \quad (2.18)$$

but not with their sum, since

$$[\hat{H}, \hat{N}_e + \hat{N}_s] = 2\hat{H}_{se} \neq 0. \quad (2.19)$$

This observation implies that both the Hamiltonian and the $\hat{\mathcal{Q}}$ operators admit the same orthonormal and complete set of eigenfunctions so that they can be simultaneously diagonalized; as we shall see in Sec. III this is not the only basis to discuss the irreducible representations (see, in particular, [39]) but it is in fact close to the standard one where the role of the Hamiltonian is played by the Casimir operator of the underlying group. The second remark is that Eq. (2.18) justifies the terminology we shall be using in the forthcoming discussions since the parametric amplification (even in a purely quantum mechanical framework) is in fact equivalent to the production of quanta. This will also be true, *a fortiori*, in the second quantized perspective that will be further scrutinized in Sec. VI.

III. THE REDUCED DENSITY OPERATORS

A. The complete density operator

The late-time density matrices follow directly from Eq. (2.17) by requiring, as repeatedly stressed, that the density operators of the system and of the environment are initially uncorrelated, as explained prior to Eq. (1.6). Therefore the initial density matrix of Eq. (2.17) shall be expressed in the following manner:

$$\hat{\rho}_{se}(t_{\text{in}}) = \hat{\rho}_s(t_{\text{in}}) \otimes \hat{\rho}_e(t_{\text{in}}). \quad (3.1)$$

At the onset of the evolution the system is in the vacuum [i.e. $\hat{\rho}_s(t_{\text{in}}) = |0\rangle\langle 0|$] while the density matrix of the environment is a mixture of states with statistical weights provided by the Bose-Einstein (geometric) distribution,

$$\hat{\rho}_e(t_{\text{in}}) = \sum_{m=0}^{\infty} \bar{p}_m(\bar{n}) |m\rangle\langle m|, \quad (3.2)$$

where $\bar{n}$ is the averaged multiplicity of the initial state and $\bar{p}_m(\bar{n})$ corresponds to the Bose-Einstein distribution³

$$\bar{p}_m(\bar{n}) = \frac{\bar{n}^m}{(\bar{n} + 1)^{m+1}}, \quad \sum_{m=0}^{\infty} \bar{p}_m(\bar{n}) = 1. \quad (3.3)$$

³Since throughout the discussion various statistical weights will progressively appear, the probability distributions associated with the initial density operator of the environment will be supplemented by an overline (e.g. $\bar{p}_m$).

In case the environment is in local thermal equilibrium the simplest possibility is that $\bar{n} = (e^{\omega_e/T_e} - 1)^{-1}$. In the presence of a chemical potential the form of $\bar{p}_m$ remains the same in terms of the averaged multiplicity but $\bar{n}$ gets modified as $\bar{n} = (e^{(\omega_e - \mu_e)/T_e} - 1)^{-1}$. This is why, incidentally, it is preferable to express the statistical weights as in Eq. (3.3); this form is also practical for further generalizations (see e.g. [40] and discussions therein). The complete expression of the total density matrix encompassing the system and the environment can then be written as

$$\hat{\rho}(t, t_{\text{in}}) = \sum_{m=0}^{\infty} \sum_{\ell=0}^{\infty} \sum_{\ell'=0}^{\infty} \mathcal{A}_{m\ell\ell'} |\ell, \ell + m\rangle \langle m + \ell', \ell'|, \quad (3.4)$$

where $\mathcal{A}_{m\ell\ell'}$ is given by

$$\mathcal{A}_{m\ell\ell'} = P_{m\ell\ell'} \sqrt{\binom{m+\ell}{m} \binom{m+\ell'}{m}}, \quad (3.5)$$

and, besides the two binomial coefficients, $P_{m\ell\ell'}$ is

$$P_{m\ell\ell'} = \bar{p}_m(\bar{n}) \frac{e^{i\alpha(\ell-\ell')}}{(\bar{n}(q) + 1)^{m+1}} \left(\frac{\bar{n}(q)}{\bar{n}(q) + 1} \right)^{(\ell+\ell')/2}, \quad (3.6)$$

with $\alpha = (\theta + \pi - \delta_e - \delta_s)$. Equation (3.6) gives the complete form of the total density operator sometimes referred to as the “proper” density operator [35] to distinguish it from the “improper” ones possibly obtainable after tracing over some of the degrees of freedom belonging either to the system or to the environment. It is straightforward but rather lengthy to show from Eq. (3.4) that

$$\text{Tr}[\hat{\rho}(t, t_{\text{in}})] = 1, \quad \text{Tr}[\hat{\rho}^2(t, t_{\text{in}})] \neq 1, \quad (3.7)$$

as it must happen in the case of a mixed state. In particular it can be shown, after simple algebra, that $\text{Tr}[\hat{\rho}^2(t, t_{\text{in}})]$ is given by

$$\frac{[\bar{n}(q) + 1]^2}{[1 + (2 + \bar{n})\bar{n}(q)][1 + 2\bar{n}(q) + \bar{n}(2 + 3\bar{n}(q))]}, \quad (3.8)$$

implying that, in the limit $\bar{n}(q) \rightarrow 0$ (no parametric amplification) $\text{Tr}[\hat{\rho}^2(t, t_{\text{in}})] = 1/(2\bar{n} + 1)$ as it is expected in the case where the statistical weights of the density matrix appear in the Bose-Einstein form. Since the present analysis involves a number of reduced density operators it is wise to check their properties, step by step, after each reduction, as we shall be consistently doing in what follows.

B. The reduced density operators

From the proper density operator of Eq. (3.4) the reduced density matrices are obtained by tracing $\hat{\rho}(t, t_{\text{in}})$ either over

the degrees of freedom of the system or of the environment (see, e.g. [35]),

$$\hat{\rho}_s(t, t_{\text{in}}) = \text{Tr}_e[\hat{\rho}(t, t_{\text{in}})], \quad (3.9)$$

$$\hat{\rho}_e(t, t_{\text{in}}) = \text{Tr}_s[\hat{\rho}(t, t_{\text{in}})], \quad (3.10)$$

where, as already stressed, Tr_e and Tr_s denote, respectively, the traces over the degrees of freedom of the environment and of the system. The explicit forms of Eqs. (3.9) and (3.10) are

$$\hat{\rho}_e(t, t_{\text{in}}) = \sum_{\ell=0}^{\infty} \sum_{m=0}^{\infty} p_{m\ell}^{(e)} |\ell + m\rangle \langle m + \ell|, \quad (3.11)$$

$$\hat{\rho}_s(t, t_{\text{in}}) = \sum_{\ell=0}^{\infty} p_{\ell}^{(s)} |\ell\rangle \langle \ell|. \quad (3.12)$$

In Eqs. (3.11) and (3.12) the statistical weights connected to the environment and to the system are indicated, respectively, by $p_{m\ell}^{(e)}$ and by $p_{\ell}^{(s)}$ and they depend upon t and t_{in} . It is convenient to characterize the final state in terms of the averaged multiplicities of the produced quanta since, for the explicit evaluation ΔS_s and ΔQ_e , this is what ultimately counts. In the quantum thermodynamical approach followed here the statistical weights only depend on the averaged multiplicities of the initial state (i.e. $\bar{n}$) and of the produced species [i.e. $\bar{n}(q)$]. In particular the statistical weight associated with the system is

$$p_{\ell}^{(s)} = \frac{\bar{n}(q)^{\ell} (1 + \bar{n})^{\ell}}{[1 + \bar{n}(q)(\bar{n} + 1)]^{\ell+1}}. \quad (3.13)$$

It can be immediately checked from Eq. (3.13) that the traces of the reduced density operator are, as expected,

$$\sum_{\ell=0}^{\infty} p_{\ell}^{(s)} = 1, \quad \text{Tr} \hat{\rho}_s = 1, \quad \text{Tr} \hat{\rho}_s^2 \neq 1. \quad (3.14)$$

The statistical weights of the environment are instead given by

$$p_{m\ell}^{(e)} = \binom{m+\ell}{m} \frac{\bar{n}^m \bar{n}(q)^{\ell}}{(\bar{n} + 1)^{m+1} [\bar{n}(q) + 1]^{m+\ell+1}}, \quad (3.15)$$

and again we can verify from Eq. (3.15) that

$$\sum_{\ell=0}^{\infty} \sum_{m=0}^{\infty} p_{m\ell}^{(e)} = 1, \quad \text{Tr} \hat{\rho}_e = 1, \quad \text{Tr} \hat{\rho}_e^2 \neq 1. \quad (3.16)$$

From Eq. (3.15) we can easily compute the probability generating function $P^{(e)}(s, w)$ for the bivariate (discrete) distribution $p_{\ell m}^{(e)}$,

$$P^{(e)}(s, w) = \sum_{\ell=0}^{\infty} \sum_{m=0}^{\infty} s^m w^{\ell} p_{\ell m}^{(e)} = \frac{1}{[1 + (1-s)\bar{n} + (1-w)\bar{N}]}. \quad (3.17)$$

From Eq. (3.17) we can immediately appreciate that in the limit $w \rightarrow 1$ we recover the probability generating function of the Bose-Einstein distribution of the initial state (with averaged multiplicity $\bar{n}$). In the limit $s \rightarrow 1$ the distribution is still of Bose-Einstein type but with averaged multiplicity $\bar{N} = (1 + \bar{n})\bar{n}^{(q)}$. This explains why the variable $\bar{N}$ appears to be very convenient in the physical discussion.

C. Symmetries of the states

The results obtained above in this section can be swiftly deduced through a judicious use of the symmetries of the underlying quantum states employed in the explicit derivation of the density operators. To clarify this suggestion we first note that the free and the interacting Hamiltonians evaluated at the onset of the dynamical evolution can be in fact expressed through three operators that shall be conventionally denoted hereunder by $\hat{K}_{\pm}$ and $\hat{K}_0$,

$$\begin{aligned} \hat{K}_{+} &= \hat{\beta}^{\dagger} \hat{e}^{\dagger}, & \hat{K}_{-} &= \hat{\beta} \hat{e}, \\ \hat{K}_0 &= (\hat{\beta}^{\dagger} \hat{\beta} + \hat{e} \hat{e}^{\dagger})/2. \end{aligned} \quad (3.18)$$

The operators of Eq. (3.18) are quadratic in the creation and annihilation operators defined in Eq. (2.7) and satisfy the commutation relations of the $SU(1, 1)$ Lie algebra (see, for instance, [38] and references therein),

$$[\hat{K}_0, \hat{K}_{\pm}] = \pm \hat{K}_{\pm}, \quad [\hat{K}_{+}, \hat{K}_{-}] = -2\hat{K}_0. \quad (3.19)$$

Furthermore, in terms of $\hat{K}_{\pm}$ and $\hat{K}_0$ the Casimir operator becomes

$$\hat{\mathcal{C}} = \hat{K}_0^2 - (\hat{K}_{+}\hat{K}_{-} + \hat{K}_{-}\hat{K}_{+})/2. \quad (3.20)$$

Thanks to the commutation relations of Eq. (3.19) the expression of $\hat{\mathcal{C}}$ can also be written as

$$\hat{\mathcal{C}} = \hat{K}_0(\hat{K}_0 - 1) - \hat{K}_{+}\hat{K}_{-}. \quad (3.21)$$

Since, by definition, the Casimir operator commutes with all the generators of the group, a possible basis for the irreducible representations of $SU(1, 1)$ corresponds to the Bargmann choice [39] where $\hat{\mathcal{C}}$ and $\hat{K}_0$ are simultaneously diagonalized. In the context of the present problem a more

convenient basis for the irreducible representation of $SU(1, 1)$ is⁴

$$|m_s, m_e\rangle = \frac{(\hat{\beta}^{\dagger})^{m_s}}{\sqrt{m_s!}} \frac{(e^{\dagger})^{m_e}}{\sqrt{m_e!}} |0_s, 0_e\rangle. \quad (3.22)$$

The density matrix encompassing the system and the environment at t_{fin} can be ultimately related to the state

$$|\delta z\rangle = \hat{\mathcal{H}}(\delta) \hat{\Sigma}_{+}(z) \hat{\Sigma}_0(z) \hat{\Sigma}_{-}(z) |m_s, m_e\rangle, \quad (3.23)$$

defined in the Schrödinger description. For an explicit expressions of $|\delta z\rangle$ we can use the Baker-Campbell-Hausdorff decomposition for the $SU(1, 1)$ group [41] (see also [42–44]) implying that the operator $\hat{\Sigma}(z)$ is factorized as the product of the exponentials of the group generators, i.e.

$$\hat{\Sigma}(z) = \hat{\Sigma}_{+}(z) \hat{\Sigma}_0(z) \hat{\Sigma}_{-}(z), \quad (3.24)$$

where $\hat{\Sigma}_{\pm}(z)$ and $\hat{\Sigma}_0(z)$ are

$$\begin{aligned} \hat{\Sigma}_{\pm}(z) &= \exp[\mp e^{\pm i\theta} \tanh r \hat{K}_{\pm}], \\ \hat{\Sigma}_0(z) &= \exp[-2 \ln(\cosh r) \hat{K}_0]. \end{aligned} \quad (3.25)$$

Thanks to Eqs. (3.24) and (3.25) we can then obtain from Eq. (3.23),

$$\begin{aligned} |\delta z\rangle &= \sum_{j=0}^{j_{\text{max}}} \sum_{\ell=0}^{\infty} \frac{e^{i\gamma(j, \ell)} (\cosh r)^{2j} (\tanh r)^{j+\ell}}{(\cosh r)^{m_s+m_e+1}} \\ &\times \mathcal{M}(j, \ell) |m_s - \ell + j, m_e - \ell + j\rangle, \end{aligned} \quad (3.26)$$

where $j_{\text{max}} = \text{Min}[m_s, m_e]$ while $\gamma(j, \ell) = [(\delta_s + \delta_e - \theta)(j - \ell) + \pi\ell - (\delta_s m_s + \delta_e m_e)]$. The explicit expression of $\mathcal{M}(j, \ell)$ is finally given by

$$\mathcal{M}(j, \ell) = \frac{\sqrt{m_s! m_e!} \sqrt{(m_s - j + \ell)! (m_e - j + \ell)!}}{j! \ell! (m_s - j)! (m_e - j)!}. \quad (3.27)$$

Thanks to Eqs. (3.26) and (3.27) the relevant initial states of the system and of the environment can be constructed and evolved in time. As repeatedly stressed in the present and in the previous sections, we are chiefly interested in the case where the system is in the vacuum while the environment is in a mixed state. In the language of Eqs. (3.26) and (3.27) the system is initially in the vacuum provided $m_s = 0$; this

⁴The connection between the Bargmann basis and the one of Eq. (3.22) is given by $m_e + m_s = (2\bar{m} - 1)$ and $m_e - m_s = (2k - 1)$. In the literature $\bar{m}$ is often denoted by m ; the notation $\bar{m}$ is preferred here to avoid potential confusions with various summation indices that may appear throughout the discussions.

means that, by definition, also $j_{\max} \rightarrow 0$ and the sum appearing in Eq. (3.26) reduces to a single contribution, i.e. $j = 0$. In practice the final state will be given by Eqs. (3.26) and (3.27) evaluated for $m_s = j = 0$ and the result becomes

$$|\Psi_{m_e}(t, t_{\text{in}})\rangle = \sum_{\ell=0}^{\infty} \frac{(\tanh r)^\ell e^{i(\alpha\ell - \delta m_e)}}{(\cosh r)^{m_e+1}} \times \sqrt{\frac{(m_e + \ell)!}{m_e! \ell!}} |\ell, m_e + \ell\rangle. \quad (3.28)$$

If Eq. (3.24) is directly applied to the initial state $|0_s m_e\rangle = |0_s\rangle |m_e\rangle$ we obtain

$$|\Psi_{m_e}(t, t_{\text{in}})\rangle = \hat{\mathcal{R}}(\delta) \hat{\Sigma}_+(z) \hat{\Sigma}_0(z) |0_s m_e\rangle, \quad (3.29)$$

which coincides with the result of Eq. (3.28) deduced from Eq. (3.27) in the limit $m_s \rightarrow 0$ and $j \rightarrow 0$. According to the previous considerations [see, in particular, Eqs. (1.6) and (3.3)] from the initial thermal state for the environment the proper density operator becomes

$$\hat{\rho}(t, t_{\text{in}}) = \sum_{m=0}^{\infty} \bar{p}_m |\Psi_m(t, t_{\text{in}})\rangle \langle \Psi_m(t, t_{\text{in}})|, \quad (3.30)$$

where, as before, $\bar{p}_m = p_m^{(\text{in})}$ is the Bose-Einstein (geometric) distribution associated with the initial density matrix of the thermal environment; note that, for simplicity, we redefined the eigenvalue as $m_e \rightarrow m$. It also follows from Eqs. (3.28)–(3.30) that the total density matrix coincides with the result of Eq. (3.4).

As we close this section, it is appropriate to comment on the representations of $SU(1, 1)$ by means of operators bilinear in the boson creation and annihilation operators $\hat{a}^\dagger$ and $\hat{a}$ satisfying canonical commutation relations $[\hat{a}, \hat{a}^\dagger] = 1$. This realization is possible because the $SU(1, 1)$ and the $Sp(2, R)$ groups are isomorphic⁵ [38]. This observation is at the heart of the first applications of two-photon coherent states in quantum optics [45–47]. There are however various physical differences between the problem analyzed here and the analog application possibly based on two-photon optics. The two-mode structure is actually essential to analyze the interaction between the system and the environment from a quantum thermodynamical viewpoint.

⁵The operators $L_+ = \hat{a}^{\dagger 2}/2$, $L_- = \hat{a}^2/2$ and $L_0 = (\hat{a}^\dagger \hat{a} + \hat{a} \hat{a}^\dagger)/4$ actually close the $SU(1, 1)$ algebra. In this case the eigenvalues of the Casimir operator correspond either to $k = 1/4$ or to $k = 3/4$. The basis of the unitary representation is, in this case, $|n\rangle = (n!)^{-1/2} (\hat{a}^\dagger)^n |0\rangle$; for n even the unitary representation corresponds to $k = 1/4$ while for n odd the unitary representation corresponds to $3/4$.

IV. THE HEAT FLOW AND THE ENTROPY

A. The heat transferred to the environment

The heat transferred to the environment and the increment of the entropy of the system shall now be carefully evaluated since both results are essential for the derivation of the entropy bounds that are quantitatively examined in Sec. V. Recalling Eqs. (1.10) and (1.11) the heat flowing to the environment is deduced from the variation of the Hamiltonian between the final and the initial states. In the Heisenberg description ΔQ_e is

$$\Delta Q_e = \langle \text{th} | \hat{H}_e(t_{\text{fin}}) | \text{th} \rangle - \langle \text{th} | \hat{H}_e(t_{\text{in}}) | \text{th} \rangle, \quad (4.1)$$

where $|\text{th}\rangle$ indicates the initial thermal state of the environment characterized by the averaged thermal multiplicity $\bar{n}$. The results obtained from Eq. (4.1) must also follow from the density matrix $\hat{\rho}_e$ of Eq. (3.10) coming from the trace over the degrees of freedom of the system [see also Eq. (1.11)] and discussion therein,

$$\Delta Q_e = \text{Tr}\{\hat{H}_e[\hat{\rho}_e(t_{\text{fin}}) - \hat{\rho}_e(t_{\text{in}})]\}, \quad (4.2)$$

where now $\hat{H}_e = \hat{H}_e(t_{\text{in}})$ is evaluated in the Schrödinger description. For the consistency of the whole approach it is interesting to check both derivations; in particular, if we start from Eq. (4.1), we obtain

$$\Delta Q_e = \omega_e \langle \text{th} | \hat{N}_e(t_{\text{fin}}) | \text{th} \rangle - \omega_e \langle \text{th} | \hat{N}_e(t_{\text{in}}) | \text{th} \rangle. \quad (4.3)$$

Since the explicit form of $\hat{N}_e(t_{\text{fin}}) = \hat{e}^\dagger(t_{\text{fin}}) \hat{e}(t_{\text{fin}})$ can be computed with the help of Eqs. (2.12), (4.3) becomes

$$\begin{aligned} \Delta Q_e &= \omega_e (\bar{n} \cosh^2 r + \sinh^2 r - \bar{n}) \\ &= \omega_e \bar{n}^{(q)} (\bar{n} + 1). \end{aligned} \quad (4.4)$$

Equation (4.4) confirms the strategy already adopted in Eqs. (3.11) and (3.12) [see also Eq. (3.17)] for the quantitative discussion of the forthcoming results. Indeed the averaged multiplicity of the created quanta [i.e. $\bar{n}^{(q)} = \sinh^2 r$] together with the thermal multiplicity $\bar{n}$ can be used, in their various combinations, as the pivotal variables of the problem. We also point out that from now on r indicates, for the sake of simplicity, the value of $q(t)$ integrated between t_{in} and t_{fin} ,

$$r = r(t_{\text{fin}}, t_{\text{in}}) = \int_{t_{\text{in}}}^{t_{\text{fin}}} q(t) dt. \quad (4.5)$$

Equations (4.4) and (4.5) clarify once more why, in Eqs. (3.13) and (3.15), we found useful to express the statistical weights directly in terms of $\bar{n}$ and $\bar{n}^{(q)}$: $\bar{n}$ defines the onset of the evolution whereas $\bar{n}^{(q)}$ characterizes the final asymptotic stages of the dynamics. Equation (4.4)

coincides with the results obtained from the reduced density matrix; more specifically along this perspective we have⁶

$$\text{Tr}[\hat{H}_e \hat{\rho}_e(t_{\text{in}})] = \omega_e [\bar{n} + 1/2], \quad (4.6)$$

$$\text{Tr}[\hat{H}_e \hat{\rho}_e(t_{\text{fin}})] = \omega_e [1/2 + \bar{n} + \bar{n}^{(q)}(\bar{n} + 1)], \quad (4.7)$$

where Eq. (4.6) is immediately obvious while Eq. (4.7) is a direct consequence of Eqs. (3.11) and (3.15). If we now subtract Eq. (4.6) from Eq. (4.7) we obtain, again, the result of Eq. (4.4). This is because, in any case, the contribution of the ground state cancels when deriving ΔQ_e ; the factor $1/2$ appearing in Eqs. (1.14) and (1.15) is therefore not essential for the result of Eq. (4.4) so that we could have defined the original free Hamiltonians by renormalizing, as occasionally done, the zero-point energy.

B. The variation of the entropy of the system

From Eq. (1.10) we can now compute the variation of the entropy of the system, namely

$$\Delta S_s = S[\hat{\rho}_s(t_{\text{fin}})] - S[\hat{\rho}_s(t_{\text{in}})], \quad (4.8)$$

where, as usual, $S[\hat{\rho}] = -\text{Tr}[\hat{\rho} \ln \hat{\rho}]$ is the von Neumann entropy which is the one customarily employed in the quantum mechanical derivation of the Landauer bound. From Eq. (4.8) we can immediately note that $S[\hat{\rho}_s^{(\text{in})}] = 0$ since the initial (pure) state of the system coincides with the vacuum. It also follows from Eqs. (3.12) and (3.13) that Eq. (4.8) can be rewritten as

$$\begin{aligned} \Delta S_s &= - \sum_{\ell=0}^{\infty} p_{\ell}^{(s)} \ln p_{\ell}^{(s)} \\ &= (\bar{N} + 1) \ln(\bar{N} + 1) - \bar{N} \ln \bar{N}, \end{aligned} \quad (4.9)$$

where $p_{\ell}^{(s)}$ has been deduced in Eq. (3.13). It is convenient to express Eq. (4.9) directly in terms of the global variable

$$\bar{N} = \bar{n}^{(q)}(\bar{n} + 1) = (\bar{n} + 1) \sinh^2 r. \quad (4.10)$$

In the limit $r \rightarrow 0$ we have $\bar{N} \rightarrow 0$ and $\Delta S_s \rightarrow 0$ so that the entropy of the system does not change in this case since the averaged multiplicity of the produced quanta vanishes (i.e. $\bar{n}^{(q)} \rightarrow 0$); recalling Eq. (4.5) this happens, in practice,

⁶The result of Eq. (4.6) simply expresses the general truism that quantum fluctuations are relevant at low temperatures since the expectation value of the Hamiltonian $\hat{H}_e$ over a thermal state characterized by a Bose-Einstein probability distribution is exactly $\langle \text{th} | \hat{H}_e | \text{th} \rangle = \omega_e (\bar{n} + 1/2)$. Furthermore, given that $\bar{n} = (e^{\omega_e/T} - 1)^{-1}$, $\langle \text{th} | \hat{H}_e | \text{th} \rangle$ goes as T for $\omega_e \ll T$ while, in the opposite limit (i.e. $\omega_e \gg T$), $\langle \text{th} | \hat{H}_e | \text{th} \rangle$ coincides in practice with the ground state energy.

when $t_{\text{fin}} \rightarrow t_{\text{in}}$ and the system remains in its initial stage of evolution without appreciable interaction with the environment. Therefore the notation of Eq. (4.10) is physically meaningful since $\bar{N}$ ultimately accounts for the total averaged multiplicity of the produced quanta.

V. THE PHYSICAL BOUNDS

A. The entropy variation and the heat flow

To clarify the nature of the bounds derived in this investigation the first useful step is to compute the ratio between the increment of the entropy of the system (i.e. $T\Delta S_s$) and the heat transferred to the environment (i.e. ΔQ_e). After constructing the ratio $T\Delta S_s/\Delta Q_e$ we shall eventually discuss if it is generically smaller than 1, as suggested in Eq. (1.2). Bearing in mind the stenographic notation spelled out in Eq. (4.10), the wanted expression is immediately obtained from Eqs. (4.4) and (4.9); the final result solely depends upon $\bar{N}$ and upon (T/ω_e) ,

$$T \frac{\Delta S_s}{\Delta Q_e} = \left(\frac{T}{\omega_e} \right) \left[\frac{\ln \bar{N}}{\bar{N}} + \left(1 + \frac{1}{\bar{N}} \right) \ln \left(1 + \frac{1}{\bar{N}} \right) \right]. \quad (5.1)$$

A simple algebraic inspection of Eq. (5.1) suggests that $T\Delta S_s < \Delta Q_e$ provided the conditions of the parametric amplification are met and this means, in particular, that $\bar{N} > 1$. The shaded areas in Fig. 1 and in the forthcoming plots correspond to the regions where $T\Delta S_s \leq \Delta Q_e$. According to the results of Fig. 1 the entropy growth connected to $\Delta S_s \geq 0$ is indeed bounded by ΔQ_e , as long as the quantum parametric amplification is operational (i.e. $\bar{N} > 1$).

A complementary perspective might however suggest that since $\bar{N} = \bar{n}^{(q)}(\bar{n} + 1)$ the condition $\bar{N} > 1$ is realized when $\bar{n}^{(q)} > 1$ while the averaged thermal multiplicity $\bar{n}$ could be both larger and smaller than 1. Therefore, to avoid confusions and to clarify the matter even further we are going to illustrate the right-hand side of Eq. (5.1) not only in the plane $(\bar{N}, \omega_e/T)$ (as already done in Fig. 1) but also in terms of the other variables that already appeared in the previous considerations of Secs. III and IV. The second point we wish to examine is what happens when $\bar{N} < 1$. In this regime the bounds derived here are likely to be violated simply because the system did not evolve from the initial vacuum state. In the following two subsections we shall separately examine the regimes $\bar{N} > 1$ and $\bar{N} < 1$.

B. The regime $\bar{N} > 1$

When $\bar{N} > 1$ the right-hand side of Eq. (5.1) can be expanded in powers of $1/\bar{N}$ and if only the leading terms of the expansion in $(1/\bar{N})$ are kept, Eq. (5.1) becomes

$$T \frac{\Delta S_s}{\Delta Q_e} = \frac{1 + \ln \bar{N}}{\bar{N} \ln(1 + 1/\bar{n})}, \quad (5.2)$$

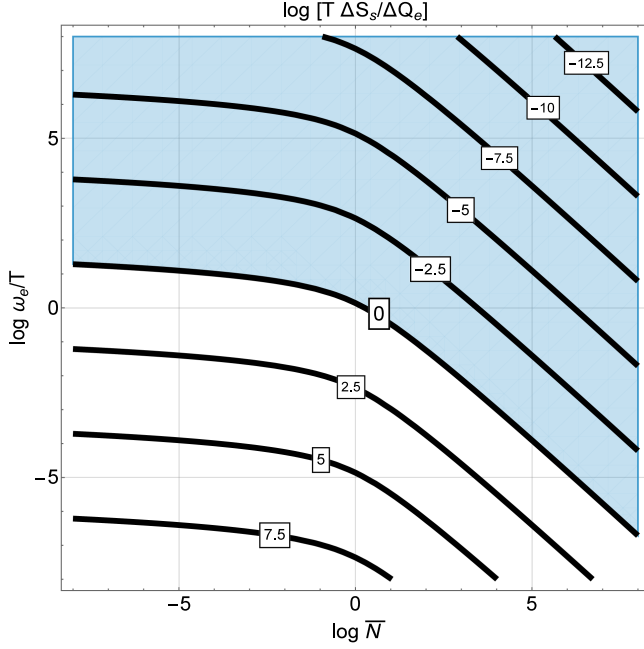


FIG. 1. In this and in the following plots of this section the shaded areas illustrate the regions where $T\Delta S_s < \Delta Q_e$. Common logarithms are employed on both axes. The labels are constant along each contour and correspond to the common logarithms of $T\Delta S_s/\Delta Q_e$ computed from the right-hand side of Eq. (5.1). In this plot the bounds stemming from Eq. (5.1) are studied in the plane $(\bar{N}, \omega_e/T)$. The white areas where the bound is not satisfied correspond to the region of large thermal multiplicities and comparatively negligible quantum amplification [i.e. $\bar{n}^{(q)} = \mathcal{O}(1)$ or smaller].

where T/ω_e has been traded for the averaged thermal multiplicity $\bar{n}$. As long as $\bar{n} \gg 1$ the logarithm in the denominator can be further expanded in the limit $(1/\bar{n}) < 1$. Moreover, in the same limit (i.e. $\bar{n} \gg 1$) Eq. (4.10) implies that $\bar{N} = \bar{n}^{(q)}\bar{n}$. Putting together the two previous observations, the following expression is finally deduced from Eq. (5.2):

$$T \frac{\Delta S_s}{\Delta Q_e} = \frac{1 + \ln(\bar{n}^{(q)}\bar{n})}{\bar{n}^{(q)}}. \quad (5.3)$$

The result of Eq. (5.3) implies that $T\Delta S_s < \Delta Q_e$ provided $\bar{n}$ and $\bar{n}^{(q)}$ satisfy the approximate inequality $\bar{n} < e^{\bar{n}^{(q)}-1}/\bar{n}^{(q)}$. Given that the amplification is operational only when the quanta are effectively produced [i.e. $\bar{n}^{(q)} > 1$], the obtained condition gets even simpler, i.e. $\bar{n}\bar{n}^{(q)} < e^{\bar{n}^{(q)}}$. Consequently, when $\bar{n}^{(q)} \gg 1$ the condition on $\bar{n}$ is always verified, in practice, due to the largeness of the exponential factor. Once more, from Fig. 2 we can appreciate that the white region for $\bar{n} > 1$ corresponds to the values of $\bar{n}^{(q)}$ that remain sufficiently small in agreement with the condition $\bar{n}\bar{n}^{(q)} < e^{\bar{n}^{(q)}}$.

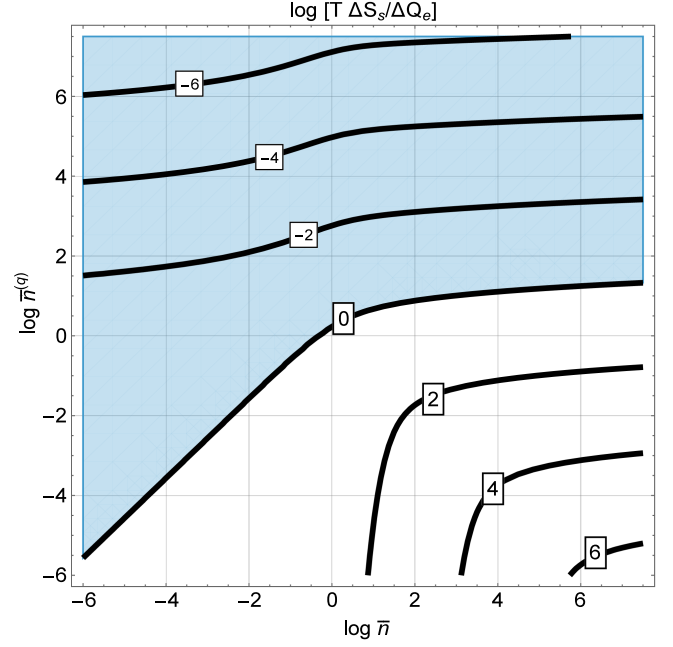


FIG. 2. As in Fig. 1 the labels appearing on the various contours correspond to the common logarithms of $T\Delta S_s/\Delta Q_e$. The bound $T\Delta S_s < \Delta Q_e$ is here scrutinized in the plane defined by the averaged multiplicities of the initial thermal state (i.e. $\bar{n}$) and of the produced quanta [i.e. $\bar{n}^{(q)}$]. The bound is always verified, in practice, when $\bar{n}^{(q)} > 1$ and in spite of the value of $\bar{n}$ except for the region of small multiplicities where the limits derived from Eq. (5.3) are effective [i.e. $\bar{n}\bar{n}^{(q)} < e^{\bar{n}^{(q)}}$].

Always under the hypothesis that $\bar{N} > 1$, we finally examine the case where $\bar{n} \ll 1$. This means that, in practice, the total averaged multiplicity is entirely generated quantum mechanically, i.e. $\bar{N} \simeq \bar{n}^{(q)}$. When $\bar{n} \ll 1$ we also have that $T/\omega_e \ll 1$ and Eq. (5.2) becomes

$$T \frac{\Delta S_s}{\Delta Q_e} = \frac{1 + \ln \bar{N}}{\bar{N}} \left(\frac{T}{\omega_e} \right). \quad (5.4)$$

We see from Eq. (5.4) that $T\Delta S_s < \Delta Q_e$ since the condition $\bar{N} \geq 1 + \ln \bar{N}$ is always verified and $T/\omega_e \ll 1$. For the sake of completeness in Fig. 3 we then illustrate the bound in the plane $(\omega_e/T, \bar{n}^{(q)})$; as in the previous plots the labels appearing in the contours indicate the common logarithm of $T\Delta S_s/\Delta Q_e$. It is also clear from Fig. 3 that the bound is even verified in the regime $\bar{n}^{(q)} \ll 1$ provided $\omega_e/T \gg 1$.

C. The regime $\bar{N} < 1$

Recalling the explicit expression of Eq. (4.10), when $\bar{N} < 1$ the quantum amplification is effectively absent. This means, in practice, that the entropy of the system does not decrease but does not increase either. Therefore in this regime we expect that the bounds are violated simply

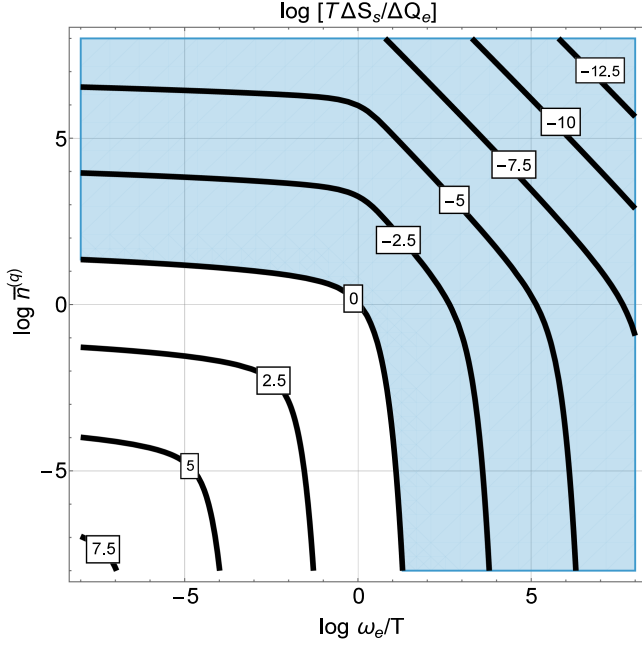


FIG. 3. We illustrate the contours of constant $T\Delta S_s/\Delta Q_e$ in the plane $(\omega_e/T, \bar{n}^{(q)})$. As usual, the labels on the various contours indicate the common logarithm of $T\Delta S_s/\Delta Q_e$. From this plot it is particularly clear that in the region $\omega_e/T \gg 1$ (i.e. $\bar{n} \ll 1$) and $\bar{n}^{(q)} > 1$ the condition $T\Delta S_s < \Delta Q_e$ is always verified, as expected from Eq. (5.4).

because the system and the environment remain close to their initial states where the corresponding density matrices are uncorrelated. Let us then go back to Eq. (5.1) and note that, without approximations, the general expression can be rephrased as

$$T \frac{\Delta S_s}{\Delta Q_e} = \frac{(\bar{N} + 1) \ln(\bar{N} + 1) - \bar{N} \ln \bar{N}}{\bar{N} \ln(1 + 1/\bar{n})}. \quad (5.5)$$

If the numerator at the right-hand side of Eq. (5.5) is expanded in the limit $\bar{N} < 1$ the resulting expression becomes

$$T \frac{\Delta S_s}{\Delta Q_e} = \frac{1 - \ln \bar{N}}{\ln(1 + 1/\bar{n})}, \quad \bar{N} < 1. \quad (5.6)$$

Again, because of Eq. (4.10) the condition $\bar{N} < 1$ must correspond to $\bar{n}^{(q)} < 1$. To investigate this portion of the parameter space, it is very convenient to recall that the averaged multiplicity of the produced quanta is, by definition, $\bar{n}^{(q)} = \sinh^2 r$ [see Eqs. (4.4) and (4.10) and discussions thereafter]. When $\bar{n}^{(q)} < 1$ the parameter r can be both negative and positive provided $|r| \leq 0.88$. In this regime the quanta are not produced but a certain class of operators (the so-called quadrature operators [13,14]) may fluctuate either above or below the quantum noise. What matters, for the present ends, is however that even in the

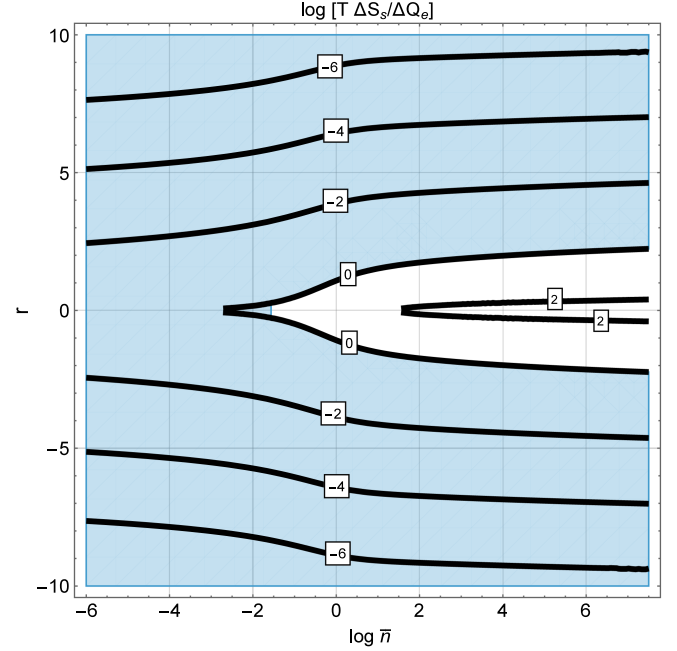


FIG. 4. We illustrate the contours of constant $T\Delta S_s/\Delta Q_e$ in the plane $(\bar{n}, r)$. We recall that since $\bar{n}^{(q)} = \sinh^2 r$, the values of r can be both positive and negative. The r -parametrization is particularly suitable for the analysis of the region $\bar{n}^{(q)} < 1$. As usual the labels on the various contours indicate the common logarithm of $T\Delta S_s/\Delta Q_e$.

regime $|r| \rightarrow 0$ the entropy variation is bounded by the heat transfer except for a tiny slice centered around $|r| \rightarrow 0$. As usual the shaded area in Fig. 4 corresponds to $T\Delta S_s \leq \Delta Q_e$ in the plane $(\bar{n}, r)$. However, for $\bar{n} \gg 1$ and $\bar{n}^{(q)} < 1$ both the system and the environment remain in their (uncorrelated) initial state as in the absence of any dynamical evolution. If we go back to Eq. (5.6) we can first consider the case $\bar{n} < 1$ so that our expression ultimately becomes⁷

$$T \frac{\Delta S_s}{\Delta Q_e} = \frac{1 - \ln \bar{n}^{(q)}}{\ln(1/\bar{n})}, \quad \bar{n} < 1. \quad (5.7)$$

Equation (5.7) implies that $T\Delta S_s < \Delta Q_e$ as long as $\ln(\bar{n}^{(q)}/\bar{n}) > 1$ which also means $e\bar{n} < \bar{n}^{(q)} < 1$. We may finally consider the opposite case, i.e. $\bar{n} \gg 1$. Now the total final multiplicity becomes $\bar{N} = \bar{n}^{(q)}(\bar{n} + 1) \simeq \bar{n}^{(q)}\bar{n}$ and the minuteness of $\bar{n}^{(q)}$ should counterbalance the largeness of $\bar{n}$ so that, overall, $\bar{N} = \bar{n}^{(q)}\bar{n} < 1$. From Eq. (5.5) we have that the analog of Eq. (5.7) is

$$T \frac{\Delta S_s}{\Delta Q_e} = \bar{n}[1 - \ln(\bar{n}^{(q)}\bar{n})], \quad \bar{n} > 1. \quad (5.8)$$

⁷To derive Eq. (5.7) we noted, as usual, that from Eq. (4.10) $\bar{N} = \bar{n}^{(q)}(\bar{n} + 1) \simeq \bar{n}^{(q)}$, while $1/\bar{n} > 1$.

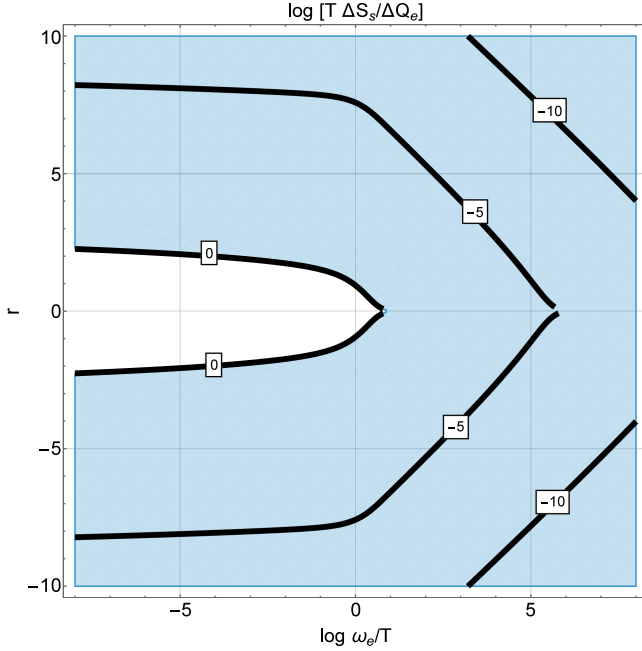


FIG. 5. We illustrate the same region discussed in Fig. 4 but in the plane $(\omega_e/T, r)$. As usual the shaded area corresponds to the region where the variation of the entropy of the system is bounded by the heat transferred to the environment.

According to Eq. (5.8) $\ln(\bar{n}^{(q)}\bar{n}) > 1 - 1/\bar{n}$ which also implies, for $\bar{n} \gg 1$, that $\bar{n}^{(q)}\bar{n} > e$. The latter condition is *incompatible* with the hypothesis $\bar{n}^{(q)}\bar{n} < 1$ that has been assumed in Eq. (5.8). The result (5.8) is expected since, in the limit $r \rightarrow 0$ and $\bar{n} \gg 1$, the environment and the system remain, in practice, in their (uncorrelated) initial states and entropy is not produced.

In Fig. 5 we illustrate, for completeness, the bound on the entropy variation in the plane $(\omega_e/T, r)$. By looking together at Figs. 4 and 5 we see that, as anticipated, the bound $T\Delta S_s < \Delta Q_e$ is always except for a slice corresponding to $|r| \rightarrow 0$ and $\bar{n} \gg 1$. In this region we have that, in practice, $\hat{H}_{se} \rightarrow 0$ since the coupling vanishes asymptotically.

D. The issue of the chemical potential

The quantum mechanical formulation of the Landauer's conjecture does not involve the presence of a chemical potential and this choice determines the density matrix of the initial state where the system is in the vacuum whereas the environment is in a thermal state characterized by a temperature T_e [see Eqs. (3.1) and (3.2) and discussions thereafter]. If the environment contains a chemical potential the statistical weights of Eq. (3.3) are formally the same [i.e. $\bar{p}_n = \bar{n}^n/(\bar{n}+1)^{n+1}$] but the averaged multiplicity depends both on T_e and μ_e , i.e.

$$\bar{n} = \frac{1}{e^{(\omega_e - \mu_e)/T_e} - 1}. \quad (5.9)$$

The presence of a chemical potential is plausible as long as the flow of energy associated with ΔQ_e is complemented by a flow of quanta measured by ΔN_e . We remind the reader, in this respect, that the total Hamiltonian of our system commutes with the difference $\hat{N}_e - \hat{N}_s$, but does not commute with $\hat{N} = \hat{N}_s + \hat{N}_e$,

$$[\hat{H}, \hat{N}] \neq 0, \quad [\hat{H}, \hat{N}_e - \hat{N}_s] = 0, \quad (5.10)$$

see also Eq. (2.18) and discussions thereafter. If a chemical potential is present the bound on the entropy variation of the system can be phrased as

$$T\Delta S_s \leq \Delta Q_e - \mu_e \Delta N_e, \quad (5.11)$$

where, consistent with the previous notations, we set $T = T_e$. As in the case $\mu_e \rightarrow 0$ the bound of Eq. (5.11) is always verified under the same conditions already deduced above this section. To reach this conclusion the simplest route is to observe that the ratio between $T\Delta S_s$ and $(\Delta Q_e - \mu_e \Delta N_e)$ is still given by

$$\left(\frac{T}{\omega_e - \mu_e} \right) \frac{(\bar{N} + 1) \ln(\bar{N} + 1) - \bar{N} \ln \bar{N}}{\bar{N}}. \quad (5.12)$$

But from Eq. (5.9) we can always trade $T/(\omega_e - \mu_e)$ for $1/\ln(1 + 1/\bar{n})$ and this means that the bound (5.11) is satisfied provided

$$\frac{(\bar{N} + 1) \ln(\bar{N} + 1) - \bar{N} \ln \bar{N}}{\bar{N} \ln(1 + 1/\bar{n})} < 1. \quad (5.13)$$

The condition (5.13) can be studied in the plane $(\bar{n}, \bar{n}^{(q)})$ and the result of this analysis coincides exactly with Fig. 2. The reason is that the condition $T\Delta S_s \leq \Delta Q_e$ deduced from Eq. (5.1) in the plane $(\bar{n}^{(q)}, \bar{n})$ takes the same form of Eq. (5.13). This is because the addition of the chemical potential in the bound is always compensated by the modified form of the averaged thermal multiplicity of Eq. (5.9). In Fig. 6 the conditions (5.11)–(5.13) are illustrated in the plane $(\bar{n}, \bar{n}^{(q)})$.

E. A complementary comment

As we close this discussion we wish to comment on the possible direct verifications of the bounds suggested here. Although it is difficult to endorse specific experimental platforms, we find it nonetheless useful to remind the reader of a few possibilities that have been repeatedly explored in the last forty years, namely the backward [48] and the forward [49] four-wave mixing as well as the parametric down-conversion [50]. In spite of the obvious differences, the quantum thermodynamical considerations discussed here are closely related to the formation of squeezed states of light (see e.g. [13,14]) that have been

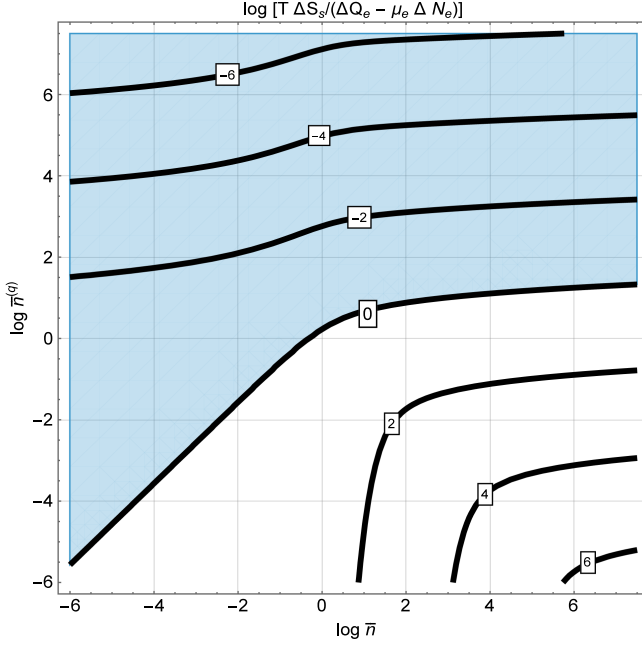


FIG. 6. We illustrate the conditions (5.12) and (5.13). The shaded area corresponds to the region where $T\Delta S_s \leq \Delta Q_e - \mu_e \Delta N_e$. This plot coincides with Fig. 2; this is because, in the presence of the chemical potential, $T/(\omega_e - \mu_e) = \ln(1 + 1/\bar{n})$ so that, ultimately, the condition (5.13) corresponds exactly to the one already deduced, for instance, in Eq. (5.5).

originally scrutinized, within complementary frameworks, forty years ago [48–50]. The two modes associated with the system and the environment can therefore be empirically identified with two appropriate modes of the electromagnetic field in a cavity or in an optical fiber. Although in the last forty years an enormous technical progress has been made in the generation of squeezed light (see e.g. [51] for a relatively recent review), the experimental platforms of nonlinear crystals, optical fibers and atomic ensembles exploited in Refs. [48–50] essentially coincide with the ones used today for generating strongly squeezed light. It seems therefore plausible that similar platforms might be effectively used for the empirical analysis of the entropic bounds derived in this investigation. Of course we are only expressing here a theoretical viewpoint (based on the similarities between the symmetries of the underlying problems) rather than a concrete empirical suggestion.

VI. THE PERSPECTIVE OF FIELD THEORY

The field theoretical perspective rests on the same physical premises of the quantum parametric amplification and the final forms of the entropy bounds are in fact very similar with the crucial difference that the averaged multiplicities, the final states and the density operators exhibit an explicit momentum dependence. For the sake of concreteness we start by examining the following

parametrization of the action of a single scalar degree of freedom $\psi(\vec{x}, \tau)$ in four space-time dimensions:

$$S = \frac{1}{2} \int d^3x \int d\tau v^2(\tau) \left[\partial_\tau \psi \partial_\tau \psi - \partial_\ell \psi \partial^\ell \psi \right], \quad (6.1)$$

where τ denotes the conformal time coordinate which is widely employed in curved backgrounds and in cosmological applications [52]; in the case of a conformally flat background metric of Friedmann-Robertson-Walker type we have that $a(\tau)d\tau = dt$. The time dependence is encoded in the function $v(\tau)$ that ultimately determines the specific form of the pump fields entering the effective Hamiltonian. Since different time parametrizations could be employed [15,16] (see also [17,18]), the introduction of τ is not strictly essential but it is convenient also in the light of potential applications. In Eq. (6.1) ∂_ℓ denotes, as usual, the spatial gradients and the related indices are Euclidean. The action (6.1) can be reduced to its canonical form after redefining the fields as $\mu(\vec{x}, \tau) = v(\tau)\psi(\vec{x}, \tau)$ so that, eventually, the Lagrangian density becomes

$$\mathcal{L}(\vec{x}, \tau) = \frac{1}{2} \left[(\partial_\tau \mu)^2 + \mathcal{F} \mu^2 - 2\mathcal{F} \mu \partial_\tau \mu - (\vec{\nabla} \mu)^2 \right], \quad (6.2)$$

where $\mathcal{F} = v'/v$ and the prime indicates hereunder a derivation with respect to the conformal time coordinate τ . The canonical momentum deduced from Eq. (6.2) is given by $\pi = (\partial_\tau \mu - \mathcal{F} \mu)$ and from Eq. (6.1) the Hamiltonian becomes

$$H(\tau) = \frac{1}{2} \int d^3x \left[\pi^2 + 2\mathcal{F} \mu \pi + (\vec{\nabla} \mu)^2 \right], \quad (6.3)$$

where $\vec{\nabla}$ denotes, as usual, the gradient in three spatial dimensions. Depending on the actual meaning of $v(\tau)$, $\mathcal{F}$ and ψ both the action (6.1) and the Hamiltonian (6.3) may correspond to diverse physical situations potentially involving the production of particles in curved backgrounds. For instance, if $\psi(\vec{x}, \tau)$ coincides with a minimally coupled scalar field in a (conformally flat) Friedmann-Robertson-Walker background, $v(\tau)$ equals the scale factor $a(\tau)$ expressed in the conformal time parametrization. In the case of the evolution of curvature inhomogeneities $\psi(\vec{x}, \tau) = \mathcal{R}(\vec{x}, \tau)$ where $\mathcal{R}(\vec{x}, \tau)$ indicates the curvature inhomogeneity on comoving orthogonal hypersurfaces. In this second situation $v(\tau) = a(\tau)\varphi'/\mathcal{H}$ where $\mathcal{H} = a'/a$ and φ is the inflaton field [52]. Last but not least, as we shall see, Eq. (6.3) may also describe the evolution of a single tensor polarization in a Friedmann-Robertson-Walker background [9] (see also Refs. [26,27] and [53]).

A. The quantum description of the problem

Following the standard approach the classical fields of Eq. (6.3) are now promoted to the status of quantum

(Hermitian) operators obeying canonical commutation relations at equal times. This means that $\mu \rightarrow \hat{\mu}$, $\pi \rightarrow \hat{\pi}$ and

$$[\hat{\mu}(\vec{x}, \tau), \hat{\pi}(\vec{y}, \tau)] = i\delta^{(3)}(\vec{x} - \vec{y}) \quad (6.4)$$

where $\hat{\mu}^\dagger = \hat{\mu}$ and $\hat{\pi}^\dagger = \hat{\pi}$. The Hamiltonian operator deduced from Eq. (6.3) becomes

$$\hat{H}(\tau) = \frac{1}{2} \int d^3x \left[\hat{\pi}^2 + \mathcal{F}(\hat{\mu} \hat{\pi} + \hat{\pi} \hat{\mu}) + (\vec{\nabla} \hat{\mu})^2 \right]. \quad (6.5)$$

Both the field operators and the Hamiltonian can be represented in Fourier space; in particular we can write

$$\hat{\mu}_{\vec{k}}(\tau) = \frac{1}{(2\pi)^{3/2}} \int e^{i\vec{k} \cdot \vec{x}} \hat{\mu}(\vec{x}, \tau) d^3x = \hat{\mu}_{-\vec{k}}^\dagger(\tau). \quad (6.6)$$

By simply replacing in Eq. (6.6) $\hat{\mu}(\vec{x}, \tau)$ and $\hat{\mu}_{\vec{k}}$ with $\hat{\pi}(\vec{x}, \tau)$ and $\hat{\pi}_{\vec{k}}$ we obtain the Fourier decomposition for the canonical momenta which are also Hermitian (i.e. $\hat{\pi}_{\vec{k}} = \hat{\pi}_{-\vec{k}}^\dagger$). From Eq. (6.4) the commutation relations in Fourier space become $[\hat{\mu}_{\vec{k}}(\tau), \hat{\pi}_{\vec{p}}(\tau)] = i\delta^{(3)}(\vec{k} + \vec{p})$. The creation and the annihilation operators connected to the canonical field operators are introduced, as usual, from

$$\begin{aligned} \hat{\mu}_{\vec{k}} &= (\hat{a}_{\vec{k}} + \hat{a}_{-\vec{k}}^\dagger)/\sqrt{2k}, \\ \hat{\pi}_{\vec{k}} &= -i\sqrt{k/2}(\hat{a}_{\vec{k}} - \hat{a}_{-\vec{k}}^\dagger), \end{aligned} \quad (6.7)$$

where, for continuous $\vec{k}$ -modes, $[\hat{a}_{\vec{k}}, \hat{a}_{\vec{p}}] = \delta^{(3)}(\vec{k} - \vec{p})$. The explicit form of the Hamiltonian operator can then be expressed either in terms of $\hat{\mu}_{\vec{k}}$ and $\hat{\pi}_{\vec{k}}$ or by means of the creation and annihilation operators. Between these two options the latter is more relevant than the former if we want to clarify the analogy with the quantum theory of parametric amplification. Thanks to Eqs. (6.6) and (6.7), the Hamiltonian of Eq. (6.5) becomes

$$\begin{aligned} \hat{H}(\tau) &= \frac{1}{2} \int d^3k k [\hat{a}_{\vec{k}}^\dagger \hat{a}_{\vec{k}} + \hat{a}_{-\vec{k}} \hat{a}_{-\vec{k}}^\dagger] \\ &+ \frac{1}{2} \int d^3k [\sigma(\tau) \hat{a}_{\vec{k}}^\dagger \hat{a}_{-\vec{k}}^\dagger + \sigma^*(\tau) \hat{a}_{\vec{k}} \hat{a}_{-\vec{k}}], \end{aligned} \quad (6.8)$$

where $\sigma = i\mathcal{F}$. Since Eq. (6.8) exhibits the two-mode structure discussed in the previous sections, the idea is now to associate half of the spectrum of the modes with the system (for instance the modes with momentum $\vec{k}$) while the remaining modes (the ones with momentum $-\vec{k}$) will belong to the environment.

B. The two-mode Hamiltonian and its symmetries

From a practical viewpoint the discussion of Eq. (6.8) is more concise if the field operators have a discrete spectrum

as it happens in a box of volume V ; in this way the explicit expressions of the quantum states are more concise, as it is well known in the context of quantum optical applications [13,14] (see also [40]). As it is well known the integrals over the wave numbers appearing in Eq. (6.8) are replaced sums and, more precisely, when the volume of the box becomes very large the connection between the two is

$$\sum_{\vec{k}} \rightarrow V \int d^3k / (2\pi)^3, \quad \mu_{\vec{k}} \rightarrow \sqrt{(2\pi)^3 / V} \mu_{\vec{k}}.$$

Similar changes apply to the creation and annihilation operators and to their commutation relations that now become $[\hat{a}_{\vec{k}}, \hat{a}_{\vec{k}}^\dagger] = \delta_{\vec{k}, \vec{p}}$ where now $\delta_{\vec{k}, \vec{p}}$ indicates the Kronecker symbol replacing the Dirac delta distribution of the continuous mode representation. Equation (6.8) can be rewritten as $\hat{H}(\tau) = \sum_{\vec{k}} \hat{H}_{\vec{k}}(\tau)$ where the two-mode Hamiltonian $\hat{H}_{\vec{k}}$ is now given by

$$\hat{H}_{\vec{k}}(\tau) = \omega_k (\hat{s}_{\vec{k}}^\dagger \hat{s}_{\vec{k}} + \hat{e}_{\vec{k}}^\dagger \hat{e}_{\vec{k}}) + g \hat{s}_{\vec{k}}^\dagger \hat{e}_{\vec{k}}^\dagger + g^* \hat{s}_{\vec{k}} \hat{e}_{\vec{k}}. \quad (6.9)$$

We note that, for consistency with the previous notations, we introduced $\omega_k^2 = k/2$ and $g = \sigma/2$; furthermore, as previously suggested, Eq. (6.9) follows from Eq. (6.8) by identifying the $\hat{a}_{\vec{k}}$ with the oscillators of the system (i.e. $\hat{a}_{\vec{k}} \rightarrow \hat{s}_{\vec{k}}$) and $\hat{a}_{-\vec{k}}$ with the ones of the environment (i.e. $\hat{a}_{-\vec{k}} \rightarrow \hat{e}_{\vec{k}}$) [9]. All the results obtained in the previous sections can be generalized to the field theory context since, besides the momentum dependence, Eqs. (1.14), (1.15), and (2.1) are analogous to Eq. (6.9). So, for instance, the evolution of the field operators follows from the total Hamiltonian $\hat{H} = \sum_{\vec{k}} \hat{H}_{\vec{k}}$,

$$\partial_\tau \hat{s}_{\vec{p}} = i[\hat{H}, \hat{s}_{\vec{p}}], \quad \partial_\tau \hat{e}_{\vec{p}} = i[\hat{H}, \hat{e}_{\vec{p}}]. \quad (6.10)$$

Recalling the explicit form of the commutation relations (i.e. $[\hat{s}_{\vec{k}}, \hat{s}_{\vec{p}}^\dagger] = [\hat{e}_{\vec{k}}, \hat{e}_{\vec{p}}^\dagger] = \delta_{\vec{k}, \vec{p}}$ and $[\hat{s}_{\vec{k}}, \hat{e}_{\vec{k}}] = 0$) the following pair of equations can be deduced from Eq. (6.10):

$$\hat{s}_{\vec{p}}' = -i\omega_p \hat{s}_{\vec{p}} - ig \hat{e}_{\vec{p}}^\dagger, \quad \hat{e}_{\vec{p}}' = -i\omega_p \hat{e}_{\vec{p}} - ig \hat{s}_{\vec{p}}^\dagger. \quad (6.11)$$

Although the time dependence of $g(\tau)$ affects Eq. (6.11) (and ultimately determines the k dependence of the averaged multiplicities) what matters for the present ends is the general form of the density operators of the system and of the environment as a function of the final multiplicities, as already described in the quantum mechanical case. We can therefore express the solution of Eq. (6.11) as

$$\begin{aligned} \hat{s}_{\vec{k}}(\tau) &= u_k(\tau, \tau_{\text{in}}) \hat{s}_{\vec{k}} - v_k(\tau, \tau_{\text{in}}) \hat{e}_{\vec{k}}^\dagger, \\ \hat{e}_{\vec{k}}(\tau) &= u_k^*(\tau, \tau_{\text{in}}) \hat{e}_{\vec{k}} - v_k^*(\tau, \tau_{\text{in}}) \hat{s}_{\vec{k}}, \end{aligned} \quad (6.12)$$

where, as in Eq. (2.7), we adopted the shorthand notations $\hat{\mathcal{J}}_{\vec{k}} = \hat{\mathcal{J}}_{\vec{k}}(\tau_{\text{in}})$ and $\hat{\mathcal{E}}_{\vec{k}} = \hat{\mathcal{E}}_{\vec{k}}(\tau_{\text{in}})$.

The evolution of $u_k(\tau, \tau_{\text{in}})$ and $v_k(\tau, \tau_{\text{in}})$ mirrors the one already analyzed in Eqs. (2.8) and (2.9),

$$\begin{aligned} u'_k &= -i\omega_k u_k + igv_k^*, \\ v'_k &= -i\omega_k v_k + igu_k^*. \end{aligned} \quad (6.13)$$

For a continuous and differentiable pump field the two (complex) functions $u_k(\tau, \tau_{\text{in}})$ and $v_k(\tau, \tau_{\text{in}})$ are anyway subjected to the condition $|u_k(\tau, \tau_{\text{in}})|^2 - |v_k(\tau, \tau_{\text{in}})|^2 = 1$; this means that $u_k(\tau, \tau_{\text{in}})$ and $v_k(\tau, \tau_{\text{in}})$ can be parametrized in terms of three real functions conventionally denoted by $\delta_k(\tau)$, $\theta_k(\tau)$, and $r_k(\tau)$ in full analogy with the quantum mechanical situation discussed before,

$$\begin{aligned} u_k(\tau, \tau_{\text{in}}) &= e^{-i\delta_k} \cosh r_k, \\ v_k(\tau, \tau_{\text{in}}) &= e^{-i(\delta_k - \theta_k)} \sinh r_k. \end{aligned} \quad (6.14)$$

The evolution of $\delta_k(\tau)$, $\theta_k(\tau)$, and $r_k(\tau)$ can be deduced from Eq. (6.13) once the expression of $g(\tau)$ is fixed.⁸ It is however not essential, for the present purposes, to analyze these equations since the density matrices of the final state depend on the parametrization of Eq. (6.14) but not on the specific functional form of the solution, exactly as in the quantum mechanical example discussed in the previous sections.

C. The total density operator of the final state

For each $\vec{k}$ -mode the basis for the irreducible representations of the $SU(1,1)$ group is analogous to the one previously introduced in the quantum mechanical analysis of the problem with the difference that the dependence on the $\vec{k}$ -mode must be consistently included,

$$|\{m^{(j)}\}\{m^{(e)}\}\rangle = \prod_{\vec{k}} |m_{\vec{k}}^{(j)} m_{\vec{k}}^{(e)}\rangle. \quad (6.15)$$

The states $|m_{\vec{k}}^{(j)} m_{\vec{k}}^{(e)}\rangle$ appearing in Eq. (6.15) are now defined by

$$|m_{\vec{k}}^{(j)} m_{\vec{k}}^{(e)}\rangle = \frac{(\hat{\mathcal{J}}^\dagger)^{m_{\vec{k}}^{(j)}} (\hat{\mathcal{E}}^\dagger)^{m_{\vec{k}}^{(e)}}}{\sqrt{m_{\vec{k}}^{(j)}!} \sqrt{m_{\vec{k}}^{(e)}!}} |0_{\vec{k}}^{(j)} 0_{\vec{k}}^{(e)}\rangle. \quad (6.16)$$

⁸So for instance in the case of a minimally coupled scalar field $g(\tau) = i\mathcal{H}/2$ and Eq. (6.14), once inserted into Eq. (6.13), implies a set of evolution equations for $\delta_k(\tau)$, $\theta_k(\tau)$ and $r_k(\tau)$. Within the present notations these equations read $r'_k = \mathcal{H} \cos(2\delta_k - \theta_k)$, $\delta'_k = \omega_k - \mathcal{H} \tanh r_k \sin(2\delta_k - \theta_k)$ and $\theta'_k = \mathcal{H} \sin(2\delta_k - \theta_k) / (\cosh r_k \sinh r_k)$. Their explicit solution is however not required to generalize the entropy bounds to the field theory setup.

The states $|m_{\vec{k}}^{(j)} m_{\vec{k}}^{(e)}\rangle$ form a convenient basis for the irreducible representations of $SU_{\vec{k}}(1,1)$ whose generators are now expressed as

$$\begin{aligned} \hat{K}_{\vec{k}}^{(+)} &= \hat{\mathcal{J}}_{\vec{k}}^\dagger \hat{\mathcal{E}}_{\vec{k}}^\dagger, & \hat{K}_{\vec{k}}^{(-)} &= \hat{\mathcal{J}}_{\vec{k}} \hat{\mathcal{E}}_{\vec{k}}, \\ \hat{K}_{\vec{k}}^{(0)} &= (\hat{\mathcal{J}}_{\vec{k}}^\dagger \hat{\mathcal{J}}_{\vec{k}} + \hat{\mathcal{E}}_{\vec{k}}^\dagger \hat{\mathcal{E}}_{\vec{k}})/2. \end{aligned} \quad (6.17)$$

The commutation relations of the generators generalize the ones already discussed in Eqs. (3.18) and (3.19),

$$\begin{aligned} [\hat{K}_{\vec{k}}^{(+)}, \hat{K}_{\vec{p}}^{(-)}] &= -2\hat{K}_{\vec{k}}^{(0)} \delta_{\vec{k}, \vec{p}}, \\ [\hat{K}_{\vec{k}}^{(0)}, \hat{K}_{\vec{p}}^{(\pm)}] &= \pm \hat{K}_{\vec{k}}^{(\pm)} \delta_{\vec{k}, \vec{p}}. \end{aligned} \quad (6.18)$$

In terms of the generators of Eq. (6.17) the operator $\hat{\Sigma}_{\vec{k}}(z_k) = \exp(z_k^* \hat{\mathcal{J}}_{\vec{k}}^\dagger \hat{\mathcal{E}}_{\vec{k}} - z_k \hat{\mathcal{J}}_{\vec{k}} \hat{\mathcal{E}}_{\vec{k}}^\dagger)$ can be written in terms of the Baker-Campbell-Hausdorff decomposition

$$\hat{\Sigma}_{\vec{k}}(z_k) = \hat{\Sigma}_{+}^{(\vec{k})}(z_k) \hat{\Sigma}_0^{(\vec{k})}(z_k) \hat{\Sigma}_{-}^{(\vec{k})}(z_k), \quad (6.19)$$

where $\hat{\Sigma}_{\pm}^{(\vec{k})}(z_k)$ and $\hat{\Sigma}_0^{(\vec{k})}(z_k)$ are defined as

$$\begin{aligned} \hat{\Sigma}_{-}^{(\vec{k})}(z_k) &= \exp \left[e^{-i\theta_k} \tanh r_k \hat{K}_{\vec{k}}^{(-)} \right], \\ &= \exp \left[(v_k^*/u_k^*) \hat{K}_{\vec{k}}^{(-)} \right] \\ \hat{\Sigma}_0^{(\vec{k})}(z_k) &= \exp \left[-2 \ln \cosh r_k \hat{K}_{\vec{k}}^{(0)} \right], \\ &= \exp \left[-2 \ln |u_k| \hat{K}_{\vec{k}}^{(0)} \right] \\ \hat{\Sigma}_{+}^{(\vec{k})}(z_k) &= \exp \left[-e^{-i\theta_k} \tanh r_k \hat{K}_{\vec{k}}^{(+)} \right] \\ &= \exp \left[-(v_k/u_k) \hat{K}_{\vec{k}}^{(+)} \right]. \end{aligned} \quad (6.20)$$

In Eq. (6.20) the Baker-Campbell-Hausdorff decomposition has been expressed both in terms of (r_k, θ_k) and in terms of (u_k, v_k) to clarify the absence of the δ_k dependence.

The analog of $|\delta z\rangle$ introduced in Eq. (3.23) is constructed in analogy with the multiparticle Fock states of Eq. (6.15),

$$|\{\delta z\}\rangle = \prod_{\vec{k}} |\delta_{\vec{k}} z_{\vec{k}}\rangle, \quad (6.21)$$

where $|\delta_{\vec{k}} z_{\vec{k}}\rangle$ is now

$$|\delta_{\vec{k}} z_{\vec{k}}\rangle = \hat{\mathcal{R}}_{\vec{k}}(\delta_{\vec{k}}) \hat{\Sigma}_{\vec{k}}(z_{\vec{k}}) |0_{\vec{k}}^{(j)} 0_{\vec{k}}^{(e)}\rangle. \quad (6.22)$$

Within the same formalism leading to Eqs. (6.15), (6.16), (6.18), and (6.21) the initial state density matrix is

$$\begin{aligned}\hat{\rho}(\tau_{\text{in}}) &= \prod_{\vec{k}} \hat{\rho}_{\vec{k}}(\tau_{\text{in}}) \\ &= \prod_{\vec{k}} \hat{\rho}_{\vec{k}}^{(s)}(\tau_{\text{in}}) \otimes \hat{\rho}_{\vec{k}}^{(e)}(\tau_{\text{in}}).\end{aligned}\quad (6.23)$$

As in the quantum mechanical case [see Eq. (3.1) and discussions therein] the initial density operator of the system is in the multiparticle vacuum [i.e. $\hat{\rho}_{\vec{k}}^{(s)}(\tau_{\text{in}}) = |0_{\vec{k}}^{(s)}\rangle\langle 0_{\vec{k}}^{(s)}|$] whereas $\hat{\rho}_{\vec{k}}^{(e)}(\tau_{\text{in}})$ is a thermal density matrix

$$\hat{\rho}_{\vec{k}}^{(e)}(\tau_{\text{in}}) = \sum_{m_{\vec{k}}^{(e)}=0}^{\infty} \bar{p}_{m_{\vec{k}}^{(e)}} \left| m_{\vec{k}}^{(e)} \right\rangle \left\langle m_{\vec{k}}^{(e)} \right|, \quad (6.24)$$

characterized by a Bose-Einstein probability distribution for each of the $\vec{k}$ -modes of the environment

$$\bar{p}_{m_{\vec{k}}^{(e)}} = \frac{\bar{n}_{\vec{k}}^{m_{\vec{k}}^{(e)}}}{(\bar{n}_{\vec{k}} + 1)^{m_{\vec{k}}^{(e)} + 1}}, \quad (6.25)$$

where, as before, $\bar{n}_{\vec{k}}$ indicates the averaged thermal multiplicity defined as

$$\bar{n}_{\vec{k}} = \frac{1}{e^{(\omega_{\vec{k}} - \mu_e)/T} - 1}. \quad (6.26)$$

While in the quantum mechanical situation the average multiplicity did only depend on the single frequency ω_e , Eq. (6.26) involves all the $\vec{k}$ -modes associated with the environment. Putting together the results of Eqs. (6.18)–(6.21) and of Eqs. (6.23)–(6.26) the density matrix of the final state can then be obtained from the initial state as

$$\hat{\rho}(\tau_{\text{fin}}, \tau_{\text{in}}) = \hat{\mathcal{U}}(\{\delta z\}) \hat{\rho}(\tau_{\text{in}}) \hat{\Sigma}^\dagger(\{z\}) \hat{\mathcal{U}}^\dagger(\{\delta z\}), \quad (6.27)$$

where $\hat{\mathcal{U}}(\{\delta z\}) = \hat{\mathcal{H}}(\{\delta\}) \hat{\Sigma}(\{z\})$. The proper density operator encompassing the system and the environment can also be written as

$$\hat{\rho}(\tau_{\text{fin}}, \tau_{\text{in}}) = \prod_{\vec{k}} \hat{\rho}_{\vec{k}}(\tau_{\text{fin}}, \tau_{\text{in}}). \quad (6.28)$$

The explicit form of $\hat{\rho}_{\vec{k}}(\tau_{\text{fin}}, \tau_{\text{in}})$ becomes

$$\hat{\rho}_{\vec{k}}(\tau_{\text{fin}}, \tau_{\text{in}}) = \hat{\mathcal{U}}_{\vec{k}}(\delta_k, z_k) \hat{\rho}_{\vec{k}}(\tau_{\text{in}}) \hat{\mathcal{U}}_{\vec{k}}^\dagger(\delta_k, z_k) \quad (6.29)$$

$$\begin{aligned}&= \sum_{m_{\vec{k}}^{(e)}=0}^{\infty} \sum_{\ell_{\vec{k}}=0}^{\infty} \sum_{\ell'_{\vec{k}}=0}^{\infty} \mathcal{A}_{m_{\vec{k}}^{(e)} \ell_{\vec{k}} \ell'_{\vec{k}}} \\ &\times \left| \ell_{\vec{k}}, \ell_{\vec{k}} + m_{\vec{k}}^{(e)} \right\rangle \left\langle m_{\vec{k}}^{(e)} + \ell'_{\vec{k}}, \ell'_{\vec{k}} \right|. \quad (6.30)\end{aligned}$$

The term $\mathcal{A}_{m_{\vec{k}}^{(e)} \ell_{\vec{k}} \ell'_{\vec{k}}}$ has been included for convenience also with the purpose of exhibiting the analogy with the quantum mechanical situation of Eq. (3.4)

$$\mathcal{A}_{m_{\vec{k}}^{(e)} \ell_{\vec{k}} \ell'_{\vec{k}}} = P_{m_{\vec{k}}^{(e)} \ell_{\vec{k}} \ell'_{\vec{k}}} \sqrt{\binom{m_{\vec{k}}^{(e)} + \ell_{\vec{k}}}{m_{\vec{k}}^{(e)}} \binom{m_{\vec{k}}^{(e)} + \ell'_{\vec{k}}}{m_{\vec{k}}^{(e)}}}, \quad (6.31)$$

where $P_{m_{\vec{k}}^{(e)} \ell_{\vec{k}} \ell'_{\vec{k}}}$ is

$$P_{m_{\vec{k}}^{(e)} \ell_{\vec{k}} \ell'_{\vec{k}}} = \frac{e^{i\alpha_{\vec{k}}(\ell_{\vec{k}} - \ell'_{\vec{k}})} \bar{p}_{m_{\vec{k}}^{(e)}}}{(\cosh r_k)^{2(m_{\vec{k}}^{(e)} + 1)}} (\tanh r_k)^{(\ell_{\vec{k}} + \ell'_{\vec{k}})}, \quad (6.32)$$

and $\alpha_{\vec{k}} = (\theta_{\vec{k}} + \pi - 2\delta_{\vec{k}})$.

D. Extensions of the entropy bound

From the total density matrix of Eqs. (6.30)–(6.32) we now derive the reduced density matrices of the system and of the environment

$$\hat{\rho}_{\vec{k}}^{(e)}(\tau, \tau_{\text{in}}) = \text{Tr}_s[\hat{\rho}_{\vec{k}}(\tau, \tau_{\text{in}})], \quad (6.33)$$

$$\hat{\rho}_{\vec{k}}^{(s)}(\tau, \tau_{\text{in}}) = \text{Tr}_e[\hat{\rho}_{\vec{k}}(\tau, \tau_{\text{in}})]. \quad (6.34)$$

The explicit form of $\hat{\rho}_{\vec{k}}^{(s)}(\tau, \tau_{\text{in}})$ is

$$\hat{\rho}_{\vec{k}}^{(s)}(\tau, \tau_{\text{in}}) = \sum_{n_{\vec{k}}^{(e)}=0}^{\infty} \langle n_{\vec{k}}^{(e)} | \hat{\rho}_{\vec{k}}(\tau, \tau_{\text{in}}) | n_{\vec{k}}^{(e)} \rangle. \quad (6.35)$$

Taking now into account the explicit results of Eqs. (6.30)–(6.32) the reduced density matrix of the system becomes

$$\hat{\rho}_{\vec{k}}^{(s)}(\tau, \tau_{\text{in}}) = \sum_{\ell_{\vec{k}}=0}^{\infty} \frac{\bar{N}_{\vec{k}}^{\ell_{\vec{k}}}}{(\bar{N}_{\vec{k}} + 1)^{\ell_{\vec{k}} + 1}} |\ell_{\vec{k}}\rangle \langle \ell_{\vec{k}}|, \quad (6.36)$$

where $\bar{N}_{\vec{k}}$ is now defined as

$$\bar{N}_{\vec{k}} = \bar{n}_{\vec{k}}^{(q)} (\bar{n}_{\vec{k}} + 1) = (\bar{n}_{\vec{k}} + 1) \sinh^2 r_k. \quad (6.37)$$

The expression of $\bar{N}_{\vec{k}}$ is fully analogous to the one already mentioned in Eq. (4.10) with the difference that now all the multiplicities include an essential k dependence. In particular Eq. (6.37) also implies that $\bar{N}_{\vec{k}} \rightarrow 0$ as $r_k \rightarrow 0$: in this limit the system remains in the vacuum and the obtained entropy variation must vanish.

To verify this important point we go back to Eqs. (6.34) and (6.35) and note that from the reduced density matrix

$\hat{\rho}_s = \prod_{\vec{k}} \hat{\rho}_{\vec{k}}^{(s)}$ the associated entropy follows from the von Neumann expression,

$$S_s = -\text{Tr}[\hat{\rho}_s \ln \hat{\rho}_s], \quad (6.38)$$

where, according to the established notations, $S_s = S[\hat{\rho}_s]$. But since $\hat{\rho}_s = \prod_{\vec{k}} \hat{\rho}_{\vec{k}}^{(s)}$ and $\text{Tr}[\hat{\rho}_{\vec{k}}^{(s)}] = 1$ the total entropy can be computed mode by mode, i.e.

$$S_s = \sum_{\vec{k}} S_{\vec{k}}^{(s)}, \quad S_{\vec{k}}^{(s)} = -\text{Tr}[\hat{\rho}_{\vec{k}}^{(s)} \ln \hat{\rho}_{\vec{k}}^{(s)}]. \quad (6.39)$$

From these expressions, recalling Eq. (6.36), we can immediately compute the total variation of the entropy of the system,

$$\begin{aligned} \Delta S_{\vec{k}}^{(s)} &= S_{\vec{k}}^{(s)}(\tau_{\text{fin}}) - S_{\vec{k}}^{(s)}(\tau_{\text{in}}) \\ &= (\bar{N}_k + 1) \ln(\bar{N}_k + 1) - \bar{N}_k \ln \bar{N}_k. \end{aligned} \quad (6.40)$$

Recalling the comment mentioned after Eq. (6.37) we see from Eq. (6.40) that $\Delta S_{\vec{k}}^{(s)} \rightarrow 0$ for $r_k \rightarrow 0$: this limit corresponds to the absence of parametric amplification [since $\bar{n}_k^{(q)} \rightarrow 0$] since under the condition $r_k \rightarrow 0$ the system remains, in practice, in the vacuum.

As in the quantum mechanical case the total variation of the heat transferred to the environment can be computed either from the reduced density matrix $\hat{\rho}_e$ [see Eq. (4.2) and discussion therein] or directly from the evolution of the multiplicities, as suggested in Eq. (4.3). The same twofold possibility arises in the present context and the final result is

$$\begin{aligned} \Delta Q_{\vec{k}}^{(e)} &= \langle \text{th} | \hat{H}_{\vec{k}}^{(e)}(\tau_{\text{fin}}) | \text{th} \rangle - \langle \text{th} | \hat{H}_{\vec{k}}^{(e)}(\tau_{\text{in}}) | \text{th} \rangle \\ &= \omega_k \Delta N_{\vec{k}}^{(e)}, \end{aligned} \quad (6.41)$$

where $|\text{th}\rangle$ is the multiparticle state computed from the thermal density matrix where each mode of the field is characterized by a k -dependent Bose-Einstein probability distribution. Moreover in Eq. (6.41) $\Delta N_{\vec{k}}^{(e)}$ indicates the flow of particles to the environment and it is given by $\Delta N_{\vec{k}}^{(e)} = \bar{N}_k = \bar{n}_k^{(q)}(\bar{n}_k + 1)$.

We can now verify, as anticipated, the generalized form of the bound already discussed at length in Sec. V,

$$T \Delta S_{\vec{k}}^{(s)} \leq \Delta Q_{\vec{k}}^{(e)} - \mu_e \Delta N_{\vec{k}}^{(e)}. \quad (6.42)$$

If we now recall the explicit expressions of Eqs. (6.40) and (6.41) for $\Delta S_{\vec{k}}^{(s)}$, $\Delta Q_{\vec{k}}^{(e)}$, and $\Delta N_{\vec{k}}^{(e)}$ we obtain that the bound of Eq. (6.42) corresponds to

$$\frac{(\bar{N}_k + 1) \ln(\bar{N}_k + 1) - \bar{N}_k \ln \bar{N}_k}{\ln(1 + 1/\bar{n}_k)} \leq 1. \quad (6.43)$$

The condition (6.43) generalizes the quantum mechanical result to the field theoretical situation and it can be analyzed in the same manner. The interesting aspect of Eq. (6.43) is that $\bar{n}_k^{(q)}$ and $\bar{n}_k$ are now different for the various k -modes. This also shows that the entropy bounds may also depend on the underlying cosmological evolution [9].

E. The case of relic gravitons

The evolution of the tensor modes of the geometry (corresponding to a massless field of spin 2 evolving in curved space-times) shares various analogies with the class of problems analyzed in this investigation, as recently pointed out [9]. The potential existence of stochastic backgrounds of relic gravitational radiation has been suggested even before the formulation of inflationary scenarios [19–21] as a genuine general relativistic effect in curved space-times. Since the evolution of the tensor modes of the geometry is not Weyl invariant [19], the associated classical and quantum fluctuations can be amplified not only in anisotropic metric but also in conformally flat background geometries [21] (see also [22]). For this reason backgrounds of relic gravitons are expected, with rather different properties, in a variety of cosmological scenarios and, in particular, during an isotropic phase of quasi-de Sitter expansion [23–25]. The second-order tensor fluctuation of the Einstein-Hilbert action in a spatially flat Friedmann-Robertson-Walker background is given by [26,27]

$$S_g = \frac{1}{8\ell_P^2} \int d^4x a^2(\tau) \eta^{\mu\nu} \partial_\mu h_{ij} \partial_\nu h^{ij}, \quad (6.44)$$

where $h_i^i = \partial_i h^{ij} = 0$ describes the tensor modes of the geometry while $\eta_{\mu\nu}$ denotes the Minkowski metric with signature $(+, -, -, -)$; as before $a(\tau)$ is the scale factor, written as a function of the conformal time coordinate τ . The rescaled canonical amplitudes and the comoving momenta analog to the ones introduced in the scalar case are now given by

$$\mu_{ij} = h_{ij} a(\tau), \quad \pi_{ij} = (\partial_\tau \mu_{ij} - \mathcal{H} \mu_{ij}) / (8\ell_P^2), \quad (6.45)$$

where, as previously mentioned, $\mathcal{H} = a'/a$ indicates the relative variation of the scale factor in terms of the conformal time coordinate. During an inflationary stage of expansion the classical inhomogeneities are quickly

ironed; the quantum fluctuations are described by the Hermitian operators $\hat{\mu}_{ij}(\vec{x}, \tau)$ and $\hat{\pi}_{ij}(\vec{x}, \tau)$,

$$\begin{aligned}\hat{\mu}_{ij}(\vec{x}, \tau) &= \sqrt{2\ell_P} \int \frac{d^3k}{(2\pi)^{3/2}} \sum_{\alpha} e_{ij}^{(\alpha)} \hat{\mu}_{\vec{k},\alpha} e^{-i\vec{k}\cdot\vec{x}}, \\ \hat{\pi}_{ij}(\vec{x}, \tau) &= \frac{1}{4\sqrt{2\ell_P}} \int \frac{d^3k}{(2\pi)^{3/2}} \sum_{\alpha} e_{ij}^{(\alpha)} \hat{\pi}_{\vec{k},\alpha} e^{-i\vec{k}\cdot\vec{x}},\end{aligned}$$

where the sums run over the two tensor polarizations $\alpha = \oplus, \otimes$, i.e.

$$e_{ij}^{(\oplus)}(\hat{k}) = (\hat{m}_i \hat{m}_j + \hat{n}_i \hat{n}_j), \quad e_{ij}^{(\otimes)}(\hat{k}) = (\hat{m}_i \hat{n}_j - \hat{m}_j \hat{n}_i). \quad (6.46)$$

Note that, in Eq. (6.46), $\hat{m}$, $\hat{n}$ and $\hat{k}$ are just a triplet of mutually orthogonal unit vectors obeying $\hat{m} \times \hat{n} = \hat{k}$. In terms of the creation and annihilation operators we have

$$\begin{aligned}\hat{\mu}_{\vec{k},\alpha} &= \left(\hat{a}_{\vec{k},\alpha} + \hat{a}_{-\vec{k},\alpha}^\dagger \right) / \sqrt{2k}, \\ \hat{\pi}_{\vec{k},\alpha} &= -i \left(\hat{a}_{\vec{k},\alpha} - \hat{a}_{-\vec{k},\alpha}^\dagger \right) \sqrt{k/2}.\end{aligned} \quad (6.47)$$

Equation (6.47) generalizes the result of Eq. (6.7) to the case of the action (6.44). Ultimately the Hamiltonian operator deduced from the action (6.44) takes then the same form of Eq. (6.8),

$$\begin{aligned}\hat{H}_g(\tau) &= \frac{1}{2} \int d^3k \sum_{\alpha} k \left[\hat{a}_{\vec{k},\alpha}^\dagger \hat{a}_{\vec{k},\alpha} + \hat{a}_{-\vec{k},\alpha} \hat{a}_{-\vec{k},\alpha}^\dagger \right] \\ &+ \frac{1}{2} \int d^3k \sum_{\alpha} \left[\sigma \hat{a}_{\vec{k},\alpha}^\dagger \hat{a}_{-\vec{k},\alpha}^\dagger + \sigma^* \hat{a}_{\vec{k},\alpha} \hat{a}_{-\vec{k},\alpha} \right],\end{aligned} \quad (6.48)$$

where $\sigma = i\mathcal{H}$. The previous quantum mechanical analysis is now easily extended to the case of the relic gravitons by bearing in mind that the modes of the field with opposite three-momenta now operate in two different Hilbert subspaces [9].

VII. CONCLUDING REMARKS

In a classical perspective the information erasure always requires an energy cost and, according to the current lore, this general conclusion should hold in spite of the actual size of the underlying physical structure. If a certain system s interacts with a thermal environment e at a temperature T the heat flowing to the environment ΔQ_e should always exceed $-T\Delta S_s$ where ΔS_s is the change of the entropy of the system between the final and the initial state. To erase one bit of information the entropy must then decrease as $\Delta S_s = -\ln 2$ and the Landauer conjecture would then imply that $\Delta Q_e \geq T \ln 2$. The Landauer bound quantifies the minimum heat cost for obliterating information but the condition $\Delta Q_e \geq -T\Delta S_s$ is only restrictive if the entropy of

the system decreases (i.e. $\Delta S_s < 0$) whereas in the opposite physical situation (i.e. $\Delta S_s \geq 0$) the increment of S_s remains unconstrained. Since both conclusions should hold true for all physical structures (i.e. regardless of their respective sizes) we found appropriate to analyze them from a quantum mechanical viewpoint.

As the quantum regime is approached, the heat transfer and the entropy variations must be computed from a set of appropriately reduced density matrices. While in statistical mechanics we typically deal with collections of many particles (of the order of the Avogadro number) in quantum thermodynamics we may even focus on a countable number of elementary oscillators. We then considered a minimal framework where both the system and the environment are described by two quantum oscillators with the difference that, while the system is initially in a pure state, the environment is described by a mixed density operator with Bose-Einstein (geometric) weights. We then demonstrated that the lack of restrictions on the acquisition of the information disappears: if $\Delta S_s \geq 0$ the growth of the entropy of the system is bounded by the heat transferred to the environment according to the bound

$$\Delta S_s \leq \Delta Q_e / T,$$

where ΔS_s is the entropy variation of the system, ΔQ_e is the heat flowing to the environment and T is the corresponding temperature. The Hermitian interaction between the system and the environment guarantees that the entropy of the system increases and if we put together the condition obtained here with the Landauer bound we can then argue that

$$-\Delta Q_e / T \leq \Delta S_s \leq \Delta Q_e / T.$$

The inequality $\Delta S_s \leq \Delta Q_e / T$ expresses the bound discussed here while $-\Delta Q_e / T \leq \Delta S_s$ is just the standard form of the Landauer conjecture of Eq. (1.1). These two conditions could also be summarized by the requirement $T^2(\Delta S_s)^2 \leq (\Delta Q_e)^2$. The bound deduced here can also include a particle flow to the environment (i.e. $\Delta N_e \geq 0$) associated with a chemical potential $\mu_e \neq 0$,

$$\Delta S_s \leq [\Delta Q_e / T - (\mu_e / T) \Delta N_e].$$

.When $\Delta S_s \geq 0$ the growth of the entropy of the system is bounded as $T\Delta S_s \leq (\Delta Q_e - \mu_e \Delta N_e)$. There is finally a natural field-theoretical extension of the quantum mechanical considerations and it is related to the production of particles with spin 0 or 2. As we showed this generalization is also relevant for the relic gravitons in cosmological backgrounds.

Although from the classical viewpoint the acquisition of information and the related growth of the entropy of the system are never constrained, the quantum mechanical

perspective pursued here demonstrates instead the opposite: the increase of the entropy of the system is always limited by the heat transferred to the environment. Since the basic features of our findings depend on the Hermitian nature of the interaction that should also amplify the entropy of the system, it would be interesting to characterize the general classes of physical structures for which an increase of the entropy of the system is always limited by the heat (and, possibly, by the particles) flowing to the environment.

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DATA AVAILABILITY

No data were created or analyzed in this study.

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