

Precision lattice calculation of the hadronic contribution to the running of the electroweak gauge couplings

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We present an update of our lattice QCD determination of the hadronic contribution to the running of the electromagnetic coupling, $\Delta\alpha_{\text{had}}^{(5)}(-Q^2)$, and of the electroweak mixing angle in the spacelike momentum region up to $Q^2 = 12 \text{ GeV}^2$. The calculation is based on coordinated lattice simulations ensembles with $N_f = 2 + 1$ flavors of $O(a)$ -improved Wilson fermions, covering five lattice spacings between 0.039 and 0.085 fm and a range of pion masses, including the physical point. A refined analysis employing a telescopic window strategy allows for a clean separation of systematic effects across Euclidean distance scales. Statistical precision is further enhanced through low-mode averaging, combined with a spectral reconstruction of the vector-vector correlator at long distances on the most chiral ensembles. We confirm significant tensions of up to 7σ at spacelike virtualities around $Q^2 = 1 \text{ GeV}^2$ between our lattice results for $\Delta\alpha_{\text{had}}^{(5)}(-Q^2)$ and the corresponding data-driven estimates based on e^+e^- cross section data. Combining our lattice data with perturbative QCD via the Euclidean split technique, we obtain at the Z pole $\Delta\alpha_{\text{had}}^{(5)}(M_Z^2) = 0.027821(34)_{\text{lat}}(35)_{\text{pQCD}}$, which is more than two times more precise than recent data-driven estimates. Our result deviates slightly, by $1 - 2\sigma$, from the value produced by global electroweak fits. For the electroweak mixing angle, we present the hadronic contribution to its running and provide a precise determination of the octet-singlet mixing component $\bar{\Pi}^{(0,8)}$, in good agreement with phenomenological models but with significantly higher precision.

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I. INTRODUCTION

The electromagnetic coupling and the weak mixing angle are fundamental parameters in the Standard Model (SM) and play a crucial role in the world-wide effort to detect signals of beyond-Standard Model (BSM) physics. Specifically, the value of the electromagnetic coupling at the Z pole, $\alpha(M_Z^2)^{-1} = 127.930 \pm 0.008$ [1] is an important input parameter for interpreting the results from high-energy

colliders. However, the relatively poor precision with which $\alpha(M_Z^2)$ is known limits the sensitivity of future SM tests. Hence, to exploit the full potential of future high-luminosity e^+e^- colliders such as the future circular collider [2] or circular electron positron collider [3], it is imperative to drastically reduce the uncertainty of $\alpha(M_Z^2)$.

A further sensitive test of the SM is provided by the running of the electroweak mixing angle, $\sin^2\theta_W$. New precision measurements in parity-violating electron-proton and electron-positron scattering are currently prepared by the P2 [4] and MOLLER [5] collaborations, respectively. These measurements will provide benchmark results for $\sin^2\theta_W$ at low energies which, when combined with the results obtained directly at the Z pole, can be compared to the energy dependence predicted by the SM. Any deviation between the observed and expected energy dependence will be attributed to BSM physics.

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The theoretical knowledge of the running of α and $\sin^2 \theta_W$ is limited by the effects of the strong interaction. In particular, the contributions to the hadronic vacuum polarization (HVP) have a sizeable influence on the energy dependence of both quantities. The traditional method to determine the hadronic contributions to the running of α and $\sin^2 \theta_W$ employs a data-driven dispersive approach [6–9], similar to the evaluation of the HVP contribution to the muon anomalous magnetic moment, a_μ^{hvp} . This method relies on experimental data for $e^+e^- \rightarrow \text{hadrons}$ cross sections, which exhibit strong, yet to be resolved tensions in the crucial $\pi^+\pi^-$ channel. As a consequence, the muon $g-2$ Theory Initiative refrained from quoting an estimate for a_μ^{hvp} from the data-driven method in their second White Paper [10], switching instead to lattice QCD calculations of this quantity. In the case of the weak mixing angle, the data-driven approach for determining the hadronic contributions to the running is complicated further by the necessity to perform a potentially model-dependent flavor separation.

Here we report on our calculation of the running of the electromagnetic coupling and the weak mixing angle, employing lattice QCD to compute the offset values used as nonperturbative input in the Euclidean split technique. A crucial new ingredient is the application of a telescopic window strategy that has allowed us to reach significantly higher spacelike virtualities. This has the advantage of a substantial reduction of the uncertainties associated with the running and matching up to the Z pole, for which we adopt an updated perturbative framework. These advances, combined with increased statistics and noise-reduction techniques on our extended set of gauge ensembles, have allowed us to improve the precision relative to our previous calculation [11] by a factor of three. Our final result for the running of α is [see also Eq. (5.9)]

$$\Delta\alpha_{\text{had}}^{(5)}(M_Z^2) = 0.027821(34)_{\text{lat}}(35)_{\text{pQCD}}. \quad (1.1)$$

With its total relative error of 0.17%, it is thus considerably more precise than recent data-driven evaluations based on e^+e^- hadronic cross sections [6–9,12,13], which quote total uncertainties that are about twice as large as ours. Our main findings and conclusions as well as a detailed summary of the prospects for future improvements are presented in the companion Letter [14].

In an effort to keep this paper self-contained, we present the calculational setup in Sec. II and describe the methodological improvements compared to our earlier determination [11] in detail in Sec. III. The evaluation of the error budget for our calculation is presented in Sec. IV. Finally, in Sec. V we report our best estimate for the running of the electromagnetic coupling and the weak mixing angle. For the sake of facilitating comparisons with other lattice calculations as well as phenomenological studies, we also parametrize the dependence of vacuum polarization

functions on the spacelike virtualities considered in our calculation in terms of rational approximants.

II. PRELIMINARIES

A. The electromagnetic coupling

The electromagnetic coupling $\alpha \equiv e^2/(4\pi)$ in the Thomson limit ($q^2 \rightarrow 0$) governs interactions at very low energies, well below the electron mass. Its precisely measured value $\alpha^{-1} = 137.035999178(8)$, as reported by the Particle Data Group [1], represents one of the most accurately known quantities in physics.¹ However, at higher energies—such as at the electroweak scale probed by the Z boson—the relevant coupling increases to $\alpha^{(5)}(M_Z) = 1/127.930(8)$ [1], a roughly 7% enhancement. This energy dependence can either be determined experimentally (see [18] for a recent proposal) or predicted from its low-energy value using the renormalization group evolution. In the on-shell renormalization scheme, one introduces an effective coupling at any timelike momentum transfer q^2 ,

$$\alpha(q^2) = \frac{\alpha}{1 - \Delta\alpha(q^2)}, \quad (2.1)$$

so that the scale dependence of the coupling is fully described by $\Delta\alpha(q^2)$. A precise knowledge of the running is a key ingredient in precision electroweak tests, and an accurate first-principles determination of this quantity is therefore crucial for ongoing and future collider programs aiming to probe effects of possible new physics through high-precision electroweak observables.

While perturbation theory provides reliable estimates for the leptonic component of $\Delta\alpha(q^2)$, the hadronic contribution at low energies is dominated by nonperturbative QCD effects. This contribution can be expressed in terms of the subtracted HVP function $\bar{\Pi}$,

$$\Delta\alpha_{\text{had}}(q^2) = 4\pi\alpha \text{Re} \bar{\Pi}(q^2), \quad \bar{\Pi}(q^2) = \Pi(q^2) - \Pi(0). \quad (2.2)$$

The HVP can be determined phenomenologically via a dispersion relation that connects it to the experimentally measured R ratio, $R(s) = \sigma(e^+e^- \rightarrow \text{hadrons})/\sigma(e^+e^- \rightarrow \mu^+\mu^-)$. Integrating $R(s)$ over the hadronic spectrum [7,13,19,20], and combining it with our previous lattice determination [11], yields the current world average $\Delta\alpha_{\text{had}}^{(5)}(M_Z^2) = 0.02783(6)$ [1].

Alternatively, the hadronic contribution to the running of α can be determined directly from first principles using lattice QCD, thereby avoiding reliance on experimental data. On the lattice, computations are naturally performed at spacelike momenta $Q^2 = -q^2$, where the HVP function

¹One should be aware, though, that there are significant tensions between the most precise measurements of α [15–17] that are yet to be resolved.

$\Pi(Q^2)$ is obtained from the two-point correlation function of the electromagnetic currents j_μ^γ [21–23]

$$j_\mu^\gamma = \frac{2}{3} \bar{u} \gamma_\mu u - \frac{1}{3} \bar{d} \gamma_\mu d - \frac{1}{3} \bar{s} \gamma_\mu s + \frac{2}{3} \bar{c} \gamma_\mu c + \dots, \quad (2.3)$$

according to

$$(Q_\mu Q_\nu - \delta_{\mu\nu} Q^2) \Pi(-Q^2) = \Pi_{\mu\nu}^{(\gamma, \gamma)}(Q) = \int d^4x e^{iQx} \langle j_\mu^\gamma(x) j_\nu^\gamma(0) \rangle. \quad (2.4)$$

As discussed in [24–26], the connection between the HVP and $\Delta\alpha_{\text{had}}$ is established through the Adler function $D(Q)$ [27]. This quantity is defined as the derivative of the HVP with respect to Q^2 , and can equivalently be expressed as a dispersion integral over the experimentally measured R ratio,

$$D(Q^2) = 12\pi^2 Q^2 \frac{d\Pi(-Q^2)}{dQ^2} = Q^2 \int_0^\infty ds \frac{R(s)}{(s+Q^2)^2}. \quad (2.5)$$

Thus, by computing the correlation function of the electromagnetic current j_μ^γ , one can access $\bar{\Pi}^{(\gamma, \gamma)}$ and thereby obtain $\Delta\alpha_{\text{had}}$ directly from QCD.

B. The electroweak mixing angle

The electroweak mixing angle (or Weinberg angle) θ_W uniquely defines the relation between the electromagnetic and weak interactions in the SM. It relates the g and g' couplings of the $\text{SU}(2)_L$ weak isospin and $\text{U}(1)_Y$ weak hypercharge interactions, respectively, with the electromagnetic coupling $\alpha = e^2/(4\pi)$ through [28,29]

$$e = g \sin \theta_W = g' \cos \theta_W, \quad \sin^2 \theta_W = \frac{g'^2}{g^2 + g'^2}. \quad (2.6)$$

Beyond tree level the value of $\sin^2 \theta_W$ depends on the renormalization scheme and energy scale, and while precise determinations exist at the Z pole [1], there is a growing interest in its low-energy value, currently known at the percent level [30–32], especially from experiments probing precision electroweak observables at $q^2 \ll M_Z^2$. In this regime, the electroweak mixing angle is sensitive to hadronic effects and can be used to probe BSM physics [33–36].

A q^2 -dependent definition of the electroweak mixing angle, linking $\sin^2 \theta_W(q^2)$ to its value in the Thomson limit takes the form [8,37–39]

$$\sin^2 \theta_W(q^2) = \left(\frac{1 - \Delta\alpha_2(q^2)}{1 - \Delta\alpha(q^2)} + \Delta\kappa_b(q^2) - \Delta\kappa_b(0) \right) \times \sin^2 \theta_W(0). \quad (2.7)$$

In this work we adopt $\sin^2 \theta_W(0) = 0.23857(5)$, quoted in [28] as the average of different evaluations [19,33,40,41]. Here $\Delta\kappa_b$ denotes the bosonic contribution as given in [40], while $\Delta\alpha_2$ is the energy running contribution to the coupling $g^2 = 4\pi\alpha_2$, defined as

$$\alpha_2(q^2) = \frac{\alpha_2}{1 - \Delta\alpha_2(q^2)}. \quad (2.8)$$

Similarly to Eq. (2.2), the hadronic contribution at low energies is computed from the HVP function

$$(\Delta\sin^2 \theta_W)_{\text{had}}(q^2) = \Delta\alpha_{\text{had}}(q^2) - \Delta\alpha_{2,\text{had}}(q^2) = -\frac{4\pi\alpha}{\sin^2 \theta_W(0)} \bar{\Pi}^{(Z, \gamma)}(q^2), \quad (2.9)$$

where $\bar{\Pi}^{(Z, \gamma)}(q^2)$ denotes the HVP mixing of the electromagnetic current j_μ^γ and the vector component of the neutral weak current

$$j_\mu^Z|_{\text{vector}} = j_\mu^{T_3}|_{\text{vector}} - \sin^2 \theta_W j_\mu^\gamma. \quad (2.10)$$

C. The TMR method

As described above, the correlation function $G_{\mu\nu}(x) = \langle j_\mu(x) j_\nu(0) \rangle$ of two vector currents computed on the lattice is the primary quantity for evaluating the hadronic running for both $\Delta\alpha_{\text{had}}$ and $(\Delta\sin^2 \theta_W)_{\text{had}}$. The relevant HVP $\bar{\Pi}^{(\gamma, \gamma)}$ and $\bar{\Pi}^{(Z, \gamma)}$ can therefore be computed as a function of Q^2 from the vector correlators by providing the currents j_μ^γ and j_μ^Z , respectively. We adopt the time-momentum representation (TMR) [42,43] to compute the subtracted HVP in the Euclidean theory through the integral

$$\bar{\Pi}^{(\alpha, \gamma)}(-Q^2) = \int_0^\infty dx_0 G^{(\alpha, \gamma)}(x_0) \left[x_0^2 - \frac{4}{Q^2} \sin^2 \left(\frac{Qx_0}{2} \right) \right], \quad \alpha = Z, \gamma. \quad (2.11)$$

Here $G^{(\alpha, \gamma)}(x_0)$ represents the zero-momentum projected correlator

$$G^{(\alpha, \gamma)}(x_0) = -\frac{1}{3} \int d^3x \sum_{k=1}^3 \langle j_k^\alpha(x_0, \vec{x}) j_k^\gamma(x_0, \vec{0}) \rangle. \quad (2.12)$$

In this work, we update our previous result [11] with the inclusion of an extended set of ensembles with improved coverage in lattice spacing and quark masses. Furthermore, inspired by techniques developed for the nonperturbative calculations of the HVP contribution to the muon anomalous magnetic moment a_μ , we introduce a new family of kernel functions designed to better control systematics across different Euclidean regions. A detailed description

of our computational strategy is given in the following section.

D. Flavor decomposition

Following the notation of [11,44,45], we conveniently express the electromagnetic current through a matrix T^a acting in flavor space,

$$j_\mu^a = \bar{\psi} T^a \gamma_\mu \psi, \quad \bar{\psi} = (\bar{u} \bar{d} \bar{s} \bar{c} \bar{b}), \quad (2.13)$$

such that the generic flavor-specific vector correlator takes the form

$$\delta_{kl} G^{(m,n)}(x_0) = - \int d^3x \langle j_k^m(x_0, \mathbf{x}) j_l^n(0) \rangle. \quad (2.14)$$

We set $T^a = \frac{\lambda^a}{2} \oplus 0_c \oplus 0_b$ for $a = 1, \dots, 8$ to describe the (u, d, s) sector through the corresponding Gell-Mann matrix λ^a and define $T^c = \text{diag}(0, 0, 0, 1, 0)$, $T^b = \text{diag}(0, 0, 0, 0, 1)$ to describe the charm and bottom sector, respectively. We also introduce the $SU(3)$ -singlet generator $T^0 = \frac{1}{2} \mathbf{1}_3 \oplus 0_c \oplus 0_b$ corresponding to the singlet current j_μ^0 . The vector correlators of interest then read

$$G^{(\gamma, \gamma)} = G^{(3,3)} + \frac{1}{3} G^{(8,8)} + \frac{4}{9} G^{(c,c)} + \frac{4}{9} G_{\text{disc}}^{(c,c)} + \frac{2}{3\sqrt{3}} G_{\text{disc}}^{(c,8)} + \frac{1}{9} G^{(b,b)} + \dots, \quad (2.15)$$

$$G^{(Z, \gamma)} = \left(\frac{1}{2} - \sin^2 \theta_w \right) G^{(\gamma, \gamma)} - \frac{1}{6\sqrt{3}} G^{(0,8)} - \frac{1}{18} G^{(c,c)} - \frac{1}{18} G_{\text{disc}}^{(c,c)} + \dots, \quad (2.16)$$

in terms of the flavor-specific building blocks, which can be computed separately. We treat the quark-connected and disconnected contributions in the heavy quark sector individually, while the ellipses represent contributions that are too small to be relevant at the current statistical precision, such as the disconnected diagrams involving bottom and top quarks.

III. GENERAL COMPUTATIONAL STRATEGY

One of the main challenges in achieving a lattice determination of the electroweak couplings with controlled and conservative uncertainties is to perform a reliable extrapolation to the physical point while accounting for the systematics involved. To address this, we follow a strategy that separates the dominant sources of uncertainty. Inspired by the splitting in short (SD), intermediate (ID)

and long (LD) distance windows introduced in the calculation of the muon $g-2$ [46], we propose the following decomposition for the subtracted HVP²

$$\bar{\Pi}(Q^2) = \hat{\Pi}(Q^2) + \hat{\Pi}(Q^2/4) + \bar{\Pi}(Q^2/16), \quad (3.1)$$

where we have defined

$$\hat{\Pi}(Q^2) \equiv \Pi(Q^2) - \Pi(Q^2/4). \quad (3.2)$$

We refer to the three terms $\hat{\Pi}(Q^2)$, $\hat{\Pi}(Q^2/4)$, and $\bar{\Pi}(Q^2/16)$ as high- (HV), mid- (MV) and low-virtuality (LV) regions, respectively. The above telescopic sum offers a clear separation of the different Euclidean regions, making it possible to disentangle the strong cutoff effects at short times from the sizeable chiral dependence expected at larger distances. This separation allows us to tailor the fit models to the specific behavior of each term in Eq. (3.1). In particular, perturbation theory becomes applicable in the HV region, where the short distance nature of the observable makes discretization effects most prominent. As a result, we can reliably achieve higher values of Q^2 compared to [11], thereby increasing the threshold energy at which the running to the Z pole is continued using QCD perturbation theory. This is a crucial aspect for improving the overall precision in $\Delta\alpha_{\text{had}}^{(5)}(M_Z^2)$, as in our previous determination [11] the perturbative piece accounted for approximately 60% of the total variance.

The finite differences of the HVP, namely $\hat{\Pi}(Q^2)$ and $\hat{\Pi}(Q^2/4)$ in the HV and MV regions are estimated via the TMR integral by convoluting the correlators of interest with the Q^2 -dependent function

$$\hat{\Pi}(Q^2) = \int_0^\infty dx_0 G(x_0) \hat{K}(x_0, Q^2), \quad (3.3)$$

$$\hat{K}(x_0, Q^2) = \frac{16}{Q^2} \sin^4 \left(\frac{Qx_0}{4} \right), \quad (3.4)$$

while the LV term $\bar{\Pi}(Q^2/16)$ is extracted from Eq. (2.11), i.e., with the standard kernel

$$K(x_0, Q^2) = x_0^2 - \frac{4}{Q^2} \sin^2 \left(\frac{Qx_0}{2} \right). \quad (3.5)$$

In Fig. 1 we plot the contributions to the integrand for the three different virtuality regions considered throughout this work. The integrands in the HV, MV, and LV regions show a behavior analogous to the SD, ID, and LD windows in the $g-2$ calculations. The separation of these regions makes it

²In Secs. III and IV, we omit the minus sign in the argument of the vacuum polarization that reminds us that we consider spacelike virtualities. The minus sign is restored in Sec. V.

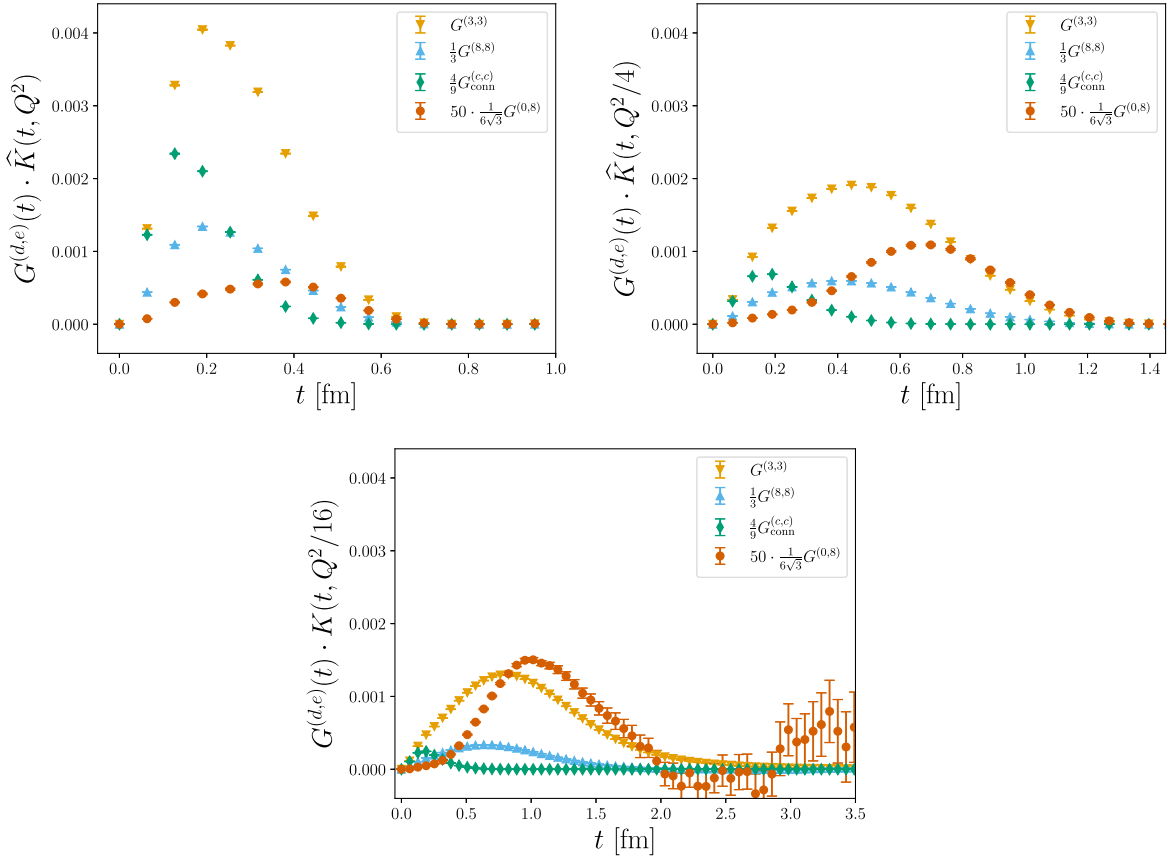


FIG. 1. Illustration of the integrands for the various contribution entering the computation of the electroweak couplings. *Upper left*: HV region integrands using the non-subtracted kernel in Eq. (3.4). *Upper right*: MV region integrands using the non-subtracted kernel in Eq. (3.4). *Bottom*: LV region integrands from the non-subtracted kernel in Eq. (3.5). Results are shown for physical pion mass ensemble E250 with $a \approx 0.064$ fm at the virtuality $Q^2 = 9 \text{ GeV}^2$.

clear that cutoff effects are most prominent in the HV regime, while the LV one is dominated by a strong chiral dependence. The LV region also provides the largest absolute contribution to the final result and, due to the long-distance nature of the observable, the signal there is more susceptible to signal-to-noise degradation. In all three regions, the isovector channel gives the dominant contribution. In what follows we describe the computational strategy adopted for each term contributing to the HVP.

A. The isovector contribution

Following the approach introduced in [44], we make use of the decomposition

$$\bar{\Pi}^{(3,3)}(Q^2) = \bar{\Pi}_{\text{sub}}^{(3,3)}(Q^2) + \bar{b}^{(3,3)}(Q^2, Q_m^2), \quad (3.6)$$

where $\bar{\Pi}_{\text{sub}}^{(3,3)}(Q^2)$ is computed from the isovector correlator $G^{(3,3)}$,

$$\bar{\Pi}_{\text{sub}}^{(3,3)}(Q^2) = \int_0^\infty dx_0 G^{(3,3)}(x_0) K_{\text{sub}}(x_0, Q^2, Q_m^2) \quad (3.7)$$

with the help of a subtracted kernel function

$$K_{\text{sub}}(x_0, Q^2, Q_m^2) = x_0^2 - \frac{4}{Q^2} \sin^2\left(\frac{Qx_0}{2}\right) - \frac{4}{3} \frac{Q^2}{Q_m^4} \sin^4\left(\frac{Q_m x_0}{2}\right), \quad (3.8)$$

and

$$\bar{b}^{(3,3)}(Q^2, Q_m^2) = \frac{Q^2}{3Q_m^2} \hat{\Pi}^{(3,3)}(4Q_m^2) \quad (3.9)$$

is to be computed in continuum perturbation theory. The above decomposition suppresses the x_0^4 behavior from the kernel function, thus reducing higher-order cutoff effects and cancelling completely the potentially dangerous lattice artifacts of $O(a^2 \log(a))$ in the very short Euclidean distance region [47,48].

Similarly, we compute the quantity $\hat{\Pi}(Q^2)$ in the HV and MV regions from the subtracted kernel function,

$$\hat{\Pi}^{(3,3)}(Q^2) = \hat{\Pi}_{\text{sub}}^{(3,3)}(Q^2) + b^{(3,3)}(Q^2, Q_m^2), \quad (3.10)$$

$$\hat{\Pi}_{\text{sub}}^{(3,3)}(Q^2) = \int_0^\infty dx_0 G^{(3,3)}(x_0) \hat{K}_{\text{sub}}(x_0, Q^2, Q_m^2), \quad (3.11)$$

$$\hat{K}_{\text{sub}}(x_0, Q^2, Q_m^2) = \frac{16}{Q^2} \sin^4\left(\frac{Qx_0}{4}\right) - \frac{Q^2}{Q_m^4} \sin^4\left(\frac{Q_mx_0}{2}\right), \quad (3.12)$$

where the second term in Eq. (3.10) is defined as

$$b^{(3,3)}(Q^2, Q_m^2) = \frac{Q^2}{4Q_m^2} \hat{\Pi}^{(3,3)}(4Q_m^2). \quad (3.13)$$

The quantities $b^{(3,3)}$ and $\bar{b}^{(3,3)}$ can be computed reliably using massless perturbation theory through the Adler function at virtualities $Q \geq 2.5$ GeV, where good convergence is expected. We further decompose the HV isovector contribution into a SD and intermediate plus long distance part using the standard Euclidean time windows as defined in [46], performing separate chiral-continuum extrapolations for each. This is beneficial, as tree-level improvement (see Sec. III D 3) is applied solely to the SD term, leading to an effective reduction of cutoff effects where they are most pronounced.

Throughout this work, we set

$$Q_m^2 = 9 \text{ GeV}^2 \quad (3.14)$$

as the spacelike momentum value at which we perform the subtraction. The HV, MV, and LV terms are computed in the energy ranges $4.0 \text{ GeV}^2 \leq Q^2 \leq 12.0 \text{ GeV}^2$, $1.0 \text{ GeV}^2 \leq Q^2/4 \leq 3.0 \text{ GeV}^2$, and $0.25 \text{ GeV}^2 \leq Q^2/16 \leq 0.75 \text{ GeV}^2$, respectively.

B. The isoscalar contribution

Similarly to the isovector contribution, we evaluate the isoscalar HVP through the decomposition in Eq. (3.1). Knowing $\hat{\Pi}^{(3,3)}(Q^2)$ in the HV region, we estimate the corresponding $\hat{\Pi}^{(8,8)}(Q^2)$ via the breakdown [44]

$$\hat{\Pi}^{(8,8)}(Q^2) = \hat{\Pi}^{(3,3)}(Q^2) + \hat{\Delta}_{Is}(Q^2), \quad (3.15)$$

where only the term $\hat{\Delta}_{Is}$ has to be computed anew. The key observation is that isovector and isoscalar correlation functions at short distances are nearly identical. As a result, their difference is parametrically suppressed, leading to a significant reduction of the statistical uncertainty and to a cancellation of log-enhanced cutoff effects. In practice, we evaluate $\hat{\Delta}_{Is}$ through the integral

$$\hat{\Delta}_{Is}(Q^2) = \int_0^\infty dx_0 [G^{(8,8)}(x_0) - G^{(3,3)}(x_0)] \hat{K}(x_0, Q^2), \quad (3.16)$$

using the unsubtracted kernel function $\hat{K}(x_0, Q^2)$ defined in Eq. (3.4). On the other hand, we extract the MV $\hat{\Pi}^{(8,8)}(Q^2/4)$ and LV $\bar{\Pi}^{(8,8)}(Q^2/16)$ contributions through a straightforward computation by employing the unsubtracted kernel introduced in Eq. (3.5).

1. $Z\gamma$ mixed isoscalar contribution

The mixed contribution $\bar{\Pi}^{(0,8)}$, relevant exclusively to the electroweak mixing angle, includes the SU(3)-singlet current and vanishes linearly in $m_s - m_l$ when approaching the SU(3)-symmetric point, ensuring that log-enhanced cutoff effects are not relevant. Therefore, no perturbative subtraction of the latter is required. Specifically, we compute this contribution in the three virtuality regions defined in Eq. (3.1) through the unsubtracted kernels in Eqs. (3.4) and (3.5).

C. Charm-connected contribution

The charm-connected contribution is computed analogously to the isovector HVP by employing the subtracted kernels introduced in Eqs. (3.12) and (3.8) for the (HV, MV) and LV regions, respectively. In the (HV, MV) cases, the subtraction function is determined according to

$$b_{\text{conn}}^{(c,c)}(Q^2) = 2b^{(3,3)}(Q^2, Q_m^2) + \Delta_{lc} b(Q^2, Q_m^2). \quad (3.17)$$

Here, the first term $b^{(3,3)}$ is evaluated in massless perturbation theory, while $\Delta_{lc} b$, defined as

$$\Delta_{lc} b(Q^2, Q_m^2) = \frac{Q^2}{4Q_m^2} (2\hat{\Pi}^{(3,3)}(4Q_m^2) - \hat{\Pi}^{(c,c)}(4Q_m^2)), \quad (3.18)$$

is computed on the lattice using the unsubtracted kernels in Eq. (3.4) convoluted with the difference between the light-connected and charm-connected correlators. A completely analogous procedure is used for the LV part, the subtraction function being obtained as the sum of the perturbatively computed $2\bar{b}^{(3,3)}(Q^2, Q_m^2)$ and the charm-mass effect $\Delta_{lc} \bar{b}$ computed on the lattice.

D. Lattice setup

1. Gauge ensembles

Our calculation is performed on coordinated lattice simulations (CLS) ensembles with $N_f = 2 + 1$ flavors [49–53] of nonperturbatively $O(a)$ -improved Wilson quarks [54] and a tree-level improved Lüscher-Weisz gauge action. The analysis is focused on a subset of CLS ensembles in which the sum of the bare quark masses is held constant as we approach the physical pion mass, ensuring that the improved bare coupling $\tilde{g}_0$ [55] also remains constant along the chiral trajectory. In addition,

TABLE I. List of CLS $N_f = 2 + 1$ ensembles used in this work. Columns show the following parameters: the bare coupling, the temporal boundary conditions, open (o) or antiperiodic (p), the lattice dimensions, the lattice spacing a in physical units based on [58], approximate pion and kaon masses, and the physical size of the lattice. Ensembles marked by a dagger belong to the second chiral trajectory where $m_s \approx m_s^{\text{phys}}$. Ensemble A654 is used exclusively for estimating isospin-breaking effects.

Id	β	bc	L/a	T/a	a [fm]	m_π (MeV)	m_K (MeV)	$m_\pi L$	L (fm)
A654	3.34	p	24	48	0.0978(5)	338	462	4.0	2.3
H101	3.40	o	32	96	0.0847(5)	424	424	5.8	2.7
H102		o	32	96		358	445	4.9	2.7
N101		o	48	128		282	468	5.8	4.1
C101		o	48	96		222	478	4.6	4.1
C102 [†]		o	48	96		225	506	4.6	4.1
D150 [†]		p	64	128		131	484	3.6	5.4
B450	3.46	p	32	64	0.0752(5)	422	422	5.1	2.4
N452		p	48	128		356	447	6.5	3.6
N451		p	48	128		291	468	5.3	3.6
D450		p	64	128		219	483	5.3	4.8
D451 [†]		p	64	128		220	510	5.3	4.8
D452		p	64	128		156	490	3.8	4.8
N202	3.55	o	48	128	0.0633(4)	418	418	6.5	3.0
N203		o	48	128		349	447	5.4	3.0
N200		o	48	128		286	468	4.4	3.0
D251		p	64	128		286	468	5.9	4.1
D200		o	64	128		202	486	4.2	4.1
D201 [†]		o	64	128		202	507	4.2	4.1
E250 [†]		p	96	192		132	495	4.1	6.1
J307	3.70	o	64	192	0.0490(3)	425	425	6.7	3.1
J306		o	64	192		350	456	5.6	3.1
J303		o	64	192		260	480	4.1	3.1
J304 [†]		o	64	192		263	530	4.3	3.1
E300		o	96	192		177	497	4.2	4.7
F300 [†]		o	128	256		136	496	4.3	6.3
J500	3.85	o	64	192	0.0385(3)	417	417	5.2	2.5
J501		o	64	192		337	450	4.2	2.5

we include four ensembles on a different chiral trajectory where the strange quark mass is close to its physical value and with a pion mass of roughly 220 MeV. This helps to account for a small mistuning in m_K^{phys} when approaching the physical pion mass [56]. Compared to our 2022 publication [11], we have substantially expanded the set of gauge ensembles, which now covers five values of the lattice spacing in the range $0.039 \text{ fm} \leq a \leq 0.087 \text{ fm}$ and pion masses in the range $130 \text{ MeV} \leq m_\pi \leq 420 \text{ MeV}$. Notably, we employ three ensembles with physical pion mass, allowing us to firmly constrain the chiral dependence. An overview of the ensembles entering this analysis is provided in Table I. Further details on the set of ensembles can be found in [11,44,45,57].

2. Renormalization and $O(a)$ improvement

We employ two discretizations of the vector current, the local (L) and point-split (C) currents, defined as

$$\begin{aligned}
 J_\mu^{(L),a}(x) &= \bar{\psi}(x) \gamma_\mu T^a \psi(x), \\
 J_\mu^{(C),a}(x) &= \frac{1}{2} (\bar{\psi}(x + a\hat{\mu})(1 + \gamma_\mu) U_\mu^\dagger(x) T^a \psi(x) \\
 &\quad - \bar{\psi}(x)(1 - \gamma_\mu) U_\mu(x) T^a \psi(x + a\hat{\mu})), \quad (3.19)
 \end{aligned}$$

with $U_\mu(x)$ denoting the gauge link at site x with direction $\hat{\mu}$. The $O(a)$ -improved versions of the currents in the massless theory are defined by

$$J_\mu^{(\alpha),a,I}(x) = J_\mu^{(\alpha),a}(x) + ac_V^{(\alpha)(g_0)} \partial_\nu \Sigma_{\mu\nu}^a(x), \quad \alpha = L, C, \quad (3.20)$$

where the local tensor current is given by $\Sigma_{\mu\nu}^a(x) = -\frac{1}{2} \bar{\psi}(x) [\gamma_\mu \gamma_\nu] T^a \psi(x)$. Adopting the prescription of [44], we replace the commonly used symmetric derivative acting on $\Sigma_{\mu\nu}^a(x)$ with

$$\tilde{\partial}_0 \Sigma_{\mu 0}^a(x) \rightarrow \frac{1}{x_0^2} [\tilde{\partial}(x_0^2 \Sigma_{\mu 0}^a(x)) - 2x_0 \Sigma_{\mu 0}^a(x)], \quad (3.21)$$

such that cutoff effects arising from the discrete derivative in the very-short Euclidean region are substantially reduced. In line with our previous works, we use two independent sets of nonperturbatively determined improvement coefficients and renormalization constants. By set 1 we denote the improvement coefficients from large-volume simulations as determined in [59], while we refer to set 2 when using c_V and Z_V from [60] and $b_V, \bar{b}_V$ from [61], determined in the Schrödinger Functional setup. In practice, an update on the value of $\tilde{b}_A$ [62], used as input for the extraction of c_V , with respect to the one entering in [59], leads to a change in the set 1 improvement coefficients. As a result in this work we consider the updated [63] (yet unpublished) values from set 1 exclusively as a cross-check of our continuum extrapolation, while the final results are based only on set 2. However, as a conservative choice, we still include set 1 in our estimate of systematic uncertainties from the model variation. Details on the renormalization pattern for the electromagnetic currents employed in this work are discussed in [11,57,64].

3. Tree-level improvement

On top of subtracting the perturbative evaluation of $b^{(d,e)}(Q^2, Q_m^2)$ to reduce lattice artifacts in short-distance observables, we may achieve an additional reduction of cutoff effects in the isovector channel by evaluating the correlators for nonzero a in massless perturbation theory at leading order. Given the tree-level computation $\mathcal{O}^{\text{tl}}(a)$ of an observable $\mathcal{O}(a)$, we perform the tree-level improvement by means of

$$\mathcal{O}(a) \rightarrow \mathcal{O}(a) \frac{\mathcal{O}^{\text{tl}}(0)}{\mathcal{O}^{\text{tl}}(a)}. \quad (3.22)$$

We also apply the same improvement schemes to the vector correlators computed in massless perturbation theory as those used in the nonperturbative calculations. The implementation of tree-level improvement significantly reduces cutoff effects and the associated systematic uncertainties, while leaving the continuum-extrapolated result unchanged within errors. For further details, we refer the reader to [44].

E. Finite-volume correction

To obtain reliable estimates for the electroweak couplings it is crucial to correct for finite-size effects in the isovector channel, arising from the finite spatial volume L^3 used in lattice QCD calculations. Following our previous studies [11,45,57], we employ two correction schemes: the Hansen-Patella (HP) approach [65,66], based on the pion's electromagnetic form factor, is particularly effective in the relatively short Euclidean distances. On the other hand, in the long-distance domain, we apply the Meyer-Lellouch-Lüscher (MLL) formalism [67,68], which relies on the timelike pion form factor.

In practice, our final estimate for finite-volume effects is obtained by applying the HP correction for times $t < t^* = (m_\pi L/4)^2/m_\pi$, and then switching to the MLL formalism beyond this threshold. The HP corrections are computed from the vector-meson dominance parametrization of the pion form factor, while the Gounaris-Sakurai formalism is used in the MLL method. In addition, we also include corrections for kaon propagation in finite volumes, non-negligible for ensembles near the SU(3)-symmetric point along the chiral trajectory $\text{Tr}(M_q) = \text{const}$. Conversely, the leading finite-volume effects cancel in the isoscalar channel and we only include contributions from $K\bar{K}$ states, treated via the HP method. Further details of our implementation of the finite-volume corrections are given in [45].

As in [45], we follow the procedure first proposed in [69] by correcting all ensembles to a common reference value of $m_\pi L$ before performing the chiral-continuum extrapolation to the physical point. To minimize the correction applied in the vicinity of the physical point, we select a reference volume that closely matches our physical pion ensembles, defined by

$$(m_\pi L)_{\text{ref}} = (m_{\pi^0})_{\text{phys}} \cdot 6.272 \text{ fm} \approx 4.290. \quad (3.23)$$

Eventually, we compute the correction from the reference to infinite volume in the continuum following the same strategy outlined in [45]. We arrive at

$$(\bar{\Pi}^{(3,3)})(L = \infty) - (\bar{\Pi}^{(3,3)})(L_{\text{ref}}) = 28.6(1.6) \times 10^{-5} \quad (3.24)$$

at $Q^2 = 9.0 \text{ GeV}^2$. As in [45], the quoted error of $\sim 10\%$ has been estimated from the variation of the resulting correction for different input parameters in the HP and MLL formalisms, noting that contributions from higher channels to the finite-volume correction are subleading. In Table II we collect results for the finite-volume corrections to $L = \infty$ at various values of Q^2 .

IV. LATTICE RESULTS

A. Physical point extrapolation

The proxies to describe the light and strange quark masses are defined in terms of the dimensionless hadronic combinations

$$\phi_2 = 8t_0 m_\pi^2, \quad \phi_4 = 8t_0 \left(m_K^2 + \frac{1}{2} m_\pi^2 \right). \quad (4.1)$$

The scale setting is performed using the gradient flow parameter t_0/a^2 [70], with its physical value $t_0^{\text{phys}} = 0.1440(7) \text{ fm}$ taken from the determination based on $f_{\pi K}$, a combination of pion and kaon decay constants, in

TABLE II. Values for the isovector finite-volume correction in the continuum from L_{ref} to $L = \infty$ at different values of Q^2 employed in this work. Results are presented in units of 10^{-5} .

Q^2 (GeV ²)	ΔL
0.25	22.1(1.3)
0.3125	23.2(1.4)
0.375	24.0(1.4)
0.4375	24.7(1.5)
0.5	25.1(1.5)
0.75	26.3(1.5)
1.0	26.9(1.6)
1.25	27.3(1.6)
1.5	27.5(1.6)
1.75	27.7(1.6)

Q^2 (GeV ²)	ΔL
2.0	27.9(1.6)
2.25	28.0(1.6)
3.0	28.2(1.6)
4.0	28.4(1.6)
5.0	28.5(1.6)
6.0	28.5(1.6)
7.0	28.6(1.6)
8.0	28.6(1.6)
9.0	28.6(1.6)
12.0	28.7(1.6)

Ref. [58]. We define the physical point in the isospin-symmetric limit by imposing $m_\pi = (m_{\pi^0})_{\text{phys}}$ and $2m_K^2 - m_\pi^2 = (m_{K^+}^2 + m_{K^0}^2 - m_{\pi^+}^2)_{\text{phys}}$ [71,72], leading to the physical masses $m_\pi = 134.9768(5)$ MeV and $m_K = 495.011(10)$ MeV for the pion and kaon, respectively, while the tuning of the charm quark mass is performed by fixing $m_{D_s} = 1968.47$ MeV.

The extrapolation to the physical point adopts a similar strategy as our previous works [44,45] and proceeds by fitting concurrently the chiral and cutoff dependence, using Symanzik effective theory to guide the continuum behavior. Denoting by $X_a^2 = a^2/(8t_0)$ the proxy for the lattice spacing, our most general fit *Ansatz* reads

$$\mathcal{O}(X_a) = \beta_2 X_a^2 + \beta_3 X_a^3 + \beta_4 X_a^4 + \delta_2 X_a^2 (\phi_2 - \phi_2^{\text{phys}}) + \delta_3 X_a^3 (\phi_2 - \phi_2^{\text{phys}}) + \epsilon_2 X_a^2 (\phi_4 - \phi_4^{\text{phys}}). \quad (4.2)$$

In our fits we always include the leading term proportional to X_a^2 and check for the significance of higher order effects by selectively dropping one or more of the terms multiplied by the parameters $\beta_i, \delta_i, \epsilon_2$. Following [73–75], we also include terms describing logarithmic enhancements of cutoff effects, modeling the leading lattice artifacts as $X_a^2 [\alpha_s(1/a)]^{\hat{\Gamma}}$. We consider two values of the anomalous dimension $\hat{\Gamma} \in [0, 0.395]$, where for each fit both choices of

$\hat{\Gamma}$ are tested to assess for the stability of the extrapolation. Similarly, the chiral dependence of observable $\mathcal{O}$ is modeled by including a linear term in ϕ_2 along with higher-order corrections,

$$\begin{aligned} \mathcal{O}(\phi_2) = & \mathcal{O}(\phi_2^{\text{phys}}) + \gamma_1 (\phi_2 - \phi_2^{\text{phys}}) \\ & + \gamma_2 (f_{\chi,1}(\phi_2) - f_{\chi,1}(\phi_2^{\text{phys}})) \\ & + \gamma_3 (f_{\chi,2}(\phi_2) - f_{\chi,2}(\phi_2^{\text{phys}})), \end{aligned} \quad (4.3)$$

where

$$\begin{aligned} f_{\chi,1} \in & \{1/\phi_2; \log(\phi_2); \phi_2 \log(\phi_2); \phi_2^2\}, \\ f_{\chi,2} \in & \{1/\phi_2; \phi_2^2\}. \end{aligned} \quad (4.4)$$

In practice, the inclusion of higher-order terms is evaluated individually based on their statistical significance, as the different HV, MV, and LV virtuality regions may not require the full set of terms.

Since the strange quark proxy ϕ_4 is close to its physical value on all ensembles, we describe it via

$$\mathcal{O}(\phi_4) = \mathcal{O}(\phi_4^{\text{phys}}) + \delta_0 (\phi_4 - \phi_4^{\text{phys}}) + \delta_1 (\phi_4^2 - \phi_4^{2,\text{phys}}), \quad (4.5)$$

where the higher-order term is only included in the isoscalar MV region.

The charm-quark mass dependence is treated separately by interpolating the data to the physical point using $\sqrt{8t_0 m_{D_s}}$ as a proxy, exploiting the approximately linear behavior in the simulated mass range, prior to the global fit.

To assess systematic uncertainties arising from the chiral-continuum extrapolations, we apply cuts to the data sets, either by excluding the coarsest lattice spacing or by removing all ensembles with $m_\pi > 400$ MeV. For the final estimate and systematic error analysis we follow our previous work and perform a weighted model average [76], with weights assigned according to the Akaike information criterion (AIC) [77]. The quality of the fits is assessed by monitoring the ratio $\chi^2/\chi_{\text{exp}}^2$, where χ_{exp}^2 denotes the expected value of the χ^2 as defined in Ref. [78], and by verifying the corresponding p values. Error analysis and propagation are performed using the Γ method of Ref. [79], as implemented in the ADERRORS package [80]. The split into statistical and systematic errors is performed according to Eq. (18) of [76], where in the estimate of the statistical error we set the derivatives of the AIC weights to zero. The chiral-continuum extrapolation plots and fit quality indicators for all channels entering the analysis are collected in Supplemental Material [81].

B. Noise reduction strategies

The rapid degradation of signal quality in both light-quark connected and disconnected correlation functions in the LD Euclidean region is one of the limiting factors for a high precision determination of the electroweak couplings. To overcome this, we employ several noise reduction strategies, focusing primarily on the isovector contribution, which accounts for roughly 50% of the total HVP. Following our recent publication [45], we adopt the improved estimator computed via low-mode averaging (LMA) [82,83] for the light-quark connected correlation function.

Additionally, in both isovector and isoscalar channels we utilize the bounding method, an established technique to tackle the signal-to-noise problem in HVP calculations [46,64,84,85]. The method consists of replacing the correlation function at $t > t_c$ with appropriate lower and upper bounds according to

$$0 \leq G^{(\alpha,\gamma)}(t_c) e^{-E_{\text{eff}}(t-t_c)} \leq G^{(\alpha,\gamma)}(t) \leq G^{(\alpha,\gamma)}(t_c) e^{-E_0(t-t_c)}, \quad (4.6)$$

where E_0 is the ground-state energy level contributing to the vector correlation function. In practice, when available we estimate E_0 from a dedicated spectroscopy study by solving a generalized eigenvalue problem (GEVP) in the isovector channel. On the other ensembles we estimate the ground-state energy through a Gounaris-Sakurai parametrization of the timelike pion form factor. In line with our previous work [45,47], we apply the same estimate for E_0 to the isoscalar correlation function, where the near degeneracy $m_\rho \lesssim m_\omega$ makes this a conservative approximation.

For the lower bound, we determine E_{eff} at some time $t_{\text{eff}} < t_c$ from the logarithmic derivative of the vector correlation function. For each ensemble, t_{eff} is set to ensure that the effective mass at that point is larger than the region where the bounding method takes over.

Eventually, we estimate the corresponding HVP contribution by averaging both bounds over an interval of roughly 0.4 fm, starting from the value of t_c where the two bounds are compatible within 0.5σ uncertainty.

Besides the $\bar{\Pi}^{(3,3)}$ and $\bar{\Pi}^{(8,8)}$ contributions, we also apply the bounding method to the mixed isoscalar $\bar{\Pi}^{(0,8)}$ term, as detailed in [11]. Unlike $G^{(3,3)}$ and $G^{(8,8)}$, the correlator $G^{(0,8)}$ lacks a strictly positive-definite spectral decomposition, which invalidates Eq. (4.6) in this case. However, it is known that $G^{(0,8)}$ shares the same ground-state energy E_0 as the isoscalar contribution, and its associated amplitude Z_0 is positive. In addition, the correlator approaches its asymptotic behavior $\sim Z_0 e^{-E_0 t}$ from below, and the effective energy E_{eff} also closes in on E_0 from below. Consequently, the role of the upper and lower bounds for $G^{(0,8)}$ are inverted compared to Eq. (4.6).

We further improve the statistical accuracy in the isovector LD region for the two ensembles D200 and E250, with pion mass of approximately 200 MeV and 130 MeV, respectively, by supplementing the LMA calculation with a dedicated spectroscopy calculation to explicitly reconstruct the tail of the $G^{(3,3)}$ correlator [64]. This involves computing the lowest energy levels E_n and amplitudes Z_n through a dedicated GEVP study. Full computational details for the physical point ensemble are provided in Appendix D of [45].

We find that for E250 the isovector correlator is saturated by the four lowest-lying energy states from approximately 1.2 fm onward, while for D200 the two lowest states are sufficient to achieve this, due to the larger pion mass on this ensemble. However, LMA continues to provide better precision at shorter time separations (below roughly 2.5 fm), therefore we switch to the reconstructed correlator only when it yields lower statistical uncertainty.

An illustration of the noise-reduction strategies applied throughout this work for the physical point ensemble E250 is shown in Fig. 2. The left panel illustrates the reconstruction of the isovector TMR integrand when including an increasing number of two-pion states in the correlator, with the signal being saturated beyond 1.2 fm. The dashed vertical line represents the source-sink separation where we switch from LMA to the spectroscopy dataset. On the right panel we show the bounding method for the isovector channel, with the green and orange points representing the lower and upper bounds, respectively. The yellow band shows the final estimate, while the dashed horizontal lines depict the result we would have obtained without the spectroscopy data.

C. The isovector contribution

The isovector channel contributes by far the biggest share to the HVP, particularly on ensembles close to the physical pion mass. As explained in Sec. IV B, LMA together with the bounding method and spectral reconstruction allow us to achieve high precision on the most important set of data close to the physical point. For instance, on ensemble E250, we were able to reduce the uncertainty from 0.6% in [11] to 0.2% in the current analysis at $Q^2 = 1.0 \text{ GeV}^2$.

At any fixed value of Q^2 in the range $0.25 \text{ GeV}^2 \leq Q^2 \leq 12 \text{ GeV}^2$, we extrapolate the isovector contribution to the physical point by exploring several functional forms, as described in Sec. IV A. During the chiral-continuum extrapolation, higher-order cutoff and chiral effects are treated independently in each Euclidean window, reflecting the different dominant sources of uncertainty in the various regions. In the HV region, we find that, even after applying tree-level improvement to the SD correlator to suppress discretization effects, higher-order terms in the lattice spacing are still required for a precise description of the data. The effect of tree-level improvement can be assessed

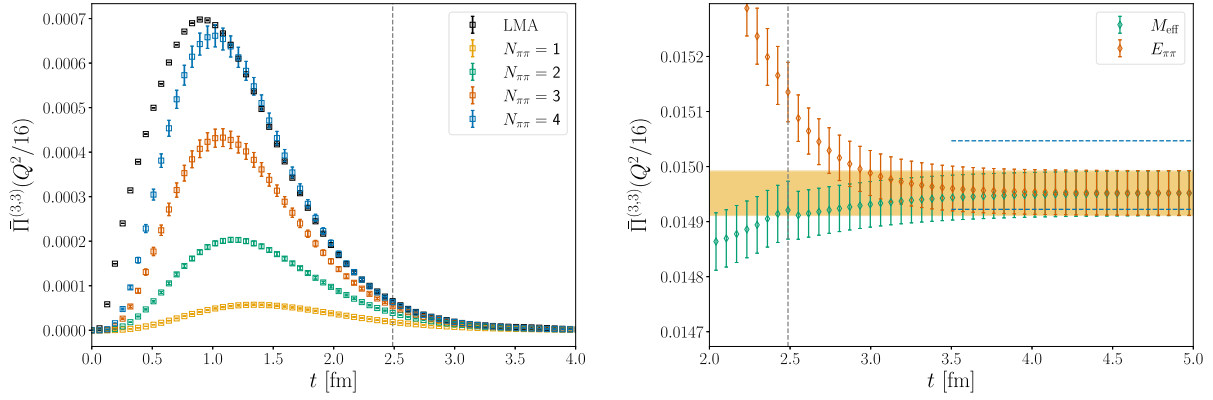


FIG. 2. Illustration of noise-reduction techniques for the $\hat{\Pi}^{(3,3)}$ channel in the LV momentum region. *Left*: integrand contribution on the physical-mass ensemble E250. Black squares correspond to the correlation function obtained using LMA, while the colored points show the reconstructed integrand from $N_{\pi\pi}$ states. The vertical dotted line marks the transition between LMA and spectroscopy data sets. *Right*: application of the bounding method on reconstructed correlator. The dashed vertical lines indicates where the correlator is replaced by spectroscopy data. Green and orange points represent the lower and upper bound, respectively, with the yellow band illustrating our final estimate. The dashed horizontal lines indicate the results we would have obtained based solely on the LMA data.

quantitatively: at $Q^2 = 9 \text{ GeV}^2$, without it, we obtain $\Pi_{\text{sub,SD}}^{(3,3)}(Q^2) = 1038(2)_{\text{stat}}(12)_{\text{syst}}$, while applying tree-level improvement yields $\Pi_{\text{sub,SD}}^{(3,3)}(Q^2) = 1043.9(1.4)_{\text{stat}}(3.4)_{\text{syst}}$. These results are fully consistent with each other, with the tree-level improvement leading to a significant reduction of the systematic uncertainty. At the same time, the light quark mass dependence is found to be mild, indicating that chiral effects are subdominant in this region.

In contrast the LV region, dominating the total isovector contribution, approaches the continuum limit with a significantly flatter behavior, consistent with what was observed in [45] for the LD contribution to the muon anomalous magnetic moment. We note that, within this

region, models incorporating only leading-order lattice artifacts tend to be favored in the model average, compared to those including higher-order cutoff effects. Fits including a^3 terms generally lead to lower central values due to the curvature introduced in the extrapolation, but the resulting shift remains well within the final systematic uncertainty assigned. However, in this region we observe an increased sensitivity to the light quark mass. In particular, to get a good description of the data close to the physical point, it is crucial to include fits with chirally divergent terms. We find that models including $1/\phi_2$ and $\log(\phi_2)$ terms dominate the average in this virtuality region, while the inclusion of a third parameter describing the chiral dependence results in a negligible shift.

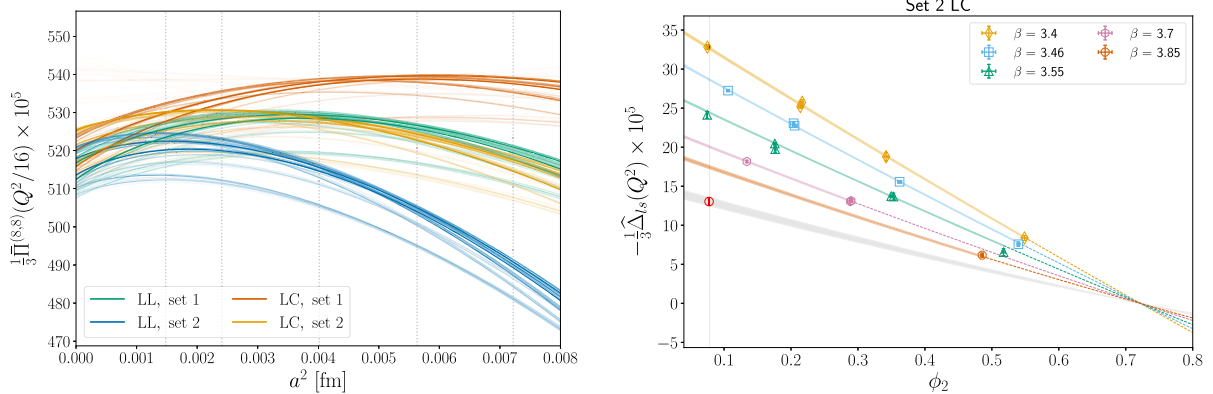


FIG. 3. Illustration of fits to the $\hat{\Pi}_{\text{sub}}^{(3,3)}(Q^2)$ isovector contribution in the HV region. Specifically, we show results for the SD Euclidean window, where tree-level improvement is applied. *Left*: continuum limit behavior for four sets of data based on different improvement schemes and discretizations of the vector current. Each line corresponds to a single fit, with the opacity associated to the weights as given by our model average prescription. Dashed vertical lines denote the lattice spacings used in this work. *Right*: chiral approach to the physical pion mass for one of the fits with the highest weight, and with $\chi^2/\chi_{\text{exp}}^2 = 0.78$. Data points are projected to ϕ_4^{phys} . Colored lines denote the chiral trajectories at finite lattice spacing, while the grey band shows the dependence on ϕ_2 in the continuum. Results are shown for $Q^2 = 9 \text{ GeV}^2$ and $Q_m^2 = 9 \text{ GeV}^2$.

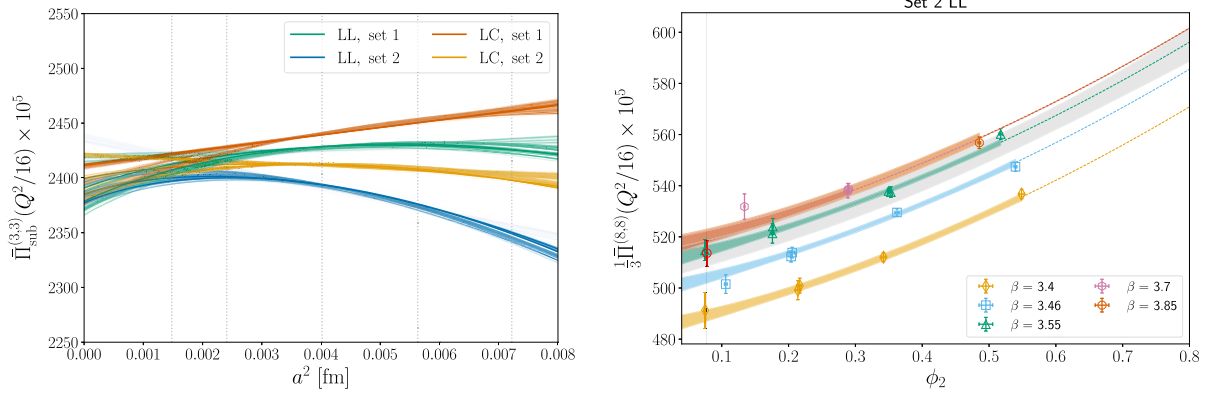


FIG. 4. Same as Fig. 3 for the isovector LV contribution $\tilde{\Pi}_{\text{sub}}^{(3,3)}(Q^2/16)$ as determined from the subtracted kernel in Eq. (3.8). Results are shown for $Q^2 = 9 \text{ GeV}^2$ and $Q_m^2 = 9 \text{ GeV}^2$, with $\chi^2/\chi_{\text{exp}}^2 = 1.1$ for the best fit model.

In Figs. 3 and 4 we show a summary of our fits for $\hat{\Pi}_{\text{sub,SD}}^{(3,3)}(Q^2)$ and $\tilde{\Pi}_{\text{sub}}^{(3,3)}(Q^2/16)$, in the HV and LV regions, respectively. The left panels show the approach to the continuum limit, projected to physical pion mass, for the two discretizations of the vector current and each set of improvement coefficients. Each line depicts a single fit, with the opacity given by the corresponding weight as determined from our information criterion. As mentioned above, we observe that higher-order lattice artifacts are required to properly describe the continuum approach for $\hat{\Pi}_{\text{sub,SD}}^{(3,3)}(Q^2)$ in the HV region, while the LV contribution $\tilde{\Pi}_{\text{sub}}^{(3,3)}(Q^2/16)$ presents a flatter behavior well described by leading-order lattice artefacts.

The right panels show the light quark mass dependence for the best fits according to our model average prescription, for two choices of improvement coefficients and current discretization. Data points are projected to ϕ_4^{phys} . Colored lines represent the chiral trajectories at finite lattice spacing, with the grey bands showing the light quark mass dependence in the continuum limit. We note that the chiral behavior is well constrained over the full range of light quark masses, with three ensembles in the vicinity of the physical point. For the HV term (see Fig. 3) we observe a mild scaling on ϕ_2 , while $\tilde{\Pi}_{\text{sub}}^{(3,3)}(Q^2/16)$ shows a stronger dependence on the light quark mass (see Fig. 4), with fits including chirally divergent terms dominating the model average.

1. Perturbative evaluation of $b^{(3,3)}(Q^2, Q_m^2)$

The subtracted piece $b^{(3,3)}$ defined in Eq. (3.13) can be computed reliably using massless perturbation theory. Similarly to [44], we evaluate the Adler function using the perturbative coefficients from [86] up to α_s^3 and from [87] for $O(\alpha_s^4)$. The computation is performed in the $N_f = 3$ massless theory with $\Lambda_{\overline{\text{MS}}} = 0.338(10) \text{ GeV}$ [88]. We observe a fast convergence of perturbation theory: taking

$Q^2 = 5 \text{ GeV}^2$ and $Q_m^2 = 9 \text{ GeV}^2$ we obtain, from the tree-level prediction to the highest perturbative order

$$b^{(3,3)}(5 \text{ GeV}^2, 9 \text{ GeV}^2) = \{243.85, 260.65, 262.55, 263.07, 263.34\} \times 10^5. \quad (4.7)$$

To estimate the total uncertainty of the perturbative calculation we add in quadrature the error from $\Lambda_{\overline{\text{MS}}}$ and the truncation error arising from the size of the last $O(\alpha_s^4)$ perturbative term. Results for several values of Q^2 employed in this work are listed in Table III.

Eventually, after combining the lattice results for $\tilde{\Pi}_{\text{sub}}^{(3,3)}$ with the perturbative evaluation of $b^{(3,3)}$, we obtain for the isovector contribution at $Q^2 = 9 \text{ GeV}^2$ in units of 10^{-5}

$$\hat{\Pi}^{(3,3)}(Q^2) = 1963.8(2.1)_{\text{stat}}(3.2)_{\text{syst}}(0)_{\text{scale}}(0)_{\text{FV}}[3.8], \quad (4.8)$$

TABLE III. Subtraction term $b^{(3,3)}(Q^2, Q_m^2)$ for the isovector correlator (in units of 10^{-5}) as obtained perturbatively from the $N_f = 3$ massless Adler function computed up to $O(\alpha_s^4)$. Results are shown for the three virtuality regions HV, MV, and LV. The first error denotes the uncertainty induced by $\Lambda_{\overline{\text{MS}}}$, while the second error represents the absolute size of the last $O(\alpha_s^n)$ included in the computation and it provides an estimate of the truncation error of the perturbative series.

Q^2 (GeV ²)	$b_{\text{HV}}^{(3,3)}(Q^2, Q_m^2)$	$b_{\text{MV}}^{(3,3)}(Q^2/4, Q_m^2)$	$\bar{b}_{\text{LV}}^{(3,3)}(Q^2/16, Q_m^2)$
4.0	210.68(20)(22)	52.669(50)(55)	17.557(16)(19)
5.0	263.35(24)(28)	65.837(61)(69)	21.946(20)(24)
6.0	316.02(29)(33)	79.004(73)(83)	26.335(24)(28)
7.0	368.69(34)(39)	92.172(86)(97)	30.724(29)(32)
8.0	421.36(39)(44)	105.34(10)(11)	35.113(33)(37)
9.0	474.03(44)(50)	118.51(11)(12)	39.502(37)(42)

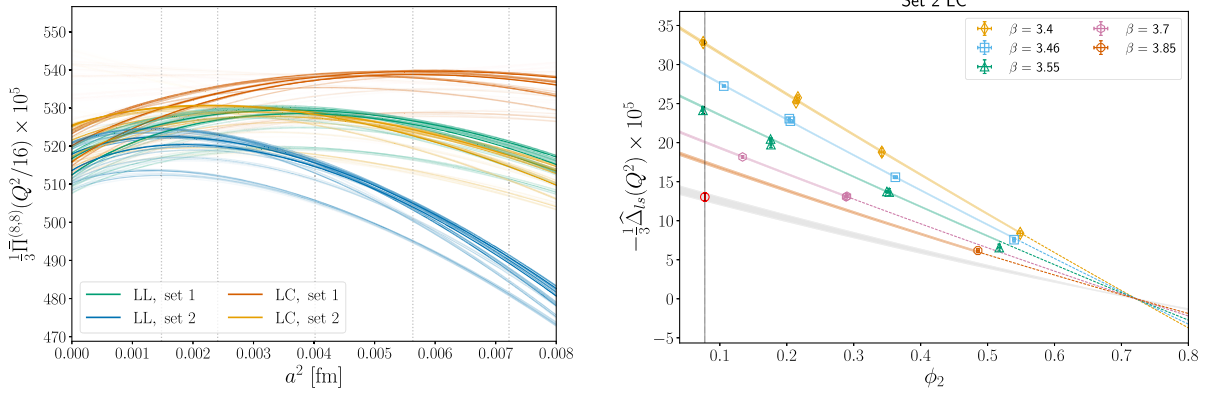


FIG. 5. Same as Fig. 3 for the term $\hat{\Delta}_{Is}(Q^2)$ used to extract the HV isoscalar contribution. Results are shown for $Q^2 = 9 \text{ GeV}^2$, with $\chi^2/\chi_{\text{exp}}^2 = 1.16$ for the best fit model.

$$\hat{\Pi}^{(3,3)}(Q^2/4) = 1961.5(3.6)_{\text{stat}}(3.5)_{\text{syst}}(0.7)_{\text{scale}}(0)_{\text{FV}}[5.1], \quad (4.9)$$

$$\bar{\Pi}^{(3,3)}(Q^2/16) = 2465(12)_{\text{stat}}(14)_{\text{syst}}(10)_{\text{scale}}(3)_{\text{FV}}[21], \quad (4.10)$$

for the HV, MH, and LV regions, respectively. Here the first error is statistical, the second is the systematic uncertainty arising from the model variation, and the third is the uncertainty due to the physical scale t_0^{phys} . The fourth error is associated with the finite volume correction from L_{ref} to $L = \infty$ as computed in Sec. III E.

D. The isoscalar contribution

The isoscalar contribution is computed through the telescopic series introduced in Eq. (3.1). In the HV piece, we employ the decomposition as given in Eq. (3.15) to extract $\hat{\Pi}^{(8,8)}(Q^2)$. In practice, only the quantity $\hat{\Delta}_{Is}$ has to be computed anew following Eq. (3.16). This quantity vanishes by definition at the SU(3)-symmetric point and is expected to scale proportionally to $m_s - m_l$ at leading order. The chiral-continuum extrapolation is therefore performed using the following parametrization

$$\begin{aligned} \hat{\Delta}_{Is}(Q^2)(\phi_\delta, \phi_4, X_a) \\ = \phi_\delta(\gamma_1 + \gamma_2\phi_\delta + \beta_2 X_a^2 + \beta_3 X_a^3 + \gamma_0\phi_4), \end{aligned} \quad (4.11)$$

where $\phi_\delta = \phi_4 - \frac{3}{2}\phi_2$. This ensures that all cutoff effects are suppressed by ϕ_δ in the proximity of the SU(3)-symmetric point. To describe the lattice spacing dependence we explore the same functional forms introduced in Sec. IV A, multiplied by the parameter ϕ_δ to describe the suppression at the SU(3)-symmetric point. Multiple fit *Ansätze* are then formed by selectively setting some of the coefficients in the parametrization to zero. An illustration of the chiral-continuum behavior for $\hat{\Delta}_{Is}$ is shown

in Fig. 5. The approach to the continuum for all the functional forms explored in this work and evaluated at the physical light and strange quark masses is given in the left panel, while the right panel shows the chiral dependence at finite lattice spacing. The intersection point of the curves reflects the $\hat{\Delta}_{Is} = 0$ constraint at the SU(3)-symmetric point. As we move further away from this point the suppression is removed and cutoff effects grow. Given the result at $Q^2 = 9 \text{ GeV}^2$ in units of 10^{-5}

$$\frac{1}{3}\hat{\Delta}_{Is}(Q^2) = -13.16(40)_{\text{stat}}(74)_{\text{syst}}(10)_{\text{scale}}[85], \quad (4.12)$$

obtained from our model average prescription, together with the result for the isovector contribution in the HV region, we quote for the HV isoscalar at the same value of Q^2

$$\frac{1}{3}\hat{\Pi}^{(8,8)}(Q^2) = 641.5(0.9)_{\text{stat}}(1.2)_{\text{syst}}(0)_{\text{scale}}[1.5]. \quad (4.13)$$

On the other hand, the MV and LV isoscalar contributions are computed in a straightforward manner using the unsubtracted kernels in Eqs. (3.4) and (3.5), respectively. At the SU(3)-symmetric point, where the quark-disconnected contribution vanishes, the isoscalar and isovector terms coincide. As outlined in Sec. IV B, we apply the bounding method to mitigate the signal-to-noise problem. This is found to be particularly effective in the LV region, where the signal extends to larger Euclidean times. In contrast, in the MV region, the bounding method leads only to a moderate reduction of the statistical uncertainty. Contrary to the isovector channel, we find that the data show a milder chiral dependence as the divergent behavior of light-connected and disconnected pieces cancels in the isoscalar channel. In both regions, we explore similar functional forms as the ones considered in the isovector channel. We observe that the chiral dependence is well

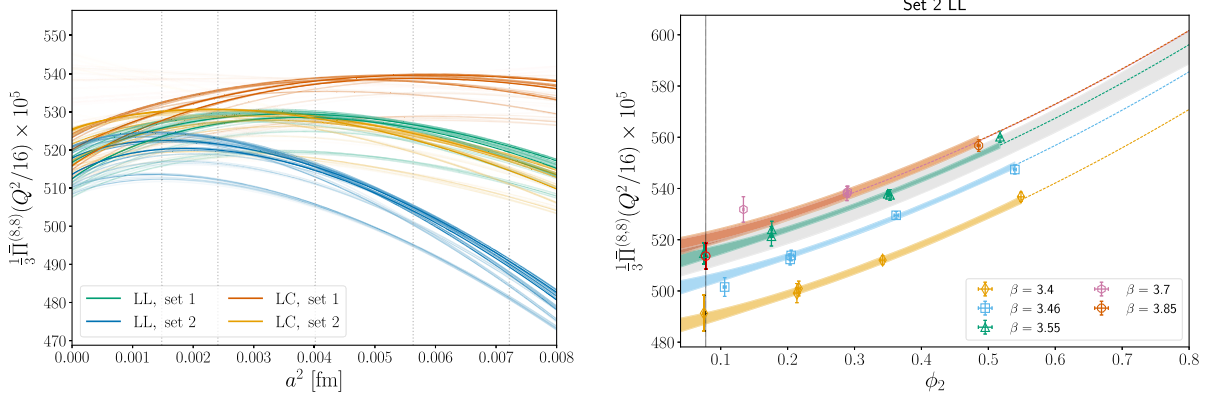


FIG. 6. Same as Fig. 3 for the isoscalar LV contributions $\bar{\Pi}^{(8,8)}(Q^2/16)$. Results are shown for $Q^2 = 9 \text{ GeV}^2$, with $\chi^2/\chi^2_{\text{exp}} = 0.82$ for the best fit model.

described by fits that do not include chirally divergent terms, and the inclusion of additional parameters describing the light quark mass dependence results in a subleading shift well within the statistical uncertainty. Fits including mass-dependent cutoff effects lead to better fit quality and dominate our model average, particularly in the MV region. Additionally we find that, in order to accurately describe ensembles along the chiral trajectory with fixed strange quark mass $m_s = m_s^{\text{phys}}$, it is necessary to include a higher-order term in ϕ_4 . Following our model average prescription, we arrive at the results

$$\frac{1}{3} \hat{\Pi}^{(8,8)}(Q^2/4) = 575.7(2.5)_{\text{stat}}(2.0)_{\text{syst}}(0.8)_{\text{scale}}[3.3], \quad (4.14)$$

$$\frac{1}{3} \bar{\Pi}^{(8,8)}(Q^2/16) = 519.3(5.4)_{\text{stat}}(5.0)_{\text{syst}}(1.7)_{\text{scale}}[7.5], \quad (4.15)$$

at $Q^2 = 9 \text{ GeV}^2$ in units of 10^{-5} .

The left-hand side of Fig. 6 shows the approach to the continuum in the LV region for the four sets of data considered, while the right panel illustrates one of the best fit for set 2 and *LL* discretization of the vector current.

E. The charm-connected contribution

The charm-connected contribution on the lattice is evaluated employing the subtracted kernel as introduced in Eq. (3.12) for the HV and MV regions, and the subtracted kernel in Eq. (3.8) in the LV region. The tuning of the charm quark mass to match the physical $m_{D_s} = 1968.47 \text{ MeV}$ meson mass is described in detail in [57,64]. Following the chiral-continuum extrapolation, a small shift is applied to adapt the tuning of the charm quark hopping parameter to the updated value of t_0^{phys} used in this work.

The charm-connected contribution shows a strong curvature as approaching the continuum that requires the inclusion of higher-order terms beyond a^2 together with mass-dependent cutoff effects. In line with our previous studies, we only include the *LC* discretization of the current in the model average as it shows significantly milder cutoff effects. When evaluating the charm contribution, we choose to use t_0^{sym} from [58] instead of the ensemble-specific t_0 to set the Q^2 input in the TMR kernel and to determine the value of ϕ_2 for each ensemble entering the chiral continuum extrapolation. While using t_0^{sym} introduces correlations among ensembles at the same lattice spacing, it significantly reduces the light-quark mass dependence of the charm-connected contribution.

Quark mass effects in the subtraction function $b^{(c,c)}$, defined in Eq. (3.17), are also computed on the lattice by evaluating nonperturbatively $\Delta_{lc}b$ using the nonsubtracted kernel in Eq. (3.4). The same quantity can also be extracted in massive perturbation theory. As previously observed in [44], the perturbative and lattice results yield similar central values, but the perturbative prediction carries a significantly larger uncertainty, dominated by the absolute size of the last $O(\alpha_s^n)$ term included in the massive perturbative expansion. In contrast, the uncertainty in the lattice result, driven by the continuum extrapolation, is found to be smaller at the value of Q_m considered in this work. Eventually, combining the nonperturbative calculations of $\Pi_{\text{sub}}^{(c,c)}$ and $\Delta_{lc}b$ with the perturbative evaluation for $b^{(3,3)}$, we obtain at $Q^2 = 9 \text{ GeV}^2$

$$\frac{4}{9} \hat{\Pi}^{(c,c)}(Q^2) = 711.9(9.4)_{\text{stat}}(9.2)_{\text{syst}}(1.8)_{\text{scale}}[13], \quad (4.16)$$

$$\frac{4}{9} \hat{\Pi}^{(c,c)}(Q^2/4) = 262.5(4.1)_{\text{stat}}(3.0)_{\text{syst}}(0.6)_{\text{scale}}[5.1], \quad (4.17)$$

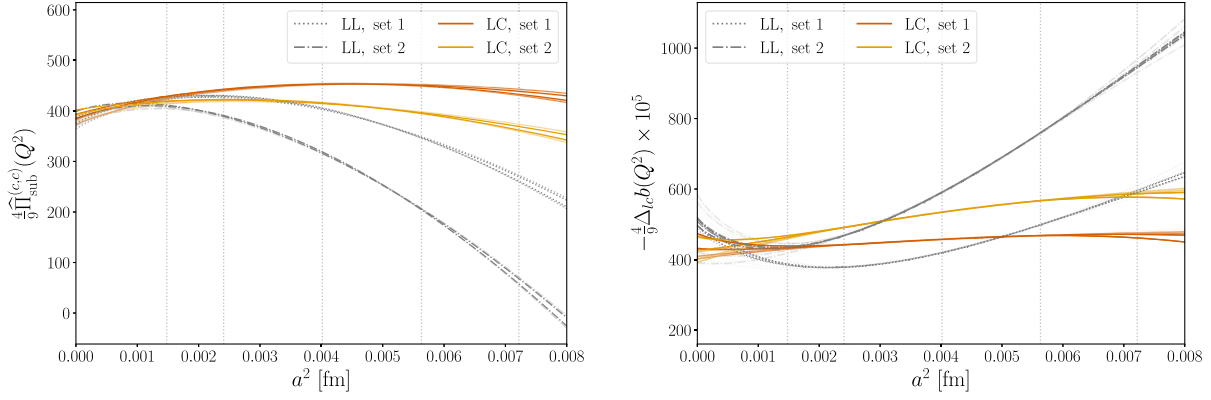


FIG. 7. Illustration of fits to the charm-connected contribution. We show the continuum limit behavior for the four sets of data based on different improvement schemes and discretizations of the vector current. Each line corresponds to a single fit, with the opacity associated to the weights as given by our model average prescription. *Left*: fits to the $\hat{\Pi}^{(c,c)}(Q^2)$ contribution in the HV region with $Q^2 = Q_m^2 = 9 \text{ GeV}^2$, with $\chi^2/\chi_{\text{exp}}^2 = 0.92$ for the best fit model. *Right*: fits for the subtraction function $\Delta_{lc}b$ computed non-perturbatively at $4Q_m^2 = 36 \text{ GeV}^2$, with $\chi^2/\chi_{\text{exp}}^2 = 0.82$ for the best fit model.

$$\frac{4}{9}\bar{\Pi}^{(c,c)}(Q^2/16) = 99.5(1.7)_{\text{stat}}(1.8)_{\text{syst}}(0.3)_{\text{scale}}[2.5], \quad (4.18)$$

for the HV, MV, and LV regions, respectively, in units of 10^{-5} .

An illustration of the continuum behavior for $\hat{\Pi}^{(c,c)}(Q^2)$ in the HV region and $\Delta_{lc}b$ is shown on the left and right panels of Fig. 7, respectively. We observe a good agreement in the continuum among the different datasets for the two quantities, despite having substantially different cutoff effects. Contrary to the other contributions, we notice that the charm-connected HVP has only a mild dependence on the light quark mass.

F. $Z\gamma$ mixing isoscalar contribution

Finally, the mixing isoscalar contribution $\bar{\Pi}^{(0,8)}$ entering the electroweak mixing angle is computed through the splitting in Eq. (3.1). At leading order this term is proportional to $m_s - m_l$, thus vanishing linearly toward the SU(3)-symmetric point and ensuring that log-enhanced cutoff effects are not present. The bounding method is applied as outlined in Sec. IV B in the MV and LV regions to mitigate the uncertainty arising from the large Euclidean distances where the disconnected contribution entering $\bar{\Pi}^{(0,8)}$ suffers from a strong signal-to-noise problem. We describe the chiral-continuum dependence along the same lines as for the HV isoscalar Δ_{ls} , by considering the fit *Ansatz*

$$\hat{\Delta}_{ls}(Q^2)(\phi_\delta, \phi_4, X_a) = \phi_\delta(\gamma_1 + \gamma_2\phi_\delta + \beta_2 X_a^2 + \gamma_0\phi_4). \quad (4.19)$$

In the LV region, we do not include an explicit dependence on the lattice spacing in the base fit model, as no

discretization effects are visible within the current statistical precision. Nonetheless, we allow for fits including $O(a^2)$ terms to probe potential systematic effects. For this observable, we fit only the local-conserved discretization, which avoids relying on the renormalized singlet local current. Figure 8 summarizes the extrapolation to the physical point for this quantity: the left panel shows the continuum limit approach in the MV region for the two available datasets, while the right panel illustrates the chiral behavior for one of the best fits in the LV region. The absence of leading $O(a^2)$ terms is consistent with the observation that data from different lattice spacings cannot be distinguished within uncertainties.

From the model average, we eventually quote for the mixing isoscalar contribution at $Q^2 = 9 \text{ GeV}^2$ in units of 10^{-5}

$$\frac{1}{6\sqrt{3}}\hat{\Pi}^{(0,8)}(Q^2) = 3.20(8)_{\text{stat}}(6)_{\text{syst}}(0)_{\text{scale}}[10], \quad (4.20)$$

$$\frac{1}{6\sqrt{3}}\hat{\Pi}^{(0,8)}(Q^2/4) = 16.20(22)_{\text{stat}}(11)_{\text{syst}}(4)_{\text{scale}}[25], \quad (4.21)$$

$$\frac{1}{6\sqrt{3}}\bar{\Pi}^{(0,8)}(Q^2/16) = 44.08(76)_{\text{stat}}(50)_{\text{syst}}(34)_{\text{scale}}[98], \quad (4.22)$$

for the HV, MV, and LV regions, respectively.

G. Isospin-breaking effects

In this section, we describe our determination of the correction to $\bar{\Pi}(Q^2)$ due to electromagnetic and strong isospin-breaking (IB) effects.

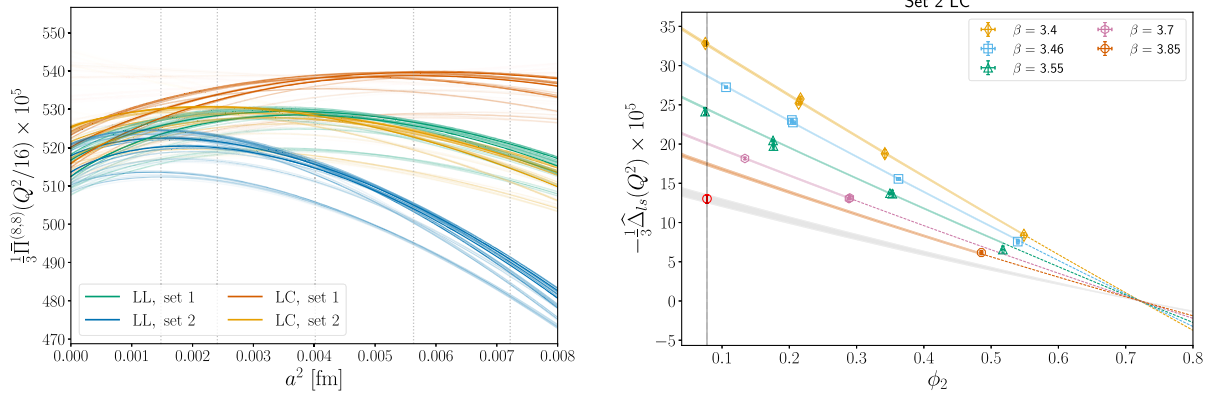


FIG. 8. Illustration of fits to the mixing isoscalar contribution at $Q^2 = 9 \text{ GeV}^2$. *Left*: continuum limit behavior for two available LC discretizations for $\hat{\Pi}^{(0,8)}(Q^2/4)$ in the MV region. Each line corresponds to a single fit, with the opacity associated to the weights as given by our model average prescription. *Right*: chiral approach of $\hat{\Pi}^{(0,8)}(Q^2/16)$ in the LV region to the physical pion mass for one of the fits with the highest weight, with $\chi^2/\chi_{\text{exp}}^2 = 0.68$ for the best fit model.

We have performed a lattice calculation of the strong and electromagnetic isospin-breaking corrections to the quark-connected contribution to $\bar{\Pi}$, taking into account only effects in the valence sector. The setup follows Refs. [11,44,45,57], where QCD + QED effects are included perturbatively around the isospin-symmetric theory using Monte Carlo reweighting combined with a leading-order expansion in the electromagnetic coupling e^2 and the quark-mass shifts Δm_u , Δm_d , and Δm_s relative to their isosymmetric values [89–93]. The additional correlation functions corresponding to quark-mass insertions and photon-exchange diagrams are computed in the non-compact formulation of lattice QED with the QED_L prescription [94] to regularize infrared divergences. The physical point of QCD + QED is defined through the meson mass combinations $m_{\pi^0}^2$, $m_{K^+}^2 + m_{K^0}^2 - m_{\pi^+}^2$, and

$m_{K^+}^2 - m_{K^0}^2 - m_{\pi^+}^2 + m_{\pi^0}^2$ together with the fine-structure constant α , following the hadronic renormalization scheme of Ref. [90].

We have performed the computation on eight gauge ensembles (A654, H102, N101, N452, N451, D450, N203, and N200), covering four values of the lattice spacing down to 0.064 fm and pion masses in the range from 350 to 220 MeV. In Fig. 9 we show the relative size of the isospin-breaking correction to the quark-connected contribution to $\bar{\Pi}(Q^2 = -5 \text{ GeV}^2)$, where, as in the isosymmetric theory, both local-local and local-conserved discretizations of the vector current are employed. Within the statistical uncertainties, the data exhibit only a mild dependence on the pion mass and the lattice spacing in the explored parameter range and point to a four per mil correction.

In addition, we pursue a complementary approach to the lattice calculations described above. A phenomenological estimate of the correction to d_μ^{hvp} from electromagnetic effects was given in [95], based on the formalism set up in [96], and an estimate of the strong isospin-breaking correction in [97]. Here we adapt those predictions to the subtracted vacuum polarization itself. We focus on two spacelike virtualities, 9 GeV^2 and 5 GeV^2 . Since the hadronic models used are meant to describe low-to-moderate energies, we will use their predictions at 5 GeV^2 and rely on QCD perturbation theory for the subsequent IB effects from 5 to 9 GeV^2 in the vacuum polarization. Indeed, for the difference $\Pi(9 \text{ GeV}^2) - \Pi(5 \text{ GeV}^2)$, which is short-distance dominated, the perturbative prediction for electro-magnetic effects amounts to a relative correction of about half a per mil [see [44], Eq. (54), which is based on the perturbative calculation [98]]. In absolute terms, this amounts to

$$\Delta_{\text{IB}}\{\Pi(9 \text{ GeV}^2) - \Pi(5 \text{ GeV}^2)\} \simeq 0.7 \times 10^{-5}. \quad (4.23)$$

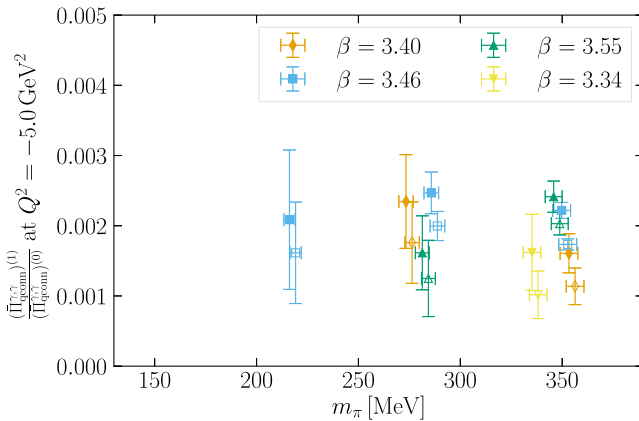


FIG. 9. Relative effect from isospin-breaking on the quark-connected contribution to $\bar{\Pi}$ at $Q^2 = -5 \text{ GeV}^2$ as described in the text. Filled symbols correspond to the local-local discretization and open symbols to the local-conserved current. The points are slightly displaced horizontally for better visibility.

TABLE IV. Corrections from IB effects to the subtracted vacuum polarization for two different virtualities, and for two different pion masses. Results based on a hadronic model, see the main text.

Contribution	$\bar{\Pi}(9 \text{ GeV}^2)$	$\bar{\Pi}(5 \text{ GeV}^2)$	$\bar{\Pi}(5 \text{ GeV}^2)$
($\times 10^5$)	$M_\pi = 135 \text{ MeV}$	$M_\pi = 135 \text{ MeV}$	$M_\pi = 291 \text{ MeV}$
π^0 pole	1.20	1.20	1.10
η pole	0.79	0.79	0.62
η' pole	1.16	1.15	1.58
η_c pole	0.51	0.40	0.40
$\pi^\pm$ loop	-9.25	-9.25	-1.7
Strong IB	13.3	13.2	12.0
Total:	7.7	7.5	14.0

We find strong isospin breaking in this interval of virtualities to be entirely negligible, as they are of order $\Delta_{\text{SIB}} R(s) = 18(\alpha_s/\pi)(4/9 - 1/9)(m_u^2 - m_d^2)/s$ in the R ratio [99].

Table IV provides the quantitative predictions of the hadronic model for $M_\pi = 291 \text{ MeV}$ and for the physical pion mass. At the latter mass, the table also indicates the results for $Q^2 = 9 \text{ GeV}^2$, showing very little change in the prediction as compared to 5 GeV^2 . The single largest contribution is the strong isospin-breaking one. The (negative) charged pion loop contribution significantly reduces the size of the correction at the physical point. It is computed with vector-meson-dominance (VMD) pion form factors and renormalized according to the FLAG scheme.³ Similarly, the pseudoscalar meson exchanges are computed using the master formula derived in [96], with a VMD Ansatz for the transition form factor.

A direct comparison between the hadronic model and our lattice data for the quark-connected contributions is possible at $M_\pi = 291 \text{ MeV}$, corresponding to gauge ensemble N451. Here, according to the model calculations, the charged pion loop contribution is still small in magnitude, the strong IB contribution dominates and disconnected diagrams are relatively small. The lattice result at finite lattice spacing, $11(3) \times 10^{-5}$ in the (u, d, s) sector, is compatible with the model prediction,⁴ 13.6×10^{-5} leaving out the η_c pole contribution.

Our final estimate for the correction to $\bar{\Pi}(9 \text{ GeV}^2)$ from IB effects in the FLAG scheme is the sum of the entry in

³Unlike in the case of a_μ^{hvp} [95], we observe little difference between the FLAG and the on-mass-shell renormalization scheme.

⁴The model for the electromagnetic corrections can approximately be adapted [95] to the fact that the lattice results contain the quark-connected diagrams only, ensuring that the leading chiral behavior of the connected diagrams is reproduced. At $M_\pi = 291 \text{ MeV}$, the model prediction for the electromagnetic effects would increase by about 1×10^{-5} .

Table IV for the correction to $\bar{\Pi}(5 \text{ GeV}^2)$ and of Eq. (4.23),

$$\Delta_{\text{IB}} \bar{\Pi}(9 \text{ GeV}^2) = (8.2 \pm 14.0) \times 10^{-5}. \quad (4.24)$$

The generous uncertainty we have assigned to this correction is based on the size of the (dominant) strong IB correction. It also covers the typical size of the quark-connected IB correction obtained in the lattice calculation.

V. HADRONIC RUNNING OF THE COUPLINGS

In this work, we evaluate the HVP functions $\bar{\Pi}^{(3,3)}, \bar{\Pi}^{(8,8)}, \bar{\Pi}^{(0,8)}, \bar{\Pi}^{(c,c)}$ at the physical point for several discrete values of the squared momentum transfer in the range $0.25 \text{ GeV}^2 \leq Q^2 \leq 12 \text{ GeV}^2$, as reported in Table V. Compared to our previous work [11], we find good agreement for the isovector, isoscalar, and charm-connected contributions. A tension at the level of about 2σ is instead observed for the octet-singlet mixing contribution $\bar{\Pi}^{(0,8)}$, primarily driven by the inclusion of an additional fine lattice spacing and a substantially improved treatment of lattice artifacts. In addition, we determine the

TABLE V. Contribution to the running for the various channel at the physical point, in units of 10^{-5} . The first quoted uncertainty corresponds to the statistical error, the second to systematics from model exploration, and the third to the scale-setting error. The final uncertainty in squared brackets is the sum in quadrature of the previous ones.

$Q^2 \text{ (GeV}^2\text{)}$	$\bar{\Pi}^{(3,3)}$	$\frac{1}{3} \bar{\Pi}^{(8,8)}$
0.5	2310(11)(14)(9)[20]	479.1(5.2)(4.9)(1.6)[7.3]
1.0	3258(12)(11)(10)[19]	740.4(6.4)(6.1)(2.9)[9.3]
2.0	4256(14)(14)(10)[23]	1041.5(6.7)(5.3)(2.4)[9.0]
3.0	4835(15)(17)(10)[25]	1228.2(7.0)(6.0)(2.5)[9.5]
4.0	5257(14)(11)(10)[20]	1364.3(6.8)(6.2)(2.9)[9.6]
5.0	5572(14)(12)(10)[21]	1468.5(6.6)(5.8)(2.0)[9.1]
6.0	5829(14)(13)(10)[21]	1552.6(6.7)(5.5)(1.9)[8.9]
7.0	6043(14)(14)(10)[22]	1622.2(6.8)(5.4)(2)[8.9]
8.0	6228(14)(15)(10)[23]	1682.1(6.8)(5.4)(2.4)[9.0]
9.0	6390(15)(15)(10)[24]	1736.5(6.9)(5.5)(2.5)[9.2]
12.0	6784(16)(17)(10)[25]	1869.9(7.1)(6.2)(2.6)[9.7]
$Q^2 \text{ (GeV}^2\text{)}$	$\frac{4}{9} \bar{\Pi}^{(c,c)}$	$\frac{1}{6\sqrt{3}} \bar{\Pi}^{(0,8)}$
0.5	88.8(1.6)(1.6)(0.2)[2.3]	41.85(76)(48)(34)[96]
1.0	173.1(3.0)(1.7)(0.5)[3.5]	52.42(87)(47)(44)[1.1]
2.0	326.3(5.8)(2.7)(0.8)[6.4]	59.44(81)(49)(39)[1.02]
3.0	463.5(7.2)(5.1)(1.2)[8.9]	61.94(82)(60)(41)[1.09]
4.0	585.7(9.3)(6.8)(1.4)[12.0]	61.91(91)(47)(43)[1.11]
5.0	699(11)(8)(2)[13]	62.66(88)(42)(44)[1.07]
6.0	803(12)(9)(2)[15]	63.25(87)(44)(40)[1.05]
7.0	900(14)(10)(2)[17]	63.24(85)(47)(38)[1.04]
8.0	990(14)(11)(2)[18]	63.24(83)(49)(38)[1.04]
9.0	1074(15)(12)(3)[19]	63.48(83)(52)(38)[1.05]
12.0	1298(18)(13)(3)[22]	63.96(84)(60)(40)[1.10]

TABLE VI. Total HVP contribution to the running of α and $\sin^2 \theta_W$ in isospin-symmetric QCD. The first quoted uncertainty is the statistical error, followed by the systematic error arising from the model exploration and the scale setting error. The final uncertainty in squared brackets is the sum in quadrature of the previous ones. Results are shown in units of 10^{-5} .

Q^2 (GeV ²)	$\Delta\alpha_{\text{had}}^{(5)}$	$(\Delta\sin^2 \theta_W)_{\text{had}}$
0.5	264.0(1.4)(1.4)(1.0)[2.2]	-268.8(1.5)(1.5)(1.0)[2.3]
1.0	382.9(1.5)(1.1)(1.3)[2.3]	-390.7(1.6)(1.3)(1.2)[2.4]
2.0	516.4(1.7)(1.5)(1.2)[2.6]	-526.7(1.8)(1.6)(1.2)[2.7]
3.0	599.4(1.9)(1.7)(1.3)[2.9]	-609.8(2.0)(1.8)(1.2)[2.9]
4.0	662.1(1.8)(1.4)(1.3)[2.7]	-672.3(1.8)(1.4)(1.2)[2.6]
5.0	711.2(1.9)(1.4)(1.3)[2.7]	-720.1(1.8)(1.5)(1.2)[2.6]
6.0	752.3(2.0)(1.5)(1.3)[2.8]	-759.6(1.9)(1.5)(1.2)[2.7]
7.0	787.6(2.1)(1.6)(1.3)[3.0]	-793.2(2.0)(1.6)(1.2)[2.8]
8.0	818.5(2.2)(1.7)(1.3)[3.1]	-822.4(2.0)(1.7)(1.2)[2.9]
9.0	846.4(2.3)(1.8)(1.4)[3.2]	-848.5(2.1)(1.8)(1.3)[3.0]
12.0	916.2(2.4)(2.0)(1.5)[3.5]	-913.1(2.2)(1.9)(1.3)[3.2]

strange-quark contribution $\bar{\Pi}^{(s,s)}$, which does not directly enter our main analysis; the corresponding results are presented in Appendix B. From these quantities, we construct the $\bar{\Pi}^{(\gamma,\gamma)}$ and $\bar{\Pi}^{(Z,\gamma)}$ HVP functions, which provide direct access to the electroweak couplings at spacelike momenta.

Our determination of $\Delta\alpha_{\text{had}}(-Q^2)$ and $(\Delta\sin^2 \theta_W)_{\text{had}}$ includes the contributions from u , d , s and c quarks. To

incorporate the missing b -quark effects, we use results from the HPQCD collaboration for the lowest four time moments of the HVP [100]. The bottom contribution is obtained by constructing Padé approximants from these moments, yielding a small correction—at most 0.4% at $Q^2 = 12 \text{ GeV}^2$ —to the total hadronic running of the coupling. This contribution is added to our lattice results to obtain a complete $N_f = 5$ flavor determination of the electroweak couplings. The final results for $\Delta\alpha_{\text{had}}^{(5)}(-Q^2)$ and $(\Delta\sin^2 \theta_W)_{\text{had}}$ in the spacelike region are listed in Table VI. Though we have performed our lattice calculation with $N_f = 2 + 1$ flavors of quarks, we have estimated the charm quenching effects along the lines of [11,44] and found them to be negligible.

Our estimates for $\Delta\alpha_{\text{had}}^{(5)}(-Q^2)$ can be directly compared with existing determinations. In Fig. 10 we plot ratios between our new results and several other lattice and phenomenological estimates. We note that our previous results [11] are up to 0.6% larger compared to this analysis, but in good agreement within errors. With respect to the Budapest-Marseille-Wuppertal calculation from 2017 [23], we observe tensions of up to 2.3σ , our results being larger by 1%–2%. A similar tension is observed with the more recent Budapest-Marseille-Wuppertal determination [69], available for $Q^2 = 1 \text{ GeV}^2$ only. Although the first lattice calculation of the quark-connected HVP contribution to the running of the electroweak couplings was reported by [21],

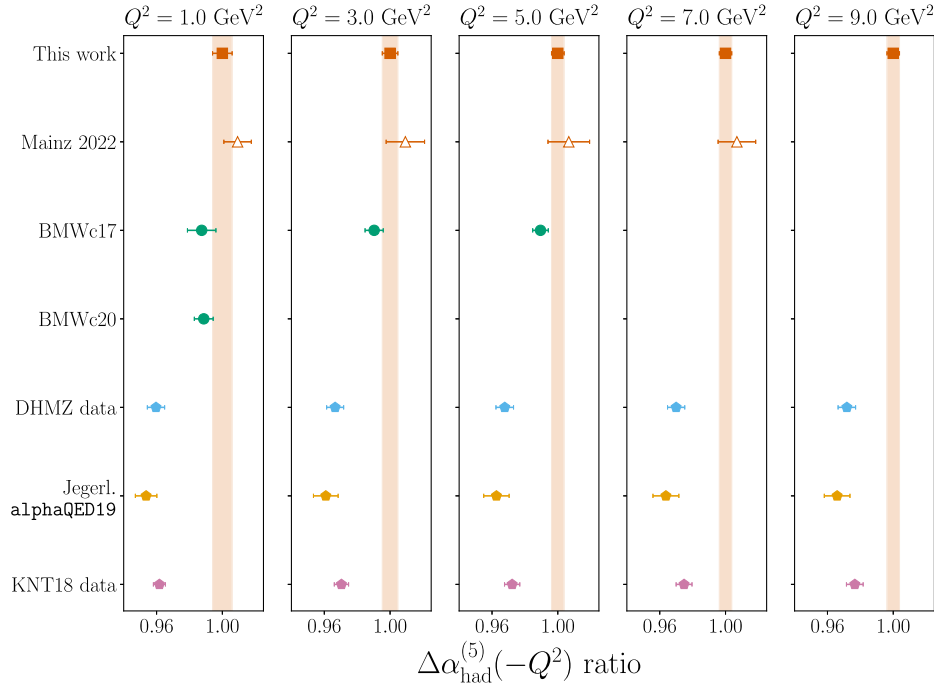


FIG. 10. Summary of results for the determination of $\Delta\alpha_{\text{had}}^{(5)}(-Q^2)$ from lattice and phenomenological estimates. We display the ratio of these determinations to the central value of our results, highlighting relative deviations. For our calculation, the orange vertical band represents the total uncertainty, including the small bottom-quark contribution. The subleading isospin-breaking effects are not included here.

we do not include this result in our comparison, since the corresponding disconnected contribution was not evaluated in that work. The phenomenological results shown in Fig. 10, labeled by “DHMZ data” [7], “Jegerl. alphaQED19” [8,9], and “KNT18 data” [6], are in good agreement within each other but lie significantly below the lattice determinations. At $Q^2 = 1 \text{ GeV}^2$, we find a discrepancy of up to seven standard deviations between the phenomenological estimates and our lattice calculation. Although the tension decreases at larger spacelike momenta, it remains sizeable, reaching about 4.5 standard deviation at $Q^2 = 9 \text{ GeV}^2$. To further investigate the origin of the tension with data-driven results, we have explored whether modifications to the experimental R -ratio $R(s)$ in some specific intervals of the center-of-mass energy $\sqrt{s}$ could reconcile the two determinations. The analysis and its implications are discussed in Appendix C.

A. Rational approximation of the running

To describe the dependence of the HVP on the space-like virtuality in a continuous and analytically tractable way, we adopt a Padé approximant, which provides a rational function representation already employed in previous studies [11,101]. The general form of the Padé approximant used is

$$\bar{\Pi}(-Q^2) \approx R_M^N(Q^2) = \frac{\sum_{j=0}^M a_j Q^{2j}}{1 + \sum_{k=1}^N b_k Q^{2k}}. \quad (5.1)$$

Here, numerator and denominator are polynomials of degree M and N , respectively. Given that the subtracted HVP function is required to vanish at zero momentum transfer $Q^2 = 0$, we impose the condition $a_0 = 0$ during the fit to capture this behavior. We find that polynomials of degree $M = 2$ and $N = 3$ provide an accurate description of the data across the accessible range of Q^2 . Attempts to include higher-order terms result in poorly determined coefficients, reflecting insufficient sensitivity in the data to resolve additional parameters.

The parameters of the Padé *Ansatz* are fitted to a range of Q^2 covering all windows of virtualities, specifically we take the values $Q^2 \in \{0.25, \dots, 12\} \text{ GeV}^2$. For the estimate of the covariance of the fit parameters of the Padé *Ansatz* we use [78], which defines a goodness-of-fit criterion for correlated data that avoids inverting the covariance matrix during minimization while still incorporating it into the error estimate.

The fitted parametrization provides a convenient analytic representation of $\bar{\Pi}^{(\gamma,\gamma)}$ (and analogously $\bar{\Pi}^{(Z,\gamma)}$) in the low-energy Euclidean region accessible nonperturbatively. This representation is intended for interpolation and phenomenological applications, and is not used in the determination of our final results. Isospin-breaking effects, found to be by far subleading at $Q^2 = 9 \text{ GeV}^2$ and approximately constant in the range 5–9 GeV^2 (see Sec. IV G), are omitted from the Padé *Ansatz* reported below. The rational approximation for $\bar{\Pi}^{(\gamma,\gamma)}$ we extract from the fit is

$$\bar{\Pi}^{(\gamma,\gamma)}(-Q^2) \approx \frac{0.102226x + 0.040299x^2}{1 + 2.074048x + 0.342168x^2 - 0.002788x^3}, \quad x = \frac{Q^2}{\text{GeV}^2}, \quad (5.2)$$

together with the covariance matrix

$$\text{cov}^{(\gamma,\gamma)} \begin{bmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 2.14338 & 1.76686 & 50.4751 & 15.7648 & -0.208102 \\ 1.76686 & 3.54353 & 77.2778 & 35.8345 & -0.524144 \\ 50.4751 & 77.2778 & 1907.63 & 748.194 & -9.87013 \\ 15.7648 & 35.8345 & 748.194 & 370.938 & -5.67152 \\ -0.208102 & -0.524144 & -9.87013 & -5.67152 & 0.0988209 \end{bmatrix} \times 10^{-6}. \quad (5.3)$$

For $\bar{\Pi}^{(Z,\gamma)}$ we find

$$\bar{\Pi}^{(Z,\gamma)}(-Q^2) \approx \frac{0.024748x + 0.010791x^2}{1 + 2.106921x + 0.393233x^2 - 0.002901x^3}, \quad x = \frac{Q^2}{\text{GeV}^2}, \quad (5.4)$$

together with the covariance matrix

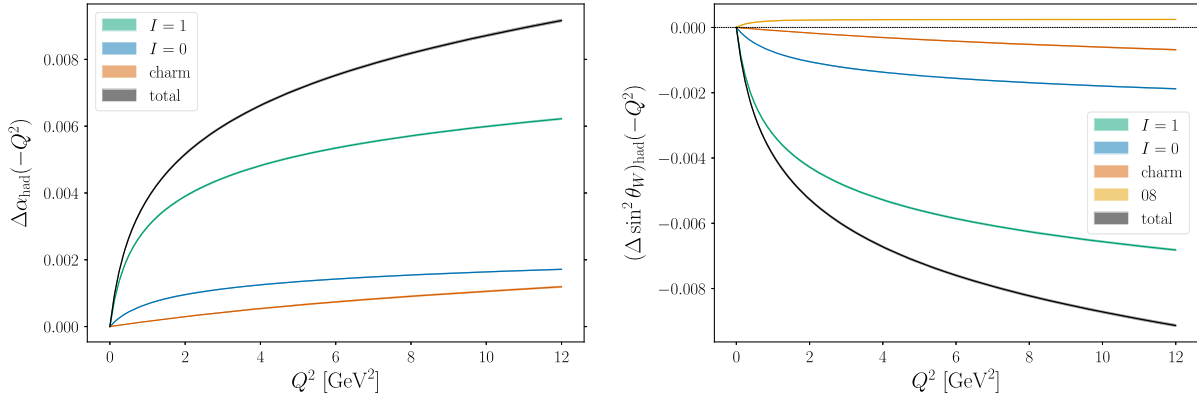


FIG. 11. Total HVP contribution to the running of α (left) and $\sin^2 \theta_W$ (right) as a function of the spacelike momentum transfer Q^2 . Colored bands correspond to the $I = 1$, $I = 0$, charm and, on the right panel, $Z\gamma$ -mixing contributions.

$$\text{cov}^{(Z,\gamma)} \begin{bmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 0.137083 & 0.136607 & 14.816 & 5.19524 & -0.0627519 \\ 0.136607 & 0.295515 & 25.7522 & 12.2846 & -0.156386 \\ 14.816 & 25.7522 & 2467.08 & 1040.29 & -12.2305 \\ 5.19524 & 12.2846 & 1040.29 & 517.894 & -6.82578 \\ -0.0627519 & -0.1563786 & -12.2305 & -6.82578 & 0.103273 \end{bmatrix} \times 10^{-6}. \quad (5.5)$$

We observe that the approximants are in very good agreement with the measured data. The deviation remains at the level of about 0.5 per mil across the entire Q^2 range, well below the overall uncertainty of our results.

The Q^2 -dependence of $\Delta\alpha_{\text{had}}(-Q^2)$ and $(\Delta \sin^2 \theta_W)_{\text{had}}(-Q^2)$ is shown in Fig. 11 as a function of the spacelike momentum Q^2 using the fit results.

B. Running of α to the Z pole

In this subsection, we describe the conversion of our results obtained at spacelike virtualities into an estimate for the hadronic contributions to the running of α at the Z boson mass, $\Delta\alpha_{\text{had}}^{(5)}(M_Z^2)$, which plays a critical role in determining the accuracy of global electroweak (EW) precision fits [8,38,102]. An established approach that provides the connection between spacelike virtualities and the Z pole is the Euclidean split technique [24,25]. This method evaluates the running entirely in the spacelike region, combining the lattice determination at low energies with the perturbative Adler function at high energies. The connection to the timelike point at the Z pole enters only through a small analytic continuation term, computed reliably in perturbation theory. Altogether, this decomposition enables us to express the hadronic contribution to the running at the Z pole as

$$\Delta\alpha_{\text{had}}^{(5)}(M_Z^2) = \Delta\alpha_{\text{had}}^{(5)}(-Q_0^2) + [\Delta\alpha_{\text{had}}^{(5)}(-M_Z^2) - \Delta\alpha_{\text{had}}^{(5)}(-Q_0^2)] + [\Delta\alpha_{\text{had}}^{(5)}(M_Z^2) - \Delta\alpha_{\text{had}}^{(5)}(-M_Z^2)]. \quad (5.6)$$

Here the first term denotes the space-like HVP as defined in Eq. (2.2), evaluated using our lattice results listed in Table V at some threshold energy Q_0^2 .

To compute the second term $\Delta\alpha_{\text{had}}^{(5)}(-M_Z^2) - \Delta\alpha_{\text{had}}^{(5)}(-Q_0^2)$, representing the high-energy contribution to the hadronic running, we adopt the approach developed in [26]: at sufficiently large Q_0^2 values, the Adler function can be computed reliably in perturbative QCD (pQCD), with only small corrections arising from nonperturbative effects [24,103]. We evaluate this term from the public software⁵ AdlerPy [104]. The code fully considers three-loop massive perturbation theory with charm and bottom quark effects included, together with the massless four- and five-loop terms for a better estimation of the high-energy tails. In addition, we include small nonperturbative corrections arising from the quark and gluon condensate as estimated in [105]. The heavy quark masses in the $\overline{\text{MS}}$ scheme and the strong coupling at the Z pole, used as input quantities, are taken from FLAG24 [88]. In particular, for the charm and bottom quark masses, we use the FLAG24 average for $N_f = 2 + 1$, obtained by averaging the results from Refs. [106–111] and [106,107], respectively. The FLAG24 average for the strong coupling $\alpha_{\overline{\text{MS}}}^{(5)}(M_Z)$ is obtained from Refs. [106,112–120].

⁵The code is available at the GitHub repository <https://github.com/rodofer2020/adlerpy>.

In addition, we cross-checked our results using the publicly available pQCDAdler software [121] to evaluate the Adler function, which had already been employed in the Mainz 2022 analysis [11] to estimate the perturbative running. In its original form, however, pQCDAdler does not allow for straightforward modifications of the input parameters. To enable a direct comparison, we updated the code so that the inputs could be set consistently with the FLAG24 values. With this modification, results from pQCDAdler move significantly closer to those obtained with AdlerPy, although perfect agreement is not expected due to the different renormalization schemes implemented in the two frameworks.

In particular, we find a noticeable upward shift in the pQCDAdler results once the input parameters are updated to reflect the FLAG24 specifications. Concerning uncertainties, pQCDAdler estimates the error by simultaneously varying input parameters within maximum bounds. On the other hand, in AdlerPy we implemented a bootstrap procedure to propagate the input uncertainties, which provides more direct control over the error budget. In addition, AdlerPy accounts for truncation errors in the light- and charm-quark contributions to the Adler function.

After computing the Adler function $D(Q^2)$ within this framework, the high-energy contribution to the running can be determined through the following integral

$$[\Delta\alpha_{\text{had}}^{(5)}(-M_Z^2) - \Delta\alpha_{\text{had}}^{(5)}(-Q_0^2)] = \frac{\alpha}{3\pi} \int_{Q_0^2}^{M_Z^2} \frac{dQ^2}{Q^2} D(Q^2), \quad (5.7)$$

with α the QED coupling in the Thomson limit. A summary of our results for several threshold energies Q_0^2 is given in Table VII. The quoted errors for AdlerPy, of the order of 0.2% at $Q_0^2 = 9 \text{ GeV}^2$, are dominated by the uncertainties of the strong coupling at the Z pole and the heavy-quark masses used as input quantities. At values of $Q_0^2 \lesssim 1 \text{ GeV}^2$ the uncertainty from nonperturbative corrections to the Adler function increases substantially, and very small Q_0^2 cannot be probed due to the breakdown of perturbation theory associated with the infrared behavior of the strong coupling.

Finally, the last piece in Eq. (5.6), provides the bridge between the space- and timelike regions at the Z pole, and we quote the pQCD estimate from Jegerlehner [8],

$$\left[\Delta\alpha_{\text{had}}^{(5)}(M_Z^2) - \Delta\alpha_{\text{had}}^{(5)}(-M_Z^2) \right] = 0.000045(2). \quad (5.8)$$

By combining these three pieces, we can now estimate the hadronic contribution to the running of α at the Z pole, using our lattice determination $\Delta\alpha_{\text{had}}^{(5)}(-Q_0^2)$ as input to Eq. (5.6). Figure 12 shows our results for $\Delta\alpha_{\text{had}}^{(5)}(M_Z^2)$ as a function of the momentum threshold Q_0^2 , where the lattice calculation is matched to the perturbative running. We

TABLE VII. Perturbative running $[\Delta\alpha_{\text{had}}^{(5)}(-M_Z^2) - \Delta\alpha_{\text{had}}^{(5)}(-Q_0^2)]$ for various threshold energies Q_0^2 . The second column is a reproduction of the results quoted in [11] using the pQCD code, while the third column is obtained using a modified version of the same code that consistently incorporates updates in the input parameters. The last column is obtained with the AdlerPy package as explained in the main text. Input parameters in the third and last columns are identical.

$Q_0^2 \text{ (GeV}^2\text{)}$	pQCD'	pQCD' updated	AdlerPy
1.0	0.023926(223)	0.024357(200)	0.024042(130)
2.0	0.022489(148)	0.022671(122)	0.022617(71)
3.0	0.021638(128)	0.021758(104)	0.021769(57)
4.0	0.021018(116)	0.021114(94)	0.021115(50)
5.0	0.020525(106)	0.020610(87)	0.020660(44)
6.0	0.020114(99)	0.020193(81)	0.020249(41)
7.0	0.019760(92)	0.019836(77)	0.019895(39)
8.0	0.019449(87)	0.019523(72)	0.019583(37)
9.0	0.019170(82)	0.019244(69)	0.019304(35)
12.0	0.018475(71)	0.018549(59)	0.018605(31)

display results obtained with the three perturbative strategies described above, along with the previous Mainz 2022 [11] determination. The orange band corresponds to the AdlerPy calculation, while the dashed lines represent the two versions of the pQCDAdler code. In all cases, the total errors is obtained by adding the lattice and perturbative errors in quadrature. We find excellent stability of the results for $Q_0^2 > 3 \text{ GeV}^2$, while for $Q_0^2 < 2 \text{ GeV}^2$ an upward trend and loss of precision signal the breakdown of perturbation theory at low energies.

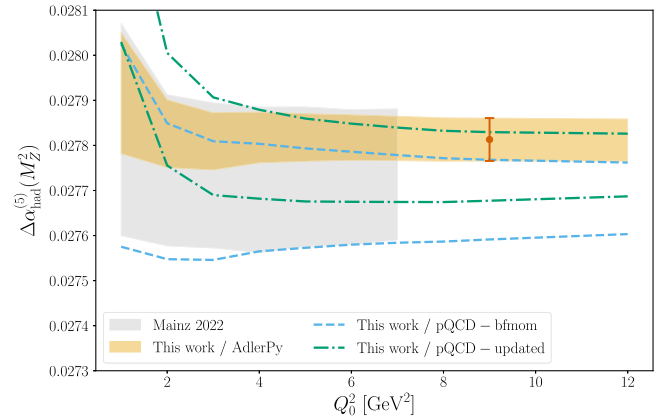


FIG. 12. Five-flavor hadronic contribution to the running coupling at the Z pole, $\Delta\alpha_{\text{had}}^{(5)}(M_Z^2)$, evaluated from Eq. (5.6) using our lattice determination of $\Delta\alpha_{\text{had}}^{(5)}(-Q_0^2)$, shown as a function of the momentum threshold Q_0^2 in the Euclidean split technique. The curves connect direct lattice determinations evaluated at discrete momenta and do not involve any Padé fit procedure.

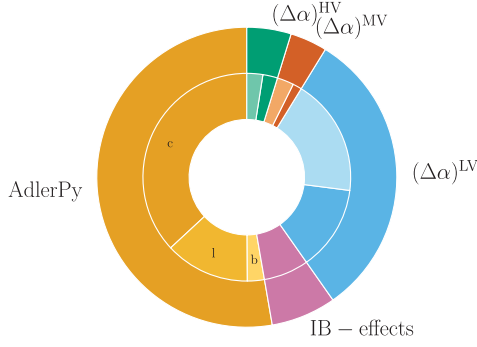


FIG. 13. Breakdown of the squared uncertainty of our final estimate for $\Delta\alpha_{\text{had}}^{(5)}(M_Z^2)$ from Eq (5.9). Each of the five contributions is decomposed into its main sources of uncertainty, shown in the inner circles. For the perturbative running, labeled AdlerPy, the uncertainty is further separated into the charm, light and bottom contributions to the Adler function, indicated by progressively lighter colours. For the three momentum windows, statistical uncertainties are shown in lighter colors, while systematic components are displayed in darker tones.

For our final determination, we adopt the AdlerPy result at $Q_0^2 = 9 \text{ GeV}^2$, where the chiral-continuum extrapolation of our lattice calculation is under good control. Choosing such a high matching scale also reduces the size and error of the perturbative contribution to the total result. Our final quoted value for the hadronic running of the QED coupling at the Z-pole, based on the AdlerPy approach and including isospin-breaking corrections as estimated in Eq. (4.24), is,

$$\Delta\alpha_{\text{had}}^{(5)}(M_Z^2) = 0.027821(34)_{\text{lat}}(35)_{\text{pQCD}}[49]. \quad (5.9)$$

The first error arises from the total uncertainty of our lattice estimate of $\Delta\alpha_{\text{had}}^{(5)}(-9 \text{ GeV}^2)$ as listed in Table VI and including isospin-breaking effects, while the second error accounts for the perturbative evaluation of the high-energy running $[\Delta\alpha_{\text{had}}^{(5)}(-M_Z^2) - \Delta\alpha_{\text{had}}^{(5)}(-Q_0^2)]$ entering Eq. (5.6), and listed in the fourth column of Table VII. Overall, we find a well-balanced distribution of uncertainties between the perturbative running and the lattice contributions. A detailed decomposition of the squared uncertainty for our final estimate at the Z pole is presented in Fig. 13. On the lattice side, the dominant source of error arises from the LV region, while the HV and MV regions, as well as isospin-breaking effects provide smaller contributions.

In Fig. 14 we compare results for $\Delta\alpha_{\text{had}}^{(5)}(M_Z^2)$ obtained from our lattice determination of the HVP, the standard dispersive approach, and global EW fits. The first symbol (red filled square) represents our main result as given in Eq. (5.9) and obtained using the AdlerPy determination of the Adler function. The following two symbols (red filled triangle) represent our results from the two version of pQCDAdler code, while the empty red triangle shows our previous determination [11]. We find tensions at the level of

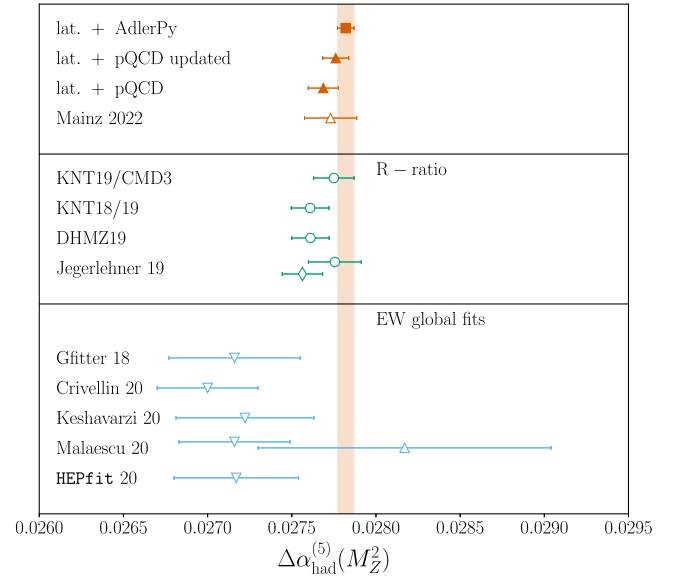


FIG. 14. Summary of results for $\Delta\alpha_{\text{had}}^{(5)}(M_Z^2)$. The first point corresponds to our determination using the perturbative results from AdlerPy listed in Table VII and including isospin-breaking corrections, with the orange band highlighting our final result. The second and third points are obtained using the pQCDAdler software package, followed by the previous Mainz 2022 result [11]. Results based on the standard dispersive approach, where the R ratio is integrated over the full energy range, are shown as green circles. Blue symbols represent values extracted from global EW fits. We refer to the main text for additional details.

$1 - 2\sigma$ with the dispersive evaluations (green points), while the determination of [122] based on CMD-3 data [123,124] is fully compatible with our estimate. The large tension with the dispersive approach observed in Fig. 10 is substantially reduced at the Z pole, due to the additional contribution to the uncertainty arising from the inclusion of the perturbative running.

Turning to global EW fits, we consider results from the Gfitter group [125], from [126] (obtained using the HEPfit code [127]), from [128,129] (obtained from the Gfitter library), and from [130]. In Fig. 14 these are shown as blue open lower triangles, obtained by fitting EW precision data with $\Delta\alpha_{\text{had}}^{(5)}(M_Z^2)$ treated as a free parameter. Such fits typically favor smaller values than both lattice and dispersive determinations, but at a reduced precision. In particular, we find a 2.7σ discrepancy with [126], while the other determinations remain compatible with our estimate within 2σ . Overall, this marks the first hint of a possible tension between lattice evaluations of the HVP contribution to the running of α and results from global EW fits. The point from [129] (blue open upper triangle), where both the Higgs mass M_H and $\Delta\alpha_{\text{had}}^{(5)}(M_Z^2)$ are fitted without priors, yields a larger central value with significantly increased uncertainty, highlighting the stabilizing role of precise M_H input.

C. Running of the electroweak mixing angle

The lattice formulation provides exact control over the quark flavor charges entering the vector currents, which allows us to separate the HVP function $\bar{\Pi}^{(Z,\gamma)}$ into individual valence-quark contributions. This feature removes the need for channel reweighting in hadronic cross-section data, therefore avoiding a key source of systematic uncertainty in phenomenological determinations of the hadronic running of $\sin^2 \theta_W$.

The results presented in Table VI at space-like momenta up to $Q^2 = 12 \text{ GeV}^2$ can be used directly to replace data-driven estimates for studies of the running of the electroweak mixing angle. For comparison, phenomenological estimates are available from Jegerlehner's ALPHAQEDC19 package [9], which provides both $\Delta\alpha_{\text{had}}$ and $\Delta\alpha_{2,\text{had}}$. The latter can be directly compared with our lattice determination for the running of the SU(2) gauge coupling α_2 . The package employs a modified SU(3)-symmetric flavor separation scheme [9,25], designed to better match earlier lattice results. A direct comparison between this estimate, our lattice data, and the previous Mainz 2022 result [11] is given in Fig. 15. We observe a good agreement between our result and Mainz 2022, while the phenomenological estimate still lies systematically below our determinations, by about 3.5% at $Q^2 = 12 \text{ GeV}^2$.

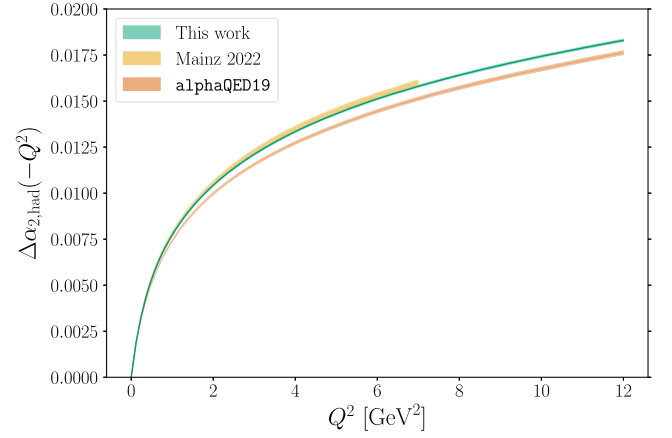


FIG. 15. Hadronic contribution to the running of the weak coupling α_2 from our analysis, shown as a function of the spacelike momentum transfer Q^2 , and compared with lattice results from [11] and with the phenomenological estimate obtained using ALPHAQEDC19 [9].

For completeness, we provide a Padé parametrization of the running of α_2 , without the inclusion of the subleading isospin-breaking effects. The rational approximant we extract from the fit is

$$\Delta\alpha_{2,\text{had}}(-Q^2) \approx \frac{0.018885x + 0.007822x^2}{1 + 2.08887x + 0.3667x^2 - 0.002853x^3}, \quad x = \frac{Q^2}{\text{GeV}^2}, \quad (5.10)$$

together with the covariance matrix

$$\text{cov}^{\Delta\alpha_{2,\text{had}}} \begin{bmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 0.0768698 & 0.0684375 & 10.1641 & 3.35298 & -0.042378 \\ 0.0684375 & 0.142874 & 16.6493 & 7.78842 & -0.105787 \\ 10.1641 & 16.6493 & 2163.42 & 878.042 & -10.8963 \\ 3.35298 & 7.78842 & 878.042 & 432.15 & -6.10704 \\ -0.042378 & -0.105787 & -10.8963 & -6.10704 & 0.0986334 \end{bmatrix} \times 10^{-6}. \quad (5.11)$$

Also here we observe that the approximant is in good agreement with the measured data and reproduces the error band very accurately.

An alternative way to estimate the hadronic contribution to the running of the electroweak mixing angle at low energies is to combine phenomenological evaluations of $\Delta\alpha_{\text{had}}(-Q^2)$ from R -ratio data with lattice input that provides exact flavor separation. From Eqs. (2.15) and (2.16), the difference between $\bar{\Pi}^{(\gamma,\gamma)}$ and $\bar{\Pi}^{(Z,\gamma)}$ is proportional to the isoscalar mixing function $\bar{\Pi}^{(0,8)}$.

Our results for the running of this channel are presented in Fig. 16. For comparison, we also display the results from our earlier Mainz 2022 analysis [11], where we observe a

tension of about 2σ with respect to the updated analysis. The difference can be largely attributed to the inclusions of significantly more ensembles with increased statistics, and an additional ensembles at the finest lattice spacing in the present work. In particular, the telescopic window decomposition allows for a clean identification of cutoff effects in the HV and MV regions that were previously obscured by statistical noise, leading to a more robust determination across all momentum regions. Finally, Fig. 16 also shows a phenomenological model estimate (detailed in Appendix D of [11]), whose uncertainty is dominated by the experimental errors on the ω and ϕ leptonic widths. The model assumes that the disconnected (s, s) and (l, s) diagrams can

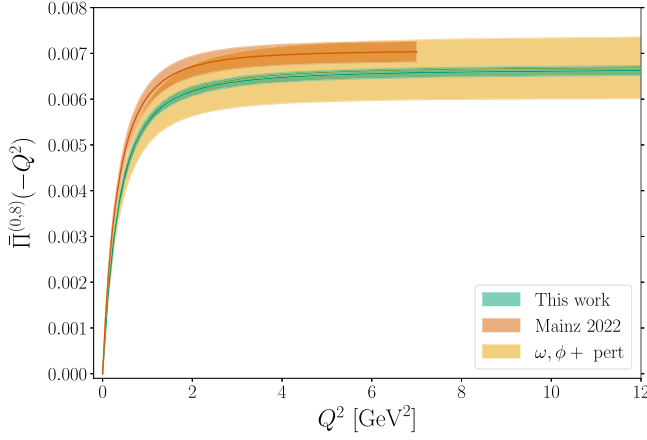


FIG. 16. Lattice results for the HVP mixing function $\bar{\Pi}^{(0,8)}(Q^2)$ as a function of Q^2 , compared with the Mainz 2022 results [11] and the phenomenological model obtained by combining the perturbative QCD contribution and the resonance one from the ω and ϕ mesons.

be neglected, an approximation known to work well in analogous models used to estimate the strange and light isoscalar contributions to $a_\mu^{\text{HVP,LO}}$. Within uncertainties, the model agrees with our results, but the lattice data are significantly more precise across the entire Q^2 range.

Using a rational approximant of order $[2/2]$ for the running of $\bar{\Pi}^{(0,8)}$ over the whole Q^2 range we obtain

$$\bar{\Pi}^{(0,8)}(-Q^2) \approx \frac{0.017001x + 0.021094x^2}{1 + 2.763262x + 3.160956x^2}, \quad x = \frac{Q^2}{\text{GeV}^2}, \quad (5.12)$$

together with the covariance matrix

$$\text{cov}^{(0,8)} \begin{bmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 0.0375666 & 0.0216062 & 3.45356 & 0.620977 \\ 0.0216062 & 1.16285 & 33.1381 & 166.475 \\ 3.45356 & 33.1381 & 1287.57 & 4612.1 \\ 0.620977 & 166.475 & 4612.1 & 24048.8 \end{bmatrix} \times 10^{-5}. \quad (5.13)$$

In the limit of $Q^2 \rightarrow \infty$, the approximant tends to the ratio $\frac{1}{6\sqrt{3}} a_2/b_2 = 64.2(1.1) \times 10^{-5}$, which agrees very well with our value at the largest $Q^2 = 12 \text{ GeV}^2$

$$\frac{1}{6\sqrt{3}} \bar{\Pi}^{(0,8)} = 64.0(1.1) \times 10^{-5}, \quad (5.14)$$

which we quote as our main result for the $I = 0$ $Z\gamma$ -mixing HVP contribution, and is only 3% larger than the result at $Q^2 = 3 \text{ GeV}^2$.

VI. CONCLUSIONS AND OUTLOOK

In this work, we have presented an updated lattice QCD determination of the HVP contribution to the running of the electromagnetic coupling $\Delta\alpha_{\text{had}}^{(5)}(-Q^2)$, and of the electroweak mixing angle in the spacelike region up to $Q^2 = 12 \text{ GeV}^2$. Our analysis benefits from a substantial increase in statistical precision and improved control of systematics, achieved through an expanded ensemble set, including additional fine lattice spacings, and a refined strategy for the chiral-continuum extrapolation.

A key innovation of this study is the implementation of a telescopic window strategy, which enables a clean separation of the HVP into low-, mid- and high-virtuality regions. This decomposition disentangles discretization effects at

short distances from the strong chiral dependence at large Euclidean times. As a result, the chiral-continuum extrapolation is much better controlled, allowing for high-precision determinations across all considered momentum regions. Complementary to this, we have employed an extensive set of noise-reduction techniques, including low-mode averaging, bounding method and explicit reconstruction of the LD correlators from spectroscopy data, further enhancing the signal quality at large Euclidean separations.

Our determination of $\Delta\alpha_{\text{had}}^{(5)}(-Q^2)$ at spacelike values of the momenta exhibits a clear and persistent tension with phenomenological estimates based on the R ratio, reaching up to 7σ at $Q^2 = 1 \text{ GeV}^2$ and remaining above 4σ up to 9 GeV^2 . This confirms, with improved precision, the discrepancy previously observed in our 2022 analysis [11].

When combining our lattice results with the perturbative determination of the Adler function through the Euclidean split technique, we obtain a high-precision determination of the five-flavor hadronic running up to the Z pole. Thanks to several improvements in our calculation, the precision of our final value for $\Delta\alpha_{\text{had}}^{(5)}(M_Z^2)$ exceeds that of state-of-the-art phenomenological determinations by a factor two. The inclusion of the perturbative running, which contributes more than half of the squared total uncertainty, substantially reduces the tension observed in the spacelike region relative to dispersive results, leaving a mild $1 - 2\sigma$ deviation.

Estimates from EW global fits, which have a significantly lower precision compared to our lattice determination, are systematically lower than our final result, but the latter generally remains within their 2σ bands.

For the electroweak mixing angle, we have computed the hadronic contribution to its running in the spacelike region up to $Q^2 = 12 \text{ GeV}^2$. Our lattice framework allows for an exact flavor separation of the mixed HVP function $\bar{\Pi}^{(Z,\gamma)}$, eliminating the need for a model-dependent flavor reweighting present in phenomenological analyses. We find overall good agreement with the ALPHAQEDC19 estimate, though our results are consistently larger by about 4%. In addition, we provide a rational representation of the octet-singlet mixing contribution $\bar{\Pi}^{(0,8)}(-Q^2)$, which asymptotically approaches a constant at large Q^2 values. We observe a tension of about 2σ compared to our previous result [11], primarily driven by the inclusion of an additional, fine lattice spacing and a far more advanced treatment of lattice artifacts.

This improved analysis demonstrates that lattice QCD provides a fully first-principles determination of the hadronic running of the electroweak couplings with competitive precision and well-controlled systematics. Future work will focus on extending the momentum range of the lattice calculation in order to reduce the uncertainty of the perturbative running, incorporating full strong isospin-breaking and QED corrections, and further reducing uncertainties associated with large Euclidean times. One particular direction to be explored is the use of the covariant coordinate-space method [131], which might reduce discretization errors on $\hat{\Pi}(-Q^2)$ at high virtualities [132].

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R -ratio data with covariance matrix, which were used to calculate tabulated data for the running of α at $Q^2 > 0$. We are grateful to our colleagues in the CLS initiative for sharing ensembles. Calculations for this project were performed on the HPC clusters Clover and HIMster-II at the Helmholtz Institute Mainz and Mogon-II and Mogon-NHR at Johannes Gutenberg-Universität (JGU) Mainz, as well as on the GCS Supercomputers JUQUEEN and JUWELS at the Jülich Supercomputing Centre (JSC), HAZELHEN and HAWK at the Höchstleistungsrechenzentrum Stuttgart (HLRS), and SuperMUC at the Leibniz Supercomputing Centre (LRZ). The authors gratefully acknowledge the support of the Gauss Centre for Supercomputing (GCS) and the John von Neumann-Institut für Computing (NIC) by providing computing time via the Projects No. HMZ21, No. HMZ23, and No. HINTSPEC at JSC, as well as Projects No. GCS-HQCD and No. GCS-MCF300 at HLRS and LRZ. We also gratefully acknowledge the scientific support and HPC resources provided by NHR-SW of Johannes Gutenberg-Universität Mainz (Project No. NHR-Gitter). This work has been supported by Deutsche Forschungsgemeinschaft (German Research Foundation, DFG) through the Collaborative Research Center 1660 “Hadrons and Nuclei as Discovery Tools,” under Grant No. HI 2048/1-2 (Project No. 399400745), and through the Cluster of Excellence “Precision Physics, Fundamental Interactions and Structure of Matter” (PRISMA+ EXC 2118/1), funded within the German Excellence strategy (Project No. 390831469). This project has received funding from the European Union’s Horizon Europe research and innovation program under the Marie Skłodowska-Curie Grant Agreement No. 101106243. A. R. was supported by the program Netzwerke 2021, an initiative of the Ministry of Culture and Science of the State of Northrhine Westphalia, in the NRW-FAIR network, funding code NW21-024-A.

DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

APPENDIX A: TABLES

This Appendix contains tables with detailed results for individual gauge ensembles. Table VIII lists the pseudoscalar masses and the gluonic observable t_0/a^2 . Tables IX and X report the HVP contributions for the various channels entering

TABLE VIII. Pseudoscalar masses in lattice units. Estimates of the gluonic observable t_0/a^2 and the two dimensionless hadronic quantities ϕ_2 and ϕ_4 are provided in the last three columns.

id	am_π	am_K	t_0/a^2	ϕ_2	ϕ_4
H101	0.18311(53)	0.18311(53)	2.8442(45)	0.7629(34)	1.1444(51)
H102	0.15439(69)	0.19161(66)	2.8779(52)	0.5488(38)	1.1197(62)
N101	0.12157(62)	0.20161(33)	2.8929(48)	0.3420(29)	1.1117(38)
C101	0.09646(52)	0.20564(32)	2.9108(40)	0.2167(22)	1.0931(33)
C102	0.09660(77)	0.21792(40)	2.8649(49)	0.2139(32)	1.1953(47)
D150	0.05645(76)	0.20874(47)	2.9462(40)	0.0751(20)	1.0645(45)
B450	0.16102(59)	0.16102(59)	3.680(18)	0.7633(52)	1.1449(78)
N451	0.11079(50)	0.17827(28)	3.6890(74)	0.3622(30)	1.1189(39)
N452	0.13548(33)	0.17031(29)	3.6748(91)	0.5396(20)	1.1225(30)
D450	0.08331(70)	0.18399(36)	3.6992(27)	0.2054(33)	1.1046(51)
D451	0.08338(29)	0.19388(20)	3.6640(33)	0.2038(13)	1.2038(20)
D452	0.05963(59)	0.18663(17)	3.7242(59)	0.1059(20)	1.0907(22)
N202	0.13433(38)	0.13433(38)	5.166(14)	0.7458(39)	1.1187(58)
N203	0.11217(28)	0.14378(23)	5.1407(53)	0.5175(24)	1.1088(37)
N200	0.09246(30)	0.15062(23)	5.1622(53)	0.3530(21)	1.1134(33)
D251	0.09198(24)	0.15046(14)	5.1661(55)	0.3496(17)	1.1104(21)
D200	0.06526(22)	0.15649(13)	5.1769(60)	0.1764(11)	1.1024(19)
D201	0.06541(37)	0.16313(21)	5.1366(66)	0.1758(19)	1.1814(31)
E250	0.04240(23)	0.159364(83)	5.2012(25)	0.07481(74)	1.0942(12)
J307	0.10519(24)	0.10519(24)	8.605(24)	0.7616(37)	1.1425(56)
J306	0.08703(33)	0.11333(34)	8.591(14)	0.5206(42)	1.1430(76)
J303	0.06472(25)	0.11965(19)	8.614(13)	0.2887(21)	1.1309(37)
J304	0.06532(19)	0.13172(18)	8.493(11)	0.2899(15)	1.3239(33)
E300	0.04408(10)	0.12399(12)	8.6178(53)	0.13399(60)	1.1269(20)
F300	0.03370(24)	0.12332(16)	8.6566(39)	0.0786(11)	1.0924(30)
J500	0.08152(22)	0.08152(22)	13.965(22)	0.7424(34)	1.1135(51)
J501	0.06591(18)	0.08787(18)	13.972(43)	0.4855(20)	1.1058(34)

TABLE IX. Values of the HVP for the isovector, isoscalar, mixed $Z\gamma$, and charm-connected contributions at $Q^2 = 9 \text{ GeV}^2$, in units of 10^{-5} . Results are shown for both local-local (LL) and local-conserved (LC) discretizations of the vector currents. For the isovector and charm-connected contributions, values obtained using the subtracted kernel are reported. In the isoscalar case, we show results combining the MV and LV momenta regions; in the HV region, the isoscalar channel is extracted from Δ_{I_s} , as described in Sec. III B. All values correspond to the improvement coefficient set 2.

id	$\bar{\Pi}_{\text{sub}}^{(3,3)}$		$\frac{1}{3}\bar{\Pi}^{(8,8)}$		$\frac{1}{6\sqrt{3}}\bar{\Pi}^{(0,8)}$	$\frac{4}{9}\bar{\Pi}_{\text{sub}}^{(c,c)}$	
	(LL)	(LC)	(LL)	(LC)	(LC)	(LL)	(LC)
H101	4503(9)	4933(8)	1256.9(2.6)	1338.7(2.3)	...	90.3(1.4)	581.7(2.5)
H102	4661(11)	5085(10)	1217.7(3.0)	1299.5(2.7)	20.38(72)	98.1(1.3)	588.5(2.5)
N101	4913(12)	5335(10)	1184.1(2.4)	1266.5(2.2)	43.6(1.8)	110.9(1.2)	599.9(2.5)
C101	5102(14)	5520(14)	1170.9(3.3)	1254.4(3.2)	56.3(1.3)	115.6(1.2)	604.0(2.3)
C102	5089(15)	5504(14)	1142.8(4.1)	1226.9(4.1)	62.9(1.5)	...	...
D150	5402(14)	5814(13)	1161.0(7.4)	1243.8(7.4)	64.7(3.6)	...	...
B450	4626(12)	4921(11)	1276.5(3.6)	1334.3(3.5)	...	302.8(1.7)	608.9(2.4)
N451	5051(9)	5341(8)	1226.2(1.9)	1284.5(1.6)	42.00(93)	...	...

(Table continued)

TABLE IX. (Continued)

id	$\bar{\Pi}_{\text{sub}}^{(3,3)}$		$\frac{1}{3}\bar{\Pi}^{(8,8)}$		$\frac{1}{6\sqrt{3}}\bar{\Pi}^{(0,8)}$	$\frac{4}{9}\bar{\Pi}_{\text{sub}}^{(c,c)}$	
	(LL)	(LC)	(LL)	(LC)	(LC)	(LL)	(LC)
N452	4861(16)	5158(15)	1253.2(2.2)	1311.3(2.1)	21.7(1.0)	...	...
D450	5266(7)	5556(5)	1204.2(2.3)	1264.0(2.1)	57.6(1.1)	334.6(1.9)	640.1(2.6)
D451	5278(8)	5571(7)	1178.2(2.5)	1238.0(2.3)	62.7(1.9)	...	...
D452	5455(9)	5742(8)	1191.0(3.8)	1250.8(3.6)	63.3(2.9)	338.7(1.4)	644.1(2.2)
N202	4857(15)	5024(14)	1335.7(4.6)	1370.5(4.6)	...	484.7(2.6)	639.1(2.9)
N203	5016(13)	5184(12)	1290.6(3.1)	1325.8(3.1)	21.96(75)	496.2(2.1)	649.2(2.4)
N200	5169(12)	5333(11)	1255.9(2.6)	1291.4(2.5)	41.1(1.1)	509.2(2.2)	662.4(2.6)
D251	5202(6)	5368(5)	1257.9(1.3)	1293.4(1.1)	40.41(80)	...	...
D200	5420(11)	5585(11)	1236.5(3.4)	1272.6(3.3)	58.4(1.5)	518.6(2.7)	671.9(2.9)
D201	5408(9)	5574(8)	1212.5(3.9)	1249.3(3.9)	62.1(3.4)	...	...
E250	5711(7)	5874(6)	1223.8(4.4)	1260.2(4.2)	69.7(2.1)	525.7(2.3)	679.6(2.6)
J307	4914(10)	4978(10)	1348.8(3.2)	1364.6(3.1)	...	...	...
J306	5100(11)	5163(10)	...	...	...	...	...
J303	5294(15)	5362(21)	1265.4(3.1)	1283.6(4.6)	46.2(1.4)	617.0(2.7)	670.1(2.7)
J304	5290(10)	5354(9)	1219.3(3.2)	1236.0(3.2)	55.1(1.3)	...	...
E300	5591(10)	5684(10)	1253.2(4.9)	1273.0(4.6)	63.7(2.2)	638.3(1.9)	686.3(2.1)
F300	5778(12)	5840(12)	...	...	...	...	...
J500	4909(13)	4934(14)	1354.1(5.7)	1362.3(5.8)	...	614.3(3.6)	628.1(3.2)
J501	5067(13)	5084(15)	1300.1(2.7)	1308.0(3.0)	23.8(1.2)	...	...

TABLE X. Values of Δ_{ls} at $Q^2 = 9 \text{ GeV}^2$ and of $\Delta_{lc}b$ at $Q^2 = 4Q_m^2 = 36 \text{ GeV}^2$, for both local-local (LL) and local-conserved (LC) discretizations of the vector currents. For ensembles at the $SU(3)$ -symmetric point, $\Delta_{ls} = 0$ by construction. Results are given in units of 10^{-5} and correspond to improvement coefficient set 2.

id	$-\frac{1}{3}\Delta_{ls}$		$-\frac{4}{9}\Delta_{lc}b$	
	(LL)	(LC)	(LL)	(LC)
H101	...	...	946.7(1.7)	571.4(1.9)
H102	9.48(26)	8.80(26)	949.5(1.8)	578.4(2.5)
N101	21.08(54)	19.61(50)	945.5(1.7)	579.9(2.0)
C101	27.55(38)	25.69(35)	948.9(1.6)	584.4(1.4)
C102	30.87(23)	28.75(20)	...	...
D150	33.86(31)	31.67(29)	...	...
B450	...	...	767.9(2.2)	566.6(2.5)
N451	16.92(13)	16.34(12)	...	...
N452	8.94(30)	8.60(30)	...	...
D450	24.11(13)	23.26(10)	758.6(1.7)	565.0(1.6)
D451	27.64(14)	26.65(12)	...	...
D452	27.96(14)	26.94(11)	760.4(1.3)	567.2(1.2)
N202	...	...	605.4(2.9)	544.8(2.7)
N203	6.73(46)	6.77(44)	597.8(2.1)	539.7(1.7)
N200	13.69(46)	13.59(44)	593.2(2.3)	535.7(2.1)
D251	14.404(99)	14.300(90)	...	...
D200	19.87(39)	19.68(38)	591.3(2.8)	535.8(2.4)
D201	22.75(15)	22.52(13)	...	...
E250	24.32(48)	24.16(46)	590.9(1.8)	534.6(1.4)
J303	12.78(38)	13.08(42)	492.0(3.0)	507.6(2.7)
J304	17.36(16)	17.46(15)	...	...
E300	18.54(17)	18.74(16)	469.2(1.4)	492.33(89)
J500	...	...	456.6(3.5)	489.9(2.6)
J501	5.05(20)	4.94(24)	...	...

the analysis, while Tables [XI–XIII](#) present the finite-volume correction to the reference volume L_{ref} in the HV, IV, and LV regions, respectively.

TABLE XI. Overview of finite-volume correction to $(m_\pi L)_{\text{ref}}$ in the HV region. The column “HP&MLL” illustrates the results obtained using the Hansen-Patella method for $t < t^*$ and the MLL formalism beyond that threshold, while the column denoted by “HP” shows results from the Hansen-Patella method only. The column “Kaon” shows the correction from the kaon, computed with HP, and already included in the pion correction for $SU(3)$ -symmetric ensembles. Finally, in the last column we list the sum of “HP&MLL” and “Kaon,” which represents the total finite-volume correction entering the isovector contribution. Results are shown for the subtracted kernel with $Q^2 = Q_m^2 = 9 \text{ GeV}^2$ and in units of 10^{-5} .

id	HP&MLL	HP	Kaon	Total
H101	−7.580(77)	−7.554(99)	...	−7.580(77)
H102	−2.337(19)	−2.345(24)	−1.151(15)	−3.488(33)
N101	−2.257(34)	−2.251(36)	−0.2606(40)	−2.518(38)
C101	−0.535(11)	−0.538(13)	−0.015 27(26)	−0.550(11)
C102	−0.580(13)	−0.583(15)	−0.011 10(24)	−0.591(13)
D150	0.805(21)	0.813(25)	0.000 3850(62)	0.806(21)
B450	−5.847(52)	−5.833(67)	...	−5.847(52)
N451	−2.034(18)	−2.037(22)	−0.2959(37)	−2.330(21)
N452	−3.962(32)	−3.960(33)	−1.723(20)	−5.685(52)
D450	−1.187(16)	−1.186(17)	−0.02194(27)	−1.209(16)
D451	−1.170(16)	−1.169(17)	−0.01405(16)	−1.184(16)
D452	0.722(14)	0.727(17)	0.001 455(18)	0.723(14)
N202	−8.124(64)	−8.114(74)	...	−8.124(64)
N203	−3.040(22)	−3.040(30)	−1.292(16)	−4.332(38)
N200	−0.4504(40)	−0.4544(54)	−0.072 83(96)	−0.5232(49)
D251	−2.373(18)	−2.372(19)	−0.2966(32)	−2.670(21)
D200	0.2371(27)	0.2375(35)	0.003 674(46)	0.2408(27)
D201	0.2407(37)	0.2409(46)	0.002 627(36)	0.2433(38)
E250	0.1992(24)	0.1989(27)	0.000 033 14(29)	0.1992(24)
J307	−8.545(89)	−8.539(94)	...	−8.545(89)
J306	−3.208(24)	−3.208(33)	−1.247(16)	−4.455(39)
J303	0.4546(32)	0.4532(43)	0.04117(52)	0.4958(36)
J304	0.2634(18)	0.2618(25)	0.01407(17)	0.2774(20)
E300	0.089 27(67)	0.089 09(84)	0.000 3658(45)	0.089 64(68)
F300	−0.030 05(55)	−0.030 11(61)	−0.000 005 23(16)	−0.030 05(55)
J500	−5.957(37)	−5.941(48)	...	−5.957(37)
J501	0.3040(26)	0.2972(32)	0.1217(19)	0.4256(43)

TABLE XII. Same as Table [XI](#) for the MV region. Results are shown for $Q^2/4 = 2.25 \text{ GeV}^2$ and in units of 10^{-5} .

id	HP&MLL	HP	Kaon	Total
H101	−23.83(32)	−23.75(38)	...	−23.83(32)
H102	−7.659(84)	−7.720(90)	−3.891(58)	−11.55(13)
N101	−7.90(12)	−7.88(12)	−0.922(15)	−8.82(13)
C101	−1.972(41)	−1.974(45)	−0.055 56(87)	−2.027(42)
C102	−2.133(49)	−2.135(53)	−0.040 24(90)	−2.174(50)
D150	3.092(80)	3.087(89)	0.001 481(23)	3.094(81)
B450	−18.03(16)	−17.90(25)	...	−18.03(16)
N451	−7.087(71)	−7.087(78)	−1.038(13)	−8.125(84)
N452	−13.19(13)	−13.19(13)	−5.901(84)	−19.09(21)
D450	−4.395(57)	−4.390(61)	−0.080 62(89)	−4.476(58)

(Table continued)

TABLE XII. (*Continued*)

id	HP&MLL	HP	Kaon	Total
D451	-4.329(57)	-4.323(60)	-0.051 43(57)	-4.380(57)
D452	2.738(55)	2.737(61)	0.005 500(59)	2.743(55)
N202	-25.42(28)	-25.41(30)	...	-25.42(28)
N203	-10.057(98)	-10.06(12)	-4.408(66)	-14.47(16)
N200	-1.527(13)	-1.559(19)	-0.2530(36)	-1.780(17)
D251	-8.344(71)	-8.338(75)	-1.047(12)	-9.391(83)
D200	0.879(11)	0.879(13)	0.013 48(16)	0.892(11)
D201	0.885(14)	0.885(16)	0.009 61(13)	0.895(14)
E250	0.7662(89)	0.765(10)	0.000 128 89(96)	0.7664(89)
J307	-26.86(39)	-26.85(40)	...	-26.86(39)
J306	-10.57(12)	-10.57(14)	-4.253(62)	-14.82(19)
J303	1.573(11)	1.574(16)	0.1444(18)	1.717(13)
J304	0.9158(65)	0.9109(91)	0.048 86(65)	0.9647(70)
E300	0.3346(28)	0.3337(31)	0.001 374(13)	0.3360(28)
F300	-0.1158(20)	-0.1159(23)	-0.000 020 38(60)	-0.1158(20)
J500	-18.53(15)	-18.37(20)	...	-18.53(15)
J501	1.063(12)	0.956(12)	0.4096(70)	1.473(16)

TABLE XIII. Same as Table XI for the LV region. Results are shown for $Q^2/16 = 0.5625 \text{ GeV}^2$ and in units of 10^{-5} .

id	HP&MLL	HP	Kaon	Total
H101	-54.8(1.0)	-53.4(1.7)	...	-54.8(1.0)
H102	-19.16(65)	-19.56(55)	-10.53(48)	-29.68(82)
N101	-26.25(61)	-24.91(77)	-2.77(11)	-29.02(69)
C101	-7.94(19)	-7.99(26)	-0.1806(64)	-8.12(20)
C102	-8.60(27)	-8.60(30)	-0.1280(54)	-8.73(27)
D150	22.14(83)	21.64(88)	0.006 19(19)	22.14(83)
B450	-39.04(63)	-38.0(1.2)	...	-39.04(63)
N451	-21.81(35)	-21.68(52)	-3.04(12)	-24.86(41)
N452	-34.85(57)	-34.47(89)	-16.50(73)	-51.4(1.2)
D450	-19.30(31)	-19.08(43)	-0.2721(88)	-19.58(31)
D451	-19.16(28)	-18.83(44)	-0.1698(56)	-19.33(28)
D452	15.90(38)	15.64(48)	0.020 88(60)	15.92(38)
N202	-57.34(88)	-56.4(1.5)	...	-57.34(88)
N203	-25.96(37)	-25.77(71)	-12.19(54)	-38.15(76)
N200	-4.367(96)	-4.55(13)	-0.722(30)	-5.09(10)
D251	-26.96(34)	-26.67(59)	-3.13(12)	-30.09(44)
D200	3.95(13)	3.84(13)	0.0451(15)	3.99(13)
D201	3.90(11)	3.78(11)	0.0316(10)	3.93(11)
E250	6.138(89)	6.04(12)	0.000 575(14)	6.139(89)
J307	-61.2(1.5)	-60.3(2.1)	...	-61.2(1.5)
J306	-26.99(59)	-26.75(91)	-11.72(50)	-38.71(90)
J303	5.032(95)	4.87(12)	0.421(16)	5.453(97)
J304	2.968(55)	2.830(68)	0.1366(52)	3.105(56)
E300	1.784(18)	1.725(31)	0.005 06(13)	1.789(18)
F300	-0.911(20)	-0.916(26)	-000 0914(35)	-0.911(20)
J500	-40.87(89)	-39.70(96)	...	-40.87(89)
J501	2.725(63)	2.280(60)	1.094(46)	3.819(79)

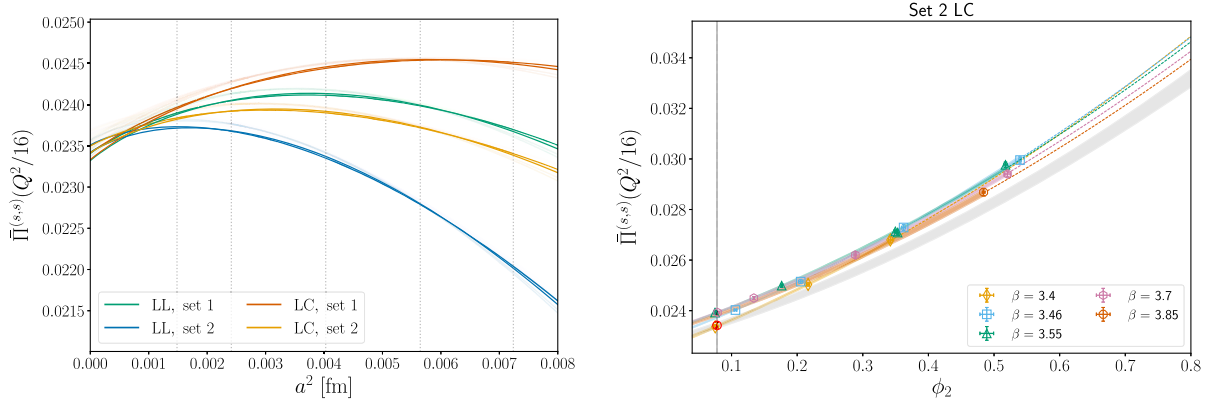


FIG. 17. Illustration of fits to the strange contribution $\hat{\Pi}^{(s,s)}(Q^2/16)$ at $Q^2 = 9 \text{ GeV}^2$ in the LV region. *Left*: continuum limit behavior for the four sets of data based on different improvement schemes and discretizations of the vector current. Each line corresponds to a single fit, with the opacity associated to the weights as given by our model average prescription. *Right*: chiral approach to the physical pion mass for one of the fits with the highest weight. Data points are projected to ϕ_4^{phys} . Colored lines denote the chiral trajectories at finite lattice spacings, while the grey band shows the dependence on ϕ_2 in the continuum.

APPENDIX B: THE STRANGE CONTRIBUTION

In addition to the other channels discussed in the main text, we also determine the strange quark contribution to the running of α in the spacelike momentum range $0.025 \text{ GeV}^2 \leq Q^2 \leq 12 \text{ GeV}^2$. This contribution is phenomenologically relevant since, together with the isovector and isoscalar results, it enables the transition from the strong isospin to the full flavor basis, as advocated in [133]. The strange contribution is evaluated following the same decomposition introduced in Eq. (3.1), with each term computed using the nonsubtracted kernels in Eqs. (3.4) and (3.5) for the HV, MV, and LV virtuality regions, respectively.

For the chiral-continuum extrapolation, we employ fit *Ansätze* analogous to those described in Sec. IV A, with the inclusion of $O(a^4)$ lattice artifacts in the HV region only. In the LV region instead, we observe enhanced sensitivity to chiral effects, though fits with chirally divergent terms are not included in our final analysis. In the same region, higher-order discretization effects of $O(a^3)$ are required to adequately describe the approach to the continuum. A summary of our fits for the LV region is shown in Fig. 17. The left panel illustrates the continuum dependence for the four lattice datasets, while the right panel displays the light-quark mass dependence to the physical point.

TABLE XIV. Contribution to the running for the strange channel at the physical point, in units of 10^{-5} . The first quoted uncertainty corresponds to the statistical error, the second to systematics from model exploration, and the third to the scale-setting error. The final uncertainty in squared brackets is the sum in quadrature of the previous ones.

$Q^2 [\text{GeV}^2]$	$\hat{\Pi}^{(s,s)}$
0.5	2329(12)(7)(30)[33]
1.0	3729(16)(7)(36)[40]
2.0	5451(18)(11)(40)[45]
3.0	6547(19)(15)(42)[49]
4.0	7245(27)(36)(38)[59]
5.0	7858(26)(29)(39)[55]
6.0	8365(25)(24)(39)[52]
7.0	8797(24)(22)(40)[51]
8.0	9173(24)(20)(40)[51]
9.0	9507(23)(19)(41)[50]
12.0	10327(23)(18)(42)[51]

A summary of our results for several values of Q^2 is given in Table XIV. In addition, we provide the analytic solution to the running of the strange component. The rational approximant we extract from the fit is

$$\hat{\Pi}^{(s,s)}(-Q^2) \approx \frac{0.062164x + 0.004783x^2}{1 + 0.764044x + 0.025822x^2}, \quad x = \frac{Q^2}{\text{GeV}^2}, \quad (\text{B1})$$

together with the covariance matrix

$$\text{cov}^{(s,s)} \begin{bmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 0.141398 & 0.0223985 & 1.93356 & 0.119207 \\ 0.0223985 & 0.0661053 & 1.62774 & 0.512815 \\ 1.93356 & 1.62774 & 55.9891 & 11.9994 \\ 0.119207 & 0.512815 & 11.9994 & 4.0041 \end{bmatrix} \times 10^{-5}. \quad (\text{B2})$$

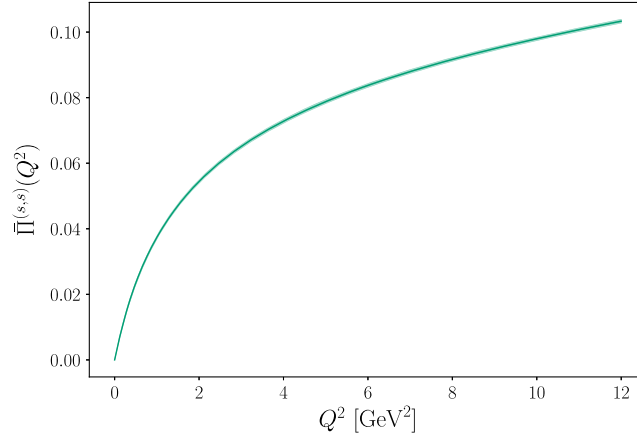


FIG. 18. Strange contribution $\bar{\Pi}^{(s,s)}$ to the running at the physical point as a function of the squared momentum transfer Q^2 .

Also here we observe that the approximant is in good agreement with the directly computed data and reproduces the error band very accurately. Our results for the running of the strange component are displayed in Fig. 18.

For completeness, we also provide the statistical correlations between the different HVP components, $\bar{\Pi}^{(3,3)}$, $\bar{\Pi}^{(8,8)}$, $\bar{\Pi}^{(0,8)}$, $\bar{\Pi}^{(c,c)}$, and $\bar{\Pi}^{(s,s)}$, at representative momenta $Q^2 = 1$ and 9 GeV^2 . These correlations can be useful for phenomenological applications where combined flavor contributions enter global analyses of the hadronic running of the electroweak couplings. At $Q^2 = 1 \text{ GeV}^2$, the correlation matrix reads

$$\text{corr} \begin{bmatrix} \bar{\Pi}^{(3,3)} \\ \bar{\Pi}^{(8,8)} \\ \bar{\Pi}^{(0,8)} \\ \bar{\Pi}^{(c,c)} \\ \bar{\Pi}^{(s,s)} \end{bmatrix} = \begin{bmatrix} 1.0 & 0.377044 & 0.260078 & 0.0615657 & 0.546812 \\ 0.377044 & 1.0 & 0.248847 & 0.0553464 & 0.36846 \\ 0.260078 & 0.248847 & 1.0 & 0.102147 & 0.38444 \\ 0.0615657 & 0.0553464 & 0.102147 & 1.0 & 0.136224 \\ 0.546812 & 0.36846 & 0.38444 & 0.136224 & 1.0 \end{bmatrix}, \quad (\text{B3})$$

while at $Q^2 = 9 \text{ GeV}^2$ we find

$$\text{corr} \begin{bmatrix} \bar{\Pi}^{(3,3)} \\ \bar{\Pi}^{(8,8)} \\ \bar{\Pi}^{(0,8)} \\ \bar{\Pi}^{(c,c)} \\ \bar{\Pi}^{(s,s)} \end{bmatrix} = \begin{bmatrix} 1.0 & 0.342392 & 0.200783 & 0.0333197 & 0.390514 \\ 0.342392 & 1.0 & 0.187271 & 0.0542744 & 0.310952 \\ 0.200783 & 0.187271 & 1.0 & 0.120811 & 0.306019 \\ 0.0333197 & 0.0542744 & 0.120811 & 1.0 & 0.156293 \\ 0.390514 & 0.310952 & 0.306019 & 0.156293 & 1.0 \end{bmatrix}. \quad (\text{B4})$$

APPENDIX C: LATTICE RESULTS FOR $\Delta\alpha_{\text{had}}(-Q^2)$ VERSUS DATA-DRIVEN ESTIMATES USING A MODIFIED R RATIO

As discussed in Sec. V, our lattice results for $\Delta\alpha_{\text{had}}(-Q^2)$ show tensions to those obtained via the data-driven approach,

$$\Delta\alpha_{\text{had}}(-Q^2)|_R = \frac{\alpha Q^2}{3\pi} \int_0^\infty ds \frac{R(s)}{s(s+Q^2)}, \quad (\text{C1})$$

where $R(s)$ denotes the so called R ratio obtained by $e^+e^- \rightarrow \text{hadron}$ cross-section data,

$$R(s) = \frac{\sigma(e^+e^- \rightarrow \gamma^* \rightarrow \text{had})}{4\pi\alpha^2(s)/(3s)}. \quad (\text{C2})$$

In this Appendix, we investigate the origin of the tension.

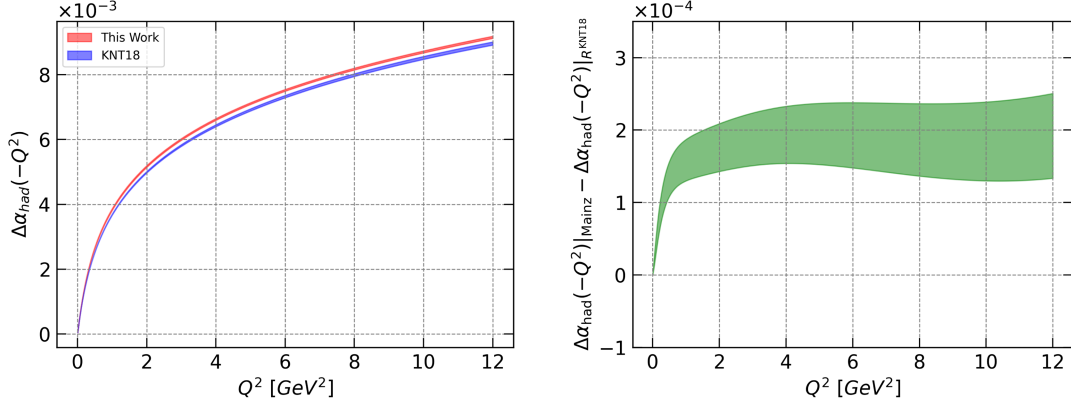


FIG. 19. *Left*: Our lattice result for $\Delta\alpha_{\text{had}}(-Q^2)$ (red band) compared with the one by the data-driven approach (blue band) [6,13]. *Right*: The difference between two methods shown in the left panel.

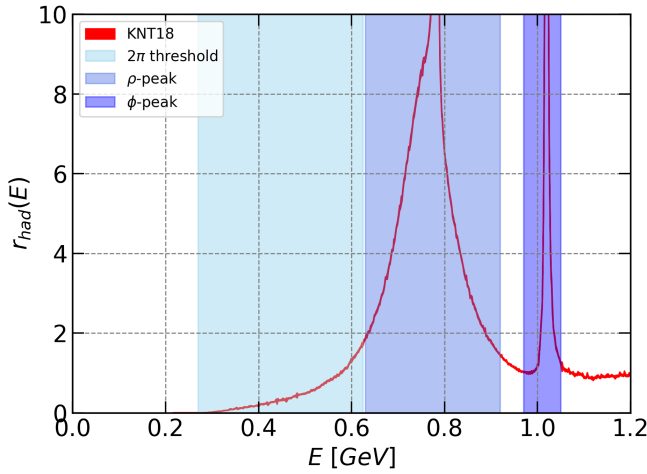


FIG. 20. R ratio provided by KNT18 and used in Refs. [6,13]. The colored region will be modified by 6%, respectively.

In the left panel of Fig. 19, the red band shows our $\Delta\alpha_{\text{had}}(-Q^2)$ obtained from the Padé fits in Eqs. (5.2) and (5.3). This is compared with the one obtained via the data-driven approach (blue band) using the R ratio (KNT18) [6,13]. Across the entire Q^2 range, our results lie systematically above the phenomenological estimates. The right panel displays the difference between the two determinations,

showing that the tension exceeds 5σ in the low-energy region $Q^2 \lesssim 1 \text{ GeV}^2$. Here, we investigate whether such a discrepancy could be accounted for by modifying the R ratio in some specific intervals of the center-of-mass energy $\sqrt{s}$.

We consider a modification for three regions in the R ratio: two-pion threshold ($E = \sqrt{s} \leq 0.63 \text{ GeV}$), ρ meson peak ($E \in [0.63, 0.92] \text{ GeV}$), and ϕ meson peak regions ($E \in [0.97, 1.05] \text{ GeV}$). The selection scheme for the range is illustrated in Fig. 20.

In Fig. 21, we show the difference between our $\Delta\alpha_{\text{had}}(-Q^2)$ and the data-driven results with a 6% modification in each range of the R ratio. In the left panel, the two-pion threshold region is modified as indicated by a suffix ($\tilde{R}_{2\pi\text{-th}}^{6\%}$). The impact of the modification is very limited; the modified data-driven results are not reconciled with our lattice data.

The middle panel of Fig. 21 shows the result after modifying the region around the ρ meson peak (suffix: $\tilde{R}_\rho^{6\%}$), which shows that the tension has disappeared for most of the Q^2 region. A small remaining tension at low Q^2 may come from the fact that $\Delta\alpha_{\text{had}}(-Q^2)|_{\text{Mainz}}$ represents the isoQCD result. Since the vector current correlator in isoQCD is dominated by neutral pions at large distances, it tends to become larger than the real-world counterpart with charged pions (slightly heavier than the neutral pions). If we take into account isospin-breaking corrections, our

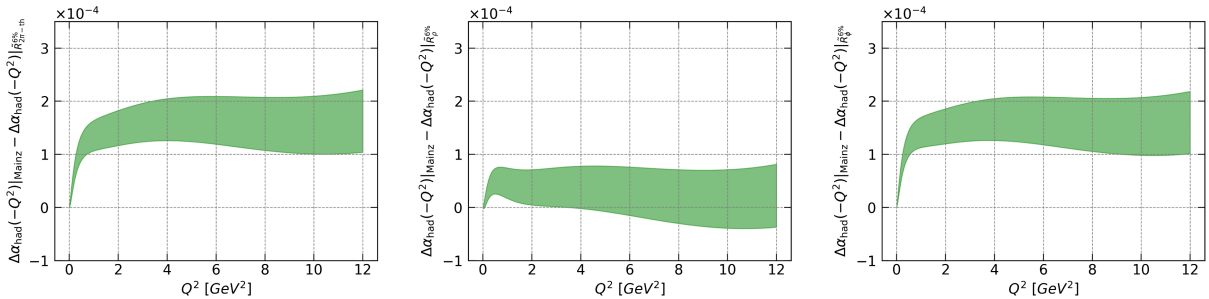


FIG. 21. Difference between our $\Delta\alpha_{\text{had}}(-Q^2)$ and the data-driven results with 6% modifications of two pion threshold (*left*), ρ meson peak (*middle*), ϕ meson peak (*right*) regions in the R ratio.

estimate for $\Delta\alpha_{\text{had}}(-Q^2)|_{\text{Mainz}}$ will be slightly suppressed in the low Q^2 region via the pion mass shift and become more consistent with the data-driven results with the ρ meson peak modification.

Finally, in the right panel of Fig. 21, we show the result with a ϕ meson peak modification. Similarly to the

two-pion threshold modification (left panel), we find a tiny effect that cannot explain the existing tension. In summary, the tension shown in Fig. 19 can be explained by modifying solely the region around the ρ peak in the R ratio. Our results are consistent with the recent work [134] based on the similar analyses.

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