

Bessel-Hagen currents for the Fierz-Pauli action

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For electromagnetism in Minkowski spacetime, the Bessel-Hagen method gives a particularly direct Noetherian derivation of the standard gauge-invariant energy-momentum tensor. The key step is to supplement the form variation generated by an infinitesimal coordinate transformation with a compensating electromagnetic gauge transformation. In this paper we ask whether the same idea can be applied to the massless spin-2 field described by the Fierz-Pauli action. We first prove that no nonzero local tensor quadratic in first derivatives of the symmetric field $h_{\mu\nu}$ can be strictly invariant under the spin-2 gauge transformation $h_{\mu\nu} \mapsto h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$; the direct electromagnetic analogue of the Bessel-Hagen construction therefore cannot exist. Once the inexact nature of the Fierz-Pauli gauge symmetry is treated correctly, however, the Bessel-Hagen construction does produce a gauge-invariant equivalence class of Noether currents. Changing the compensating spin-2 gauge parameter changes the current only by terms proportional to the Fierz-Pauli field equations; performing an independent spin-2 gauge transformation on $h_{\mu\nu}$ changes the current only by a trivial current given by the divergence of an antisymmetric superpotential plus field-equation terms. This provides the natural spin-2 analogue of Bessel-Hagen's electromagnetic construction, but only in the quotient space of conserved currents, and not as a preferred local gauge-invariant energy-momentum tensor.

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I. INTRODUCTION

It is often stated that the Fierz-Pauli action [1] is the unique special relativistic action for a free massless spin-2 field $h_{\mu\nu} = h_{\nu\mu}$. More precisely, it is unique up to an overall constant and a total divergence among Lorentz-invariant Lagrangian densities quadratic in first derivatives of $h_{\mu\nu}$ whose action is invariant under the gauge transformation

$$h_{\mu\nu} \rightarrow h'_{\mu\nu} = h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu. \quad (1)$$

Here $\xi_\mu(x)$ is an arbitrary covector gauge parameter on Minkowski spacetime; in the present discussion it is not to be interpreted as the generator of a coordinate transformation.

The question considered here is whether the Bessel-Hagen method [2], which gives the physical electromagnetic

energy-momentum tensor directly from Noether's first theorem, can similarly be used to obtain a gauge-invariant energy-momentum tensor for the Fierz-Pauli field. Since two different vector fields occur in the discussion, we shall write infinitesimal coordinate transformations as

$$x'^\mu = x^\mu + \zeta^\mu(x), \quad (2)$$

whereas the spin-2 gauge transformation will be parametrized by $\xi_\mu(x)$. We restrict attention to global Poincaré transformations, for which

$$\zeta^\mu(x) = a^\mu + \omega^\mu{}_\nu x^\nu, \quad \omega_{\mu\nu} = -\omega_{\nu\mu}. \quad (3)$$

We use the metric signature $(+ - - -)$ throughout.

The result obtained below has two parts. First, there is no local, pointwise gauge-invariant tensor $\tau_{\mu\nu} \sim \partial h \partial h$ for the Fierz-Pauli field. The closest spin-2 precedent for this statement is Deser and Henneaux [3], who showed that the Abelian spin-2 stress tensor is conserved on the linearized Einstein shell but is neither gauge invariant nor locally improvable to a gauge-invariant tensor at the usual derivative order, while the integrated Poincaré charges remain gauge invariant. They also emphasise that genuine local

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gauge-invariant quantities must be built from the linearized Riemann tensor, for instance the linearized Bel-Robinson tensor. Magnano and Sokolowski [4] later established a closely related no-go theorem for the metric, or variational, energy-momentum tensor of a gauge-invariant spin-2 field, on which Padmanabhan relies in his review [5]; see also the textbook discussions in Refs. [6,7]. We give a short and explicit proof in Sec. IV via the observation that ∂h at a point can be set to zero by a spin-2 gauge transformation, which fixes the geometric content of the obstruction and connects directly to the Bessel-Hagen calculation.

Second, once the inexact gauge symmetry of the Fierz-Pauli density is treated correctly, the Bessel-Hagen method nevertheless produces a well-defined gauge-invariant equivalence class of Noether currents. The equivalence relation employed here, namely currents modulo divergences of antisymmetric superpotentials and terms proportional to the field equations, is the standard one in the characteristic local cohomology of Barnich, Brandt and Henneaux (BBH) [8–10]. At linearized order the non-uniqueness of the canonical,¹ Hilbert [12], Belinfante [13,14] and Landau-Lifshitz [15] expressions modulo such terms has been examined in detail by Baker [16–19], who show that infinitely many “improved” expressions can be obtained from the canonical tensor by appending superpotential and on-shell terms; a complementary parametric classification of Noetherian energy-momentum tensors for linearized gravity has been given by Taylor and Baker [11].

The new content of the present paper is the identity² $J_{\zeta,\xi}^\mu = J_{\zeta,0}^\mu - 2\mathcal{E}^{\mu\nu}\xi_\nu$, where $\mathcal{E}^{\mu\nu}$ is the Euler-Lagrange expression of the Fierz-Pauli action, i.e. the linearized Einstein tensor up to overall normalization. This identity is obtained from the inexact Bessel-Hagen boundary term K_ξ^μ , together with its interpretation as a gauge-invariant current class rather than a preferred local tensor. To the best of our knowledge, the literature does not contain the corresponding *inexact* Bessel-Hagen calculation for the Fierz-Pauli action, in which a single compensating spin-2 gauge parameter ξ_μ is permitted in the form variation and the non-vanishing gauge boundary term K_ξ^μ is included consistently in the Noether construction. Two features of this calculation are worth flagging. (i) The ξ -dependent piece $-2\mathcal{E}^{\mu\nu}\xi_\nu$ has precisely the weakly vanishing form familiar from the “improvement” literature, but here it has a manifestly Noetherian origin; it comes from the gauge boundary term K_ξ^μ of the inexact gauge symmetry, not from

an *ad hoc* improvement appended to the canonical tensor. (ii) Within this construction, the compensating gauge parameter ξ_μ cannot select a preferred gauge-invariant representative: it merely interchanges representatives of the same current class.

Baker *et al.* [18], in Sec. 5 of their paper, identify the application of the inexact-symmetry Bessel-Hagen method to Fierz-Pauli as the subject of future work; Taylor and Baker [11], in their Sec. 4.7, likewise explicitly leave the treatment of non-exact gauge symmetries by Bessel-Hagen’s method to future work. The present paper supplies that calculation. A close analogue had previously been carried out by Baker and Kuzmin [20] for the exactly gauge-invariant case of linearized Gauss-Bonnet gravity, where the higher-derivative, curvature-based structure of the action allows the Bessel-Hagen procedure to yield a strictly gauge-invariant energy-momentum tensor even though the corresponding spin-2 form variation is not itself gauge invariant. The Fierz-Pauli case differs essentially because the action and current are built at the first-derivative level, where ∂h is connection-like rather than curvature-like.

II. ELECTROMAGNETIC CASE

For electromagnetism in Minkowski spacetime, the action is

$$S = \int \mathcal{L}_{\text{EM}} d^4x, \quad \mathcal{L}_{\text{EM}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \quad (4)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Under an infinitesimal coordinate transformation $x'^\mu = x^\mu + \zeta^\mu(x)$, the form variation of the potential is

$$\delta_0^{(\zeta)} A_\mu = -\zeta^\nu \partial_\nu A_\mu - A_\nu \partial_\mu \zeta^\nu, \quad (5)$$

which leaves the action invariant when ζ^μ is given by (3). The corresponding Noether current is

$$J^\mu = \frac{\partial \mathcal{L}_{\text{EM}}}{\partial(\partial_\mu A_\sigma)} \delta_0^{(\zeta)} A_\sigma + \zeta^\mu \mathcal{L}_{\text{EM}}. \quad (6)$$

For translations $\zeta^\mu = a^\mu$, one has $J^\mu = -a^\nu t^\mu{}_\nu$, where

$$t^\mu{}_\nu = -F^{\mu\sigma} \partial_\nu A_\sigma + \frac{1}{4} \delta_\nu^\mu F^{\rho\sigma} F_{\rho\sigma} \quad (7)$$

is the canonical energy-momentum tensor, which is not invariant under the gauge transformation

$$A_\mu \rightarrow A_\mu + \partial_\mu \alpha. \quad (8)$$

¹For the usual $\Gamma\Gamma$ representative of the quadratic Einstein/Fierz-Pauli action, the canonical Noether tensor coincides with the Einstein pseudotensor at linearized order; see e.g., Taylor and Baker [11] just below their Eq. (5).

²The use of ζ and ξ (and, later, χ and η) as subscripts or superscripts denotes functional dependence on these parameters and should not be confused with tensor indices.

The Bessel-Hagen observation is that the form variation appearing in Noether's first theorem may be replaced by any variation that leaves the action invariant, possibly up to a surface term. Since the electromagnetic action is also invariant under (8), one may instead use

$$\delta_0 A_\mu = -\zeta^\nu \partial_\nu A_\mu - A_\nu \partial_\mu \zeta^\nu + \partial_\mu \alpha. \quad (9)$$

Choosing

$$\alpha = A_\nu \zeta^\nu \quad (10)$$

gives the manifestly gauge-invariant form variation

$$\delta_0 A_\mu = \zeta^\nu F_{\mu\nu}. \quad (11)$$

The Noether current then becomes

$$J^\mu = \zeta^\nu \left(F^{\mu\sigma} F_{\nu\sigma} - \frac{1}{4} \delta_\nu^\mu F^{\rho\sigma} F_{\rho\sigma} \right) = -\zeta^\nu \tau^\mu{}_\nu, \quad (12)$$

where

$$\tau^\mu{}_\nu = - \left(F^{\mu\sigma} F_{\nu\sigma} - \frac{1}{4} \delta_\nu^\mu F^{\rho\sigma} F_{\rho\sigma} \right) \quad (13)$$

is the standard physical energy-momentum tensor of the electromagnetic field (the sign reflecting our metric convention). The important point is that the compensating gauge transformation makes the form variation itself gauge invariant. Since the other quantities appearing in the electromagnetic current are also gauge invariant, the resulting current is gauge invariant as a local expression.

Changing α in (9) also generates an equivalence class of conserved currents. Writing $J_{\zeta,\alpha}^\mu$ for the Noether current associated with a given α , and using the exact gauge invariance of the electromagnetic action,

$$J_{\zeta,\alpha}^\mu - J_{\zeta,0}^\mu = -F^{\mu\nu} \partial_\nu \alpha = -\partial_\nu (F^{\mu\nu} \alpha) + (\partial_\nu F^{\mu\nu}) \alpha, \quad (14)$$

which is the divergence of an antisymmetric superpotential $F^{\mu\nu} \alpha$ plus a term proportional to the Maxwell equations of motion. Changing α therefore shifts the current only within an equivalence class of Noether currents. What is special about the electromagnetic case is that the natural choice (10) produces a unique representative that is strictly gauge invariant as a local tensor. As we shall see, the Fierz-Pauli analogue treated below also admits an equivalence class of Noether currents but no such gauge-invariant local representative.

III. FIERZ-PAULI ACTION

For the massless spin-2 field we take the Fierz-Pauli density in the form³ [the Singh-Hagen tuned mass term [21], which gives the massive theory at the cost of breaking gauge invariance, is omitted throughout]:

$$\mathcal{L}_{\text{FP}} = \frac{1}{4} (-\partial_\lambda h \partial^\lambda h + \partial_\lambda h_{\alpha\beta} \partial^\lambda h^{\alpha\beta} - 2\partial_\lambda h_{\alpha\beta} \partial^\beta h^{\alpha\lambda} + 2\partial_\alpha h \partial_\beta h^{\beta\alpha}), \quad (15)$$

where $h = h^\mu{}_\mu$. Its variation may be written as

$$\delta \mathcal{L}_{\text{FP}} = \mathcal{E}^{\alpha\beta} \delta h_{\alpha\beta} + \partial_\mu \theta^\mu (\delta h), \quad (16)$$

where $\mathcal{E}^{\alpha\beta} \equiv \delta \mathcal{L}_{\text{FP}} / \delta h_{\alpha\beta}$ is the Euler-Lagrange expression⁴ of the Fierz-Pauli action, given explicitly in (19) below (it is the linearized Einstein tensor, whose vanishing gives the linearized Einstein equation for $h_{\alpha\beta}$), and

$$\theta^\mu (\delta h) = \Pi^{\mu|\alpha\beta} \delta h_{\alpha\beta}, \quad \Pi^{\mu|\alpha\beta} \equiv \frac{\partial \mathcal{L}_{\text{FP}}}{\partial (\partial_\mu h_{\alpha\beta})}. \quad (17)$$

Direct differentiation of (15) gives

$$\begin{aligned} \Pi^{\mu|\alpha\beta} &= \frac{1}{2} \partial^\mu h^{\alpha\beta} - \frac{1}{2} \eta^{\alpha\beta} \partial^\mu h - \partial^{(\alpha} h^{\beta)\mu} \\ &\quad + \frac{1}{2} \eta^{\alpha\beta} \partial_\lambda h^{\lambda\mu} + \frac{1}{2} \eta^{\mu(\alpha} \partial^{\beta)} h, \end{aligned} \quad (18)$$

where parentheses around indices denote symmetrization with unit weight, $A^{(\alpha\beta)} = \frac{1}{2} (A^{\alpha\beta} + A^{\beta\alpha})$. Since $\mathcal{L}_{\text{FP}}$ has no explicit dependence on $h_{\alpha\beta}$, the Euler-Lagrange expression is

$$\begin{aligned} \mathcal{E}^{\mu\nu} &= -\partial_\sigma \Pi^{\sigma|\mu\nu} = -\frac{1}{2} \square h^{\mu\nu} + \partial_\sigma \partial^{(\mu} h^{\nu)\sigma} - \frac{1}{2} \partial^\mu \partial^\nu h \\ &\quad + \frac{1}{2} \eta^{\mu\nu} (\square h - \partial_\sigma \partial_\rho h^{\rho\sigma}), \end{aligned} \quad (19)$$

which is the linearized Einstein tensor. The gauge invariance of the action under

³The form of $\mathcal{L}_{\text{FP}}$ given in (15) is unique modulo total divergences and the one-parameter family of linear field redefinitions $h_{\mu\nu} \rightarrow h_{\mu\nu} + \lambda \eta_{\mu\nu} h$ that mix the trace into the tensor. Such a redefinition rescales the trace as $h \rightarrow (1 + 4\lambda)h$ in four dimensions and correspondingly alters the coefficients of the second-order (kinetic) operator $\mathcal{O}$ defined by $\mathcal{E}^{\alpha\beta} = \mathcal{O}^{\alpha\beta\gamma\delta} h_{\gamma\delta}$ [cf. Eq. (19)], while leaving the massless spin-2 content unchanged. It is invertible except at the trace-removing value $\lambda = -\frac{1}{4}$, where $h \rightarrow 0$ and the map degenerates; this value is therefore excluded.

⁴Strictly, the variational-derivative notation $\delta/\delta h_{\alpha\beta}$ applies to functionals rather than to the local density $\mathcal{L}_{\text{FP}}$; here it is used as a shorthand for the Euler-Lagrange expression $\partial \mathcal{L}_{\text{FP}} / \partial h_{\alpha\beta} - \partial_\mu (\partial \mathcal{L}_{\text{FP}} / \partial (\partial_\mu h_{\alpha\beta}))$.

$$D_\xi h_{\alpha\beta} \equiv \partial_\alpha \xi_\beta + \partial_\beta \xi_\alpha \quad (20) \quad \xi_\mu \text{ just chosen),}$$

implies the Noether identity

$$\partial_\alpha \mathcal{E}^{\alpha\beta} \equiv 0. \quad (21)$$

Under a Poincaré coordinate transformation $x'^\mu = x^\mu + \zeta^\mu$, the form variation of $h_{\alpha\beta}$ is

$$\delta_0^{(\zeta)} h_{\alpha\beta} = -\zeta^\rho \partial_\rho h_{\alpha\beta} - h_{\rho\beta} \partial_\alpha \zeta^\rho - h_{\alpha\rho} \partial_\beta \zeta^\rho. \quad (22)$$

As in the electromagnetic case, the Bessel-Hagen observation is that the form variation appearing in Noether's first theorem may be replaced by any variation that leaves the action invariant. Since the Fierz-Pauli action is also invariant under the spin-2 gauge transformation (1), one may augment (22) by the compensating piece $D_\xi h_{\alpha\beta} = \partial_\alpha \xi_\beta + \partial_\beta \xi_\alpha$ and use

$$\begin{aligned} \delta_0 h_{\alpha\beta} = & -\zeta^\rho \partial_\rho h_{\alpha\beta} - h_{\rho\beta} \partial_\alpha \zeta^\rho - h_{\alpha\rho} \partial_\beta \zeta^\rho \\ & + \partial_\alpha \xi_\beta + \partial_\beta \xi_\alpha. \end{aligned} \quad (23)$$

We shall write this combined form variation compactly as

$$\Delta_{\alpha\beta}^{\zeta, \xi} \equiv -\mathcal{L}_\zeta h_{\alpha\beta} + D_\xi h_{\alpha\beta}, \quad (24)$$

where $\mathcal{L}_\zeta h_{\alpha\beta} = \zeta^\rho \partial_\rho h_{\alpha\beta} + h_{\rho\beta} \partial_\alpha \zeta^\rho + h_{\alpha\rho} \partial_\beta \zeta^\rho$ is the Lie derivative of $h_{\alpha\beta}$ regarded as a (0, 2) tensor on Minkowski spacetime, so that $-\mathcal{L}_\zeta h_{\alpha\beta} = \delta_0^{(\zeta)} h_{\alpha\beta}$ recovers the Poincaré piece (22).

The most immediate analogue of the electromagnetic choice (10) is

$$\xi_\mu = h_{\mu\nu} \zeta^\nu. \quad (25)$$

A short calculation gives

$$\Delta_{\alpha\beta}^{\zeta, h\zeta} = \zeta^\rho (\partial_\alpha h_{\beta\rho} + \partial_\beta h_{\alpha\rho} - \partial_\rho h_{\alpha\beta}) = 2\zeta^\rho \Gamma_{\rho\alpha\beta}^{(1)}, \quad (26)$$

where $\Gamma_{\rho\alpha\beta}^{(1)} \equiv \frac{1}{2}(\partial_\alpha h_{\beta\rho} + \partial_\beta h_{\alpha\rho} - \partial_\rho h_{\alpha\beta})$ is the linearized Christoffel symbol. This is the closest spin-2 analogue of the electromagnetic result; it coincides with the linearized Christoffel form variation studied by Baker and Kuzmin [20] in the linearized Gauss-Bonnet context and isolated in Sec. 4.3 of Taylor and Baker [11] within their general parametric classification of form variations for linearized gravity. There is, however, a crucial difference between electromagnetism (EM) and the Fierz-Pauli action: $F_{\mu\nu}$ is gauge invariant, whereas $\Gamma_{\rho\alpha\beta}^{(1)}$ is not. Indeed, under an independent spin-2 gauge transformation $h_{\mu\nu} \rightarrow h_{\mu\nu} + D_\chi h_{\mu\nu}$ with parameter χ_μ (distinct from the compensating

$$\Gamma_{\rho\alpha\beta}^{(1)} \rightarrow \Gamma_{\rho\alpha\beta}^{(1)} + \partial_\alpha \partial_\beta \chi_\rho. \quad (27)$$

IV. LOCAL NO-GO RESULT

The lack of gauge invariance just described is not merely the result of a poor choice (25) of compensating gauge parameter. No local tensor quadratic in first derivatives of $h_{\mu\nu}$ can be strictly gauge invariant. The argument is the linearized gauge-theoretic analogue of the familiar fact that all Christoffel symbols vanish at a point in Riemann normal coordinates.

To see this, work at a point p . The first derivative $\partial_\lambda h_{\mu\nu}$ at p is equivalent to specifying the linearized connection $\Gamma_{\rho\mu\nu}^{(1)}$ at p , since

$$\partial_\lambda h_{\mu\nu} = \Gamma_{\mu\lambda\nu}^{(1)} + \Gamma_{\nu\lambda\mu}^{(1)}. \quad (28)$$

Under the spin-2 gauge transformation,

$$\delta \Gamma_{\rho\mu\nu}^{(1)} = \partial_\mu \partial_\nu \xi_\rho. \quad (29)$$

At the point p the symmetric second derivatives $\partial_\mu \partial_\nu \xi_\rho(p)$ are arbitrary independent quantities. Hence, given any value of $\Gamma_{\rho\mu\nu}^{(1)}(p)$, one may choose ξ_ρ so that

$$\Gamma_{\rho\mu\nu}^{(1)}(p) \rightarrow 0, \quad (30)$$

and therefore also $\partial_\lambda h_{\mu\nu}(p) \rightarrow 0$.

Suppose now that $\tau_{\mu\nu}$ is a local pointwise expression depending only on $\eta_{\mu\nu}$ and $\partial_\lambda h_{\mu\nu}$, homogeneous quadratic in ∂h , transforming as a Lorentz tensor, and strictly invariant under the spin-2 gauge transformation. Since every value of ∂h at p is gauge-equivalent to zero,

$$\tau_{\mu\nu}(\partial h) = \tau_{\mu\nu}(0). \quad (31)$$

But $\tau_{\mu\nu}$ is quadratic in ∂h , so $\tau_{\mu\nu}(0) = 0$. Therefore

$$\tau_{\mu\nu} \equiv 0. \quad (32)$$

This proves the claim: there is no nonzero local, pointwise gauge-invariant tensor $\tau_{\mu\nu} \sim \partial h \partial h$ for the Fierz-Pauli field. The first genuinely gauge-invariant local field strength is the linearized Riemann tensor,

$$R_{\mu\nu\rho\sigma}^{(1)} \sim \partial \partial h, \quad (33)$$

so strictly gauge-invariant local quantities can be built from $R_{\mu\nu\rho\sigma}^{(1)}$ and its derivatives, for instance the linearized Bel-Robinson tensor quadratic in $R^{(1)}$, but not from $\partial h \partial h$ alone [3].

V. INEXACT BESSEL-HAGEN CURRENT

Although the preceding result precludes a strict electromagnetic analogue, it does not rule out a weaker Noetherian result. The important point is that the Fierz-Pauli gauge symmetry is an inexact symmetry of the Lagrangian density: the action is invariant, but the density changes by a total divergence. Although this distinction is invisible at the level of the action, it is essential at the level of the current. Noether's construction builds the conserved current from the density-level transformation, so the boundary term K_ξ^μ defined below enters the current explicitly. For electromagnetism the corresponding boundary term vanishes identically, since $\delta_\alpha \mathcal{L}_{\text{EM}} = 0$; for Fierz-Pauli it does not, and it is precisely this term that produces the ξ -dependent piece of the current.

For a pure spin-2 gauge transformation $\delta h_{\alpha\beta} = D_\xi h_{\alpha\beta}$, the variational identity (16) gives

$$\delta_\xi \mathcal{L}_{\text{FP}} = \mathcal{E}^{\alpha\beta} D_\xi h_{\alpha\beta} + \partial_\mu \theta^\mu (D_\xi h). \quad (34)$$

Using (21), the first term may be written as

$$\mathcal{E}^{\alpha\beta} D_\xi h_{\alpha\beta} = 2\mathcal{E}^{\alpha\beta} \partial_\alpha \xi_\beta = \partial_\alpha (2\mathcal{E}^{\alpha\beta} \xi_\beta). \quad (35)$$

Thus,

$$\delta_\xi \mathcal{L}_{\text{FP}} = \partial_\mu K_\xi^\mu, \quad (36)$$

which determines K_ξ^μ only up to the divergence of an antisymmetric superpotential. The natural representative, obtained by reading the divergence directly from the steps above, is

$$K_\xi^\mu = \theta^\mu (D_\xi h) + 2\mathcal{E}^{\mu\nu} \xi_\nu. \quad (37)$$

Any other choice differs from this by an antisymmetric superpotential divergence and shifts the resulting Bessel-Hagen current within its BBH equivalence class.

The calculation below is first carried out for an arbitrary external gauge parameter $\xi_\mu(x)$. Field-dependent Bessel-Hagen choices such as $\xi_\mu = h_{\mu\nu} \zeta^\nu$ are then obtained by substitution (the derivation uses only $\partial_\mu \mathcal{E}^{\mu\nu} \equiv 0$ and is therefore insensitive to whether ξ_μ depends on h); their field-dependence is accounted for explicitly when an independent gauge variation δ_χ is later taken in Sec. VI.

For a combined Poincaré and spin-2 gauge transformation $\delta_0 h = \Delta^{\zeta, \xi}$ in (24), the variation of the density is $\delta_0 \mathcal{L}_{\text{FP}} = -\partial_\mu (\zeta^\mu \mathcal{L}_{\text{FP}}) + \partial_\mu K_\xi^\mu$, since $\partial_\mu \zeta^\mu = 0$ for Poincaré. Combining with (16) gives

$$\partial_\mu [\theta^\mu (\Delta^{\zeta, \xi}) + \zeta^\mu \mathcal{L}_{\text{FP}} - K_\xi^\mu] = -\mathcal{E}^{\alpha\beta} \Delta_{\alpha\beta}^{\zeta, \xi}. \quad (38)$$

Since $\mathcal{E}^{\alpha\beta} \simeq 0$ on the Fierz-Pauli equations of motion—we use $\simeq$ throughout for weak equality, i.e. equality modulo

the field equations—the right-hand side vanishes on-shell, identifying the bracket on the left as the conserved Bessel-Hagen current corresponding to the combined transformation:

$$J_{\zeta, \xi}^\mu = \theta^\mu (\Delta^{\zeta, \xi}) + \zeta^\mu \mathcal{L}_{\text{FP}} - K_\xi^\mu. \quad (39)$$

Substituting (37) and using the linearity of θ^μ in its argument gives the simple form

$$J_{\zeta, \xi}^\mu = \theta^\mu (-\mathcal{L}_\zeta h) + \zeta^\mu \mathcal{L}_{\text{FP}} - 2\mathcal{E}^{\mu\nu} \xi_\nu. \quad (40)$$

On-shell, this becomes

$$J_{\zeta, \xi}^\mu \simeq \theta^\mu (-\mathcal{L}_\zeta h) + \zeta^\mu \mathcal{L}_{\text{FP}}. \quad (41)$$

This is the central result. Equivalently, writing $J_{\zeta, 0}^\mu \equiv \theta^\mu (-\mathcal{L}_\zeta h) + \zeta^\mu \mathcal{L}_{\text{FP}}$ for the $\xi = 0$ representative, Eq. (40) reads

$$J_{\zeta, \xi}^\mu = J_{\zeta, 0}^\mu - 2\mathcal{E}^{\mu\nu} \xi_\nu. \quad (42)$$

Note first that (42) is an off-shell equality of local tensors, not merely an on-shell equivalence: although the right-hand side has the same general form as the equations-of-motion-improvement freedom of the BBH equivalence relation, the specific shift $-2\mathcal{E}^{\mu\nu} \xi_\nu$ is an off-shell prediction of the Bessel-Hagen construction, not a freedom merely permitted by that equivalence relation. Indeed, although K_ξ^μ admits superpotential additions, the only freedom propagating through the construction is that of the symplectic potential θ^μ , which shifts $J_{\zeta, \xi}^\mu$ and $J_{\zeta, 0}^\mu$ identically and so cancels in their difference; the ξ -dependence is thus $-2\mathcal{E}^{\mu\nu} \xi_\nu$ independently of that choice. This term is not itself a superpotential: by $\partial_\mu \mathcal{E}^{\mu\nu} \equiv 0$ its divergence is $\partial_\mu (-2\mathcal{E}^{\mu\nu} \xi_\nu) = -\mathcal{E}^{\mu\nu} D_\xi h_{\mu\nu}$, the (on-shell-vanishing) Euler-Lagrange contraction, whereas a superpotential is identically conserved. The shift therefore lies in the weakly-vanishing sector of the trivial currents rather than the superpotential sector, which is the precise sense in which (42) is an off-shell statement and not merely a relation within the BBH class. Two further consequences of (42) are worth recording. First, the identity holds for any local choice $\xi_\mu[h, \zeta]$, since its derivation uses only $\partial_\mu \mathcal{E}^{\mu\nu} \equiv 0$; the electromagnetic-like choice $\xi_\mu = h_{\mu\nu} \zeta^\nu$ is singled out only because it makes the form variation connection-like, $\Delta_{\mu\nu}^{\zeta, h\zeta} = 2\zeta^\rho \Gamma_{\rho\mu\nu}^{(1)}$, not at the level of the current class. Second, the ξ -dependent piece $-2\mathcal{E}^{\mu\nu} \xi_\nu$ is *not* an *ad hoc* improvement: it arises from the gauge boundary term K_ξ^μ via Eq. (37) and is therefore built into the Bessel-Hagen construction itself. The ξ -family parametrized by (42) spans only the equation-of-motion half of the equivalence class; the external superpotential plus on-shell freedom of Refs. [16–19] enters via the BBH equivalence relation invoked in Sec. VI.

The current (40) is conserved only weakly, $\partial_\mu J_{\zeta,\xi}^\mu \simeq 0$, as is standard for Noether currents and as already holds for the Maxwell tensor of Sec. II; the off-shell content of the construction resides in the local identity (42) rather than in the conservation law. The physically meaningful object is the conserved charge $Q_\zeta = \int_\Sigma J_{\zeta,\xi}^0 d^3x$, which is conserved on solutions and is independent of the compensating parameter ξ , since the shift $-2\mathcal{E}^{\mu\nu}\xi_\nu$ vanishes on-shell. The attendant nonuniqueness of the local current—its determination only up to the equivalence class—is thus not a limitation peculiar to this construction but the expected feature of gravitational energy-momentum, with the charge as the well-defined invariant.

Relative to the parametric classification of Taylor and Baker [11], the point is therefore the following. The linearized-Christoffel form variation (26) is one member of their family of admissible form variations, but the current-level identity (42) depends on the inexact gauge boundary term K_ξ^μ , which lies outside their exact-symmetry classification.

Choosing the $\xi = 0$ representative of the class and specializing to translations $\zeta^\mu = a^\mu$, one obtains

$$J_{a,0}^\mu = -a^\rho \Pi^{\mu|\alpha\beta} \partial_\rho h_{\alpha\beta} + a^\mu \mathcal{L}_{\text{FP}} = -a^\rho t^\mu{}_\rho, \quad (43)$$

where

$$t^\mu{}_\rho = \Pi^{\mu|\alpha\beta} \partial_\rho h_{\alpha\beta} - \delta_\rho^\mu \mathcal{L}_{\text{FP}} \quad (44)$$

is the canonical energy-momentum tensor of the Fierz-Pauli field. This tensor is not gauge invariant as a local expression, but the current class it defines is gauge invariant in the sense made precise in Sec. VI below.

We have displayed the translation current because it is the part of the Poincaré current conventionally identified with energy-momentum. Nothing in the preceding derivation is restricted to translations, however. For Lorentz transformations $\zeta^\mu = \omega^\mu{}_\nu x^\nu$, the same identity (42) holds and gives the corresponding angular-momentum current, schematically $J_\omega^\mu = \frac{1}{2} \omega_{\rho\sigma} (x^\rho t^{\mu\sigma} - x^\sigma t^{\mu\rho} + S^{\mu|\rho\sigma})$, where $S^{\mu|\rho\sigma}$ is the spin current associated with the tensor indices of $h_{\mu\nu}$. Since the present paper is concerned with the energy-momentum tensor, we spell out only the translation specialization.

VI. GAUGE INVARIANCE OF THE CURRENT CLASS

Section V showed that varying the compensating Bessel-Hagen parameter ξ , with the field $h_{\mu\nu}$ held fixed, keeps the current within a fixed equivalence class. We now show that this class is itself gauge invariant under an independent transformation $h_{\mu\nu} \rightarrow h_{\mu\nu} + D_\chi h_{\mu\nu}$ of the field. The current (39) satisfies the off-shell identity

$$\partial_\mu J_{\zeta,\xi}^\mu = -\mathcal{E}^{\alpha\beta} \Delta_{\alpha\beta}^{\zeta,\xi}, \quad (45)$$

which is just Noether's identity for the combined Poincaré and spin-2 gauge variation. Now perform an independent spin-2 gauge transformation with parameter χ_μ , distinct from the compensating ξ_μ already appearing in $\Delta^{\zeta,\xi}$:

$$\delta_\chi h_{\mu\nu} = D_\chi h_{\mu\nu}. \quad (46)$$

Since the Fierz-Pauli Euler expression is gauge invariant, $\delta_\chi \mathcal{E}^{\alpha\beta} = 0$, we obtain from (45)

$$\partial_\mu \delta_\chi J_{\zeta,\xi}^\mu = -\mathcal{E}^{\alpha\beta} \delta_\chi \Delta_{\alpha\beta}^{\zeta,\xi}. \quad (47)$$

By (47), the induced change in the current is controlled entirely by the change in the form variation $\Delta^{\zeta,\xi}$, which we now evaluate. Writing $\Delta_{\mu\nu}^{\zeta,\xi} = -\mathcal{L}_\zeta h_{\mu\nu} + D_\xi h_{\mu\nu}$, the independent transformation δ_χ acts through two channels: directly on the field $h_{\mu\nu}$ in the transport term $-\mathcal{L}_\zeta h$, and on the field-dependent compensating parameter $\xi_\mu[h, \zeta]$ in $D_\xi h$. Together these give

$$\delta_\chi \Delta_{\mu\nu}^{\zeta,\xi} = -\mathcal{L}_\zeta (D_\chi h_{\mu\nu}) + D_{\delta_\chi \xi} h_{\mu\nu}. \quad (48)$$

The first channel can be simplified: for the Poincaré transformations considered here, $\partial_\mu \partial_\nu \zeta^\rho = 0$, so that the Lie derivative along ζ commutes with the spin-2 gauge action up to a relabeling of the parameter, $\chi \rightarrow \mathcal{L}_\zeta \chi$, and hence

$$\mathcal{L}_\zeta (D_\chi h_{\mu\nu}) = D_{\mathcal{L}_\zeta \chi} h_{\mu\nu}, \quad (49)$$

where $\mathcal{L}_\zeta \chi$ denotes the Lie derivative of χ_μ regarded as a covector gauge parameter on Minkowski spacetime, $(\mathcal{L}_\zeta \chi)_\mu = \zeta^\rho \partial_\rho \chi_\mu + \chi_\rho \partial_\mu \zeta^\rho$. Defining the effective spin-2 gauge parameter $\eta_\mu \equiv \delta_\chi \xi_\mu - (\mathcal{L}_\zeta \chi)_\mu$, which combines the induced change of the compensating $\xi[h]$ under the χ -transformation of the field and the failure of the Poincaré and gauge actions to commute on h , it follows that

$$\delta_\chi \Delta_{\mu\nu}^{\zeta,\xi} = D_\eta h_{\mu\nu}. \quad (50)$$

Equation (50) is the crucial step: the gauge variation of the form variation is itself a pure spin-2 gauge variation. It is precisely because $\delta_\chi \Delta^{\zeta,\xi}$ is again of pure-gauge form that the induced change in the current is a trivial (pure-gauge) current, as we now show. The Noether current associated with the pure spin-2 gauge variation $D_\eta h_{\mu\nu}$ is trivial:

$$J_\eta^\mu = \theta^\mu (D_\eta h) - K_\eta^\mu = -2\mathcal{E}^{\mu\nu} \eta_\nu. \quad (51)$$

Indeed, using the Noether identity $\partial_\mu \mathcal{E}^{\mu\nu} \equiv 0$, its divergence is

$$\partial_\mu J_\eta^\mu = -2\mathcal{E}^{\mu\nu}\partial_\mu \eta_\nu = -\mathcal{E}^{\mu\nu}D_\eta h_{\mu\nu}. \quad (52)$$

Equations (47) and (50) therefore imply

$$\partial_\mu(\delta_\chi J_{\zeta,\xi}^\mu - J_\eta^\mu) \equiv 0. \quad (53)$$

Thus, locally in a topologically trivial region, the difference is an identically conserved current and hence is the divergence of an antisymmetric superpotential,⁵

$$\delta_\chi J_{\zeta,\xi}^\mu - J_\eta^\mu = \partial_\nu U_{\zeta,\chi}^{[\mu\nu]}. \quad (54)$$

Substituting (51) gives

$$\delta_\chi J_{\zeta,\xi}^\mu = \partial_\nu U_{\zeta,\chi}^{[\mu\nu]} - 2\mathcal{E}^{\mu\nu}[\delta_\chi \xi_\nu - (\mathcal{L}_\zeta \chi)_\nu]. \quad (55)$$

Consequently, on shell

$$\delta_\chi J_{\zeta,\xi}^\mu \simeq \partial_\nu U_{\zeta,\chi}^{[\mu\nu]}. \quad (56)$$

This is precisely the expected notion of gauge invariance for a Noether current. Currents which differ by

$$J^\mu \rightarrow J^\mu + \partial_\nu U^{[\mu\nu]} + V^\mu{}_{\alpha\beta} \mathcal{E}^{\alpha\beta} \quad (57)$$

define the same conserved charge under the usual boundary conditions. We denote the resulting equivalence class by $[\cdot]$. The object produced by the Bessel-Hagen method for the Fierz-Pauli action is therefore not a single gauge-invariant local tensor, but the equivalence class

$$[J_\zeta^\mu] = [\theta^\mu(-\mathcal{L}_\zeta h) + \zeta^\mu \mathcal{L}_{\text{FP}}]. \quad (58)$$

VII. DISCUSSION

The contrast with electromagnetism is now clear. In electromagnetism the gauge-invariant field strength is first order in the potential, $F_{\mu\nu} \sim \partial A$, and the Bessel-Hagen choice $\alpha = A_\mu \zeta^\mu$ turns the form variation itself into the gauge-invariant expression $\zeta^\nu F_{\mu\nu}$. The Noether current therefore produces a strictly gauge-invariant local representative of its equivalence class: the standard Maxwell energy-momentum tensor.

For the Fierz-Pauli field, the first derivative ∂h is connection-like. It can be made to vanish at a point by a spin-2 gauge transformation. The first local gauge-invariant field strength is curvature-like and contains two derivatives of $h_{\mu\nu}$. Thus no Bessel-Hagen choice can give a nonzero local tensor of the form $\partial h \partial h$ that is strictly gauge invariant.

⁵This is the local algebraic Poincaré lemma for identically conserved currents. Equivalently, it is the standard trivial-current equivalence relation used in characteristic cohomology [8–10]: currents differing by superpotentials and weakly vanishing currents represent the same conservation law.

This explains why the numerous canonical, Belinfante, Hilbert, Landau-Lifshitz and related expressions for gravitational energy in linearized gravity are either gauge dependent, defined only after gauge fixing, or equivalent only after quotienting by superpotentials and field-equation terms.

The calculation above shows, however, that the Bessel-Hagen method does not simply fail for Fierz-Pauli theory. Rather, once the inexact gauge symmetry of the Fierz-Pauli density is treated correctly, the method naturally lands in the quotient space of Noether currents.⁶ The compensating spin-2 gauge parameter ξ_μ cannot be used to select a preferred gauge-invariant local representative, but it also cannot alter the Noether current class. This provides a precise spin-2 counterpart of the Bessel-Hagen construction in electromagnetism, albeit in a weaker sense.

It may seem at first sight that the construction has merely returned the canonical Noether energy-momentum tensor. The point, however, is not that (44) is a new local expression; it is not. Rather, the inexact Bessel-Hagen construction shows why no choice of compensating spin-2 gauge parameter can improve this representative into a preferred gauge-invariant local tensor: all such choices give $J_{\zeta,\xi}^\mu = J_{\zeta,0}^\mu - 2\mathcal{E}^{\mu\nu}\xi_\nu$, and hence the same current class. The canonical tensor appears here only as a convenient representative of the Bessel-Hagen current class. The nontrivial result is the identification of this class, and the demonstration that the Bessel-Hagen mechanism degenerates, for Fierz-Pauli, into equivalence-class invariance rather than strict local invariance.

The two broad ingredients of the result, namely the no-go for a strictly gauge-invariant local $\partial h \partial h$ tensor and the gauge-invariant equivalence class $[J_\zeta^\mu]$ that survives, are not new in isolation: the obstruction to a strictly gauge-invariant local spin-2 energy-momentum tensor is already present in Deser and Henneaux [3] and in the related theorem of Magnano and Sokolowski [4], while the quotient-space interpretation is the standard characteristic-cohomology framework of BBH [8–10]. What is new is that the present analysis brings these two ingredients together in a single Bessel-Hagen treatment: the no-go result explains why no strictly invariant local tensor can emerge, while Eq. (42) shows that the surviving Bessel-Hagen object is a current class, with the particular field-equation freedom associated with the compensating gauge parameter acquiring a Noetherian origin in the boundary

⁶A concrete illustration is provided by the Butcher tensor of Refs. [22–24], originally introduced in harmonic gauge in Ref. [22] and later generalized to other gauges in Ref. [24]. For its original harmonic-gauge form, Barker *et al.* [25] showed that it is obtained from the linearized Einstein pseudotensor (in the same gauge) by subtracting an identically conserved current, i.e. the divergence of an antisymmetric superpotential. This is therefore a relocalization within the equivalence class of Noether currents considered in the present paper.

term K_ξ^μ of the inexact gauge symmetry rather than the status of an external improvement step.

Three nested classes are useful to distinguish here. (i) The ξ -family $\{J_{\xi,\epsilon}^\mu\}$ has members differing from $J_{\xi,0}^\mu$ only by terms proportional to the field equations. (ii) Closing (i) under independent gauge transformations δ_χ of $h_{\mu\nu}$ adds the specific superpotentials $\partial_\nu U_{\xi,\chi}^{[\mu\nu]}$ of Sec. VI. (iii) The standard BBH equivalence class of $J_{\xi,0}^\mu$ treats any antisymmetric superpotential and any term proportional to the field equations as trivial. These satisfy (i) $\subset$ (ii) $\subset$ (iii). The ξ -family in (i), and its gauge orbit in (ii), should therefore not be confused with the full improvement freedom. Named improved expressions such as the Belinfante [13,14], Hilbert [12], Landau-Lifshitz [15], and Møller [26] tensors are best regarded, when related to the canonical tensor by superpotentials and field-equation terms, as representatives of the broader BBH class (iii). They need not be generated by varying the compensating Bessel-Hagen parameter, nor by the specific gauge-induced superpotentials of Sec. VI. The wider improvement freedom of Refs. [16–19] therefore belongs to the full current-equivalence relation, not to (i) or (ii).

The Bessel-Hagen construction is, in this sense, not a generator of the full improvement family of spin-2 energy-momentum tensors; its role is more diagnostic. It shows that the compensating gauge freedom which is decisive in electromagnetism becomes, for the Fierz-Pauli action, a weakly vanishing ambiguity. The remaining differences among the familiar gravitational energy-momentum representatives listed above arise from choices that lie outside the Bessel-Hagen mechanism: relocalization, Lagrangian representative, background covariantization, and choice of pseudotensor superpotential. The Bessel-Hagen analysis is orthogonal to these: it identifies the Noether current class but does not, on its own, select a preferred local representative.

The gauge-fixing approach of Butcher *et al.* [22–24] addresses the gauge ambiguity in a complementary way to the Bessel-Hagen construction: rather than identify a gauge-invariant equivalence class of currents, one selects a particular local representative by imposing a gauge condition on $h_{\mu\nu}$. A related first-derivative construction is Tóth’s Fierz-tensor formulation of linearized gravity [27], in which an energy-momentum tensor with favourable properties, including satisfaction of the dominant energy condition in a class of gauges containing the transverse-traceless gauge, is obtained from the Fierz tensor built from first derivatives of $h_{\mu\nu}$. This is complementary to the present result: it belongs to the gauge-fixed strategy rather than to the equivalence-class strategy of the present paper.

The Bessel-Hagen analysis presented here is also the linearized counterpart of the manifestly covariant Bessel-Hagen construction of Hobson *et al.* [28] for the gauge theories of Weyl and of extended Weyl gravity, in which the

local field strengths are themselves gauge covariant and the construction succeeds in the strict, rather than quotient, sense. A different generalization of the Bessel-Hagen method, to fourth-order conformally invariant theories of gravity, has recently been given by Faria [29], who obtains a (gauge-dependent) pseudotensor; the higher-derivative structure of that case differs essentially from the second-order Fierz-Pauli setting considered here. For broader context on the structure of massive and massless spin-2 theory, including gauge-invariance issues, see the review by Hinterbichler [30].

VIII. CONCLUSIONS

In summary, the spin-2 Bessel-Hagen calculation produces no strictly gauge-invariant local $\partial h \partial h$ tensor, because ∂h can be set to zero pointwise by a spin-2 gauge transformation. Once the inexact nature of the Fierz-Pauli gauge symmetry is treated correctly, however, the construction identifies not a single tensor but an entire equivalence class of Noether currents—gauge invariant as a class, not pointwise. The identity $J_{\xi,\epsilon}^\mu = J_{\xi,0}^\mu - 2\mathcal{E}^{\mu\nu}\xi_\nu$ of Eq. (42) describes the ξ_μ -generated part of this class: changing the compensating Bessel-Hagen gauge parameter moves the current only by a weakly vanishing term. Gauge transformations of the field further add superpotential terms, and the full conserved-current class is the standard BBH class of $J_{\xi,0}^\mu$ modulo superpotentials and field equations. The canonical Noether tensor, which at linearized order coincides with the Einstein pseudotensor in the usual $\Gamma\Gamma$ representation of the action, is a convenient representative of this class. Other improved expressions in the literature, when they differ by superpotentials and field-equation terms, represent the same conservation law. The standard BBH current-equivalence relation places these representatives on the same footing, while the Bessel-Hagen construction identifies this class from within a single inexact Noether calculation. This is precisely the structure that the cohomological treatment of BBH and the improvement-tensor literature had suggested must be present, here derived directly at the Lagrangian level, with the inexact gauge boundary term K_ξ^μ supplying a Noetherian origin for the field-equation freedom that is usually introduced by hand. The Bessel-Hagen method therefore has a precise spin-2 counterpart for the Fierz-Pauli action, but only as a gauge-invariant equivalence class of Noether currents, not as a preferred local gauge-invariant energy-momentum tensor.

In closing, we note that the Bessel-Hagen construction presented here also extends to higher spins via the Fronsdal action [31], the generalization of (massless) Fierz-Pauli to totally symmetric rank- s tensor fields. The spin-3 case will be set out in a future paper, in which a unique two-parameter compensating gauge choice reduces the form variation to a generalised Christoffel symbol and the inexact-gauge boundary term retains its Noetherian form,

with the coefficient $-2\mathcal{E}^{\mu\nu}\xi_\nu$ of (42) replaced by $-3\mathcal{E}^{\mu\nu\rho}\xi_{\nu\rho}$. The structural conclusions of the present paper are further expected to carry over to arbitrary integer spin, but this is a topic for future research.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

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