

Charmonium radiative transitions to dileptons from lattice QCD: The case of $h_c \rightarrow \eta_c \ell^+ \ell^-$ and $\chi_{c1} \rightarrow J/\psi \ell^+ \ell^-$

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We present a lattice QCD study of dilepton production in charmonium transitions, specifically focusing on the $1^{+-} \rightarrow 0^{++}$ and $1^{++} \rightarrow 1^{--}$ processes: $h_c \rightarrow \eta_c \ell^+ \ell^-$ and $\chi_{c1} \rightarrow J/\psi \ell^+ \ell^-$, where $\ell = e, \mu$. The relevant hadronic matrix elements are computed using gauge field configurations generated by the Extended Twisted Mass Collaboration with $N_f = 2 + 1 + 1$ dynamical Wilson-Clover twisted-mass fermions at four lattice spacings. Simulations are performed at physical dynamical u, d, s , and c quark masses, except for the coarsest lattice, where the lightest sea quark mass corresponds to a slightly heavier pion mass. A controlled continuum extrapolation is carried out. In the continuum limit for the h_c decays, we obtain $\Gamma(h_c \rightarrow \eta_c e^+ e^-) = 5.45(19)$ keV, and $\Gamma(h_c \rightarrow \eta_c \mu^+ \mu^-) = 0.635(22)$ keV. For the χ_{c1} decays, we find $\Gamma(\chi_{c1} \rightarrow J/\psi e^+ e^-) = 2.869(90)$ keV and $\Gamma(\chi_{c1} \rightarrow J/\psi \mu^+ \mu^-) = 0.1993(72)$ keV. Our results for the χ_{c1} decays show good compatibility with experimental data. However, our prediction for the $h_c \rightarrow \eta_c e^+ e^-$ decay rate is approximately 3σ larger than the BESIII result. We also present predictions for the differential decay widths as functions of the dilepton invariant mass, q^2 , and for angular observables sensitive to longitudinal transition form factors, which are inaccessible in radiative decays with real photon emission. These results constitute the first fully dynamical lattice QCD predictions for dilepton decay rates in h_c and χ_{c1} charmonium transitions, including their differential distributions and angular observables. They provide benchmark predictions for future experimental studies.

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I. INTRODUCTION

In recent years, electromagnetic transitions in heavy quarkonia have attracted significant phenomenological and theoretical interest, as they offer a sensitive probe of the internal structure and dynamics of heavy quark-antiquark bound states. Radiative decays such as $h_{c(b)} \rightarrow \eta_{c(b)} \gamma$, $J/\psi \rightarrow \eta_c \gamma$, and $\Upsilon \rightarrow \eta_b \gamma$ are governed by electric and magnetic multipole amplitudes and they can be described using effective field theories of QCD in the nonrelativistic regime, such as nonrelativistic QCD (NRQCD) and potential nonrelativistic QCD (pNRQCD) [1–4]. However, these

approaches rely on an expansion in the heavy-quark velocity and, in practice, require truncation, which introduces systematic uncertainties that are often difficult to quantify. By contrast, lattice QCD provides a first-principles determination of the relevant hadronic matrix elements directly from the QCD Lagrangian, without relying on a velocity expansion or model assumptions, and with systematically improvable uncertainties.

Radiative quarkonium transitions have already been studied in lattice QCD for several channels [5–14], yielding in some cases high-precision determinations of the corresponding multipole form factors. Building on these advances, it is natural to extend this first-principles framework to transitions mediated by an off-shell photon.

Beyond purely radiative channels, electromagnetic decays involving a virtual photon that subsequently converts into a lepton-antilepton pair provide an additional probe of quarkonium structure. In these dilepton decays,

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also known as Dalitz decays, the invariant mass of the lepton pair ranges from threshold ($2m_\ell$) up to the mass difference between the initial and final quarkonium states. Because the photon is off shell, the decay amplitude includes contributions from longitudinal virtual-photon polarization, which are absent in the on-shell limit ($q^2 = 0$). Dilepton decays thus grant access to additional structures of the hadronic matrix elements.

Experimentally, dilepton decays of charmonium states have become accessible at modern charm factories. The branching fractions for the transitions $\chi_{c1} \rightarrow J/\psi e^+ e^-$ and $\chi_{c1} \rightarrow J/\psi \mu^+ \mu^-$ have been measured with good precision, enabling detailed studies of integrated decay rates and quantitative tests of lepton universality. The BESIII Collaboration reported [15,16]¹

$$\left. \frac{\text{Br}(\chi_{c1} \rightarrow J/\psi e^+ e^-)}{\text{Br}(\chi_{c1} \rightarrow J/\psi \gamma)} \right|_{\text{BESIII}} = (10.1 \pm 0.3 \pm 0.5) \times 10^{-3}, \quad (1)$$

$$\left. \frac{\text{Br}(\chi_{c1} \rightarrow J/\psi \mu^+ \mu^-)}{\text{Br}(\chi_{c1} \rightarrow J/\psi e^+ e^-)} \right|_{\text{BESIII}} = (6.73 \pm 0.51 \pm 0.50) \times 10^{-2}. \quad (2)$$

More recently, experimental studies have been extended to dilepton transitions in other charmonium states with different spin configurations. In particular, the BESIII Collaboration reported the first observation of the electromagnetic Dalitz decay $h_c \rightarrow \eta_c e^+ e^-$ [17], measuring²

$$\left. \frac{\text{Br}(h_c \rightarrow \eta_c e^+ e^-)}{\text{Br}(h_c \rightarrow \eta_c \gamma)} \right|_{\text{BESIII}} = (0.59 \pm 0.10 \pm 0.04) \times 10^{-2}, \quad (3)$$

with a statistical significance exceeding 5 s.d. This result demonstrates that dilepton transitions involving the h_c are experimentally accessible and provide new input for the study of electromagnetic transitions in the charmonium sector.

In addition to the standard electromagnetic contribution, dilepton decays of quarkonia may, in principle, receive contributions from nonstandard interactions associated with new mediators. Precise Standard Model predictions for decay rates, differential distributions, and angular observables are therefore essential for interpreting experimental measurements and identifying possible deviations from Standard Model expectations.

Motivated by this physics potential, Dalitz decays of heavy quarkonia have recently been investigated within

phenomenological frameworks that exploit heavy-quark spin symmetry to relate different transitions and infer the corresponding form factors from experimental data [18]. While such approaches provide valuable phenomenological insights, they necessarily rely on symmetry assumptions and model input, thereby motivating first-principles lattice QCD determinations of the relevant transition form factors.

Despite this growing experimental and theoretical interest, a quantitative understanding of these decays directly from QCD remains incomplete. Radiative transitions such as $h_c \rightarrow \eta_c \gamma$ and $\chi_{c1} \rightarrow J/\psi \gamma$ have already been investigated in lattice QCD [5–9], including recent high-precision determinations of the associated multipole form factors [10,11]. In contrast, analogous first-principles studies of dilepton channels remain limited, as their theoretical description requires the study of the full timelike q^2 -dependence of the form factors. Knowledge of this momentum-transfer dependence is crucial for reliable predictions of total decay rates and differential spectra in the dilepton invariant mass.

In this paper, we present the first unquenched lattice QCD study of the dilepton decays

$$h_c \rightarrow \eta_c \ell^+ \ell^-, \quad \chi_{c1} \rightarrow J/\psi \ell^+ \ell^-, \quad (4)$$

with $\ell = e, \mu$. The calculation uses gauge field configurations with $N_f = 2 + 1 + 1$ dynamical Wilson-Clover twisted-mass (TM) fermions generated by the Extended Twisted Mass Collaboration (ETMC), at four lattice spacings and with physical dynamical u , d , s , and c quark masses, except for the coarsest ensemble where the lightest pion mass is $m_\pi \simeq 175$ MeV. From our lattice determination of the relevant electromagnetic transition form factors over the full kinematic range in q^2 , we compute the total decay rates for final states containing either an electron or a muon pair. We also present predictions for the differential decay widths in the dilepton invariant mass and for angular observables that provide direct sensitivity to longitudinal transition form factors, which are absent in the limit of real-photon emission.

We summarize our main results for the total decay rates as follows:

$$\begin{aligned} \Gamma(h_c \rightarrow \eta_c e^+ e^-) &= 5.45(19) \text{ keV}, \\ \Gamma(h_c \rightarrow \eta_c \mu^+ \mu^-) &= 0.635(22) \text{ keV}, \\ \Gamma(\chi_{c1} \rightarrow J/\psi e^+ e^-) &= 2.869(90) \text{ keV}, \\ \Gamma(\chi_{c1} \rightarrow J/\psi \mu^+ \mu^-) &= 0.1993(72) \text{ keV}, \end{aligned} \quad (5)$$

which can be compared with the results quoted by the Particle Data Group (PDG) [19] for $\chi_{c1} \rightarrow J/\psi \ell^+ \ell^-$, as follows:

¹The quantum numbers of χ_{c1} and J/ψ are $J^{PC} = 1^{++}$ and 1^{--} , respectively.

²The respective quantum numbers of h_c and η_c are $J^{PC} = 1^{+-}$ and 0^{-+} .

$$\begin{aligned}\Gamma^{\text{PDG}}(\chi_{c1} \rightarrow J/\psi e^+ e^-) &= 2.91(24) \text{ keV}, \\ \Gamma^{\text{PDG}}(\chi_{c1} \rightarrow J/\psi \mu^+ \mu^-) &= 0.196(26) \text{ keV},\end{aligned}\quad (6)$$

where we have used $\Gamma^{\text{PDG}}(\chi_{c1}) = 0.84(4) \text{ MeV}$ to convert the experimental branching fractions into decay widths. With this choice of the full width $\Gamma(\chi_{c1})$, we observe excellent agreement with the experimental determinations for both the electron and muon channels. For a more detailed phenomenological discussion, see Sec. IV. In the case of $h_c \rightarrow \eta_c \ell^+ \ell^-$, only the electron channel has been measured, and the PDG quotes

$$\Gamma^{\text{PDG}}(h_c \rightarrow \eta_c e^+ e^-) = 2.7(1.1) \text{ keV}, \quad (7)$$

obtained using $\text{Br}(h_c \rightarrow \eta_c e^+ e^-) = (3.5 \pm 0.7) \times 10^{-3}$ and $\Gamma^{\text{PDG}}(h_c) = 0.78 \pm 0.28 \text{ MeV}$. We thus find a difference of about 2.5σ with respect to our result (5).

$$\begin{aligned}\langle J/\psi(k, \varepsilon) | J_{\text{em}}^\mu | \chi_{c1}(p, \eta) \rangle &= \frac{iQ_c}{\sqrt{2}\lambda(m_{\chi_{c1}}^2, m_{J/\psi}^2, q^2)} \epsilon^{\mu\nu\rho\sigma} (p-k)_\sigma \times \left\{ 2m_{\chi_{c1}}^2 [E_1(q^2) + M_2(q^2)] (\eta \cdot k) \varepsilon_\nu^*(p+k)_\rho \right. \\ &\quad + 2m_{\chi_{c1}}^2 \frac{m_{J/\psi}}{m_{\chi_{c1}}} [E_1(q^2) - M_2(q^2)] (\varepsilon^* \cdot p) \eta_\nu (p+k)_\rho + C_1(q^2) [-\lambda(m_{\chi_{c1}}^2, m_{J/\psi}^2, q^2) \eta_\nu \varepsilon_\rho^* \\ &\quad \left. + [(m_{\chi_{c1}}^2 - m_{J/\psi}^2 + q^2) (\eta \cdot k) \varepsilon_\nu^* + (m_{\chi_{c1}}^2 - m_{J/\psi}^2 - q^2) (\varepsilon^* \cdot p) \eta_\nu] (p+k)_\rho] \right\},\end{aligned}\quad (8)$$

$$\langle \eta_c(k) | J_{\text{em}}^\mu | h_c(p, \varepsilon) \rangle = 2iQ_c m_{h_c} \left\{ F_1(q^2) \left(\varepsilon^\mu - \frac{\varepsilon \cdot q}{q^2} q^\mu \right) + \tilde{F}_2(q^2) (\varepsilon \cdot q) \left[\frac{q^\mu}{q^2} - \frac{(p+k)^\mu}{m_{h_c}^2 - m_{\eta_c}^2} \right] \right\}, \quad (9)$$

where J_{em}^μ is the electromagnetic current,

$$J_{\text{em}}^\mu(x) = \sum_{f=u,d,s,c} J_f^\mu(x) = \sum_{f=u,d,s,c} Q_f \bar{q}_f(x) \gamma^\mu q_f(x), \quad (10)$$

with the sum running over all quark flavors, $q_f(x)$ being the quark field, and Q_f its electric charge in units of $e = \sqrt{4\pi\alpha_{\text{em}}}$. In Eq. (8), p and k are the momenta of χ_{c1} and J/ψ , respectively, while η and ε are their respective polarization vectors. $E_1(q^2)$, $M_2(q^2)$, and $C_1(q^2)$ are the form factors parametrizing the $\chi_{c1} \rightarrow J/\psi \gamma^*$ matrix element, which are functions of $q^2 = (p-k)^2$, while $\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2yz - 2xz$ is the Källén function. In Eq. (9), instead, p and k are the momenta of the h_c and η_c , respectively, $q = p - k$, while ε is the polarization of the h_c meson. The nonperturbative form factors relevant for the $h_c \rightarrow \eta_c \ell^+ \ell^-$ decays are $F_1(q^2)$ and $\tilde{F}_2(q^2)$, which obey the $q^2 = 0$ constraint $F_1(0) = \tilde{F}_2(0)$. For later convenience, we also introduce the longitudinal form factor

The remainder of this paper is organized as follows: in Sec. II we introduce the form-factor decompositions and the lattice observables used to extract them. For clarity, the discussion is organized channel by channel: we first present the formalism for $\chi_{c1} \rightarrow J/\psi \ell^+ \ell^-$ and then for $h_c \rightarrow \eta_c \ell^+ \ell^-$. The corresponding numerical analyses are presented in the same order in Sec. III. In Sec. IV we discuss the resulting decay rates, differential distributions, and angular observables, comparing them with available experimental results where possible. Finally, in Sec. V, we summarize our findings and outline prospects for future improvements and extensions to other decay channels.

II. FORM FACTORS AND LATTICE SETUP

We adopt the following decomposition of the transition matrix elements describing the decay $\chi_{c1} \rightarrow J/\psi \gamma^*$ and $h_c \rightarrow \eta_c \gamma^{*3}$:

$$\begin{aligned}F_L(q^2) &= \frac{1}{2q^2} \left[\frac{\lambda(m_{h_c}^2, m_{\eta_c}^2, q^2)}{m_{h_c}^2 - m_{\eta_c}^2} \tilde{F}_2(q^2) \right. \\ &\quad \left. - (m_{h_c}^2 - m_{\eta_c}^2 + q^2) F_1(q^2) \right].\end{aligned}\quad (11)$$

In terms of the form factors $F_1(q^2)$ and $F_L(q^2)$, the decay rate for $h_c \rightarrow \eta_c \ell^+ \ell^-$ is simply the sum of two contributions proportional to $|F_1(q^2)|^2$ and $|F_L(q^2)|^2$, without interference terms (see Sec. IV).

In the case of real photon emission considered in Refs. [10,11], only the form factors $E_1(0)$, $M_2(0)$, $F_1(0)$ contribute. In this paper we extend these calculations, and

³We take this opportunity to correct a transcription typo in Eq. (3) of Ref. [10]. Specifically, the factor $m_{\chi_{c1}}^2$ multiplying the $(E_1 + M_2)$ and $(E_1 - M_2)$ terms is missing in that equation. In addition, compared to Eq. (3) of Ref. [10], we redefine $C_1(q^2)/\sqrt{q^2} \rightarrow C_1(q^2)$. With this definition, $C_1(q^2)$ is nonzero in the limit $q^2 \rightarrow 0$. We also remind the reader that the original decomposition of the matrix element was first presented in Ref. [5], where a derivation of this covariant decomposition is provided in the Appendix.

TABLE I. $N_f = 2 + 1 + 1$ ETMC gauge ensembles used in this work. a is the lattice spacing, L/a the spatial extent of the lattice, am_c is the bare charm quark mass, and Z_V the vector current renormalization constant [22]. With the exception of A48 (which corresponds to a pion mass $m_\pi \simeq 175$ MeV), all the ensembles have been generated at physical values of the light, strange, and charm quark masses. At each lattice spacing, the charm quark mass m_c has been tuned to reproduce $m_{D_s} = 1968$ MeV.

ID	L/a	a fm	Z_V	am_c
A48	48	0.0907(5)	0.68700(15)	0.2620
B64	64	0.07948(11)	0.706354(54)	0.23157
C80	80	0.06819(14)	0.725440(33)	0.19840
D96	96	0.056850(90)	0.744132(31)	0.16490

evaluate $E_1(q^2)$ and $M_2(q^2)$ and $F_1(q^2)$ over the full kinematic range allowed, and determine the additional form factors that contribute to virtual photon emission only, namely $C_1(q^2)$ (in the case of the χ_{c1} decay) and $\tilde{F}_2(q^2)$ (in the case of the h_c decay).

For this computation we use the gauge field configurations produced by the ETMC, with $N_f = 2 + 1 + 1$ dynamical Wilson-Clover TM fermions which guarantee the automatic $\mathcal{O}(a)$ improvement of physical on-shell observables [20,21]. Basic information regarding the four lattice ensembles used in this work is collected in Table I, and further details can be found in Ref. [22]. To extract the form factors relevant to the $\chi_{c1} \rightarrow J/\psi \ell^+ \ell^-$ and $h_c \rightarrow \eta_c \ell^+ \ell^-$ decays, we work in the rest frame of the decaying meson (either the h_c or the χ_{c1}), i.e., we set $\mathbf{p} = 0$.

The core nonperturbative input needed to extract the $E_1(q^2)$, $M_2(q^2)$, and $C_1(q^2)$ form factors parametrizing the $\chi_{c1} \rightarrow J/\psi \ell^+ \ell^-$ decay is given by the following Euclidean three-point Euclidean function:

$$C_{3\chi}^{ijk}(t_\chi; t_J, \mathbf{k}) = \sum_{\mathbf{x}, \mathbf{y}, \mathbf{z}} e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})} \langle 0 | \mathcal{O}_{J/\psi}^i(\mathbf{x}, 0) J_{\text{em}}^j(\mathbf{y}, -t_J) \times \mathcal{O}_{\chi_{c1}}^{k\dagger}(\mathbf{z}, -t_\chi) | 0 \rangle, \quad (12)$$

where $\mathcal{O}_{J/\psi}^i$ and $\mathcal{O}_{\chi_{c1}}^k$ are the interpolating operators of the J/ψ and χ_{c1} , respectively, which, following previous studies, we take to be the Gaussian-smeared ones defined by

$$\mathcal{O}_{J/\psi}^i(\mathbf{x}, t) = \sum_{\mathbf{y}} \bar{q}_c(\mathbf{x}, t) G_t^n(\mathbf{x}, \mathbf{y}) \gamma^i q_c(\mathbf{y}, t),$$

$$\mathcal{O}_{\chi_{c1}}^k(\mathbf{x}, t) = \sum_{\mathbf{y}} \bar{q}_c(\mathbf{x}, t) G_t^n(\mathbf{x}, \mathbf{y}) \gamma^k \gamma^5 q_c(\mathbf{y}, t), \quad (13)$$

where n is an integer,

$$G_t(\mathbf{x}, \mathbf{y}) = \frac{1}{1 + 6\kappa} [\delta_{\mathbf{x}, \mathbf{y}} + \kappa H_t(\mathbf{x}, \mathbf{y})], \quad (14)$$

κ is a smearing parameter (see below for more details) and $H_t(\mathbf{x}, \mathbf{y})$ is the Gaussian smearing operator,

$$H_t(\mathbf{x}, \mathbf{y}) = \sum_{\mu=1}^3 [U_\mu^\star(\mathbf{x}, t) \delta_{\mathbf{x}+\hat{\mu}, \mathbf{y}} + U_\mu^{\star\dagger}(\mathbf{x} - \hat{\mu}, t) \delta_{\mathbf{x}-\hat{\mu}, \mathbf{y}}], \quad (15)$$

with $U_\mu^\star(x)$ being the so-called APE-smeared links; cf. Ref. [7]. The spatial momentum $\mathbf{k}$ is injected along the third spatial direction $\mathbf{k} = (0, 0, k_z)$, and we compute the three-point function for the following values of k_z , as follows:

$$k_z = 0, \pm 2.4 \times 10^{-2}, 0.175, 0.275, 0.337, 0.3894 \text{ GeV}. \quad (16)$$

This allows us to cover the full kinematic range: from $q^2 = 0$ corresponding to

$$|\mathbf{k}| = \frac{m_{\chi_{c1}}^2 - m_{J/\psi}^2}{2m_{\chi_{c1}}} \simeq 389.4 \text{ MeV}, \quad (17)$$

up to $q_{\text{max}}^2 = (m_{\chi_{c1}} - m_{J/\psi})^2 \sim (0.41 \text{ GeV})^2$ corresponding to $|\mathbf{k}| = 0$. Note however that, as in the case of semileptonic weak transition between pseudoscalar states, e.g. $D \rightarrow \pi \ell \nu$, the three-point function at $|\mathbf{k}| = 0$ is not sufficient by itself to extract all the form factors exactly at q_{max}^2 .⁴ For this reason we have additionally evaluated the three-point function $C_{3\chi}^{ijk}$ at two very tiny values of $k_z \sim \pm 10^{-2} \text{ GeV}$, which will allow us to effectively predict all three form factors at $q^2 \sim q_{\text{max}}^2$. This will be explained in the following subsection.

In the case of the $h_c \rightarrow \eta_c \ell^+ \ell^-$ transition, one needs to compute

$$C_{3h}^{ij}(t_h; t_J, \mathbf{k}) = \sum_{\mathbf{x}, \mathbf{y}, \mathbf{z}} e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})} \langle 0 | \mathcal{O}_{\eta_c}^i(\mathbf{x}, 0) J_{\text{em}}^j(\mathbf{y}, -t_J) \times \mathcal{O}_{h_c}^{i\dagger}(\mathbf{z}, -t_h) | 0 \rangle, \quad (18)$$

where the interpolators of the η_c and h_c mesons are defined as

⁴As it will be clear in the next subsection, only the form factor E_1 can be evaluated directly at q_{max}^2 from the only knowledge of the three-point function at $|\mathbf{k}| = 0$.

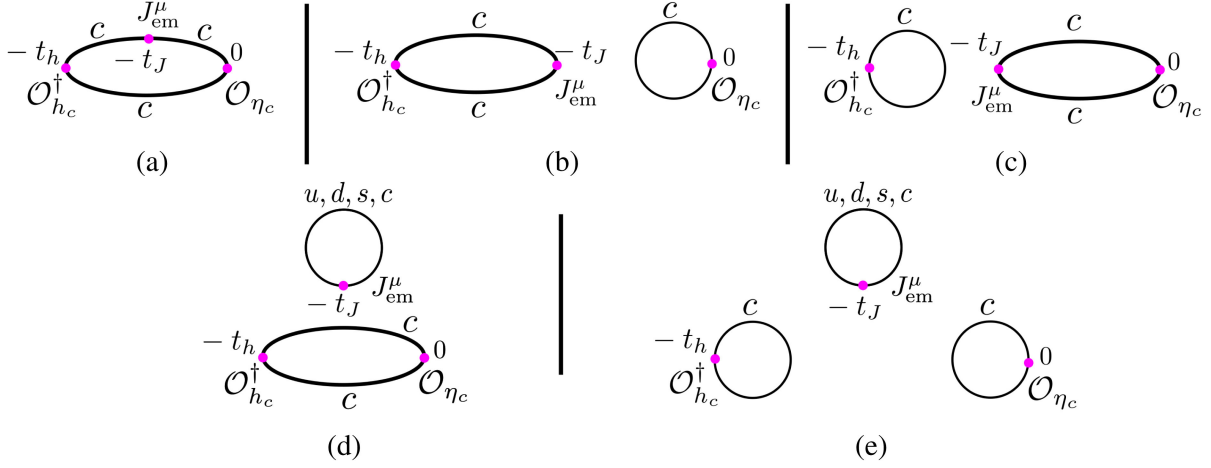


FIG. 1. Wick contractions relevant to the correlation function (18). Diagram (a) corresponds to the connected contribution, while (b)–(e) correspond to the disconnected ones. Diagrams (b), (c), and (e) are Zweig suppressed. For each diagram, we have indicated the flavor associated to each quark line. Light quark (u, d, s) contributions only appear in diagram (d) and diagram (e) where the photon is emitted by a sea quark. These two diagrams are suppressed in the SU(3)-limit $m_u = m_d = m_s$, while the effect of the c sea quark should be negligible.

$$\begin{aligned}\mathcal{O}_{\eta_c}(\mathbf{x}, t) &= \sum_{\mathbf{y}} \bar{q}_c(\mathbf{x}, t) G_t^n(\mathbf{x}, \mathbf{y}) \gamma^5 q_c(\mathbf{y}, t), \\ \mathcal{O}_{h_c}^i(\mathbf{x}, t) &= \sum_{\mathbf{y}} \bar{q}_c(\mathbf{x}, t) G_t^n(\mathbf{x}, \mathbf{y}) \epsilon^{ikl} \sigma_{kl} q_c(\mathbf{y}, t).\end{aligned}\quad (19)$$

Again, we inject the spatial momentum along the third direction, $\mathbf{k} = (0, 0, k_z)$. In the case of the $h_c \rightarrow \eta_c$ transition, the spatial momenta we considered are

$$k_z = 0, \pm 1.2 \times 10^{-6}, 0.125, 0.250, 0.375, 0.469, 0.5 \text{ GeV}, \quad (20)$$

allowing us to cover the full phase space, from $q^2 = 0$ corresponding to

$$|\mathbf{k}| = \frac{m_{h_c}^2 - m_{\eta_c}^2}{2m_{h_c}} \simeq 0.50 \text{ GeV}, \quad (21)$$

up to $q_{\max}^2 = (m_{h_c} - m_{\eta_c})^2 \simeq (0.54 \text{ GeV})^2$, corresponding to $\mathbf{k} = 0$. As in the case of the χ_{c1} decay, also here only the form factor $F_1(q_{\max}^2)$ can be extracted directly from simulations at $\mathbf{k} = 0$.

In analogy with what we did for the $\chi_{c1} \rightarrow J/\psi$ decay, we have computed the three-point functions for two very tiny values of $k_z \simeq \pm 10^{-6} \text{ GeV}$, which then allow us to obtain $\bar{F}_2(q_{\max}^2)$, as will be discussed in the next subsection. For all the meson interpolators discussed above, in analogy with our previous studies [10,11], we used the smearing parameter $\kappa = 0.4$, and fixed on each ensemble the number of steps n in Eq. (13) to have a smearing radius $r_0 = a\sqrt{n}/\sqrt{\kappa^{-1} + 6} \simeq 0.15 \text{ fm}$.

The Wick contractions of the three-point functions C_{3h}^{ij} and $C_{3\eta_c}^{ij}$ in Eqs. (12) and (18) give rise to quark-connected

and quark-disconnected contributions, cf. Fig. 1 where we sketch Feynman diagrams corresponding to the Wick contractions of C_{3h}^{ij} (the case of $C_{3\eta_c}^{ij}$ is identical). In the following, we will refer to these contributions as *connected* and *disconnected*, respectively. Diagram (a) corresponds to the dominant (connected) contribution, while (b)–(e) correspond to the disconnected ones. The contributions corresponding to (b), (c), and (e) are expected to be very small due to the Zweig suppression. Furthermore, diagrams (d) and (e) respect the SU(3) suppression in which the contributions of u, d , and s cancel out in the $m_u = m_d = m_s$ limit. In this work we focus on the evaluation of the dominant (connected) diagram, leaving the evaluation of the disconnected contributions for future works.⁵ Clearly, only the charm quark component J_c^μ of the electromagnetic current J_{em}^μ contributes to the connected part of the correlation functions C_{3h}^{ijk} and $C_{3\eta_c}^{ij}$.

If only the quark-connected contribution is considered, it is possible to use partially quenched twisted boundary conditions to tune the spatial momentum $\mathbf{k}$ to the values given in Eqs. (16) and (20), which are in general not integer multiples of $2\pi/L$. This is implemented by twisting the gauge links $U_\mu(x)$ on which one of the charm quark propagators is computed [24–26]. More explicitly,⁶

$$U_\mu(x) \rightarrow U_\mu^\theta(x) = e^{i\pi a \theta_\mu / L} U_\mu(x), \quad \theta_\mu = (0, \vec{\theta}), \quad (22)$$

with the twist angle $\vec{\theta}$ set to

⁵A lattice QCD study of Ref. [23] shows that the contributions arising from disconnected diagrams indeed give a tiny contribution to the masses of charmonia.

⁶In Refs. [10,11], the phase in the corresponding formulas for the twisted gauge link should include an additional factor of π .

$$\vec{\theta} = (0, 0, \theta_z^c), \quad \theta_z^c = \frac{L}{\pi} k_z, \quad (23)$$

where L is the spatial extent of the lattice. Note that the physical timelike kinematic region relevant for the Dalitz decays cannot be accessed in a sufficiently dense way using only the standard Fourier lattice momenta. For this reason, we employ partially twisted boundary conditions, which allow us to continuously tune the hadron momenta and thereby map the full timelike q^2 dependence of the form factors. This is different from what has been done in previous works on the subject.

We are now in a position to discuss our specific regularization of quark bilinears, as adopted for the calculation of the quark-connected contribution to both $C_{3\chi}^{ijk}$ and C_{3h}^{ij} . As discussed in Refs. [10,11], when evaluating the connected diagrams using twisted-mass fermions, there are multiple options for discretizing the quark bilinears, differing in the choice of the Wilson parameter of the quark field. Specifically, the quark-connected contributions to the three-point functions can be derived from a mixed-action setup, in which—alongside the simulated twisted-mass u , d , s , and c sea quarks—the following valence-quark action is introduced:

$$S_{\text{val}} = \sum_{f=c,c',c''} \sum_x \bar{q}_f(x) [\gamma_\mu \bar{\nabla}_\mu [U] - i r_f \gamma^5 (W^{\text{cl}}[U] + m_{\text{cr}}) + m_c] q_f(x), \quad (24)$$

where $W^{\text{cl}}[U]$ is the Wilson-Clover term [27], m_{cr} is the critical mass, m_c is the charm-quark mass, and $r_{f=c,c',c''} = \pm 1$ is the sign of the twisted-Wilson parameter of the valence quark field q_f . The primed fields are thus replica charm-quark fields differing only in the choice of the twisted-Wilson parameter, while all other bare parameters are kept identical.

In this mixed action lattice setup, the quark-connected contribution to the three-point functions in Eqs. (12) and (18) can be obtained by representing the quark bilinear

meson interpolating operators as follows:

$$\begin{aligned} \bar{q}_c \Gamma_{\chi_{c1}} q_c &\rightarrow \bar{q}_{c'} \Gamma_{\chi_{c1}} q_c, & \bar{q}_c \Gamma_{J/\psi} q_c &\rightarrow \bar{q}_{c'} \Gamma_{J/\psi} q_{c''}, \\ \bar{q}_c \Gamma_{h_c} q_c &\rightarrow \bar{q}_{c'} \Gamma_{h_c} q_c, & \bar{q}_c \Gamma_{\eta_c} q_c &\rightarrow \bar{q}_{c'} \Gamma_{\eta_c} q_{c''}, \end{aligned} \quad (25)$$

where $\bar{q}_c \Gamma_P q_c$ denotes the interpolating operator of the meson P . The charm-quark component of the electromagnetic current, J_c^μ , is discretized as $J_c^\mu = Q_c \bar{q}_c \gamma^\mu q_{c''}$. The full Euclidean correlation function evaluated with these operators corresponds to the quark-connected contribution, up to an overall factor of 2, accounting for the missing charge-conjugated diagram in which the electromagnetic current is inserted in the antiquark line.

With twisted-mass fermions, the parameters r_c , $r_{c'}$, and $r_{c''}$ can be freely chosen as ± 1 . We restrict ourselves to $r_c = r_{c''} = 1$. This still leaves open the choice between $r_c = r_{c'}$ and $r_c = -r_{c'}$. The first choice corresponds to the Osterwalder-Seiler (OS) regularization of the meson interpolating fields, while the second corresponds to the TM regularization of the same fields. In Refs. [10,11], we demonstrated that for the χ_{c1} decay (resp. h_c decay), the OS (resp. TM) regularization is preferable, because, due to $H(3)$, charge conjugation and (only for the TM case) parity times G -parity symmetries of the valence quark action, it avoids unwanted mixing of the interpolated meson states with lighter states of different J^{PC} quantum numbers. For the present calculation, we adopt the same choices as in Refs. [10,11], using OS regularization for the $\chi_{c1} \rightarrow J/\psi \ell^+ \ell^-$ decay and TM regularization for the $h_c \rightarrow \eta_c \ell^+ \ell^-$ decay. Since the electromagnetic current is discretized in both cases as $\bar{q}_c \gamma^i q_{c''}$ with $r_c = r_{c''} = 1$, the current renormalizes with the vector renormalization constant $Z_V(\beta)$, whose values are collected in Table I.

In our mixed-action setup, after denoting by $S_c^\pm(x, y)$ and $S_c^{\pm, \theta}(x, y)$ the charm-quark propagators evaluated with $r = \pm 1$ on the background field configurations corresponding to $U_\mu(x)$ and $U_\mu^\theta(x)$, respectively, the quark-connected contribution to $C_{3\chi}^{ijk}(t_\chi; t_J, \mathbf{k})$ and $C_{3h}^{ij}(t_h; t_J, \mathbf{k})$ is evaluated as⁷

$$\begin{aligned} C_{3\chi}^{ijk}(t_\chi; t_J, \mathbf{k}) &= 2Q_c \sum_{\mathbf{x}, \mathbf{y}, \mathbf{w}} \langle \text{Tr}[S_c^+(x, w) \gamma^k \gamma^5 S_c^+(w, y) \gamma^j S_c^{+\theta}(y, x) \gamma^i] \rangle, \\ C_{3h}^{ij}(t_h; t_J, \mathbf{k}) &= 2Q_c \sum_{\mathbf{x}, \mathbf{y}, \mathbf{z}} \langle \text{Tr}[S_c^-(x, z) \epsilon^{ikl} \sigma_{kl} S_c^+(z, y) \gamma^j S_c^{+\theta}(y, x) \gamma^5] \rangle, \end{aligned} \quad (26)$$

where $x = (\mathbf{x}, 0)$, $y = (\mathbf{y}, -t_J)$, $w = (\mathbf{w}, -t_\chi)$, and $z = (\mathbf{z}, -t_h)$. The trace $\text{Tr}[\dots]$ is taken over color and Dirac indices, and $\langle \dots \rangle$ denotes the average over the SU(3) gauge field configurations $U_\mu(x)$. The factor of 2 in Eq. (26) accounts for the contribution of the charge-conjugated diagram. The sum over $\mathbf{x}$ is evaluated stochastically by inverting the Dirac charm-quark operator on N_s spatial stochastic sources placed at time $t = 0$. The number

of gauge configurations and stochastic sources used for the evaluation of the three-point functions differs slightly between the h_c and χ_{c1} cases and will be detailed in the corresponding numerical sections. We now proceed to the

⁷For simplicity, we consider local interpolating operators, i.e., $\kappa = 0$. In practice, we implement Gaussian smearing as discussed in the text.

extraction of the relevant form factors from the three-point functions, beginning with the χ_{c1} case.

A. Extraction of the $\chi_{c1} \rightarrow J/\psi \ell^+ \ell^-$ form factors E_1 , M_2 , and C_1

We first consider the two-point functions of the interpolating fields $\mathcal{O}_{J/\psi}^i$ and $\mathcal{O}_{\chi_{c1}}^k$ to extract their masses/couplings and energies:

$$\begin{aligned} C_{2,J/\psi}^i(t, \mathbf{k}) &= \sum_{\mathbf{x}} e^{i\mathbf{k}\mathbf{x}} \langle 0 | \mathcal{O}_{J/\psi}^i(\mathbf{x}, t) \mathcal{O}_{J/\psi}^{i\dagger}(\mathbf{0}, 0) | 0 \rangle, \\ C_{2,\chi_{c1}}^k(t) &= \sum_{\mathbf{x}} \langle 0 | \mathcal{O}_{\chi_{c1}}^k(\mathbf{x}, t) \mathcal{O}_{\chi_{c1}}^{k\dagger}(\mathbf{0}, 0) | 0 \rangle, \end{aligned} \quad (27)$$

neglecting the Zweig-suppressed disconnected contributions. In the limit of large Euclidean time t , the two-point correlation functions behave as

$$\begin{aligned} C_{2,J/\psi}^i(t, \mathbf{k}) &= \frac{|Z_{J/\psi}^i(\mathbf{k})|^2}{2E_{J/\psi}(\mathbf{k})} (e^{-E_{J/\psi}(\mathbf{k})t} + e^{-E_{J/\psi}(\mathbf{k})(T-t)}) + \dots, \\ C_{2,\chi_{c1}}^k(t) &= \frac{|Z_{\chi_{c1}}^k|^2}{2m_{\chi_{c1}}} (e^{-m_{\chi_{c1}}t} + e^{-m_{\chi_{c1}}(T-t)}) + \dots, \end{aligned} \quad (28)$$

with $m_{\chi_{c1}}$ and $E_{J/\psi}(\mathbf{k})$ being the mass and energy of the corresponding ground states, and

$$\begin{aligned} Z_{J/\psi}^i(\mathbf{k}) &= \langle 0 | \mathcal{O}_{J/\psi}^i | J/\psi(\mathbf{k}, \varepsilon_i) \rangle, \\ Z_{\chi_{c1}}^k &= \langle \chi_{c1}(\mathbf{0}, \eta_k) | \mathcal{O}_{\chi_{c1}}^{k\dagger} | 0 \rangle, \end{aligned} \quad (29)$$

where the dependence of $Z_{J/\psi}^i$ on the momentum $\mathbf{k}$ arises from the smearing. For χ_{c1} , the interpolator $\mathcal{O}_{\chi_{c1}}^k$ ($k = 1, 2, 3$) creates a χ_{c1} state with polarization $\eta_k^\mu \propto \delta_k^\mu$. For J/ψ , the interpolator $\mathcal{O}_{J/\psi}^i$ ($i = 1, 2$) annihilates a J/ψ state with polarization $\varepsilon_i^\mu \propto \delta_i^\mu$, while for $i = 3$ it annihilates a longitudinally polarized J/ψ with $\varepsilon_3^\mu = (\alpha, 0, 0, \sqrt{1+\alpha^2})$ and $\alpha = |\mathbf{k}|/m_{J/\psi}$. The ellipses in Eq. (28) denote contributions that vanish in the large-time limit, $0 \ll t \ll T$, and are further suppressed by the smearing [cf. Eq. (14)]. With our choice of kinematics, we have $Z_{\chi_{c1}}^1 = Z_{\chi_{c1}}^2 = Z_{\chi_{c1}}^3$ and $Z_{J/\psi}^1(\mathbf{k}) = Z_{J/\psi}^2(\mathbf{k})$.

Using Eq. (8), we combine the above quantities to construct the following estimators:

$$\begin{aligned} \bar{E}_1(t_\chi; t_J, \mathbf{k}) &= \frac{iZ_V(\beta)}{\sqrt{2}Q_c} 4E_{J/\psi}(\mathbf{k}) e^{E_{J/\psi}(\mathbf{k})t_J} e^{m_{\chi_{c1}}(t_\chi - t_J)} \times \left[\frac{C_{3\chi}^{213}(t_\chi; t_J, \mathbf{k})}{Z_{J/\psi}^2(\mathbf{k})Z_{\chi_{c1}}^3} - \frac{C_{3\chi}^{312}(t_\chi; t_J, \mathbf{k})}{Z_{J/\psi}^3(\mathbf{k})Z_{\chi_{c1}}^2} \right], \\ \bar{M}_2(t_\chi; t_J, \mathbf{k}) &= \frac{iZ_V(\beta)}{\sqrt{2}Q_c} 4E_{J/\psi}(\mathbf{k}) e^{E_{J/\psi}(\mathbf{k})t_J} e^{m_{\chi_{c1}}(t_\chi - t_J)} \times \left[\frac{C_{3\chi}^{213}(t_\chi; t_J, \mathbf{k})}{Z_{J/\psi}^2(\mathbf{k})Z_{\chi_{c1}}^3} + \frac{C_{3\chi}^{312}(t_\chi; t_J, \mathbf{k})}{Z_{J/\psi}^3(\mathbf{k})Z_{\chi_{c1}}^2} \right], \\ \bar{C}_1(t_\chi; t_J, \mathbf{k}) &= \frac{iZ_V(\beta)}{\sqrt{2}Q_c} 4E_{J/\psi}(\mathbf{k}) \frac{m_{\chi_{c1}}}{|\mathbf{k}|} e^{E_{J/\psi}(\mathbf{k})t_J} e^{m_{\chi_{c1}}(t_\chi - t_J)} \times \left[\frac{C_{3\chi}^{102}(t_\chi; t_J, \mathbf{k})}{Z_{J/\psi}^1(\mathbf{k})Z_{\chi_{c1}}^2} - \frac{C_{3\chi}^{201}(t_\chi; t_J, \mathbf{k})}{Z_{J/\psi}^2(\mathbf{k})Z_{\chi_{c1}}^1} \right], \end{aligned} \quad (30)$$

which, for large Euclidean time separations, yield the desired form factors E_1 , M_2 , and C_1 , as follows:

$$\begin{aligned} \bar{E}_1(t_\chi; t_J, \mathbf{k}) &\xrightarrow[t_\chi - t_J \rightarrow \infty]{t_J \rightarrow \infty} E_1(q^2), & \bar{M}_2(t_\chi; t_J, \mathbf{k}) &\xrightarrow[t_\chi - t_J \rightarrow \infty]{t_J \rightarrow \infty} M_2(q^2), & \bar{C}_1(t_\chi; t_J, \mathbf{k}) &\xrightarrow[t_\chi - t_J \rightarrow \infty]{t_J \rightarrow \infty} C_1(q^2). \end{aligned} \quad (31)$$

The relation between $|\mathbf{k}|$ and q^2 is given by

$$q^2 = m_{\chi_{c1}}^2 + m_{J/\psi}^2 - 2m_{\chi_{c1}} \sqrt{m_{J/\psi}^2 + |\mathbf{k}|^2}. \quad (32)$$

In $C_{3\chi}^{ijk}(t_\chi; t_J, \mathbf{k})$, the interpolating field of J/ψ is placed at $t = 0$, the charm-quark component of the electromagnetic current is inserted at $-t_J < 0$, and the time $-t_\chi \ll -t_J$, at which the χ_{c1} is created, corresponds to the *sink* of the correlation function. This setup allows us to monitor the dominance of the χ_{c1} state. Since t_J is fixed in our computational setup, it must be chosen large enough to isolate the J/ψ state. Placing the J/ψ interpolator at the

source of the correlation function is advantageous, as t_J can be chosen reasonably large: the signal-to-noise ratio (SNR) of $C_{3\chi}^{ijk}(t_\chi; t_J)$ for fixed $|t_\chi - t_J|$ decreases only as $e^{-(E_{J/\psi} - m_{\eta_c})t} \simeq e^{-\Delta_c t}$, where $\Delta_c \simeq 113$ MeV is the (experimental) hyperfine splitting. We considered two values of $t_J \simeq 1.6, 2.4$ fm and observed only minor differences in our results, which we include in the final systematic error.

We introduce two improvements to the estimators for $M_2(q^2)$ and $C_1(q^2)$ that significantly reduce their statistical uncertainties:

- (1) Case of $C_1(q^2)$: the three-point correlation functions $C_{3\chi}^{102}$ and $C_{3\chi}^{201}$ vanish in the limit $\mathbf{k} \rightarrow 0$ [see

Eq. (12)]. However, on a finite statistical sample, they vanish only within statistical uncertainties. For small but nonzero $\mathbf{k}$, the noise associated with the correlation function is expected to be similar to that at $\mathbf{k} = 0$. We therefore use

$$\begin{aligned} C_{3\chi}^{201}(t_\chi; t_J, \mathbf{k}) &\rightarrow C_{3\chi}^{201}(t_\chi; t_J, \mathbf{k}) - C_{3\chi}^{201}(t_\chi; t_J, \mathbf{0}), \\ C_{3\chi}^{102}(t_\chi; t_J, \mathbf{k}) &\rightarrow C_{3\chi}^{102}(t_\chi; t_J, \mathbf{k}) - C_{3\chi}^{102}(t_\chi; t_J, \mathbf{0}), \end{aligned} \quad (33)$$

since part of the statistical noise cancels in the correlated differences. This subtraction is applied to all considered momenta $\mathbf{k}$ and is particularly effective at very small momenta.

At $\mathbf{k} = 0$, the form factor C_1 cannot be evaluated directly. However, by performing a correlated subtraction between the three-point function at very small momentum $|\mathbf{k}| \simeq 10^{-2}$ GeV and that at vanishing momentum, we obtain an estimate of $C_1(q^2 \simeq q_{\max}^2)$.

- (2) Case of $M_2(q^2)$: for vanishing momentum $\mathbf{k} = 0$, the relation $C_{3\chi}^{213}(t_\chi; t_J, \mathbf{0}) = -C_{3\chi}^{312}(t_\chi; t_J, \mathbf{0})$ holds, implying $M_2(q_{\max}^2) = 0$. Since both $C_{3\chi}^{213}$ and $C_{3\chi}^{312}$ are even functions of the spatial momentum, it follows that for small $|\mathbf{k}|$, $M_2(q^2 \sim q_{\max}^2) \propto |\mathbf{k}|^2$. We perform a zero-momentum subtraction to reduce the statistical uncertainty:

$$\bar{M}_2(t_\chi; t_J, \mathbf{k}) \rightarrow \bar{M}_2(t_\chi; t_J, \mathbf{k}) - \bar{M}_2(t_\chi; t_J, \mathbf{0}). \quad (34)$$

This procedure is applied to all momenta k_z except for the smallest momentum $k_z = 2.4 \times 10^{-2}$ GeV, for which a further improved estimator is constructed by exploiting the behavior $M_2(q^2 \sim q_{\max}^2) \propto |\mathbf{k}|^2$. In this case, we use

$$\bar{M}_2(t_\chi; t_J, \mathbf{k}) \rightarrow \frac{1}{2} [\bar{M}_2(t_\chi; t_J, \mathbf{k}) + \bar{M}_2(t_\chi; t_J, -\mathbf{k})]. \quad (35)$$

This subtraction, involving $\pm\mathbf{k}$ together with $\mathbf{k} = 0$, is more effective than the zero-momentum subtraction of Eq. (34), as it cancels noise components that are either momentum-independent or linear in $\mathbf{k}$. At very small momentum, this improvement is crucial, as the signal scales as $|\mathbf{k}|^2$ and would otherwise be overwhelmed by statistical noise.

In the limit $\mathbf{k} \rightarrow 0$, the improved estimator divided by $|\mathbf{k}|^2$ directly provides an estimate of the second derivative of the form factor with respect to

$|\mathbf{k}|$, allowing a clean extraction of the slope

$$\alpha = \lim_{|\mathbf{k}| \rightarrow 0} \frac{M_2(q^2(\mathbf{k}))}{|\mathbf{k}|^2}. \quad (36)$$

We emphasize that these improvements are effective only if the zero-momentum subtraction and the averaging in Eq. (35) are performed in a fully correlated manner, using identical gauge configurations and stochastic sources for all terms.

Having described our strategy for determining the form factors $E_1(q^2)$, $M_2(q^2)$, and $C_1(q^2)$, we now turn to the $h_c \rightarrow \eta_c \ell^+ \ell^-$ case.

B. Extraction of the $h_c \rightarrow \eta_c \ell^+ \ell^-$ form factors $F_1(q^2)$ and $\tilde{F}_2(q^2)$

We now discuss the extraction of the two-point functions for the interpolating fields $\mathcal{O}_{\eta_c}$ and $\mathcal{O}_{h_c}^i$, which are necessary to determine their masses/energies and couplings. We define these two-point functions as

$$\begin{aligned} C_{2,\eta_c}(t, \mathbf{k}) &= \sum_{\mathbf{x}} e^{i\mathbf{k}\mathbf{x}} \langle 0 | \mathcal{O}_{\eta_c}(\mathbf{x}, t) \mathcal{O}_{\eta_c}^\dagger(\mathbf{0}, 0) | 0 \rangle, \\ C_{2,h_c}^i(t) &= \sum_{\mathbf{x}} \langle 0 | \mathcal{O}_{h_c}^i(\mathbf{x}, t) \mathcal{O}_{h_c}^{i\dagger}(\mathbf{0}, 0) | 0 \rangle, \end{aligned} \quad (37)$$

neglecting the Zweig-suppressed disconnected contributions to $C_{2,\eta_c}(t, \mathbf{k})$ and $C_{2,h_c}^i(t)$.

In the limit of large Euclidean time t , the two-point correlation functions behave as

$$\begin{aligned} C_{2,\eta_c}(t, \mathbf{k}) &= \frac{|Z_{\eta_c}(\mathbf{k})|^2}{2E_{\eta_c}(\mathbf{k})} (e^{-E_{\eta_c}(\mathbf{k})t} + e^{-E_{\eta_c}(\mathbf{k})(T-t)}) + \dots, \\ C_{2,h_c}^i(t) &= \frac{|Z_{h_c}^i|^2}{2m_{h_c}} (e^{-m_{h_c}t} + e^{-m_{h_c}(T-t)}) + \dots, \end{aligned} \quad (38)$$

where m_{h_c} and E_{η_c} are the mass and energy of the corresponding mesons, and

$$Z_{\eta_c}(\mathbf{k}) = \langle 0 | \mathcal{O}_{\eta_c} | \eta_c(\mathbf{k}) \rangle, \quad Z_{h_c}^i = \langle h_c(\mathbf{0}, \epsilon_i) | \mathcal{O}_{h_c}^{i\dagger} | 0 \rangle. \quad (39)$$

The dependence of Z_{η_c} on $\mathbf{k}$ is due to the smearing of the interpolator. With our choice of kinematics, $Z_{h_c}^1 = Z_{h_c}^2 = Z_{h_c}^3 = Z_{h_c}$.

The ellipses in Eq. (38) denote contributions that vanish in the large time limit, $0 \ll t \ll T$. The interpolating field $\mathcal{O}_{h_c}^{i\dagger}$ in Eq. (19) creates an h_c meson with polarization $\epsilon_i^\mu \propto \delta_i^\mu$. Given our choice of kinematics, $\mathbf{p} = 0$ and $\mathbf{k} = (0, 0, k_z)$, we use Eq. (9) to define the following estimators:

$$\begin{aligned}\bar{F}_1(t_h; t_J, \mathbf{k}) &= \frac{iZ_V(\beta)}{2Q_c} \frac{4E_{\eta_c}(\mathbf{k})}{Z_{\eta_c}(\mathbf{k})Z_{h_c}} e^{E_{\eta_c}(\mathbf{k})t_J} e^{m_{h_c}(t_h-t_J)} C_{3h}^{11}(t_h; t_J, \mathbf{k}), \\ \bar{R}_1(t_h; t_J, \mathbf{k}) &= \frac{iZ_V(\beta)}{2Q_c} \frac{4E_{\eta_c}(\mathbf{k})}{Z_{\eta_c}(\mathbf{k})Z_{h_c}} e^{E_{\eta_c}(\mathbf{k})t_J} e^{m_{h_c}(t_h-t_J)} C_{3h}^{33}(t_h; t_J, \mathbf{k}),\end{aligned}\quad (40)$$

which, for large Euclidean time separations, allow us to extract the form factors $F_1(q^2)$ and $\tilde{F}_2(q^2)$, as follows:

$$\begin{aligned}\bar{F}_1(t_h; t_J, \mathbf{k}) &\xrightarrow[t_J \rightarrow \infty]{t_h - t_J \rightarrow \infty} F_1(q^2), \\ \bar{R}_1(t_h; t_J, \mathbf{k}) &\xrightarrow[t_J \rightarrow \infty]{t_h - t_J \rightarrow \infty} \left(1 + \frac{|\mathbf{k}|^2}{q^2}\right) F_1(q^2) - \left[\frac{1}{(m_{h_c}^2 - m_{\eta_c}^2)} + \frac{1}{q^2}\right] |\mathbf{k}|^2 \tilde{F}_2(q^2).\end{aligned}\quad (41)$$

For convenience, we define an estimator that converges directly to $\tilde{F}_2(q^2)$ as follows:

$$\begin{aligned}\bar{F}_2(t_h; t_J, \mathbf{k}) &\equiv \frac{\bar{R}_1^{\text{sub}}(t_h; t_J, \mathbf{k})}{\delta(q^2)} \\ &\equiv \frac{\bar{R}_1(t_h; t_J, \mathbf{k}) - \beta(q^2)\bar{F}_1(t_h; t_J, \mathbf{k})}{\delta(q^2)},\end{aligned}\quad (42)$$

with

$$\beta(q^2) = 1 + \frac{|\mathbf{k}|^2}{q^2}, \quad \delta(q^2) = -|\mathbf{k}|^2 \left[\frac{1}{(m_{h_c}^2 - m_{\eta_c}^2)} + \frac{1}{q^2} \right]. \quad (43)$$

We remind the reader that, in our setup, the relation between $|\mathbf{k}|$ and q^2 is given by

$$q^2 = m_{h_c}^2 + m_{\eta_c}^2 - 2m_{h_c}\sqrt{m_{\eta_c}^2 + |\mathbf{k}|^2}. \quad (44)$$

In analogy with the $\chi_{c1} \rightarrow J/\psi \ell^+ \ell^-$ transition discussed in Sec. II A, we compute $C_{3h}^{ij}(t_h; t_J, \mathbf{k})$ by placing the interpolating field of η_c at $t = 0$, inserting the charm-quark component of the electromagnetic current at a fixed time $-t_J < 0$, while the time $-t_h \ll -t_J$, at which the h_c state is created, corresponds to the *sink* of the correlation function. This setup allows us to monitor the onset of the h_c ground-state dominance.

The insertion time t_J must be chosen large enough to isolate the η_c ground state. The advantage of our setup is that t_J can be taken reasonably large without significant loss of signal, since the SNR of $C_{3h}^{ij}(t_h; t_J, \mathbf{k})$ for fixed time separation $|t_h - t_J|$ scales as

$$e^{-(E_{\eta_c}(\mathbf{k}) - m_{\eta_c})t_J} \simeq e^{-|\mathbf{k}|^2 t_J / (2m_{\eta_c})}, \quad (45)$$

with $|\mathbf{k}|^2 / (2m_{\eta_c}) \leq 40$ MeV. The SNR therefore decreases only slowly as t_J increases for fixed $|t_h - t_J|$. As in the previous case, we consider two values, $t_J \simeq 1.6$ and 2.4 fm, and verify that the results for $\bar{F}_{1(2)}(t_h; t_J, \mathbf{k})$ are consistent within statistical uncertainties.

To further reduce statistical errors, we employ additional numerical improvements, particularly for $\tilde{F}_2(q^2)$. This procedure closely parallels the strategy used for the $M_2(q^2)$ form factor. In the limit $\mathbf{k} \rightarrow 0$, we see from Eq. (41) that $\bar{R}_1 = \bar{F}_1$, and with $\beta(q^2) \rightarrow 1$, the estimator $\bar{R}_1^{\text{sub}}(t_h; t_J, \mathbf{0})$ vanishes. However, with finite statistics, this cancellation is not exact, and the noise of $\bar{R}_1(t_h; t_J, \mathbf{0})$ is strongly correlated with that of $\bar{R}_1(t_h; t_J, \mathbf{k})$, at least for moderately small momenta $\mathbf{k}$. For this reason, the zero-momentum subtraction,

$$\bar{F}_2(t_h; t_J, \mathbf{k}) \rightarrow \frac{\bar{R}_1^{\text{sub}}(t_h; t_J, \mathbf{k}) - \bar{R}_1^{\text{sub}}(t_h; t_J, \mathbf{0})}{\delta(q^2)}, \quad (46)$$

leads to a significantly reduced statistical noise.

The zero-momentum subtraction is applied to all values of $\mathbf{k}$ in Eq. (20), except for the very small momentum $k_z \simeq 10^{-6}$ GeV. In this case, we introduce a further improvement by subtracting noise components that are momentum-independent or linear in $\mathbf{k}$, as follows:

$$\bar{F}_2(t_h; t_J, \mathbf{k}) \rightarrow \frac{1}{2} [\bar{F}_2(t_h; t_J, \mathbf{k}) + \bar{F}_2(t_h; t_J, -\mathbf{k})], \quad (47)$$

exploiting the fact that $\bar{F}_2$ is an even function of $\mathbf{k}$. This correlated subtraction is essential for determining

TABLE II. Number, N_g , of gauge configurations considered for the calculation of the $\chi_{c1} \rightarrow J/\psi \ell^+ \ell^-$ form factors, for each of the $N_f = 2 + 1 + 1$ ETMC gauge ensembles in Table I. N_s^0 is the number of (spin and color diluted) stochastic sources adopted in Ref. [10] for the calculation of the E_1 and M_2 form factors at $q^2 = 0$, while N_s is the number of stochastic sources considered for the present calculation in which the photon is generally off shell.

ID	N_g	N_s^0	N_s
A48	800	140	140
B64	400	32	32
C80	640	24	8
D96	300	35	8

$\tilde{F}_2(q_{\max}^2)$, which would otherwise suffer from very large statistical uncertainties.

III. NUMERICAL RESULTS

A. Lattice results for the $\chi_{c1} \rightarrow J/\psi \ell^+ \ell^-$ form factors

Having described in Sec. II A the extraction strategy for the $\chi_{c1} \rightarrow J/\psi \ell^+ \ell^-$ form factors, we now turn to the corresponding numerical analysis. In Ref. [10], we determined the form factors $E_1(0)$ and $M_2(0)$ through a high-statistics analysis of the relevant two- and three-point correlation functions, employing a large number of gauge field configurations and stochastic sources. The statistics used in that work are summarized for each ensemble in Table II, where N_s^0 denotes the number of stochastic sources employed in that calculation.

To extend the analysis to the case of a virtual photon, we performed additional simulations for all momentum values listed in Eq. (16), using the same gauge configurations as in Ref. [10] but with newly generated stochastic sources, whose number is denoted by N_s . For the computationally more demanding C80 and D96 ensembles, a reduced number of stochastic sources was employed; the corresponding values are reported in Table II. To benefit from the high-statistics computation of Ref. [10] within the present analysis, we implemented the following improvements:

First, the mass and couplings of the χ_{c1} are always determined from the high-statistics dataset, achieving higher precision. Moreover, for the form factors $E_1(q^2)$ and $M_2(q^2)$, we introduce the following preconditioning of the estimators $\bar{E}_1(t_\chi; t_J, \mathbf{k})$ and $\bar{M}_2(t_\chi; t_J, \mathbf{k})$:

$$\begin{aligned} \bar{E}_1(t_\chi; t_J, \mathbf{k}) &= \bar{E}_1^{N_s^0}(t_\chi; t_J, \mathbf{k}_{\max}) \frac{\bar{E}_1^{N_s}(t_\chi; t_J, \mathbf{k})}{\bar{E}_1^{N_s}(t_\chi; t_J, \mathbf{k}_{\max})}, \\ \bar{M}_2(t_\chi; t_J, \mathbf{k}) &= \bar{M}_2^{N_s^0}(t_\chi; t_J, \mathbf{k}_{\max}) \frac{\bar{M}_2^{N_s}(t_\chi; t_J, \mathbf{k})}{\bar{M}_2^{N_s}(t_\chi; t_J, \mathbf{k}_{\max})}, \end{aligned} \quad (48)$$

where the superscripts N_s^0 and N_s indicate determinations obtained with N_s^0 and N_s stochastic sources, respectively.

The rationale behind this strategy is that results obtained for a generic momentum $|\mathbf{k}|$ and at the reference momentum $|\mathbf{k}_{\max}| \simeq 389.4$ MeV are partially correlated, with the correlation increasing as $\mathbf{k}$ approaches $\mathbf{k}_{\max}$. This allows the ratio between form factors at q^2 and at $q^2 = 0$ to be determined using a reduced set of stochastic sources without significant loss of precision. The high-statistics determinations of $E_1(0)$ and $M_2(0)$ obtained with N_s^0 stochastic sources can then be used to reconstruct $E_1(q^2)$ and $M_2(q^2)$ over the full kinematic range.

This preconditioning procedure proves highly effective in reducing statistical uncertainties and, more generally, can be advantageous for reducing computational costs whenever form factors must be evaluated across the allowed kinematic region. In particular, when accessing different kinematic points requires additional inversions of the Dirac operator, as in calculations employing twisted boundary conditions, it is convenient to determine the form factors with high precision at a reference kinematic point (here $q^2 = 0$) and then compute ratios with respect to this reference using a smaller number of stochastic sources.

In Fig. 2, we show, as an illustrative example, the estimators of the form factors $E_1(q^2)$, $M_2(q^2)/|\mathbf{k}|^2$, and $C_1(q^2)$ on the finest lattice-spacing ensemble (D96) for a fixed value of $t_J \simeq 1.6$ fm.⁸ We observe that the q^2 dependence of E_1 , $M_2/|\mathbf{k}|^2$, and C_1 is smooth, and the quality of the plateaus remains comparable when moving away from $q^2 = 0$. As $q_{\max}^2$ is approached, an increase in statistical noise is observed for E_1 and M_2 . This can be traced back to the reduced efficiency of the preconditioning in Eq. (48), since the correlation between $q^2 = 0$ and $q^2 \simeq q_{\max}^2$ becomes weaker. Note also that the longitudinal form factor, $C_1(q^2)$, is new. It was not considered in our study of $\chi_{c1} \rightarrow J/\psi \gamma$ [10] since it does not contribute in the real-photon limit. The estimators for $C_1(q^2)$ are also shown in Fig. 2.

The form factors are extracted from correlated constant fits to the effective estimators in the plateau region. For illustration, the vertical dashed lines in Fig. 2 approximately delimit the fit intervals adopted for the D96 ensemble, namely $t \in [1.1, 1.5]$ fm for E_1 , $t \in [1.0, 1.4]$ fm for C_1 , and $t \in [0.65, 1.1]$ fm for M_2 . Similar intervals are used for the other ensembles and other kinematic points, with variations of at most about 0.2 fm. In the case of $C_1(q^2)$, the constant fit yields a statistical precision of approximately 2%.

The dependence of the results on the insertion time of the electromagnetic current, t_J , is found to be very small, in

⁸Note that for small values of the J/ψ three-momentum, $\mathbf{k}$, in the χ_{c1} rest frame, $M_2(q^2) \propto |\mathbf{k}|^2$. We factorize the leading $|\mathbf{k}|^2$ dependence and present results for the dimensionless combination $M_2(q^2)|\mathbf{k}_{\max}|^2/|\mathbf{k}|^2$, where $|\mathbf{k}_{\max}| \simeq 389.4$ MeV, which exhibits a much weaker dependence on q^2 .

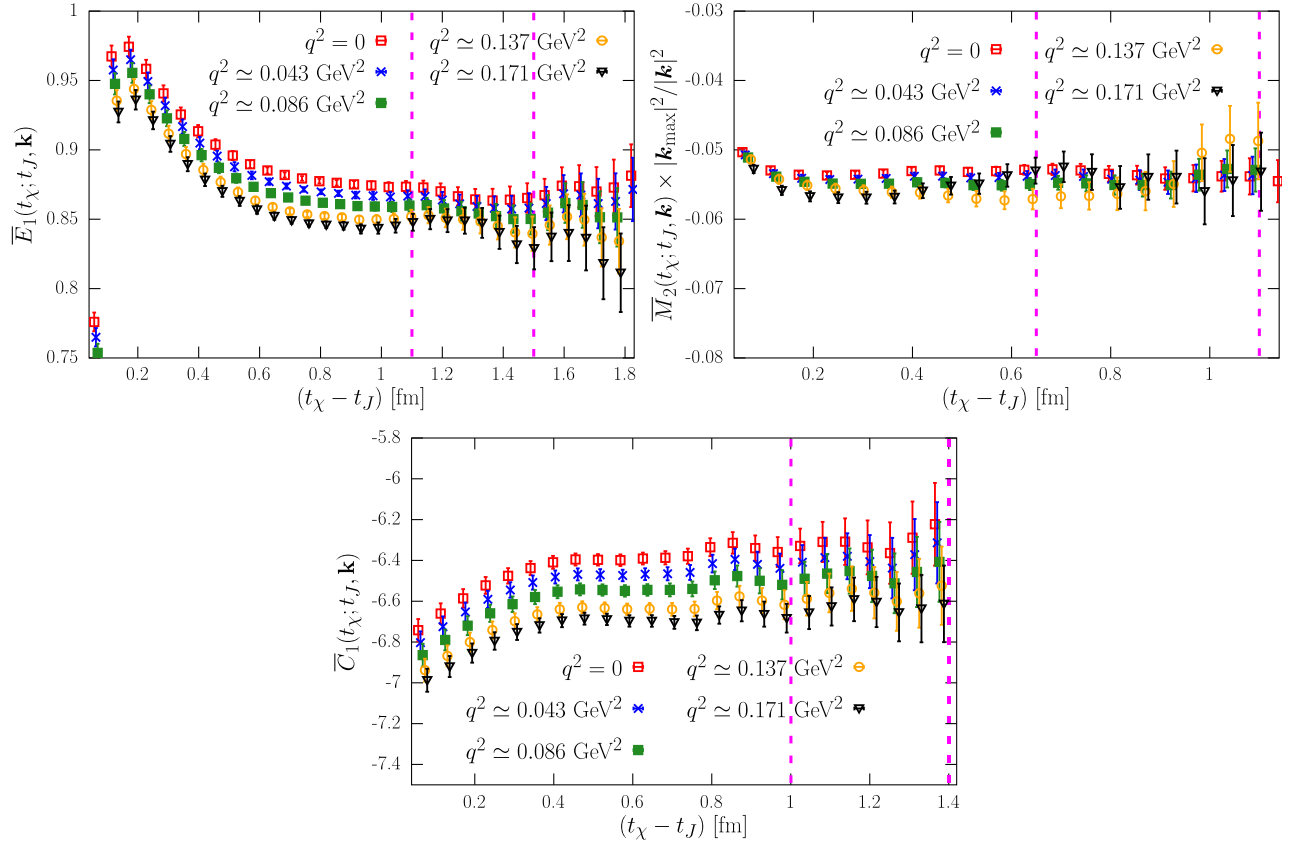


FIG. 2. Estimators $\bar{E}_1(t_\chi; t_J, \mathbf{k})$, $\bar{M}_2(t_\chi; t_J, \mathbf{k})/|\mathbf{k}|^2$, and $\bar{C}_1(t_\chi; t_J, \mathbf{k})$ for a fixed value of $t_J \simeq 1.6$ fm on the finest lattice spacing ensemble adopted for this study (ensemble D96). In each plot, the different curves correspond to the different simulated values of q^2 . The data points at the different q^2 have been slightly shifted horizontally for better visualization. The magenta vertical lines (approximately) delimit the time interval in which we perform a constant fit to extract the form factors.

agreement with our previous study [10]. This investigation was performed on a single gauge ensemble (the B64 ensemble), for which all three-point correlation functions were computed using two values of t_J , namely $t_J \simeq 1.6$ fm and $t_J \simeq 2.4$ fm.

A comparison between the estimators of the form factors obtained with these two choices of t_J is shown in Fig. 3 for an intermediate value $q^2 \simeq (0.293 \text{ GeV})^2$. The behavior observed in all other cases is qualitatively the same. Within statistical uncertainties, no significant dependence on t_J is observed in the plateau regions, indicating that our smallest choice, $t_J \simeq 1.6$ fm, is already sufficient to render finite- t_J effects negligible at the present level of precision.

To estimate any residual t_J dependence, we associate a systematic uncertainty to the results obtained at $t_J \simeq 1.6$ fm, defined as

$$\Sigma_F = |F(t_J \simeq 2.4 \text{ fm}) - F(t_J \simeq 1.6 \text{ fm})|,$$

$$\Delta F = \Sigma_F \text{erf}\left(\frac{\Sigma_F}{\sqrt{2}\sigma_{\Sigma_F}}\right), \quad F = E_1(q^2), M_2(q^2), C_1(q^2), \quad (49)$$

where σ_{Σ_F} denotes the statistical uncertainty on Σ_F and $\text{erf}(x)$ is the error function. The quantities ΔE_1 , ΔM_2 , and ΔC_1 therefore represent the observed spreads weighted by the probability that they do not originate from statistical fluctuations.

As for the lattice-spacing dependence of the results, we illustrate in Fig. 4 the form factors as functions of a^2 for an intermediate $q^2 \simeq (0.293 \text{ GeV})^2$ and for a value close to q_{max}^2 . The lattice-spacing dependence of the $E_1(q^2)$ and $M_2(q^2)$ form factors closely resembles that observed at $q^2 = 0$, cf. Ref. [10]. While $E_1(q^2)$ displays a rather mild dependence on a^2 , we again observe a more pronounced cutoff dependence for $M_2(q^2)$. As $q^2 \rightarrow q_{\text{max}}^2$, the statistical uncertainties increase, particularly for the form factor $M_2(q^2)$.

The longitudinal form factor $C_1(q^2)$, instead, exhibits sizable cutoff effects of order $\mathcal{O}(10\% - 15\%)$, which are nevertheless well described by an a^2 scaling. To extrapolate our results to the continuum limit, we adopt the same strategy as in Ref. [10]. Specifically, for each form factor and each value of q^2 , we perform two linear fits in a^2 : the first using the full dataset and the second excluding the coarsest lattice spacing. The fits obtained from the full

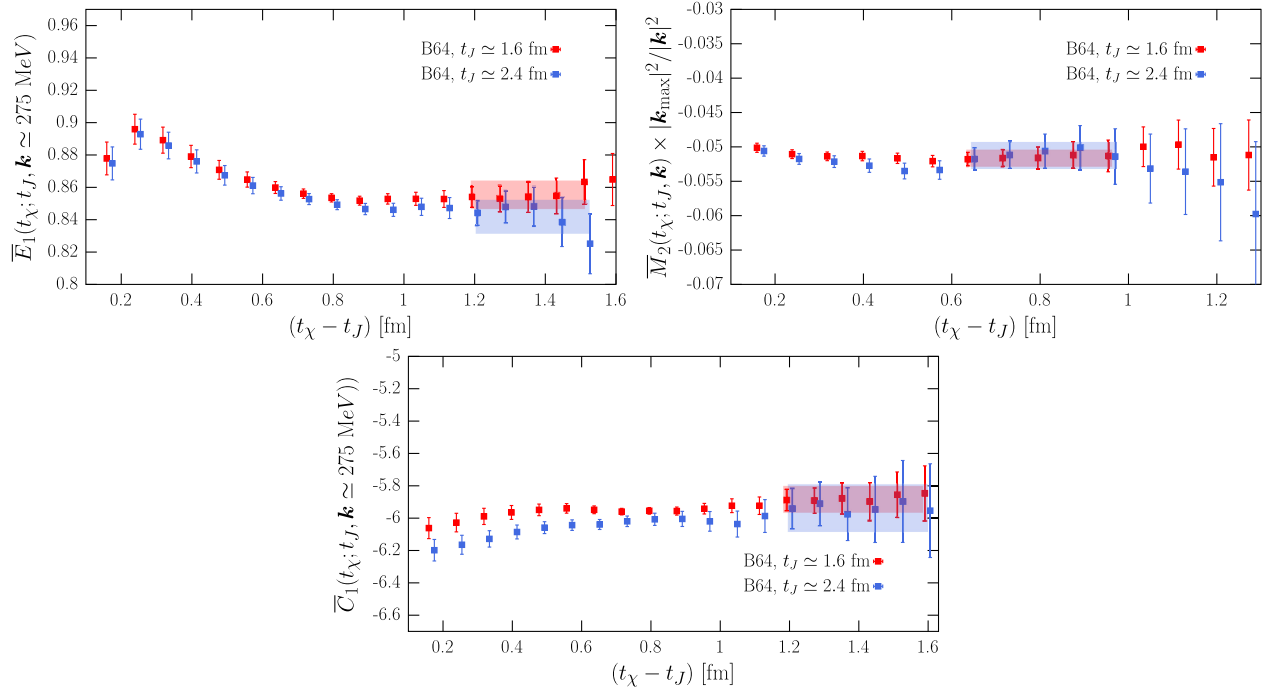


FIG. 3. Estimators $\bar{E}_1(t_\chi; t_J, \mathbf{k})$, $\bar{M}_2(t_\chi; t_J, \mathbf{k})|\mathbf{k}_{\text{max}}|^2/|\mathbf{k}|^2$, and $\bar{C}_1(t_\chi; t_J, \mathbf{k})$ are shown for two values of $t_J \simeq 1.6, 2.4$ fm. The results correspond to our determination on the B64 ensemble for $k_z = 275$ MeV, which corresponds to $q^2 \simeq 0.086 \text{ GeV}^2$. The colored bands indicate the results of constant fits to the estimators over the selected time intervals, from which the form factors are determined.

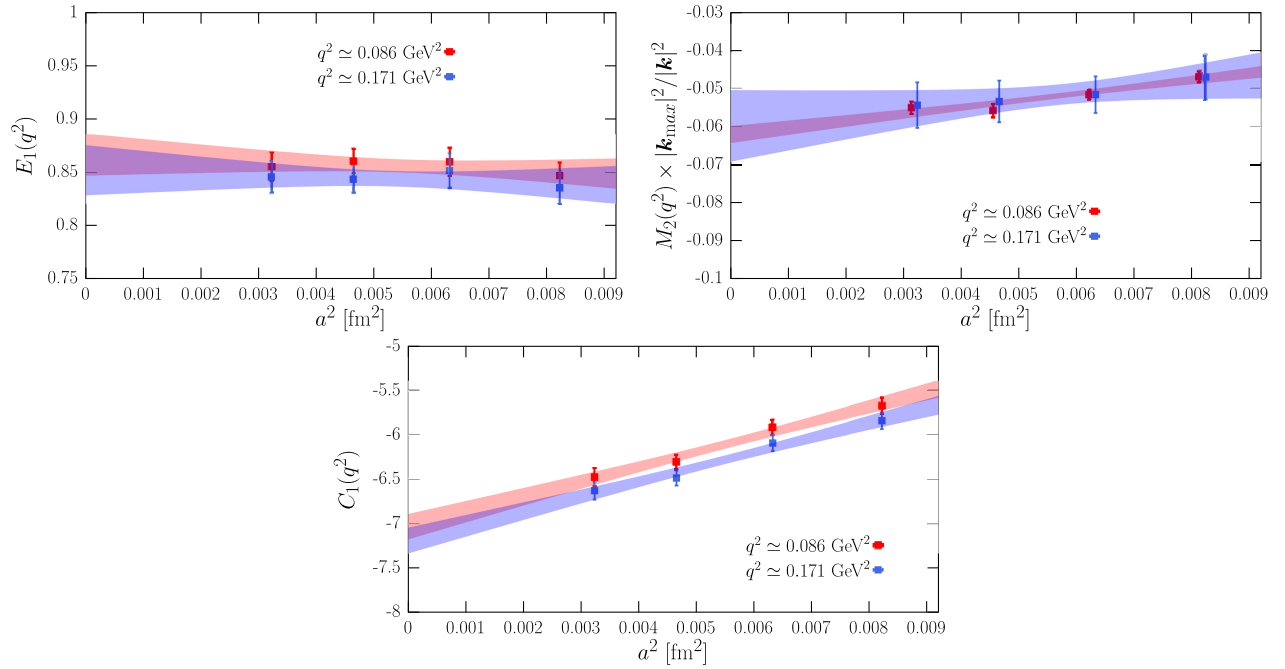


FIG. 4. Lattice spacing dependence of the form factors $E_1(q^2)$, $M_2(q^2)$, and $C_1(q^2)$ for two values of the three-momentum transfer $\mathbf{k}$ corresponding to $q^2 \simeq 0.086 \text{ GeV}^2$ and $q^2 \simeq 0.171 \text{ GeV}^2 \simeq q_{\text{max}}^2$.

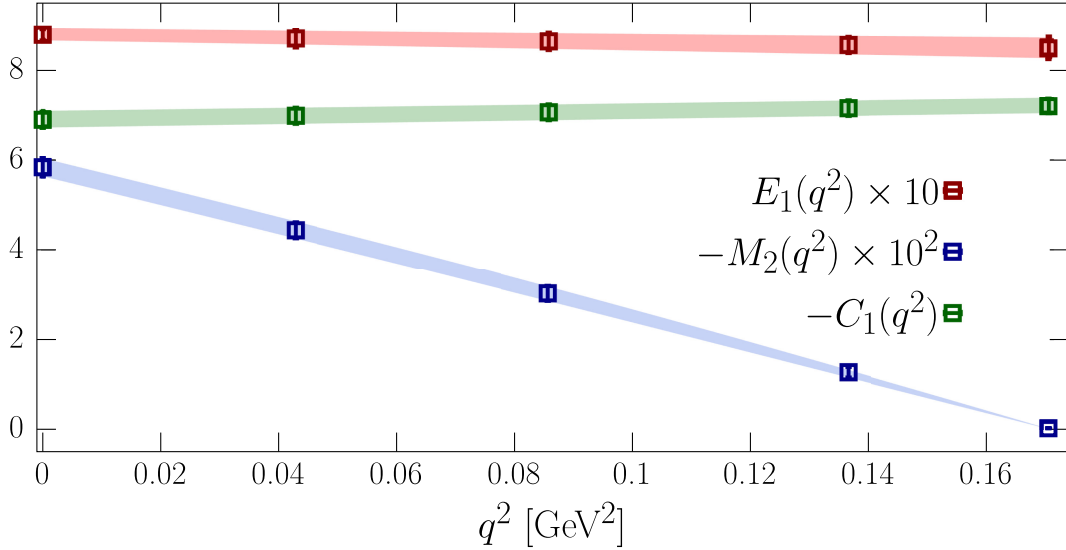


FIG. 5. Three form factors $E_1(q^2)$, $M_2(q^2)$, and $C_1(q^2)$ describing the $\chi_{c1} \rightarrow J/\psi \ell^+ \ell^-$ decay as a function of q^2 in the range $q^2 \in [0, q_{\max}^2]$. For illustrative purposes, we have rescaled the form factors or reversed their sign to fit them in a single plot. Note that this time we show directly $M_2(q^2)$ and not $M_2(q^2) \times |\mathbf{k}_{\max}|^2/|\mathbf{k}|^2$. The colored bands correspond to the results of the q^2 fits performed according to Eq. (50).

dataset are shown in Fig. 4 as colored bands. In all cases, the fit quality is satisfactory, with reduced χ^2 values of order $\mathcal{O}(1)$.

Our final results are obtained by averaging the two fits using the Bayesian Akaike Information Criterion (BAIC).⁹ The resulting continuum-extrapolated form factors $E_1(q^2)$, $M_2(q^2)$, and $C_1(q^2)$ are displayed in Fig. 5, and the values are given in Table III.

We observe that the q^2 -dependence of $E_1(q^2)$, $M_2(q^2) \times |\mathbf{k}_{\max}|^2/|\mathbf{k}|^2$, and $C_1(q^2)$ is smooth, and we fit them to

$$F(q^2) = C_F[1 + D_F q^2], \quad F \in \left\{ E_1, M_2 \frac{|\mathbf{k}_{\max}|^2}{|\mathbf{k}|^2}, C_1 \right\}, \quad (50)$$

where C_F and D_F are the fit parameters. The resulting fit curves are shown in Fig. 5 as colored bands, while the values of $F(0)$ and D_F , together with their correlations, can be found in Table IV for each form factor. We note that the q^2 dependence is very mild, with $|D_F q^2| \ll 1$ over the full physical range, which is densely covered by our lattice data. Alternative smooth parametrizations, such as pole or dipole forms, would therefore lead to numerically very similar interpolations.

Note that only in the case of the longitudinal form factor $C_1(q^2)$ do we observe a nonzero slope in q^2 . The form factor $M_2(q^2)$ is, to a very good approximation, proportional to $|\mathbf{k}|^2$, and no statistically significant slope is

observed when fitting the ratio $M_2(q^2)/|\mathbf{k}|^2$. In Sec. IV, we use the linear q^2 parametrization of the form factors to determine the decay rates and differential observables for $\chi_{c1} \rightarrow J/\psi e^+ e^-$ and $\chi_{c1} \rightarrow J/\psi \mu^+ \mu^-$.

We now turn to the numerical results for the form factors $F_1(q^2)$ and $\tilde{F}_2(q^2)$ relevant for the $h_c \rightarrow \eta_c \ell^+ \ell^-$ decay.

B. Lattice results for the $h_c \rightarrow \eta_c \ell^+ \ell^-$ form factors

We now present the numerical analysis of the $h_c \rightarrow \eta_c \ell^+ \ell^-$ transition, following the extraction strategy described in Sec. II B. For the calculation of the two- and

TABLE III. Continuum extrapolated values of the form factors $E_1(q^2)$, $M_2(q^2) \times |\mathbf{k}_{\max}|^2/|\mathbf{k}|^2$, and $C_1(q^2)$ for all values of q^2 considered.

q^2 [GeV ²]	$E_1(q^2)$	$M_2(q^2) \times \mathbf{k}_{\max} ^2/ \mathbf{k} ^2$	$C_1(q^2)$
0	0.879(12)	-0.0584(21)	-6.90(18)
0.043	0.869(20)	-0.0593(24)	-6.98(18)
0.086	0.863(19)	-0.0605(34)	-7.05(18)
0.137	0.855(18)	-0.0631(57)	-7.15(17)
0.171	0.849(24)	-0.0591(91)	-7.20(17)

TABLE IV. Values of the fit parameters C_F and D_F for the three form factors describing the $\chi_{c1} \rightarrow J/\psi \ell^+ \ell^-$ decay.

Parameter	E_1	$M_2 \times \mathbf{k}_{\max} ^2/ \mathbf{k} ^2$	C_1
C_F	0.879(14)	-0.0584(20)	-6.90(19)
D_F [GeV ⁻²]	-0.20(10)	0.39(35)	0.258(61)
Corr($F(0)$, D_F)	0.31	-0.42	-0.49

⁹See Eqs. (21) and (22) of Ref. [11] for details.

TABLE V. Number of gauge configurations (N_g) and stochastic sources (N_s) used for the calculation of the $h_c \rightarrow \eta_c \ell^+ \ell^-$ form factors, for each of the $N_f = 2 + 1 + 1$ ETMC gauge ensembles in Table I.

ID	N_g	N_s
A48	300	32
B64	203	8
C80	609	4
D96	150	8

three-point correlation functions required to determine the form factors $F_1(q^2)$ and $\tilde{F}_2(q^2)$ of the $h_c \rightarrow \eta_c \ell^+ \ell^-$ decay, we employed the same number of gauge configurations and stochastic sources for each of the $N_f = 2 + 1 + 1$ gauge ensembles listed in Table I as in our previous study [11]. This information is summarized in Table V. Note that the number of stochastic sources used for this calculation is significantly smaller than that used for the $\chi_{c1} \rightarrow J/\psi \ell^+ \ell^-$ transition. Given the relatively lower computational cost, we did not implement the preconditioning described in the previous subsection [Eq. (48)], but instead computed the two- and three-point correlation functions using the same number of gauge configurations and sources for all kinematic points, corresponding to both $q^2 = 0$ and $q^2 \neq 0$.

Unlike in Ref. [11], where the form factor $F_1(0)$ relevant to the $h_c \rightarrow \eta_c \gamma$ decay was computed including the E112 ensemble ($a \simeq 0.049$ fm), we did not perform simulations on this ensemble in the present calculation, as cutoff effects in this channel were found to be very small. Nevertheless, some of the results obtained in Ref. [11] at $q^2 = 0$ can still be reused in the present analysis, as discussed below.

We recall that the two form factors $F_1(q^2)$ and $\tilde{F}_2(q^2)$ are extracted from the estimators defined in Eqs. (40)–(42), after applying the improvements described in Eqs. (46) and (47). In Fig. 6, we show the improved estimators of the

form factors $F_1(q^2)$ and $\tilde{F}_2(q^2)$ on the finest lattice-spacing ensemble used in this study (the D96 ensemble), for a fixed value of the electromagnetic-current insertion time $t_J \simeq 1.6$ fm. The figure shows clear and stable plateaus for both form factors at all kinematic points.

The q^2 dependence of the form factors $F_1(q^2)$ and $\tilde{F}_2(q^2)$ is very mild, varying by about 5% between $q^2 = 0$ and $q^2 = q_{\max}^2 \simeq 0.293$ GeV². We also investigated finite- t_J effects by performing simulations at two values, $t_J \simeq 1.6$ fm and $t_J \simeq 2.4$ fm, on a single ensemble (B64). The comparison, shown in Fig. 7 for an intermediate $q^2 \simeq 0.219$ GeV², reveals excellent agreement between the two choices. This indicates that $t_J \simeq 1.6$ fm is already sufficient to suppress finite- t_J effects for both the F_1 and $\tilde{F}_2$ form factors at the current level of precision.

For all ensembles other than B64, the three-point correlation functions are computed at a single value $t_J \simeq 1.6$ fm, as done for the $\chi_{c1} \rightarrow J/\psi \ell^+ \ell^-$ analysis. The estimators $\bar{F}_1$ and $\bar{F}_2$ are fitted to a constant in the interval $t_h - t_J \in [1, 1.4]$ fm, where a clear plateau is observed, in order to extract the form factors $F_1(q^2)$ and $\tilde{F}_2(q^2)$.

For the extrapolation to the continuum limit, we proceed slightly differently compared to the previous subsection. To exploit the $q^2 = 0$ simulations performed in Ref. [11], which included an ultrafine lattice-spacing ensemble with $a \simeq 0.049$ fm (the E112 ensemble), we first extrapolate to the continuum limit the ratios

$$R_1(q^2) \equiv \frac{F_1(q^2)}{F_1(0)}, \quad R_2(q^2) \equiv \frac{\tilde{F}_2(q^2)}{\tilde{F}_2(0)}, \quad (51)$$

which are, by definition, equal to unity at $q^2 = 0$ [recall that in the on-shell photon limit, one has $F_1(0) = \tilde{F}_2(0)$]. The continuum extrapolation of these ratios is performed using the four gauge ensembles A48, B64, C80, and D96. We then combine the extrapolated ratios with our determination

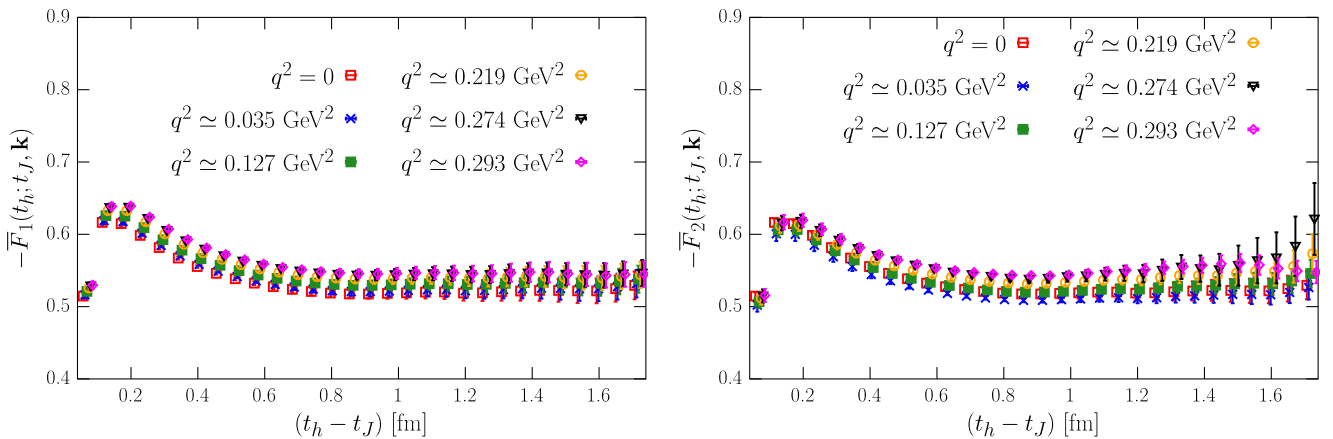


FIG. 6. Estimators $\bar{F}_1(t_h; t_J, \mathbf{k})$ and $\bar{F}_2(t_h; t_J, \mathbf{k})$ for a fixed value of $t_J \simeq 1.6$ fm on the finest lattice spacing ensemble (D96). In each plot, the different curves correspond to the different simulated values of q^2 . The data points corresponding to the different q^2 values have been slightly shifted horizontally for better visualization.

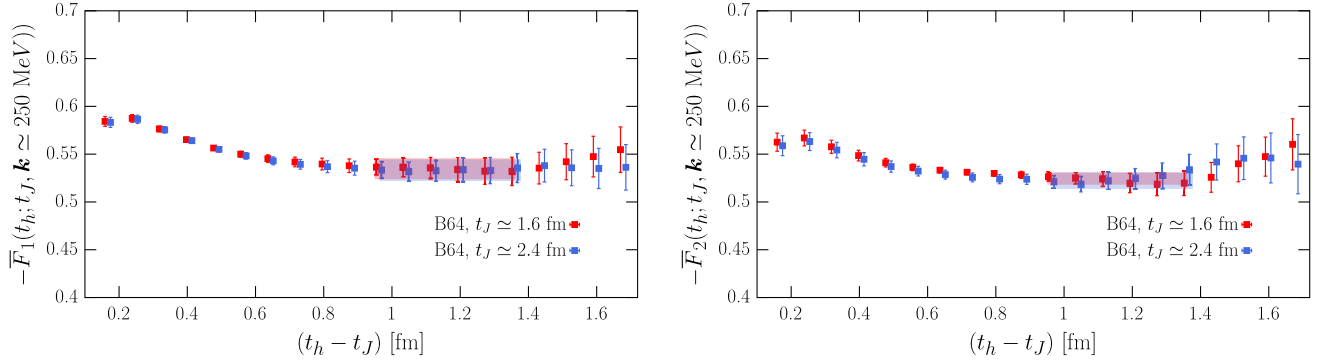


FIG. 7. Estimators $\bar{F}_1(t_h; t_J, \mathbf{k})$ and $\bar{F}_2(t_h; t_J, \mathbf{k})$ for two values of $t_J \simeq 1.6, 2.4$ fm. The results correspond to our determination on the B64 ensemble for $k = 250$ MeV, which corresponds to $q^2 \simeq 0.219$ GeV². The colored bands indicate the results of constant fits to the estimators over the selected time intervals, from which the form factors are determined.

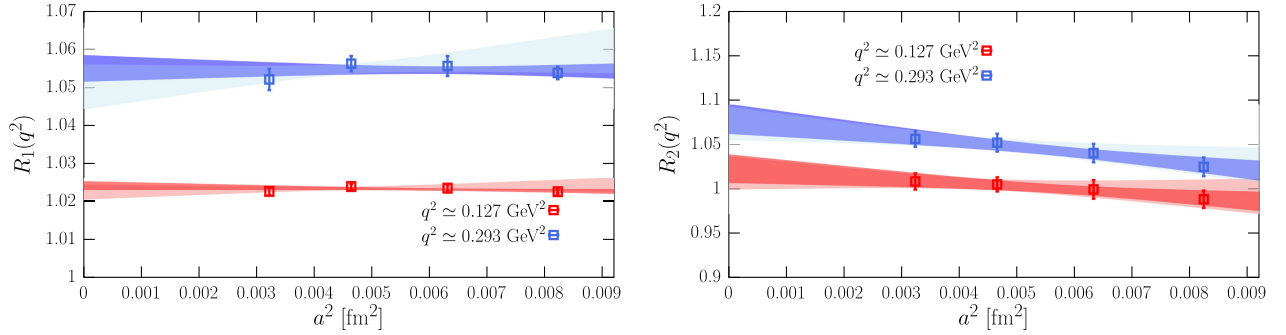


FIG. 8. Lattice spacing dependence of the ratios $R_1(q^2)$ and $R_2(q^2)$ for two values of the momentum transfer corresponding to $q^2 \simeq 0.127$ GeV² and $q^2 \simeq q_{\text{max}}^2 \simeq 0.293$ GeV².

$F_1(0) = -0.522(10)$ from Ref. [11], obtained from a continuum extrapolation including five lattice spacings (i.e., including the E112 ensemble), in order to reconstruct both $F_1(q^2)$ and $\tilde{F}_2(q^2)$.

In Fig. 8, we illustrate the ratios $R_1(q^2)$ and $R_2(q^2)$ as functions of the squared lattice spacing a^2 for an intermediate $q^2 = 0.127$ GeV² and for a value close to $q_{\text{max}}^2 = 0.293$ GeV². The lattice spacing dependence is very mild, and for the extrapolation of $R_1(q^2)$ and $R_2(q^2)$ to the continuum limit, we follow the same procedure as in the $\chi_{c1} \rightarrow J/\psi \ell^+ \ell^-$ case, i.e., we perform two linear fits in a^2 : one using the full dataset and a second excluding the coarsest lattice spacing. The results of both fits are shown in Fig. 8, where the lighter colored bands correspond to fits excluding the coarsest lattice spacing.

For all fits, the reduced χ^2 values do not exceed $\chi^2/\text{d.o.f.} = 1.1$. Our results for $R_1(q^2)$ and $R_2(q^2)$ are obtained by averaging the two fits using the BAIC. Our final results for $F_1(q^2)$ and $\tilde{F}_2(q^2)$ are shown in Fig. 9, where, as mentioned above, we use $F_1(0) = \tilde{F}_2(0) = -0.522(10)$ from Ref. [11] to determine $F_1(q^2)$ and $\tilde{F}_2(q^2)$ from the knowledge of the ratios $R_1(q^2)$ and $R_2(q^2)$. The continuum-extrapolated values of the ratios $R_1(q^2)$ and $R_2(q^2)$ are collected in Table VI.

As in the case of the $\chi_{c1} \rightarrow J/\psi \ell^+ \ell^-$ form factors, having determined $F_1(q^2)$ and $\tilde{F}_2(q^2)$ at discrete q^2 values over the full kinematic range, we fit them using a simple polynomial in q^2 . We again use a linear ansatz in q^2 to describe the form factor $F_1(q^2)$ and $\tilde{F}_2(q^2)$. We perform a joint fit given by

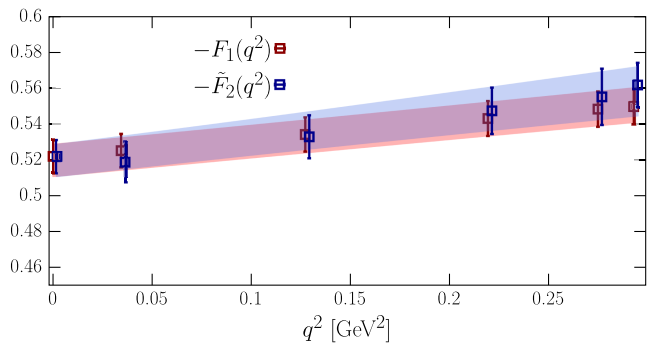


FIG. 9. Two form factors $F_1(q^2)$ and $\tilde{F}_2(q^2)$ describing the $h_c \rightarrow \eta_c \ell^+ \ell^-$ decay as a function of q^2 in the range $q^2 \in [0, q_{\text{max}}^2]$, where $q_{\text{max}}^2 \simeq 0.293$ GeV². The colored bands correspond to the results of the q^2 fits performed according to Eq. (52).

TABLE VI. Continuum extrapolated values of the ratios $R_1(q^2)$ and $R_2(q^2)$, defined in Eq. (51), for all values of q^2 considered in this work.

q^2 [GeV ²]	$R_1(q^2)$	$R_2(q^2)$
0.035	1.00635(41)	0.994(15)
0.127	1.0235(15)	1.021(16)
0.219	1.0404(27)	1.049(18)
0.274	1.0504(38)	1.063(24)
0.293	1.0532(45)	1.076(16)

TABLE VII. Values of the fit parameters entering Eq. (52) and associated correlation matrix.

Parameter	Value	Correlation matrix		
		C	D_{F_1}	D_{F_2}
$-C$	0.520 ± 0.010	1	-0.13	0.10
D_{F_1} [GeV ⁻²]	0.202 ± 0.022	-0.13	1	-0.63
D_{F_2} [GeV ⁻²]	0.252 ± 0.058	0.10	-0.63	1

$$\begin{aligned} F_1(q^2) &= C \times [1 + D_{F_1} q^2], \\ \tilde{F}_2(q^2) &= C \times [1 + D_{F_2} q^2], \end{aligned} \quad (52)$$

where C , D_{F_1} , and D_{F_2} are free fit parameters. The result of the global fit is given by the colored bands in Fig. 9, while in Table VII we provide the values of the fit parameters, including their correlations, which can be easily implemented in phenomenological analyses. As in the case of the $\chi_{c1} \rightarrow J/\psi \ell^+ \ell^-$ form factors, the q^2 dependence is very mild, with $|D_{F_{1(2)}} q^2| \ll 1$ over the full physical range.

The results for the longitudinal form factor $F_L(q^2)$, introduced in Eq. (11), are shown in Fig. 10.

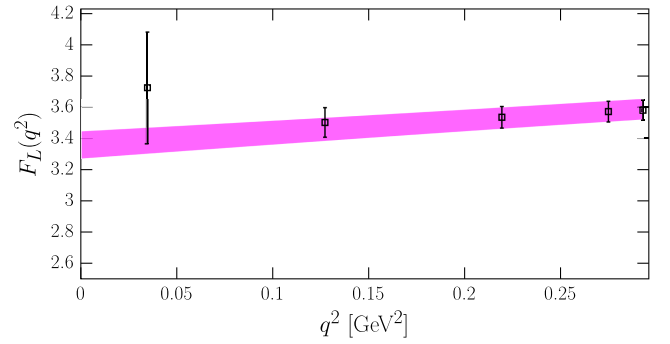


FIG. 10. Form factor $F_L(q^2)$, obtained from our data for $F_1(q^2)$ and $\tilde{F}_2(q^2)$ through Eq. (11).

IV. COMPARISON WITH EXPERIMENTAL DATA

In this section, we compare our predictions for decay rates and differential distributions with the available experimental data and discuss observables with varying sensitivity to the underlying form factors. The existing measurements include extensive studies by the BESIII Collaboration of the $\chi_{c1} \rightarrow J/\psi e^+ e^-$ and $\chi_{c1} \rightarrow J/\psi \mu^+ \mu^-$ decays [15,16], as well as the recent observation of the $h_c \rightarrow \eta_c e^+ e^-$ transition [17].

A. $\chi_{c1} \rightarrow J/\psi \ell^+ \ell^-$ decay

The decay rate for $\chi_{c1} \rightarrow J/\psi \ell^+ \ell^-$, after summing over the J/ψ polarizations and averaging over the χ_{c1} polarizations, can be expressed as

$$\Gamma(\chi_{c1} \rightarrow J/\psi \ell^+ \ell^-) = \int_{4m_\ell^2}^{q_{\max}^2} dq^2 \int_{-1}^1 d\cos\theta \Gamma_{\chi_{c1}}^{\ell\ell}(q^2, \cos\theta), \quad (53)$$

where θ is the angle between ℓ^+ and χ_{c1} in the dilepton rest frame, $q_{\max}^2 = (m_{\chi_{c1}} - m_{J/\psi})^2$, and the doubly differential decay rate is given by

$$\begin{aligned} \Gamma_{\chi_{c1}}^{\ell\ell}(q^2, \cos\theta) &= \frac{\alpha_{\text{em}}^2 Q_c^2 \beta_\ell \sqrt{\lambda(m_{\chi_{c1}}^2, m_{J/\psi}^2, q^2)}}{48\pi q^2 m_{\chi_{c1}}^3} [A(q^2) + B(q^2) \cos^2\theta] \\ &= \Gamma_{\chi_{c1},A}^{\ell\ell}(q^2) + \Gamma_{\chi_{c1},B}^{\ell\ell}(q^2) \cos^2\theta, \end{aligned} \quad (54)$$

where $\lambda(x, y, z)$ is the Källén function, $\beta_\ell = \sqrt{1 - 4m_\ell^2/q^2}$, m_ℓ is the lepton mass, and the functions $A(q^2)$ and $B(q^2)$ are defined as

$$\begin{aligned} A(q^2) &= q^2 |C_1(q^2)|^2 + m_{\chi_{c1}}^2 \left(1 + \frac{4m_\ell^2}{q^2}\right) (|E_1(q^2)|^2 + |M_2(q^2)|^2), \\ B(q^2) &= -\beta_\ell^2 [q^2 |C_1(q^2)|^2 - m_{\chi_{c1}}^2 (|E_1(q^2)|^2 + |M_2(q^2)|^2)]. \end{aligned} \quad (55)$$

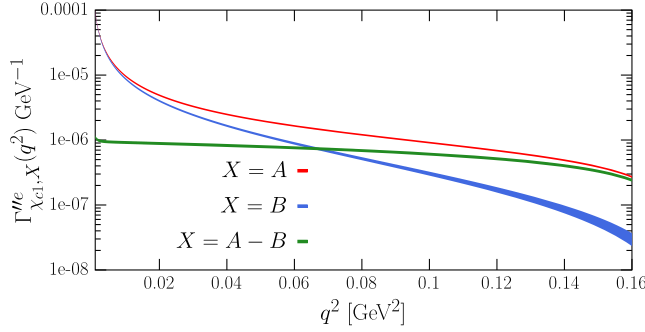


FIG. 11. Our results for the $A(q^2)$ and $B(q^2)$ contributions to $\Gamma_{\chi_{c1}}^{\ell\ell}(q^2, \cos \theta)$ of the $\chi_{c1} \rightarrow J/\psi e^+ e^-$ decay, as defined in Eq. (54). We also show the behavior of $\Gamma_{\chi_{c1}, A-B}^{\ell\ell}(q^2)$, defined in Eq. (57), which is primarily sensitive to the longitudinal form factor $C_1(q^2)$.

The q^2 -differential decay rate is then given by

$$\begin{aligned} \Gamma_{\chi_{c1}}^{\ell\ell}(q^2) &\equiv \int_{-1}^1 d\cos\theta \Gamma_{\chi_{c1}}^{\ell\ell}(q^2, \cos\theta) \\ &= \frac{\alpha_{\text{em}}^2 Q_c^2 \beta_\ell \sqrt{\lambda(m_{\chi_{c1}}^2, m_{J/\psi}^2, q^2)}}{48\pi} \frac{q^2 m_{\chi_{c1}}^3}{q^2 m_{\chi_{c1}}^3} \\ &\quad \times \left[2A(q^2) + \frac{2}{3}B(q^2) \right]. \end{aligned} \quad (56)$$

In Fig. 11, we show our determination of the $A(q^2)$ and $B(q^2)$ contributions to $\Gamma_{\chi_{c1}}^{\ell\ell}(q^2, \cos \theta)$ for the electron ($\ell = e$) mode, as defined in Eq. (54).

In Fig. 11, we also show the result for the following angular observable:

$$\Gamma_{\chi_{c1}, A-B}^{\ell\ell}(q^2) \equiv \Gamma_{\chi_{c1}, A}^{\ell\ell}(q^2) - \Gamma_{\chi_{c1}, B}^{\ell\ell}(q^2), \quad (57)$$

which, in the massless limit, is only sensitive to the longitudinal form factor $C_1(q^2)$, since

$$A(q^2) - B(q^2) = 2q^2 |C_1(q^2)|^2 + \mathcal{O}(m_\ell^2). \quad (58)$$

Since both $A(q^2)$ and $B(q^2)$ can, in principle, be extracted experimentally from the angular analysis of the decay products, measuring the quantity in Eq. (57) for final-state electrons allows for a direct measurement of the longitudinal form factor $C_1(q^2)$, which does not contribute in the case of real photon emission.

In Fig. 12, we show our determination of the q^2 -differential decay rate in Eq. (56) for both the electron ($\ell = e$) and muon ($\ell = \mu$) modes. All the differential rates, as expected, peak at $q^2 \gtrsim 4m_\ell^2$ and decrease rapidly as q^2 approaches q_{max}^2 . After integrating the differential rates over q^2 , we obtain the following values for the total decay rates:

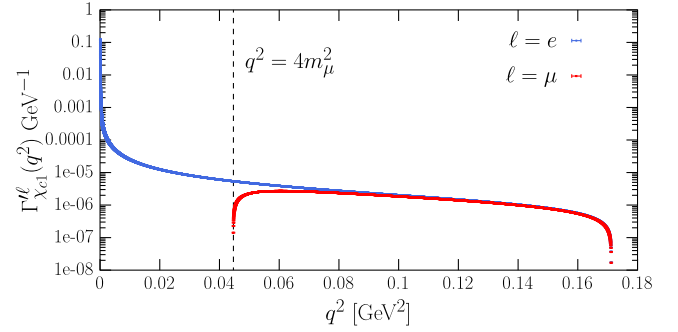


FIG. 12. Our results for the q^2 -differential decay rates $\Gamma_{\chi_{c1}}^{\ell\ell}(q^2)$ for both the electron mode (blue band) and muon mode (red band).

$$\Gamma(\chi_{c1} \rightarrow J/\psi e^+ e^-) = 2.869(90) \text{ keV},$$

$$\Gamma(\chi_{c1} \rightarrow J/\psi \mu^+ \mu^-) = 0.1993(72) \text{ keV}, \quad (59)$$

achieving a precision of about 3.1% and 3.6% for the electron and muon modes, respectively.

To compare with experiment, we convert the PDG results for the branching fractions [19]

$$\text{Br}^{\text{PDG}}(\chi_{c1} \rightarrow J/\psi e^+ e^-) = (3.46 \pm 0.24) \times 10^{-3},$$

$$\text{Br}^{\text{PDG}}(\chi_{c1} \rightarrow J/\psi \mu^+ \mu^-) = (2.33 \pm 0.29) \times 10^{-4}, \quad (60)$$

to decay widths using $\Gamma(\chi_{c1}) = 0.84(4) \text{ MeV}$ [19].

Our results are in excellent agreement with, and significantly more precise than, the experimental results in both channels, as shown in Fig. 13. A similar comparison with the direct experimental results quoted by BESIII [15,16],

$$\text{Br}^{\text{BESIII}}(\chi_{c1} \rightarrow J/\psi e^+ e^-) = (3.73 \pm 0.09 \pm 0.25) \times 10^{-3},$$

$$\text{Br}^{\text{BESIII}}(\chi_{c1} \rightarrow J/\psi \mu^+ \mu^-) = (2.51 \pm 0.18 \pm 0.20) \times 10^{-4}, \quad (61)$$

leads to results that are still compatible with our Standard Model predictions within 1 s.d. (see Fig. 13).

Alternatively, after dividing our lattice predictions for the partial decay widths in Eq. (59) by the experimentally measured branching fractions, we extract the total decay width of the χ_{c1} . We obtain

$$\Gamma(\chi_{c1})_{e^+ e^-} = 0.77(6) \text{ MeV} [0.83(6) \text{ MeV}],$$

$$\Gamma(\chi_{c1})_{\mu^+ \mu^-} = 0.79(9) \text{ MeV} [0.86(11) \text{ MeV}], \quad (62)$$

where the values in square brackets are obtained using the PDG branching fractions [Eq. (60)], while the others are obtained using the BESIII results [Eq. (61)].

A more direct comparison with the BESIII measurements, which avoids relying on the PDG value of the total decay width of χ_{c1} , can be made by comparing our results for

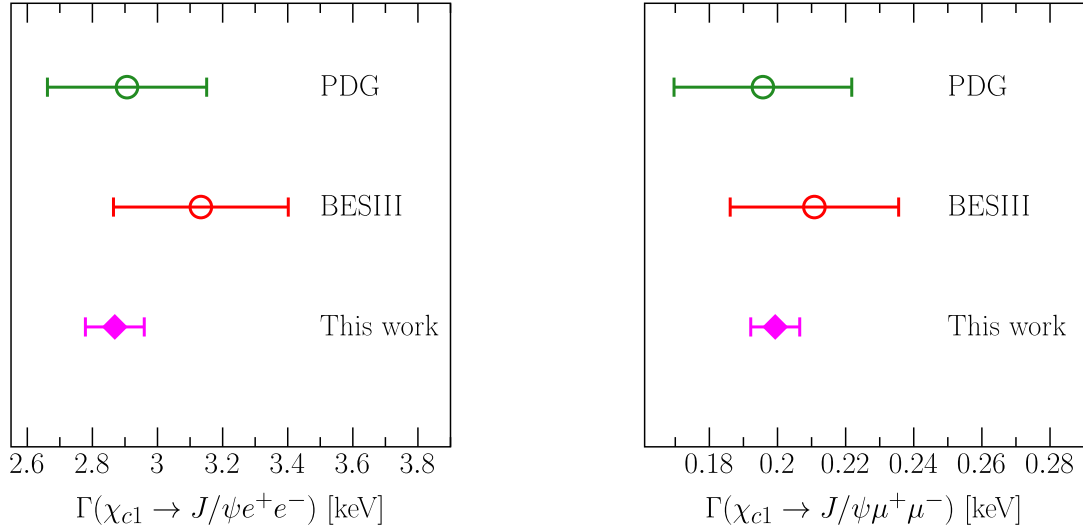


FIG. 13. Comparison between our lattice QCD predictions for the decay rates $\Gamma(\chi_{c1} \rightarrow J/\psi e^+ e^-)$ (left) and $\Gamma(\chi_{c1} \rightarrow J/\psi \mu^+ \mu^-)$ (right) and the results quoted by the PDG [19] and by BESIII [15,16].

$$\frac{\text{Br}(\chi_{c1} \rightarrow J/\psi \mu^+ \mu^-)}{\text{Br}(\chi_{c1} \rightarrow J/\psi e^+ e^-)} = 6.95(10) \times 10^{-2}, \quad (63)$$

$$\frac{\text{Br}(\chi_{c1} \rightarrow J/\psi e^+ e^-)}{\text{Br}(\chi_{c1} \rightarrow J/\psi \gamma)} = 8.78(2) \times 10^{-3}. \quad (64)$$

While the lepton-flavor-universality ratio is in excellent agreement with the experimental result given in Eq. (2), our result in Eq. (64) is smaller by about 2.3σ than the BESIII determination in Eq. (1).

A more detailed comparison between our predictions and the experimental data can be made for the q^2 -differential decay rate. BESIII provided such results for the electron channel in Ref. [15], where the number of events was measured from threshold up to $\sqrt{q^2} \simeq 0.3675$ GeV using short bins of width 5 MeV.

We first integrate our differential decay rate over the same bins, divide our results by the total decay width in Eq. (59), and then multiply the resulting values by the total number of events recorded by BESIII ($N_{\text{events}} = 1980$). The comparison is shown in Fig. 14, where the orange data points correspond to the BESIII measurements and the blue ones to our theoretical predictions.

An important aspect of this comparison is that the bin-by-bin event yields reported in Ref. [15] are obtained after applying selection criteria. Therefore, a meaningful comparison requires efficiency corrections. The authors of Ref. [15] kindly provided us with the corresponding correction factors for each bin. The efficiency-corrected event yields are shown in Fig. 14 as magenta data points.

As illustrated in the figure, the efficiency correction is essential to achieve agreement in the small- q^2 region, where a large discrepancy between lattice QCD predictions and experimental data would otherwise be observed. After

applying the efficiency corrections, the BESIII data are found to be in reasonably good agreement with our lattice QCD results, although some differences remain in individual bins.

In Fig. 15, we display the bin-by-bin relative difference between the BESIII measurements and our lattice QCD predictions in the small- q^2 region, where the event statistics are higher and the uncertainties are more reliable. In practice, we restrict the comparison to bins containing more than 20 events. The pull $\Delta(\sqrt{q^2})$ is defined as

$$\Delta(\sqrt{q^2}) = \frac{|N_{\text{ev}}^{\text{BESIII}}(\sqrt{q^2}) - N_{\text{ev}}^{\text{LQCD}}(\sqrt{q^2})|}{\sqrt{(\sigma^{\text{BESIII}}(\sqrt{q^2}))^2 + (\sigma^{\text{LQCD}}(\sqrt{q^2}))^2}}, \quad (65)$$

where $N_{\text{ev}}^{\text{BESIII}}(\sqrt{q^2})$ denotes the efficiency-corrected

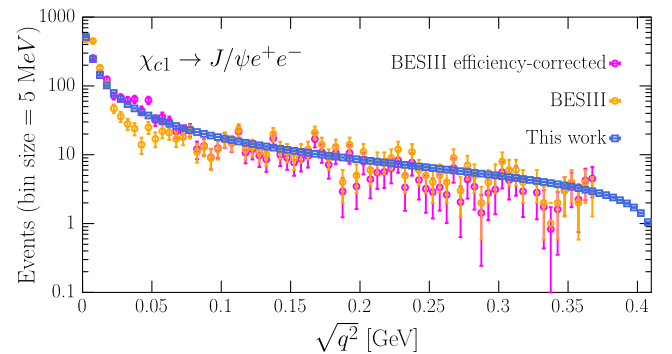


FIG. 14. Number of bin-per-bin events measured by BESIII in Ref. [15] before (orange data points) and after (magenta data points) efficiency corrections. Our corresponding theoretical predictions are shown by the blue data points. Each bin has a size of 5 MeV.

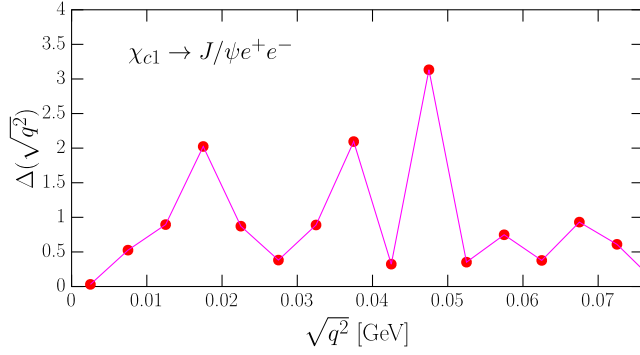


FIG. 15. Relative difference $\Delta(\sqrt{q^2})$ in Eq. (65) in the low- q^2 region up to $\sqrt{q^2} \simeq 0.075$ GeV.

number of events measured by BESIII in a given $\sqrt{q^2}$ bin, and $N_{\text{ev}}^{\text{LQCD}}(\sqrt{q^2})$ the corresponding lattice QCD prediction. The quantities $\sigma^{\text{BESIII}}(\sqrt{q^2})$ and $\sigma^{\text{LQCD}}(\sqrt{q^2})$ represent the associated uncertainties.

As the figure shows, there is overall agreement between our lattice QCD predictions and the experimental results, with the exception of the bins $\sqrt{q^2} \simeq 17, 37$, and 47 MeV. In these bins, we observe a tension of about 2.2σ at 17 and 37 MeV, and approximately 3.2σ at 47 MeV. Although no statistically significant deviation from the Standard Model is observed, the excess in the 17 MeV bin is qualitatively compatible with previously reported hints from the Atomki [28] and Padme [29] experiments, which should be further experimentally scrutinized. A similar comparison cannot be performed using the more recent BESIII data reported in Ref. [30], which provide a precise measurement of the $\chi_{c1} \rightarrow J/\psi e^+ e^-$ decay width in the small- q^2 region. The published event yields correspond to data after the selection criteria have been applied, but the corresponding efficiency corrections were not available to us.

B. $h_c \rightarrow \eta_c \ell^+ \ell^-$ decay

Similarly to the case of the $\chi_{c1} \rightarrow J/\psi \ell^+ \ell^-$ decay discussed in the previous section, the decay rate for $h_c \rightarrow \eta_c \ell^+ \ell^-$, after averaging over the h_c polarizations, can be written as

$$\Gamma(h_c \rightarrow \eta_c \ell^+ \ell^-) = \int_{4m_\ell^2}^{q_{\text{max}}^2} dq^2 \int_{-1}^1 d\cos\theta \Gamma_{h_c}^{\ell\ell}(q^2, \cos\theta), \quad (66)$$

where $q_{\text{max}}^2 = (m_{h_c} - m_{\eta_c})^2$, and θ is the angle between ℓ^+ and the decaying h_c in the dilepton rest frame. The doubly differential decay rate is given by

$$\begin{aligned} \Gamma_{h_c}^{\ell\ell}(q^2, \cos\theta) &= \frac{\alpha_{\text{em}}^2 Q_c^2 \beta_\ell \sqrt{\lambda(m_{h_c}^2, m_{\eta_c}^2, q^2)}}{12\pi} \frac{1}{q^2 m_{h_c}^3} \\ &\times [A(q^2) + B(q^2) \cos^2\theta] \\ &= \Gamma_{h_c,A}^{\ell\ell}(q^2) + \Gamma_{h_c,B}^{\ell\ell}(q^2) \cos^2\theta, \end{aligned} \quad (67)$$

where $A(q^2)$ and $B(q^2)$ are defined similarly to Eq. (55), but with the replacements

$$|C_1(q^2)|^2 \rightarrow |F_L(q^2)|^2, \quad |E_1(q^2)|^2 + |M_2(q^2)|^2 \rightarrow |F_1(q^2)|^2. \quad (68)$$

The q^2 -differential decay rate is then given by

$$\begin{aligned} \Gamma_{h_c}^{\ell\ell}(q^2) &\equiv \int_{-1}^1 d\cos\theta \Gamma_{h_c}^{\ell\ell}(q^2, \cos\theta) \\ &= \frac{\alpha_{\text{em}}^2 Q_c^2 \beta_\ell \sqrt{\lambda(m_{h_c}^2, m_{\eta_c}^2, q^2)}}{12\pi} \frac{1}{q^2 m_{h_c}^3} \left[2A(q^2) + \frac{2}{3}B(q^2) \right]. \end{aligned} \quad (69)$$

We define the following angular observable, analogous to the $\chi_{c1} \rightarrow J/\psi \ell^+ \ell^-$ case, as follows:

$$\Gamma_{h_c,A-B}^{\ell\ell}(q^2) \equiv \Gamma_{h_c,A}^{\ell\ell}(q^2) - \Gamma_{h_c,B}^{\ell\ell}(q^2), \quad (70)$$

which, in the massless limit, is only sensitive to the longitudinal form factor F_L , since

$$A(q^2) - B(q^2) = 2q^2 |F_L(q^2)|^2 + \mathcal{O}(m_\ell^2). \quad (71)$$

In Fig. 16, we show our determination of $\Gamma_{h_c,A}^{\ell\ell}(q^2)$, $\Gamma_{h_c,B}^{\ell\ell}(q^2)$, and $\Gamma_{h_c,A-B}^{\ell\ell}(q^2)$ for the electron mode. In Fig. 17, we show the q^2 -differential decay rate $\Gamma_{h_c}^{\ell\ell}(q^2)$ for both the electron and muon channels ($\ell = e, \mu$).

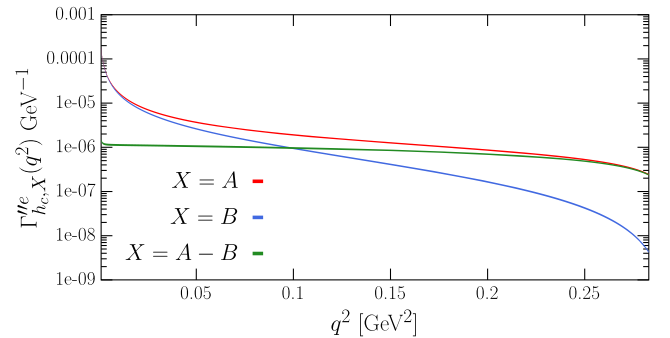


FIG. 16. Our results for the $A(q^2)$ and $B(q^2)$ contributions to $\Gamma_{h_c}^{\ell\ell}(q^2, \cos\theta)$ in Eq. (67), as well as the results for the angular observable $\Gamma_{h_c,A-B}^{\ell\ell}(q^2)$ in Eq. (70), which is primarily sensitive to the longitudinal form factor F_L .

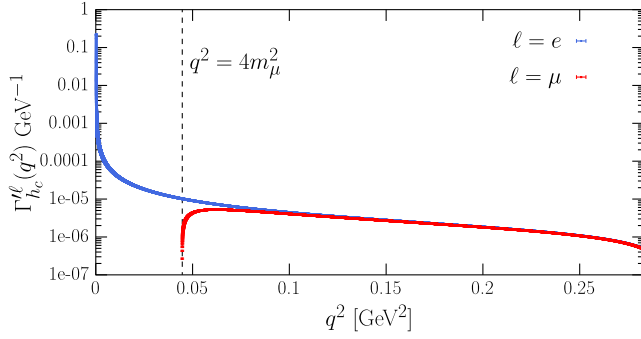


FIG. 17. Our results for the q^2 -differential decay rates $\Gamma_{h_c}^\ell(q^2)$ for both the electron (red band) and muon modes (blue band).

After integrating $\Gamma_{h_c}^\ell(q^2)$, we obtain

$$\begin{aligned}\Gamma(h_c \rightarrow \eta_c e^+ e^-) &= 5.45(19) \text{ keV}, \\ \Gamma(h_c \rightarrow \eta_c \mu^+ \mu^-) &= 0.635(22) \text{ keV},\end{aligned}\quad (72)$$

achieving a precision similar to that obtained in the case of $\chi_{c1} \rightarrow J/\psi \ell^+ \ell^-$, namely about 3.1% for the electron mode and about 3.5% for the muon mode.

To compare our result for the electron channel with the experimentally measured branching fraction [19],

$$\text{Br}^{\text{PDG}}(h_c \rightarrow \eta_c e^+ e^-) = (3.5 \pm 0.7) \times 10^{-3}, \quad (73)$$

we use the total width of the h_c quoted by PDG, $\Gamma(h_c) = 0.78(28) \text{ MeV}$ [19], and obtain

$$\Gamma^{\text{PDG}}(h_c \rightarrow \eta_c e^+ e^-) = 2.7(1.1) \text{ keV}, \quad (74)$$

which is in tension with our result in Eq. (72) by about 2.5 s.d.

We arrive at a similar tension by directly comparing our result with the BESIII result for the ratio [17]¹⁰

$$\left. \frac{\text{Br}(h_c \rightarrow \eta_c e^+ e^-)}{\text{Br}(h_c \rightarrow \eta_c \gamma)} \right|_{\text{BESIII}} = (0.59 \pm 0.10 \pm 0.04) \times 10^{-2}. \quad (75)$$

Indeed, using

$$\Gamma(h_c \rightarrow \eta_c \gamma) = \frac{2Q_c^2}{3} \alpha_{\text{em}} \frac{m_{h_c}^2 - m_{\eta_c}^2}{m_{h_c}} |F_1(0)|^2, \quad (76)$$

we obtain

¹⁰Note that the experimental result quoted in Ref. [17], also given in Eq. (75), is a weighted average of two different modes to produce h_c . In the first it comes from $\Psi(2S) \rightarrow h_c \pi^0$ and the ratio is measured $0.46(12)(5)\%$, while in the second h_c is produced by $e^+ e^- \rightarrow h_c \pi^+ \pi^-$, and the ratio is $0.89(18)(9) \times 10^{-2}$.

$$\frac{\text{Br}(h_c \rightarrow \eta_c e^+ e^-)}{\text{Br}(h_c \rightarrow \eta_c \gamma)} = 0.9341(14)\%, \quad (77)$$

which is higher than the BESIII result by 3.2 s.d. In the future, it will be important to improve the experimental determination to monitor whether this difference persists or disappears. Unfortunately, in the case of $h_c \rightarrow \eta_c \mu^+ \mu^-$, no experimental information is currently available.

V. CONCLUSIONS

In this work, we have presented the first fully dynamical lattice QCD determination of the electromagnetic transition form factors governing the decays $h_c \rightarrow \eta_c \ell^+ \ell^-$ and $\chi_{c1} \rightarrow J/\psi \ell^+ \ell^-$, where $\ell = e, \mu$, thereby providing new information on the nonperturbative dynamics of charmonium systems. The calculation was performed using $N_f = 2 + 1 + 1$ dynamical Wilson-Clover twisted-mass fermions at four lattice spacings, allowing for a controlled continuum extrapolation at (near-)physical quark masses. The relevant matrix elements were computed including only quark-connected contributions.

We have determined all relevant form factors as functions of the momentum transfer q^2 over the full kinematic range. In the case of $\chi_{c1} \rightarrow J/\psi \ell^+ \ell^-$, this includes the multipole form factors $E_1(q^2)$, $M_2(q^2)$, and the longitudinal form factor $C_1(q^2)$, which does not contribute in the real-photon limit. Similarly, for $h_c \rightarrow \eta_c \ell^+ \ell^-$, we have computed both $F_1(q^2)$ and $\tilde{F}_2(q^2)$, the latter being accessible only for off-shell photons. The q^2 -dependence of all form factors is smooth, and the continuum extrapolations are well described by an a^2 scaling behavior. The statistical precision after continuum-limit extrapolation is at the few-percent level, increasing slightly as q^2 approaches q_{max}^2 .

Using the continuum-extrapolated form factors, we have computed the partial decay widths reported in Eq. (59) for the χ_{c1} channel. These results are in very good agreement with the experimental determinations obtained from the PDG branching fractions in Eq. (60) or from the BESIII branching fractions in Eq. (61).

In addition to the comparison of absolute decay widths, we have considered ratios of branching fractions, which provide a direct comparison with experiment without relying on external input for the total width of the χ_{c1} . In particular, we have compared our lattice determination of the ratio between the Dalitz decay and the corresponding radiative decay,

$$\frac{\text{Br}(\chi_{c1} \rightarrow J/\psi e^+ e^-)}{\text{Br}(\chi_{c1} \rightarrow J/\psi \gamma)}, \quad (78)$$

given in Eq. (64), with the experimental result quoted in Eq. (2). We find a difference of about 2σ . The lepton-flavor universality ratio reported in Eq. (63) is, instead, in agreement with experiment.

We have also compared our predictions for the q^2 -differential distribution in the $\chi_{c1} \rightarrow J/\psi e^+ e^-$ decay with the corresponding BESIII measurements. After applying the efficiency corrections, we find overall agreement between lattice QCD and experimental data. However, a few bins in the low- q^2 region exhibit tensions at the level of approximately $2 - 3\sigma$. More precise experimental measurements in this kinematic region would therefore be highly valuable to assess the significance of these differences.

Turning to the $h_c \rightarrow \eta_c e^+ e^-$ decay, our prediction for the total decay width, reported in Eq. (72), exceeds the experimental determination quoted in Eq. (75) by approximately $2.5 - 3\sigma$. Given the current experimental uncertainties on both the branching fraction and the total width of the h_c , improved measurements of this channel would be particularly important to clarify whether this difference persists with increased precision.

The availability of form factors over the full kinematic range enables a variety of phenomenological analyses. In particular, our results provide the nonperturbative QCD input required for refined studies of possible contributions from light new mediators in charmonium Dalitz decays. In analogy with the recent analysis performed by the BESIII Collaboration in Ref. [30], where bounds on dark-photon couplings were extracted from $\chi_{cJ} \rightarrow J/\psi e^+ e^-$ data, our lattice determination of the transition form factors allows for a fully first-principles treatment of the hadronic matrix elements entering such searches. This will make it possible to reassess and potentially sharpen constraints on dark-sector scenarios from charmonium decays. A dedicated phenomenological study along these lines will be presented elsewhere.

A natural extension of the present work is the study of the transition $\Psi(2S) \rightarrow \chi_{c1} \gamma$ together with its Dalitz counterpart in which the virtual photon converts into a lepton pair. The abundant production of $\Psi(2S)$ at BESIII would allow for precise experimental measurements of these processes. From the lattice QCD perspective, the $\Psi(2S)$ state presents additional challenges, as it is not the ground state in the corresponding spin and flavor channel. However, established techniques, such as the generalized eigenvalue method [31], can be employed to face the problem. This extension will be investigated in future work.

The open-source packages `tmLQCD` [32–35], `Lemon` [36], `DD- $\alpha$ AMG` [37–40], `QPhiX` [41,42], and `QUDA` [43–45] have been used in the ensemble generation.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

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