

Extracting edge modes: Reduction of 3D and 2D gravities

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We investigate the boundary reduction of 3D Einstein gravity and Jackiw-Teitelboim gravity into their respective edge mode theories—namely, the Liouville and Alekseev-Shatashvili, and Schwarzian models. By examining the roles of boundary conditions, canonical transformations, and differing formulations—metric versus Chern-Simons—we clarify how physical degrees of freedom become localized at the boundary and resolve several longstanding ambiguities in the reduction procedure.

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In gauge theories on manifolds with boundaries, gauge symmetries are typically broken at boundaries, where would-be gauge redundancies acquire physical meaning and manifest as genuine physical degrees of freedom. These would-be gauge modes or edge modes are frequently overlooked, as their contributions to physical observables are generally considered negligible compared to those from the bulk. Given the recent surge of interest in anti-de Sitter/conformal field theory (AdS/CFT) and defect field theories, it is timely to reexamine the role of edge modes, whose contributions could be critical to understanding boundary-related phenomena.

In topological field theories—such as Chern-Simons (CS) or BF theory—where no local propagating degrees of freedom exist in the bulk, the system's dynamical content is entirely encoded in their edge modes. Notable examples are the free chiral scalar or chiral Wess-Zumino-Witten (WZW) theory in two dimensions, which can be obtained from Abelian and non-Abelian CS theories by Hamiltonian reduction [1–4]. Edge modes in topological gauge theory can be generalized to higher dimensions [7–9].

In this paper, we revisit edge modes arising in three-dimensional gravity and Jackiw-Teitelboim (JT) gravity. These gravity theories are topological and can be reformulated as 3D CS and 2D BF theories, respectively. We regard these low-dimensional models as a stepping stone toward uncovering gravitational edge modes in higher dimensions, and accordingly, we develop a readily extensible framework. Before entering the details, let us review the vast

literature related to this subject.[10] The role of edge modes in gravity was first noted by Balachandran *et al.* [11], who found the boundary charge algebra using the symplectic structure of the edge modes.[12] These results are closely related to the asymptotic symmetries of gravity [16], which were among the precursors of AdS/CFT. In modern terms, these derivations can be obtained most straightforwardly using the CS formulation of gravity. By combining the two copies of $SL(2, \mathbb{R})$ WZW actions—which arise from the Hamiltonian reduction of 3D CS gravity with a specific boundary term—into a single $SL(2, \mathbb{R})$ WZW action and imposing constraints that amount to the asymptotic AdS condition, Coussaert *et al.* showed that one obtains Liouville theory [17],

$$S = \frac{\ell}{32\pi G} \int_{\partial\mathcal{M}} d^2x \left(\frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - 8e^\varphi \right), \quad (1)$$

where φ parametrizes the diagonal element of $SL(2, \mathbb{R})$. For more details, see [18,19] and the recent works [20,21].

Shortly after the work of [17], a new perspective based on the AdS/CFT correspondence began to dominate studies of gravity, where one of the key objects is the on-shell gravity action. Working in the Fefferman-Graham gauge, the on-shell reduction of gravity was systematically analyzed, notably by [22–24], and it was understood how the breaking of radial diffeomorphisms is connected to the Weyl anomaly of the dual CFT—the holographic Weyl anomaly. For the computation of this anomaly, one introduces the radial diffeomorphism in the on-shell gravity action and interprets it as a *wiggling* boundary position. The resulting position variable, a boundary scalar φ , is essentially a Goldstone boson and thus corresponds to the edge mode. In the case of AdS_3 , it was observed that the anomalous energy-momentum tensor of φ takes the same form as the energy-momentum tensor [25–27] of the action,

$$S = -\frac{\ell}{32\pi G} \int_{\partial\mathcal{M}} d^2x \sqrt{-g} \left(\frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - R\varphi \right). \quad (2)$$

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Many other works, such as [28–32], have made similar observations, but it was in [33] where the physical meaning of the Liouville action as part of the gravitational degrees of freedom was clearly articulated. In that work, it was also pointed out that the two derivations of the Liouville action—the Hamiltonian reduction (Coussaert *et al.*) and the on-shell reduction (holography)—display a significant discrepancy. The former includes the potential term e^φ while the latter does not. In fact, there is another crucial difference between the two actions that has not been addressed in the literature—the overall sign of the action.

These two computations use very different procedures, making the origin of the discrepancy elusive. One notable distinction is that they employ different boundary data. The Hamiltonian reduction uses the standard boundary conditions and boundary terms in CS gauge theory, which are rather odd in gravity because the boundary term $\text{Tr}(A \wedge \star A)$ —or, the boundary term that one obtains after integrating out A_0 —involves a flat metric through Hodge star $\star$ in addition to the genuine metric hidden inside A . It also employs an unnatural gauge choice in gravity: rather than solving the algebraic torsion constraint, it solves differential constraints. The computation can be performed whether the boundary lies at a finite distance or at infinity; the asymptotic nature of the boundary is enforced through additional constraints that ultimately reduce the WZW action into Liouville theory. In contrast, the holographic reduction uses the metric formulation with the Gibbons-Hawking-York terms plus counterterms as boundary terms and imposes Dirichlet boundary conditions, and works in the Fefferman-Graham gauge. Here, the on-shell action diverges, and it requires the holographic renormalization—evaluates at a finite cutoff with counterterms, then sends the boundary to infinity. Despite the remarks of [33], these discrepancies went unaddressed for about two decades.

In recent years, gravitational edge modes have attracted renewed interest due to major progress on the SYK-JT duality [34–36]. This two-dimensional theory parallels the three-dimensional case. JT gravity is topological and can be reformulated as an $SL(2, \mathbb{R})$ BF gauge theory. Maldacena *et al.* first showed that JT gravity with a wiggling boundary can be reduced to the Schwarzian theory [37]; a partial on-shell evaluation of BF theory later yielded the same conclusion [38]. Although the analyses of JT gravity are closely analogous to those of 3D gravity, no discrepancy analogous to the potential term identified in [33] has been noted.

More recently, Cotler and Jensen applied this reduction scheme to 3D gravity [39]. Using Hamiltonian reduction, they imposed the asymptotic AdS condition separately on each chiral WZW action instead of merging them, obtained the Alekseev-Shatashvili-type (AS) action [40] and argued that it is the correct edge-mode action of 3D gravity rather than the Liouville theory.

In both the Schwarzian and AS theories, the temperature-sensitive quadratic term is analogous to the Liouville potential; these terms are introduced through a reparametrization.

Later, it was understood that this term was originally absent because the Gauss decomposition was compatible only with the zero-temperature case. By using the Iwasawa decomposition, it was shown [41] that the temperature-dependent quadratic term can be obtained directly from the BF or CS analysis. This observation raises another puzzle; the original derivation [17] of the Liouville theory with potential term actually used the Gauss decomposition, seemingly at odds with the AS theory [39].

To analyze all of these issues in a systematic manner, we adopt the following methods. First, we work with the radial Hamiltonian method, in which the radial coordinate r is treated as the Hamiltonian evolution parameter. This brings the gravity action into an Arnowitt-Deser-Misner-like form, but the role of time t is played by r . This method is commonly used in the on-shell reduction of gravity theories in AdS/CFT (see, e.g., [24,42]) and is sometimes generalized to a Hamilton-Jacobi formulation [43]. In this setup, issues of boundary data are controlled by canonical transformations, since the boundary terms are nothing but generating functions of canonical transformations [44,45]. The role of canonical transformations in holographic renormalization, i.e., the counterterms, was explored in [46,47]. Our viewpoint is a bit more general: we regard the canonical transformation as a tool to control the most general boundary conditions and boundary terms of a theory—that is, the most general edge-mode theory. The related issues require more detailed explanations and will be treated in a separate paper.

Here, we take the following Hamiltonian action as our starting point.

$$\begin{aligned} S_{\text{ren}}[A] &= S_{\text{CS}}[A] - \frac{\ell}{16\pi G} \int_{\partial\mathcal{M}} (\epsilon_{\alpha\beta} e^\alpha \wedge f^\beta - b \wedge \omega) \\ &= -\frac{\ell}{8\pi G} \int_{\mathcal{M}} \text{dr} [\epsilon_{\alpha\beta} f^\alpha \wedge \partial_r e^\beta - b \wedge \partial_r \omega \\ &\quad + \epsilon_{\alpha\beta} (n^\alpha (\text{D}f^\beta - b \wedge f^\beta) - m^\alpha (\text{D}e^\beta + b \wedge e^\beta)) \\ &\quad - n(\text{d}\omega + \epsilon_{\alpha\beta} f^\alpha \wedge e^\beta) - m(\text{d}b + e^\alpha \wedge f_\alpha)], \quad (3) \end{aligned}$$

where $\text{D}V^\alpha = \text{d}V^\alpha + \omega \wedge \epsilon^{\alpha\beta} V_\beta$ is the 2D Lorentz covariant derivative and $\alpha, \beta = 0, 1$. Here, the level of Chern-Simons action is chosen as $k = \frac{\ell}{4\pi G}$. The $so(2, 2)$ connection A is decomposed as

$$\begin{aligned} A &= (e^\alpha + n^\alpha \text{d}r) P_\alpha + (f^\alpha + m^\alpha \text{d}r) K_\alpha \\ &\quad + (\omega + m \text{d}r) J + (b + n \text{d}r) D, \quad (4) \end{aligned}$$

in terms of the generators of conformal algebras with nontrivial Lie brackets,

$$\begin{aligned} [P_\alpha, D] &= +P_\alpha, & [P_\alpha, J] &= \epsilon_{\alpha\beta} P^\beta, \\ [K_\alpha, D] &= -K_\alpha, & [K_\alpha, J] &= \epsilon_{\alpha\beta} K^\beta, \\ [P_\alpha, K_\beta] &= \eta_{\alpha\beta} D + \epsilon_{\alpha\beta} J. \quad (5) \end{aligned}$$

Here, e^α , f^α , ω , and b are one-forms on constant- r hyperplanes, while the Lagrange multipliers n^α , n , m^α , and m are scalars. Note here that the system of constraints is nothing but the frame formulation of conformal geometry, see, e.g., [48] for a review.

Note that our boundary term differs from $\text{Tr}(A \wedge \star A)$ used in the usual CS formulation: this term preserves $so(2,2)$ gauge symmetry but breaks diffeomorphisms, whereas ours preserves diffeomorphisms but breaks a part of $so(2,2)$. However, for the flat boundary metric $e^\alpha = e^r dx^\alpha$, the two boundary terms coincide. Therefore our boundary term can be viewed as a geometric generalization of the usual boundary term of CS formulation.

As the subscript of S_{ren} hints, our action already contains all necessary counterterms via a suitable canonical transformation that brings the gravity action into the above form. The variation of the action gives

$$\delta S_{\text{ren}} = (\text{EoM}) - \frac{\ell}{8\pi G} \int_{\partial\mathcal{M}} \epsilon_{\alpha\beta} f^\alpha \wedge \delta e^\beta - b \wedge \delta \omega, \quad (6)$$

and we choose the boundary conditions $\delta e^\alpha = 0$ and $b = 0$. Under the $so(2,2)$ gauge symmetry,

$$\delta S_{\text{ren}} = \frac{\ell}{8\pi G} \int_{\partial\mathcal{M}} \epsilon_{\alpha\beta} \kappa^\alpha D e^\beta + \sigma d\omega, \quad (7)$$

where κ^α and σ are, respectively, the K_α and D gauge transformation parameters, and hence those parts of gauge symmetry are broken at the boundary. Under a finite κ^α transformation, b transforms as $b \rightarrow b + e^\alpha \kappa_\alpha$. Since we set $b = 0$ at the boundary, even though κ_α is broken at the boundary, we still have the (optional) freedom to gauge-fix $b = 0$ in the bulk using the unbroken bulk part of the κ_α gauge symmetry; this amounts to imposing the Fefferman-Graham gauge. In the end, the D gauge symmetry is broken at the boundary, and the corresponding gauge parameter σ becomes the edge mode.

To isolate the edge-mode action, we use the Stueckelberg transformation $A = g^{-1}(d + \tilde{A})g$, which amounts to rendering a flat boundary into a wiggling one. With $g = e^{\sigma D} e^{\kappa^\alpha K_\alpha}$ at the boundary, we require the boundary condition $b = 0$ to be preserved even after the Stueckelberg shift: $\tilde{b} = 0$. This fixes κ^α in terms of σ as $\kappa_\alpha[\sigma] = -e^{-\sigma} \tilde{e}_\alpha{}^\mu \partial_\mu \sigma$ and yields

$$\begin{aligned} e^\alpha &= e^\sigma \tilde{e}^\alpha, \\ f^\alpha &= e^{-\sigma} \tilde{f}^\alpha + \tilde{D}\kappa^\alpha - \frac{1}{2} e^\sigma \kappa^\beta \kappa_\beta \tilde{e}^\alpha, \\ \omega &= \tilde{\omega} + e^\sigma \epsilon_{\alpha\beta} \tilde{e}^\alpha \kappa^\beta. \end{aligned} \quad (8)$$

Using the finite gauge transformation of CS action,

$$\begin{aligned} S_{\text{CS}}[A] &= S_{\text{CS}}[\tilde{A}] - \frac{\ell}{16\pi G} \int_{\partial\mathcal{M}} \text{tr}(g^{-1} dg \wedge \tilde{A}) \\ &\quad - \frac{\ell}{48\pi G} \int_{\mathcal{M}} \text{tr}(g^{-1} dg)^3, \end{aligned} \quad (9)$$

we find

$$S_{\text{ren}}[A] = S_{\text{ren}}[\tilde{A}] + \frac{\ell}{16\pi G} \int_{\partial\mathcal{M}} \sqrt{-\tilde{g}} (-\tilde{g}^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma - \sigma \tilde{R}), \quad (10)$$

where we solved the torsion constraint $\tilde{D}\tilde{e}^\alpha = 0$ to write the final edge-mode action in metric form. By construction, the above action is invariant, aside from all other gauge transformations, under the D Stueckelberg transformation,

$$\sigma \rightarrow \sigma + \varsigma, \quad \tilde{A} \rightarrow g_\varsigma^{-1}(d + \tilde{A})g_\varsigma, \quad (11)$$

with $g_\varsigma = e^{\varsigma D} e^{K_\alpha \kappa^\alpha[\varsigma]}$.

If we solve the bulk equations for $\tilde{A}$, the remaining action is the Liouville action but without exponential potential term. This reproduces the result obtained in the holographic method [26] and leads to the puzzle between the two methods as remarked in [33] and the beginning of this paper.

In this paper, we adopt a new boundary term in the CS formulation to bridge the two methods. We use the Stueckelberg formulation, which is equivalent to introducing a wiggly-boundary; however, the distinction between the original field A and its Stueckelberg-shifted counterpart $\tilde{A}$ is kept explicit. This allows us to make the following observations.

First, the boundary condition should be imposed on the original A rather than on $\tilde{A}$. Accordingly we must express $\tilde{e}^\alpha$ in terms of e^α and σ at the boundary. This brings the action into the form

$$S_{\text{ren}}[A] = S_{\text{ren}}[\tilde{A}] + \frac{\ell}{16\pi G} \int_{\partial\mathcal{M}} \sqrt{-g} (g^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma - \sigma R), \quad (12)$$

whose only difference from (10) is that the kinetic term for the edge mode has a positive sign. This sign is the correct one from the unitarity perspective as well as in comparison with the Hamiltonian reduction result. Thus, imposing the boundary condition on the correct field fixes the sign but does not generate a potential term.

Second, the on-shell reduction of $S_{\text{ren}}[\tilde{A}]$ becomes more delicate due to the boundary condition imposed on A . If we solve all the bulk equations of motion in $\tilde{F} = 0$ and extend them to boundary, the boundary condition subjects also the edge mode σ on-shell. In order to keep it off-shell—which is important for the quantization of the theory—we should not impose $\tilde{F}_{0r} = 0$ and the J part of $\tilde{F}_{01} = 0$, at least close enough to the boundary. Gauge fixing the Lagrange multipliers $\tilde{n}^\alpha, \tilde{m}^\alpha, \tilde{m}$ to zero and $\tilde{n} = 1$ —which amounts to imposing the asymptotic AdS condition—the equations $\tilde{F}_{1r} = 0$ become as simple as

$$\begin{aligned} \partial_r \tilde{e}_1^\alpha &= \tilde{e}_1^\alpha, & \partial_r \tilde{f}_1^\alpha &= -\tilde{f}_1^\alpha, \\ \partial_r \tilde{b}_1 &= 0, & \partial_r \tilde{\omega}_1 &= 0. \end{aligned} \quad (13)$$

With this gauge choice and the equations (13), the bulk part of the action is reduced to $S_{\text{ren}}[\tilde{A}] = \frac{\ell}{8\pi G} \int_{\partial\mathcal{M}} d^2x \epsilon_{\alpha\beta} \tilde{f}_1^\alpha \tilde{e}_0^\beta$, up to the linearly divergent term in r proportional to the total-derivative term $\int_{\partial\mathcal{M}} d\tilde{\omega}$. For simplicity, we impose a flat two-dimensional metric boundary condition, $e^\alpha = e^r dx^\alpha$, then the action becomes $S_{\text{ren}}[\tilde{A}] = -\frac{\ell}{8\pi G} \int_{\partial\mathcal{M}} d^2x \mu(x) e^{-\sigma}$ with $\mu(x) = e^r \tilde{f}_1^1(r, x)$. Using the Stueckelberg symmetry,

$$\mu \rightarrow e^{-\varsigma} \left(\mu - (\partial_1)^2 \varsigma + \partial_1 \varsigma \partial_1 \varsigma - \frac{1}{2} \partial^\rho \varsigma \partial_\rho \varsigma \right), \quad (14)$$

one can set $\mu(x)$ to a constant which cannot be removed as it would require a physics-changing *large* Stueckelberg transformation. Eventually, we end up with the Liouville action,

$$S_{\text{ren}}[A] \simeq \frac{\ell}{16\pi G} \int_{\partial\mathcal{M}} d^2x [\partial_\mu \sigma \partial^\mu \sigma - 2\mu e^{-\sigma}], \quad (15)$$

where $\simeq$ stands for the fact that we solved a part of bulk equations of motion [49].

Next, we consider a different reduction procedure. The decomposition $so(2, 2) = sl(2, \mathbb{R})_L \oplus sl(2, \mathbb{R})_R$, where each $sl(2, \mathbb{R})_{L/R}$ is generated by $P_\pm, K_\mp, D_\pm = D \pm J$, can lead to the decomposition of the action, $S_{\text{ren}}[A = A^L + A^R] = S_{\text{ren}}[A^L] - S_{\text{ren}}[A^R]$, thanks to the boundary condition $b = 0$. Focussing the half of the action $S_{\text{ren}}[A^{L/R}]$, we introduce Stueckelberg fields for all three gauge symmetries while imposing two conditions $\tilde{b}_1 \pm \tilde{\omega}_1 = 0$ and $\tilde{e}_1^\pm = \pm e^r$. Note that these conditions are only about the dx^1 components. This leaves one edge mode at each half of the action. We introduce this edge mode as the Stueckelberg fields, [51]

$$g_\pm = e^{\pm \varsigma} (e^r P_\pm + e^{-r} \mathcal{L}_\pm K_\mp) e^{\sigma_\pm D_\pm} e^{e^{-r} \kappa^\mp K_\mp}, \quad (16)$$

satisfying the conditions $\tilde{b}_1 \pm \tilde{\omega}_1 = 0$ and $\tilde{e}_1^\pm = \pm e^r$

$$\begin{aligned} e^{-\sigma_\pm} &= 1 \pm \partial_1 \varsigma^\pm \\ e^r f^\pm_1 &= e^{-\sigma_\pm} \mathcal{L}_\pm (\mp 1 + \partial_1 \varsigma^\pm) + \partial_1 \kappa^\pm \mp (\kappa^\pm)^2, \\ 0 &= \partial_1 \sigma^\pm \mp 2\kappa^\mp. \end{aligned} \quad (17)$$

These determine $\kappa^\pm$ in terms of $\partial_1 \sigma^\pm$ and $\sigma^\pm$ in terms of $\partial_1 \varsigma^\pm$. Substituting these into the action $S_{\text{ren}}[A^{L/R}]$ yields

$$\begin{aligned} S_{\text{ren}}[A^{L/R}] &= S_{\text{ren}}[\tilde{A}^{L/R}] - \frac{\ell}{8\pi G} \int_{\partial\mathcal{M}} d^2x \tilde{f}_1^\mp \tilde{e}_0^\pm \\ &\mp \frac{\ell}{16\pi G} \int_{\partial\mathcal{M}} d^2x \left[\frac{\partial_\mp \partial_1 \chi^{L/R} \partial_1^2 \chi^{L/R}}{(\partial_1 \chi^{L/R})^2} \right. \\ &\left. + 4\mathcal{L}_\mp \partial_\mp \chi^{L/R} \partial_1 \chi^{L/R} \right], \end{aligned} \quad (18)$$

where the second line is the AS-like action $\pm S_{\text{AS}}^{L/R}[\chi^{L/R}]$ found in [39], and $\chi^{L/R} = \pm\varphi + \xi^\pm$ is the edge mode.

Let us remark that this action already contains a potential term, namely the $\mathcal{L}_\mp$ -dependent quadratic term, even though we have not yet reduced the bulk part. Moreover, the on-shell reduction of the first line vanishes upon substituting the pure-gauge solution (16); using a more general solution produces an additional edge mode that eventually combines with $\chi^{L/R}$. Finally, we confirm that the 3D gravity reduces to two copies of AS theories

$$S_{\text{ren}}[A] \simeq S_{\text{AS}}^L[\chi^L] - S_{\text{AS}}^R[\chi^R]. \quad (19)$$

We have seen that the 3D gravity can be reduced both to Liouville and AS theory by Stueckelberg shift and solving a part of bulk equations. Since we can always work with the unitary gauge where Stueckelberg fields vanish, the reduction to Liouville and AS theory is possible if one solves just the right part of bulk equations, which in fact contains the constraint in the usual temporal Hamiltonian. Therefore our derivation should be compatible with the Hamiltonian reduction: this relation is straightforward for AS reduction, but rather involved for Liouville as it requires the steps which breaks manifest 2D Lorentz. In our derivation, the potential term appears only for the background bulk solution with nontrivial boundary topology. This is in contrast with [17].

In JT gravity, which is the 2D analog of the 3D gravity, the above derivations can be repeated in an analogous but simpler way. The starting point is the action,

$$\begin{aligned} S_{\text{ren}}[A, \Phi] &= S_{\text{BF}}[A, \Phi] - \frac{\ell}{8\pi G} \int_{\partial\mathcal{M}} (\phi_P f - \phi_D b) \\ &= \frac{\ell}{8\pi G} \int_{\mathcal{M}} dr \left[\phi_K \partial_r e - f \partial_r \phi_P - \frac{1}{2} b \partial_r \phi_D \right. \\ &\quad \left. + n(d\phi_K + \phi_K b - \phi_D f) + m(d\phi_P - \phi_P b + \phi_D e) \right. \\ &\quad \left. + \frac{1}{2} l(d\phi_D + 2\phi_P f - 2\phi_K e) \right], \end{aligned} \quad (20)$$

where $S_{\text{BF}}[A, \Phi] = \frac{\ell}{16\pi G} \int_{\mathcal{M}} \text{tr}(\Phi F)$ with $F = dA + A \wedge A$, and the $sl(2, \mathbb{R})$ gauge connection A and scalar field Φ are given by

$$\begin{aligned} A &= (e + ndr)P + (f + mdr)K + (b + ldr)D, \\ \Phi &= \phi_P P + \phi_K K + \phi_D D. \end{aligned} \quad (21)$$

We impose the boundary condition $\delta e = 0 = \delta \phi_P$ and $b = 0$ and we find K and D symmetries are broken at the boundary.

Similar to the derivation of the Liouville action, we introduce Stueckelberg fields for the broken symmetry via $g = e^{\sigma D} e^{\kappa K}$ and impose $\tilde{b} = 0$ at the boundary. This fixes $\tau = \frac{1}{2} e^{-1} \partial_l \sigma$ and leads to relations analogous to the first

two of (8). With this we find

$$S_{\text{ren}} = S_{\text{BF}}[\tilde{\Phi}, \tilde{A}] - \frac{\ell}{8\pi G} \int dt \frac{\phi_P}{2e_t} \left(\frac{1}{2} (\partial_t \sigma)^2 + \partial_t^2 \sigma + \tilde{f}_t e^{-\sigma} \right). \quad (22)$$

Thus, even before reducing the action we obtain a Liouville-like edge-mode action with a potential term. Solving the full bulk equations of motion would put the edge mode on shell. We therefore impose only the equation $\tilde{F} = 0$, which suffices to reduce the bulk term to zero.

A procedure analogous to the derivation of the AS-like action is also available. We introduce all three Stueckelberg fields ξ , σ , κ and impose the two conditions $\tilde{e}_t = e_t$ and $\tilde{b} = 0$. Setting $e_t = e^r$ and $\tilde{f} = e^{-r} \mathcal{L}$ yields relations analogous to (17) and produces a Schwarzian action,

$$S_{\text{ren}}[A, \Phi] \simeq -\frac{\ell}{8\pi G} \int dt \varphi_P \left(\frac{\ddot{\chi}}{\dot{\chi}} - \frac{3}{2} \left(\frac{\ddot{\chi}}{\dot{\chi}} \right)^2 - 2\mathcal{L}\dot{\chi}^2 \right), \quad (23)$$

where $\chi = t + \xi$ with boundary condition $\phi_P = 2e^r \varphi_P$. This Schwarzian action differs from [34,38] by an overall sign, a convention that is not physical because it can be absorbed into a redefinition of the dilaton boundary value ϕ_P . Nevertheless, the overall sign issue is analogous to three-dimensional Chern-Simons gravity, where the corresponding

sign choice is physically meaningful. Specifically, the wiggly-boundary prescription of [34] is equivalent to imposing boundary conditions on Stückelberg-shifted variables rather than on the original fields, and the BF formulation of [38] imposes the partial on-shell condition $F = 0$ instead of solving the Hamiltonian constraints. Because these procedures differ from the derivation used here, a different overall sign is obtained. See also [52] for other discussion of subtleties in deriving the Schwarzian action from a wiggling boundary, where the boundary conditions are still imposed on Stückelberg-shifted variables.

We expect our analysis to make manifest and to clarify subtleties in the derivation of gravitational edge-mode actions in three and two dimensions, and to provide insight into edge-mode descriptions of higher-dimensional gravity and higher-spin theories.

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