

Nonfactorizing interface in the two-dimensional long-range Ising model

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The factorization proposal claims that the codimension one “pinning defect,” on which a local relevant operator is integrated, factorizes the space into two halves in general conformal field theories in the infrared limit. In this paper, we study a two-dimensional long-range Ising model at criticality with a line defect or an interface, which physically corresponds to changing the local temperature on it. We show that in the perturbative regime, it is not factorizing even in the infrared limit. An intuitive explanation of the nonfactorization is that the long-range Ising model is equivalent to a local conformal field theory in higher dimensions. In this picture, the space is still connected through the “extra dimension” across the defect line.

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I. INTRODUCTION

What happens if a heater (or cooler) is applied on a codimension one hyperplane in the critical Ising model? Will the criticality remain on this interface, or will it be destroyed? In the two-dimensional critical short-range Ising model, changing the local temperature on the line is marginal; Affleck and Oshikawa gave an exact description, where the defect line remains critical for a whole range of temperatures [1,2]. Furthermore, they showed that a localized renormalization group (RG) flow triggered by a relevant deformation from the local magnetic field makes the entire space factorize into two halves in the infrared (IR), with no energy exchange between them [3].

In the three-dimensional Ising model, changing the temperature or applying the magnetic field locally on the interface is relevant, and we expect that the interface will factorize the space into halves in the IR. Since so far the bulk theory can be studied only through numerical analysis such as the Monte-Carlo simulations [4], conformal bootstrap [5–9], or fuzzy sphere [10,11], the explicit demonstration may be difficult.¹ Nonetheless, in the

closely related $O(N)$ model with a planar interface, some evidence of exhibiting such a factorization has been reported in [15,16].

More recently, Popov and Wang proposed that the factorization property should hold for a conformal field theory (CFT) in general dimensions with a codimension one “pinning defect” [17].² They gave a formal proof and claimed that an RG flow triggered by the pinning field deformation shall lead the interface to factorize the space in the IR limit.

These developments motivate us to ask if the factorization is an intrinsic property for a pinning-type interface CFT [1,2,42–45] in the IR limit. We pay attention to a special type of CFT in two dimensions, the long-range Ising (LRI) model, which is nonlocal and does not possess the Virasoro symmetry.³ Surprisingly, we will show that changing the local temperature on the interface does not factorize the space. An intuitive explanation of the nonfactorization is that the LRI model is equivalent to a local CFT in higher dimensions. In this picture, the space is still connected through the “extra dimension” across the defect line.

The primary focus of our letter is to study the Landau-Ginzburg (LG) description of the LRI model with the introduction of a localized massive deformation on the codimension one defect

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¹There has been interesting works studying the boundary three-dimensional Ising model from the conformal bootstrap perspective appeared in [12], and recent works using the fuzzy sphere can be found in [13,14].

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²A partial list of “pinning” type models can be found in [18–41].

³The absence of the Virasoro symmetry is due to the lack of the local stress tensor: For this reason, it is sometimes called conformal theory instead of conformal *field* theory. The proof of the Virasoro symmetry by Polchinski [46–48] does not apply due to the absence of the local stress tensor.

$$S = \frac{\mathcal{N}_s}{2} \int d^d y \phi \mathcal{L}_s \phi + \frac{h_0}{4!} \int d^d y \phi^4 + \frac{g_0}{2} \int d^r z \phi^2, \quad (1)$$

where the kinetic term is the “fractional Laplacian” $\mathcal{L}_s = (-\partial^2)^{s/2}$. We assume s is slightly above $d/2$, $s = d/2 + \varepsilon$ ($0 < \varepsilon \ll 1$), such that the interaction ϕ^4 is weakly relevant and can be studied perturbatively in ε .⁴ The classical dimension of the LG field is $\Delta_\phi = \frac{d-s}{2}$. Eventually, we will be interested only in the codimension one case $r = d - 1$. For the deformation by ϕ^2 on the defect to be relevant $r > d - s$, this only leaves us with the choice $r = d - 1 = 1$ or $r = d - 1 = 2$ given $d, r \in \mathbb{Z}^+$ and $d < 4$. In this paper, we focus on the former choice so that the coupling h_0 and g_0 can be perturbative simultaneously.⁵ This choice of $d = 2$, $r = 1$ physically corresponds to the two-dimensional LRI model with a line defect. Note that within physical dimensions, we can still tune ε so that perturbative computation is valid.

While the LRI model has a nonlocal kinetic term, the so-called Caffarelli-Silvestre (CS) trick [54] can be applied to map the same system to a *local* conformal field theory in $(2-s)$ -higher dimensions with (multiple) defects. With this trick, the LG field $\phi(y)$ is lifted to the field $\Phi(x_\perp, y)$, where $\phi(y)$ can be viewed as the Dirichlet boundary condition $\phi(y) = \Phi(x_\perp = 0, y)$. The equivalent description is

$$S' = \frac{1}{2} \int d^D x (\partial_M \Phi)^2 + \frac{h_0}{4!} \int_{x_\perp=0} d^d y \Phi^4 + \frac{g_0}{2} \int_{x_\perp=0, y_\perp=0} d^r z \Phi^2, \quad (2)$$

where $D = d + 2 - s = \frac{d}{2} + 2 - \varepsilon$. The coordinates here are denoted as $x = (x_\perp, y)$ where $x_\perp$ are the “extra” dimensions to the original d -dimensional LRI theory, and $y = (y_\perp, z)$, where $y_\perp$ are the transverse directions to the defect in the LRI model.⁶ In this paper, we use this equivalent formulation to employ the local field theory techniques.

⁴The model without the line defect is the LRI model, for instance, reviewed in [49] and earlier works [50,51]. For $d/2 < s < s^*$, the critical theory is nontrivial and non-Gaussian, which is supposed to be in the same universality class as the critical point of the LRI lattice model. We are interested in the region where s is slightly above $d/2$, while an up-to-date discussion on the transition from the LRI model to the short-range Ising (SRI) model where s approaches s^* from below appeared in [52,53].

⁵We thank Tatsuma Nishioka for discussion on the latter possibility.

⁶This model, in the end, coincides with the composite defect model studied in [55] by one of the authors.

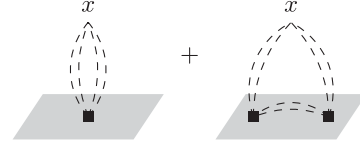


FIG. 1. Diagrams contributing to the one-loop order of the one-point function $\langle \Phi^4(x) \rangle$, with the solid square being the vertex of the quartic coupling h_0 .

II. LONG-RANGE ISING MODEL

To start with, we turn off the localized massive deformation g_0 for the moment and study the bulk theory. Such a theory can be renormalized by keeping the following one-point function finite as $\varepsilon \rightarrow 0$

$$\langle [\Phi^4](x) \rangle = \text{finite}, \quad (3)$$

⁷ which involves two diagrams up to the second order in the coupling h_0 , as in Fig. 1. In a free bulk, the wave function renormalization is trivial, such that the bulk operator $\Phi^4 = [\Phi^4]$ up to normal ordering. From now on, normal ordering of the composite operators is always assumed: we do not include any self-contractions of Φ in perturbative expansions. The renormalization condition, for which we use minimal subtraction (MS) scheme, gives rise to the RG beta function for the coupling h

$$\beta_h = -2\varepsilon h + \frac{3\Gamma(\frac{d}{4})^2}{16\pi^2\Gamma(\frac{d}{2})} h^2 + \dots, \quad (4)$$

which allows a nontrivial IR fixed point,

$$h_* = \frac{32\pi^2\Gamma(\frac{d}{2})}{3\Gamma(\frac{d}{4})^2} \varepsilon + O(\varepsilon^2). \quad (5)$$

At this fixed point, the theory describes the critical d -dimensional LRI model, which is conformally invariant [49]. The LG field ϕ and composite operators ϕ^n living on the plane $x_\perp = 0$ are subject to the Dirichlet boundary conditions $\phi^n(y) = \Phi^n(y, x_\perp = 0)$. They acquire an anomalous dimension at the LRI fixed point, which can be obtained in terms of their multiplicative renormalization factor Z_{ϕ^n} , defined as $\phi^n \equiv Z_{\phi^n}[\phi^n]$ with $[\phi^n]$ being the renormalized composite operators. Perturbatively the renormalization factor can be computed as

$$Z_{\phi^n} = 1 - \frac{(n-1)n\Gamma(\frac{d}{4})^2}{64\pi^2\Gamma(\frac{d}{2})} \frac{h}{\varepsilon} + \dots, \quad (6)$$

which gives the anomalous dimension at the fixed point

⁷For the parent field Φ , as the wave function renormalization is trivial, we do not distinguish the bare field and the renormalized one.

$$\begin{aligned}\gamma_{\phi^n}^* &= \left. \frac{\partial \ln Z_{\phi^n}}{\partial \ln \mu} \right|_{h=h_*} = \frac{(n-1)n\Gamma(\frac{d}{2})^2}{32\pi^2\Gamma(\frac{d}{2})} h_* + \dots \\ &= \frac{(n-1)n}{3} \varepsilon + O(\varepsilon^2).\end{aligned}\quad (7)$$

Among those composite operators, there are two protected ones, $[\phi]$ and $[\phi^3]$ [49]. At this fixed point, (i) $\gamma_{\phi}^* = 0$ vanishes to all-loop order, due to the nonlocality of its kinetic term, such that wave function renormalization is simply the identity (proved in [56]); (ii) $\gamma_{\phi^3}^* = 2\varepsilon$ to all-loop order, this is a result of the e.o.m. from the D -dimensional local field theory point of view

$$\square\Phi(x) = \frac{h}{3!} \delta^{(D-d)}(x_\perp) [\phi^3](y) \Rightarrow \Delta_{\phi^3} = \frac{d}{2} - \frac{s}{2}. \quad (8)$$

These two operators $[\phi]$ and $[\phi^3]$ form a shadow pair in the d -dimensional LRI theory [52],

$$\Delta_{\phi^3} + \Delta_{\phi} = 2\Delta_{\phi} + s = d. \quad (9)$$

III. NONLOCAL GAUSSIAN CFT

It is also interesting to consider the localized massive deformation alone by turning off the quartic coupling h_0 . In this case, the model (2) becomes a D -dimensional free scalar theory with a localized mass term and d becomes fictitious. This is a Gaussian theory that can be solved completely. Using a similar renormalization procedure by requiring $\langle[\Phi^2](x)\rangle = \text{finite}$ as $\varepsilon \rightarrow 0$, we sum up all the looplike diagrams depicted in Fig. 2

$$\begin{aligned}\langle[\Phi^2](x)\rangle &= -g_0 \int \frac{d^D p}{(2\pi)^D} \int \frac{d^D p'}{(2\pi)^{D-r}} \frac{e^{i(p-p')x}}{p^2} \\ &\times \frac{1}{1 + g_0 \frac{a_{D,r}}{\bar{p}^{2+r-D}}} \frac{\delta^{(r)}(\bar{p}' - \bar{p})}{p'^2}.\end{aligned}\quad (10)$$

Note that since the bulk is a free theory, this bulk operator does not have any wave function renormalization, namely $\Phi^2 = [\Phi^2]$ up to normal ordering. The “barred” momenta are the parallel components in the r -dimensional space, and $a_{D,r} \equiv (4\pi)^{\frac{r-D}{2}} \Gamma(\frac{2+r-D}{2})$. We focus on the case $\varepsilon = 2 - D + r > 0$ so that the localized “mass” deformation is relevant. The renormalization condition (10) leads to the exact beta function for the localized “mass” deformation

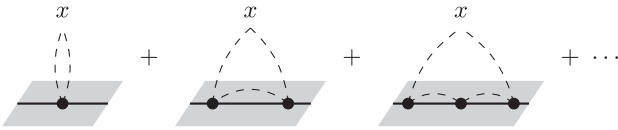


FIG. 2. Looplike diagrams contributing to the one-point function $\langle\Phi^2(x)\rangle$, with the solid dot representing the vertex of the quadratic coupling g_0 .

$$g_0 = \mu^\varepsilon \frac{g}{1 - \frac{g}{2\pi\varepsilon}} \Rightarrow \beta_g = \frac{g(g - 2\pi\varepsilon)}{2\pi}, \quad (11)$$

admitting an exact nontrivial IR fixed point at

$$g_* = 2\pi\varepsilon. \quad (12)$$

At this fixed point, the field $\hat{\phi}$, defined as the Dirichlet boundary condition of the field Φ on the r -dimensional hyperplane $\hat{\phi}(z) = \Phi(z; x_\perp = 0, y_\perp = 0)$, acquires an anomalous dimension, though the kinetic term remains nonlocal. Its anomalous dimension can be obtained by calculating the two-point function of the renormalized field $[\hat{\phi}]$. To all-loop order, this gives

$$\langle[\hat{\phi}](z_1)[\hat{\phi}](z_2)\rangle = \frac{1}{Z_{\hat{\phi}}^2} \int \frac{d^r \bar{p}}{(2\pi)^r} \frac{e^{iz_{12}\bar{p}}}{\frac{\bar{p}^{2+r-D}}{a_{D,r}} + g_0}. \quad (13)$$

This calculation can be done, first in the parallel momentum space using the propagator (I.4) and then Fourier transforms back to the position space (see Sec. I of the Supplemental Materials [57]). The multiplicative renormalization factor to all-loop order is obtained as

$$Z_{\hat{\phi}} = 1 - \frac{g}{2\pi\varepsilon} = \frac{\mu^\varepsilon g}{g_0}, \quad (14)$$

using the fixed point value, (12) gives the anomalous dimension of $\hat{\phi}$

$$\gamma_{\hat{\phi}}^* = \varepsilon. \quad (15)$$

This is consistent with the large-distance behavior⁸ of the all-loop correlator (13). By expanding it for small parallel momentum $\bar{p} \ll \mu$,

$$\langle[\hat{\phi}](z_1)[\hat{\phi}](z_2)\rangle \stackrel{z_{12} \gg 1/\mu}{\sim} \frac{\Gamma(r/2)}{4\pi^{1+r/2}} |z_{12}|^{-r-\varepsilon}. \quad (16)$$

The anomalous dimension can be read immediately, giving the same value as (15). Alternatively, the anomalous dimension of $\hat{\phi}$ can be read directly from the e.o.m. $\square\Phi(x) = g\delta^{(D-r)}(x_\perp; y_\perp)[\hat{\phi}](z)$, which gives (15) as well, since the operator $\hat{\phi}$ is protected. Noting that the operators Φ and $\hat{\phi}$ satisfy a shadow relation in the r -dimensional theory at the IR fixed point,

$$\Delta_{\Phi} + \Delta_{\hat{\phi}} = r. \quad (17)$$

One can also obtain the anomalous dimension for the composite operators $\hat{\phi}^n$

⁸This integral can also be explicitly performed by using the Mittag-Leffler function [58] (in particular when $r = 1$).

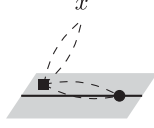


FIG. 3. The mixed diagram contributing to $\langle \Phi^2(x) \rangle$ at the second order in the couplings, with the square and dot representing the vertices of h_0 and g_0 respectively.

$$Z_{\hat{\phi}^n} = (Z_{\hat{\phi}})^n \Rightarrow \gamma_{\hat{\phi}^n}^* = n\epsilon. \quad (18)$$

As the two-point function of $\hat{\phi}^n$ factorizes into the n th power of the two-point function of $\hat{\phi}$, $\langle \hat{\phi}^n(z_1) \hat{\phi}^n(z_2) \rangle = (\langle \hat{\phi}(z_1) \hat{\phi}(z_2) \rangle)^n$.

Let us briefly discuss the physical implications. Unlike the short-range Gaussian theory, adding localized mass term does not gap out the defect line, and a nontrivial critical behavior remains. In the fictitious d -dimensional nonlocal Gaussian theory, we can show that the interface (i.e., $d = 2$ and $r = 1$) is not factorizing, for instance setting $h = 0$ in Sec. II of the supplemental materials [57]. The essence of such nonfactorization roots in the fractional kinetic term, as can be seen in a simpler quantum mechanical system illustrated in Sec. III of the Supplemental Materials [57]. This should be contrasted with the short-range Gaussian theory, where the massive deformation on the interface factorizes the space in the IR limit as expected from the factorization proposal.

IV. LONG-RANGE ISING WITH A LINE DEFECT

By turning on simultaneously the quartic coupling h_0 and the localized massive deformation g_0 , we can now study if the codimension one defect will factorize the space in halves in the LRI model at the criticality. While the beta function for h remains unchanged as in (4), the beta function for g receives corrections from h . At the second order in the couplings, an extra mixed diagram (Fig. 3) contributes to the beta function of g (taking $r = d - 1 = 1$)

$$\beta_g = -\epsilon g + \frac{gh + 8g^2}{16\pi}. \quad (19)$$

Then the IR fixed point of g is shifted by the fixed point value of h in (5)

$$g_* = 2\pi\epsilon - \frac{h_*}{8} = \frac{2\pi\epsilon}{3} + O(\epsilon^2). \quad (20)$$

The bulk composite operators ϕ^n remain the same as in the LRI model, but for those composite defect operators $\hat{\phi}^n$ living on the line, their anomalous dimensions get further corrected by the quartic coupling as compared to the nonlocal Gaussian case. At the new fixed point,

$$\gamma_{\hat{\phi}^n}^* = \frac{(n-1)n\Gamma(\frac{d}{4})^2}{32\pi^2\Gamma(\frac{d}{2})} h_* + \frac{ng_*}{2\pi} = \frac{n^2}{3}\epsilon + O(\epsilon^2), \quad (21)$$

scales as n^2 at the leading order in ϵ .

Given these composite operators on the line defect, we now study the bulk-defect two-point functions. This gives a perturbative check of the conformal invariance of the line defect, which will be detailed in the next section. Assuming the system is conformally invariant, these correlators should have the following form⁹

$$\langle [\phi^n](y) [\hat{\phi}^n](z) \rangle = \frac{\mathcal{N}_{\phi^n} \mathcal{N}_{\hat{\phi}^n} \mathcal{B}_{\phi^n \hat{\phi}^n}}{|y_\perp|^{\gamma_{\phi^n} - \gamma_{\hat{\phi}^n}} |y - z|^{2\Delta_{\hat{\phi}^n}}}, \quad (22)$$

where $\mathcal{N}_{\phi^n}$ and $\mathcal{N}_{\hat{\phi}^n}$ are the two-point function normalizations for ϕ^n and $\hat{\phi}^n$, respectively,

$$\langle [\phi^n](y_1) [\phi^n](y_2) \rangle = \frac{\mathcal{N}_{\phi^n}^2}{|y_{12}|^{2(n\Delta_\phi + \gamma_{\phi^n})}}, \quad (23)$$

$$\langle [\hat{\phi}^n](z_1) [\hat{\phi}^n](z_2) \rangle = \frac{\mathcal{N}_{\hat{\phi}^n}^2}{|z_{12}|^{2(n\Delta_{\hat{\phi}} + \gamma_{\hat{\phi}^n})}}. \quad (24)$$

Here the two-point function for ϕ^n is the one without the presence of the line defect. The additional factor $\mathcal{B}_{\phi^n \hat{\phi}^n}$ encodes the bulk-to-defect OPE data. The explicit computation shows that to the first order corrections in ϵ , the bulk-defect two-point function exhibits the conformal form, with the related conformal data given as

$$\mathcal{N}_{\phi^n}^2 = \frac{n!}{(4\pi)^n} \left(1 - \frac{n(2n-5)\epsilon}{6} (\ln(4\pi) + \gamma) \right), \quad (25)$$

$$\mathcal{N}_{\hat{\phi}^n}^2 = \frac{n!}{(4\pi)^n} \left(1 - \frac{n(2n-3)\epsilon}{6} (\ln(4\pi) + \gamma) \right), \quad (26)$$

$$\mathcal{B}_{\phi^n \hat{\phi}^n} = 1 - \frac{n\epsilon}{3} \ln 2, \quad (27)$$

where γ is the Euler-Mascheroni constant. This provides the first nontrivial check of conformal invariance of the line defect, whose formal proof will be given in the following section.

V. NONFACTORIZING CONFORMAL INTERFACE

The two-dimensional LRI with a line defect discussed above gives rise to a scale-invariant interface theory, as indicated by the existence of a nontrivial IR fixed point

⁹Note here that the scaling operators $[\phi]^n$ are inserted away from the interface. A special case is $n = 4$, where a true scaling operator is a mixture of $[\phi]^4(y)$ and $\delta(y_\perp)[\hat{\phi}^2](z)$ discussed in detail in Sec. IVA of the Supplemental Materials [57].

(20) with the line deformation. In fact, such an interface theory is conformally invariant, *albeit* only in the global sense due to its nonlocality. The conformal invariance of the two-dimensional LRI model itself was proved in [49]. Adding an interface breaks part of its translational symmetries as well as special conformal symmetries. However, the residual conformal symmetries, as we shall show, are still preserved on the interface. Starting directly from the LRI model would be hard, as a lack of a conserved local stress tensor [52]. From the higher-dimensional theory perspective, however, the conformal invariance of the LRI model with the line defect becomes relatively straightforward.

We follow the procedures in [49,59]. The idea is that the conformal Ward-Takahashi identities are broken along the RG flow, but should be restored at the fixed point due to the lack of the candidate for the “virial current” that is necessary to realize scale invariance without conformal invariance. This suggests that the brokenness is likely to be proportional to the beta functions of the couplings. With the detailed steps illustrated in Sec. IV of the supplemental materials [57], the conformal Ward-Takahashi identities on the projected 2-dimensional space ($x_\perp = 0$) can be written as

$$\sum_{i=1}^n [y_i \cdot \partial_{y_i} + \Delta_\phi] G_n(y_1, \dots, y_n) = \frac{\beta_h}{4!} \mu^{2\epsilon} \int d^2 y G_n(y_1, \dots, y_n; [\phi^4]) + \frac{\beta_g}{2} \mu^\epsilon \int d^2 y \delta(y_\perp) G_n(y_1, \dots, y_n; [\hat{\phi}^2]), \quad (28)$$

$$\sum_{i=1}^n \left[(2y_i^\mu y_i^a - \delta^{\mu a} y_i^2) \frac{\partial}{\partial y_i^\mu} + 2\Delta_\phi y_i^a \right] G_n(y_1, \dots, y_n) = 2 \frac{\beta_h}{4!} \mu^{2\epsilon} \int d^2 y y^a G_n(y_1, \dots, y_n; [\phi^4]) + 2 \frac{\beta_g}{2} \mu^\epsilon \int d^2 y y^a \delta(y_\perp) G_n(y_1, \dots, y_n; [\hat{\phi}^2]), \quad (29)$$

where $G_n(y_1, \dots, y_n)$ is the renormalized n -point function of the LG field $[\phi]$. The μ , ν , and a indices run over the 2-dimensional bulk and the interface respectively. The first equality is related to the scaling transformation, while the second one shows the response to the special conformal transformation along the interface. At the IR fixed point, the vanishing beta functions restore the scale and special conformal Ward-Takahashi identities, hence proving the conformal invariance.¹⁰

Now we pay special attention to the displacement operator of the interface [44,63,64]. Though a locally conserved stress tensor is absent in the LRI model, the displacement operator can still be constructed from the higher-dimensional theory. It appears in the divergence of the stress tensor as follows

$$\partial^M T_{MN} = -E_N + \delta^{(D-d)}(x_\perp) D_N + \delta^{(D-r)}(x_\perp; y_\perp) \bar{D}_N, \quad (30)$$

with N running over the full D -dimensional space. The first term E_N vanishes on-shell, and the two delta-localized terms give rise to two displacement operators D_N and $\bar{D}_N$, of which $\bar{D}_N$ is the relevant one for the LRI interface, while D_N displaces the full interface LRI CFT inside the ambient

space. All three are renormalized operators whose explicit expressions are given as

$$E_N = \partial_N \Phi \left(-\square \Phi + \delta^{(D-d)}(x_\perp) \frac{h}{3!} [\phi^3] + \delta^{(D-r)}(x_\perp; y_\perp) g[\hat{\phi}] \right), \quad (31)$$

$$D_N = \left(\frac{h}{4!} [\partial_N \phi^4] + \delta^{(d-r)}(y_\perp) \frac{g}{2} [\partial_N \hat{\phi}^2] \right) \delta_{N,n_1}, \quad (32)$$

$$\bar{D}_N = \frac{g}{2} [\partial_N \hat{\phi}^2] \delta_{N,n_2}. \quad (33)$$

The terms inside the bracket of (31) are e.o.m. related, n_1 denotes the normal direction towards the ambient space while n_2 denotes the normal direction to the LRI interface. The two displacement operators are protected with exact dimensions,

$$\Delta_{D_N} = d + 1 = 3, \quad \Delta_{\bar{D}_N} = r + 1 = 2, \quad (34)$$

which agree with the perturbative checks through their anomalous dimensions. Considering the renormalization factor of $\partial_{n_2} \hat{\phi}^2 = Z_{\bar{D}_N} [\partial_{n_2} \hat{\phi}^2]$, at the linear order in the couplings

$$Z_{\bar{D}_N} = 1 - \frac{g}{2\pi\epsilon} - \frac{h}{32\pi\epsilon} \Rightarrow \gamma_{\bar{D}_N}^* = \epsilon, \quad (35)$$

¹⁰One might worry about a potential “ ∞ ” coming from the integral part, such that $0 \times \infty \sim O(1)$. In [49], it was argued that a cancellation should happen between the UV and IR parts to ensure that the integral times the scale is actually of $O(1)$.

consistent with its protected dimension. Renormalizing $\partial_{n_1}\phi^4$ involves a mixture of the renormalized operators $[\partial_{n_1}\phi^4]$ and $[\partial_{n_1}\hat{\phi}^2]$

$$\begin{pmatrix} \frac{1}{4!}\partial_{n_1}\phi^4 \\ \delta(y_\perp)\frac{1}{2}\partial_{n_1}\hat{\phi}^2 \end{pmatrix} = Z_{\text{mix}} \begin{pmatrix} \frac{1}{4!}[\partial_{n_1}\phi^4] \\ \delta(y_\perp)\frac{1}{2}[\partial_{n_1}\hat{\phi}^2] \end{pmatrix}, \quad (36)$$

with the renormalization factor matrix given as

$$Z_{\text{mix}} = \begin{pmatrix} 1 - \frac{3h}{32\pi\epsilon} & -\frac{g}{32\pi\epsilon} \\ 0 & 1 - \frac{g}{2\pi\epsilon} \end{pmatrix}. \quad (37)$$

The off-diagonal entry cancels exactly the mixed coupling term from the renormalization factor in g_0 at this order, such that D_N remains finite even when written in terms of the bare fields and couplings. The eigenvalues of the anomalous dimension matrix $\gamma_{\text{mix}} = \partial \ln Z_{\text{mix}} / \partial \ln \mu$ give the anomalous dimension of D_N and $[\partial_{n_1}\hat{\phi}^2]$, 2ϵ and $\epsilon/3$ respectively, in the leading order in ϵ . This leading correction is consistent with the protected dimension of D_N as $(\frac{D}{2} - 1) \times 4 + 1 + 2\epsilon = 3 = \Delta_{D_N}$.

Finally, we would like to study the (non)factorizing property of our interface in the IR limit. The residual conformal symmetries fix the bulk two-point function of scalar operators on the two sides of the interface to be $\langle O(y_1)O(y_2) \rangle = \frac{1}{|y_{1,\perp}|^\Delta |y_{2,\perp}|^\Delta} G(\xi)$, where $G(\xi)$ is a function of the cross-ratio $\xi = \frac{(y_1 - y_2)^2}{4|y_{1,\perp}||y_{2,\perp}|}$ [65–69] and factorization requires $G(\xi) = \text{const.}$ However, it is hardly satisfied for any bulk operators in the LRI theory.

Taking $\mathcal{O}(y) = [\phi^n](y)$, the perturbative calculations show $G(\xi) = \frac{C_{\mathcal{O}\mathcal{O}}}{\xi^\Delta} + \mathcal{O}(h, g)$ as in Sec. II of the supplemental materials [57], where the h -term shifts the dimension Δ and the g -term in the $\xi \rightarrow 0$ limit is related to the leading correction to the bulk-to-defect coefficient $\mathcal{B}_{\phi^n \hat{\phi}^n}$ as in (26). Obviously the factorization condition cannot be satisfied perturbatively. Thus our interface seems not to comply with the factorization proposal. The technical reason why the proof in [17] does not apply here is that the relevant deformation $g\phi^2$ has a nontrivial fixed point at a finite value of g ; taking $g \rightarrow \infty$ is not guaranteed as what has been assumed in their proof. More physically, the space is not factorized because the two halves can be still connected via the “extra dimensions.”

Let us comment on the relation between the displacement operator and the factorization property. The explicit computation shows that the two-point function coefficient of the displacement operator is perturbatively small of $\mathcal{O}(\epsilon^2)$, which should be in accordance with the nonfactorization property of our interface. We suspect that the Zamolodchikov norm of the displacement operator (called C_D in the literature) may serve as a diagnostic of the factorization property of the interface. One can expect once this coefficient saturates its upper bound, the translational symmetry would be maximally broken, hence the factorization. It is the case in two-dimensional interface CFTs with local Virasoro symmetries, where factorization indicates that $C_D = 2(c_L + c_R)$ [3,70], saturating its upper bound. However, how this criterion can apply to a nonlocal and/or higher-dimensional interface CFT requires further exploration.

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DATA AVAILABILITY

No data were created or analyzed in this study.

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