

Stress stability criterion of $U(1)$ gauged nontopological solitons in the $(3+1)$ -dimensional $O(3)$ sigma model

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We study the energy-momentum tensor of the spherically symmetric nontopological solitons of the $O(3)$ nonlinear sigma model with a standard kinetic term and with a symmetry breaking potential in $(3+1)$ -dimensional flat spacetime. We evaluate the distributions of the corresponding energy density, shear forces, and pressure and study the stability criteria for these solutions. We argue that the presence of domains with negative energy density and violation of the energy conditions most likely do not lead to destabilization of solitons.

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I. INTRODUCTION

Much attention has been devoted over the last few decades to the study of soliton solutions that arise in various nonlinear systems. They represent regular localized lumps of energy. These field configurations emerge in many different areas of physics, e.g., physics of condensed matter [1,2], solid-state physics [3], nonlinear optics [4], biophysics [5], and cosmology [6]. In many cases, existence of the solitons is related to topological properties of the system; see, e.g., [7,8]. There are two different classes of soliton configurations: topological and nontopological solitons. While topological solitons are characterized by a certain topological invariants related to the vacuum structure of the theory, nontopological solitons possess a conserved Noether charge related to some symmetry of the Lagrangian of the model. However, there are theories that support both types of solitons.

An interesting example of such a model is the well-known nonlinear sigma model [9] and its modifications. The field of the $O(3)$ model is a triplet $\phi^a = (\phi^1, \phi^2, \phi^3)$ restricted to a two-sphere, $(\phi^a)^2 = 1$, and it asymptotes to the vacuum $\phi_\infty^a = (0, 0, 1)$. In the simplest $(2+1)$ -dimensional case, corresponding soliton solutions belong

to an equivalence class characterized by the homotopy group $\pi_2(S^2) = \mathbb{Z}$. Soliton solutions of this conformally invariant self-dual planar theory without a potential were constructed in [10].

Further, in three spacial dimensions there also are topological soliton solutions of the $O(3)$ model classified by the linking number, the first Hopf map $\phi^a: S^3 \mapsto S^2$. However, according to Derrick's theorem [11], to support stable soliton configurations in $(3+1)$ dimensions, the Lagrangian of the $O(3)$ model should also include, apart from the usual kinetic term, a Skyrme-type term that is quartic in derivatives [12–14]. In addition, a potential term, which does not contain derivatives, can be included in the Lagrangian. Although the structure of the potential term of the Faddeev-Skyrme model is largely arbitrary, its particular choice determines different ways of symmetry breaking; see, e.g., [15–17].

Another mechanism of stabilization of solitons, which allows one to circumvent Derrick's theorem, is related to internal rotations of the configuration. Such time-dependent solitons were constructed both in the planar $O(3)$ model with a symmetry breaking potential [18–20] and in the CP^2 model [21,22]. Evidently, there is a certain similarity between these internally rotating topological solitons [23] and Q balls, which are time-dependent lumps of a complex scalar field with a stationary oscillating phase [24–27].

Note that, in general, internally rotating stationary solitons cannot be approximated as a rigid body. Moreover, topological arguments alone cannot guarantee stability of solitons; they can shrink or expand unless balanced by other physical mechanisms and interactions.

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The deformations of topological solitons may result in instability of the configuration with respect to decay into constituents [28,29]. Depending on the structure of the potential and the values of the parameters of the model, there can be stable, metastable, and unstable solutions.

It was pointed out that static nontopological solitons may exist in $(3 + 1)$ dimensions in the $O(3)$ sigma model with a scalar potential that includes a difference of two positive-definite terms [30]. Since such potentials are not being positive everywhere, one expects violation of the energy conditions in the system. A prototype of such potential is the well-known Lennard-Jones potential, which combines long- and short-range interactions.

Further modification of the potential allows for the existence of internally rotating nontopological $O(3)$ solitons [31], making such configurations similar to the usual Q balls. Such potential breaks the $O(3)$ symmetry of the model to the $SO(2)$ subgroup; furthermore, by analogy with the similar situation in the planar baby Skyrme model [32–35], one can gauge this Abelian subgroup and add the Maxwell term to the Lagrangian of the system [31,36–43]. The presence of a long-range Coulomb field in the system can significantly affect the properties of the soliton, and the configuration coupled to the Abelian gauge field shows critical thresholds based on coupling parameters.

The problem of stability of various solitons can be analyzed by studying of the matrix elements of energy-momentum tensor and related spatial distributions of the forces acting in the interior of the configuration [44–51]. This approach supplements the Vakhitov-Kolokolov criteria of stability of solitons [52]. It originates from the study of form factors of the energy-momentum tensor of hadrons [44,45] and evaluation of the so-called “D-term,” the quantity that is related to the mechanical deformations of the configuration. Note that this method can be applied for both topological [49,50] and nontopological [46–48] solitons. More generally, all finite energy regular localized configurations satisfy certain criteria for the distribution of the shear force and the pressure. Further, it has been shown that the electrostatic repulsion may destabilize Q balls [48].

In this paper, we extend this approach to the case of $U(1)$ gauged spherically symmetric nontopological solitons of the $O(3)$ sigma model with a symmetry breaking potential in flat space. We study the energy-momentum tensor of the system and discuss corresponding distributions of the pressure and shear forces, acting in the interior of the soliton.

This paper is organized as follows. In Sec. II, we introduce the model including the field equations, conserved Noether current and the stress-energy tensor. Scaling-type arguments are then used to establish the existence of regular electrically charged solitonic solutions. Here, we describe the spherically symmetric parametrization of the fields and the boundary conditions imposed on the configuration. The internal mechanical properties of the $U(1)$ gauged nontopological solitons of the $O(3)$ sigma

model are discussed in Sec. III, where, in particular, we study the shear forces and the pressure in the interior of these localized configurations. Here we also discuss the stability condition of the solitons that follow from the conservation of the energy-momentum tensor of the systems. Numerical results are presented in Sec. IV. Conclusions and remarks are formulated in Sec. V.

II. THE MODEL AND FIELD EQUATIONS

A. The action and the field equations

We consider the $U(1)$ gauged nonlinear $O(3)$ sigma model in a $(3 + 1)$ -dimensional Minkowski spacetime, defined by the Lagrangian density

$$L_m = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(D_\mu\phi^a)^2 - U(\phi), \quad (1)$$

where the real triplet of the scalar fields ϕ^a , $a = 1, 2, 3$, is constrained as $(\phi^a)^2 = 1$, and $U(\phi)$ is a symmetry breaking potential. We introduced the usual Maxwell term with the field strength tensor defined as $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The covariant derivative of the field ϕ^a , which minimally couples the scalar field to the gauge potential, is

$$\begin{aligned} D_\mu(\phi^1 + i\phi^2) &= \partial_\mu(\phi^1 + i\phi^2) - ieA_\mu(\phi^1 + i\phi^2), \\ D_\mu\phi^3 &= \partial_\mu\phi^3, \end{aligned} \quad (2)$$

with e being the gauge coupling.

Following previous discussions in [30,31,53], we will consider the potential $U(|\phi|)$, which breaks $O(3)$ symmetry to the subgroup $SO(2) \sim U(1)$ and is taken as the difference of two positive-definite terms (see Fig. 1),

$$U(\phi^a) = \mu^2(1 - \phi^3) - \beta(1 - \phi^3)^4, \quad (3)$$

where μ and β are two real parameters. Clearly, rescaling of the model $\beta \rightarrow \beta/\mu^2$ allows us to set $\mu = 1$.

Importantly, the potential (3), apart from the common weakly attractive “pion mass” term, also contains a term that yields an additional short-range interaction [42,54–56]. Potentials of that type may support localized nontopological soliton solutions in $(3 + 1)$ -dimensional flat spacetime [30,31]; they exist for $\beta > \beta_c = \frac{1}{8}$, as the potential (3) possesses a global minimum at the antivacuum $\phi^3 = -1$, where $U(\phi) < 0$. The vacuum $\phi^3 = 1$ corresponds to the local minimum, as $\beta = \beta_c = 1/8$, the vacuum becomes degenerated, and the field configuration becomes an infinitely thin spherical wall separating two vacua.¹

¹Note there is a significant difference between the soliton solutions of the nonlinear $O(3)$ sigma model and the usual Q balls in a complex scalar field theory [24–27], for which the vacuum value of the field is $\phi^3 = 0$.

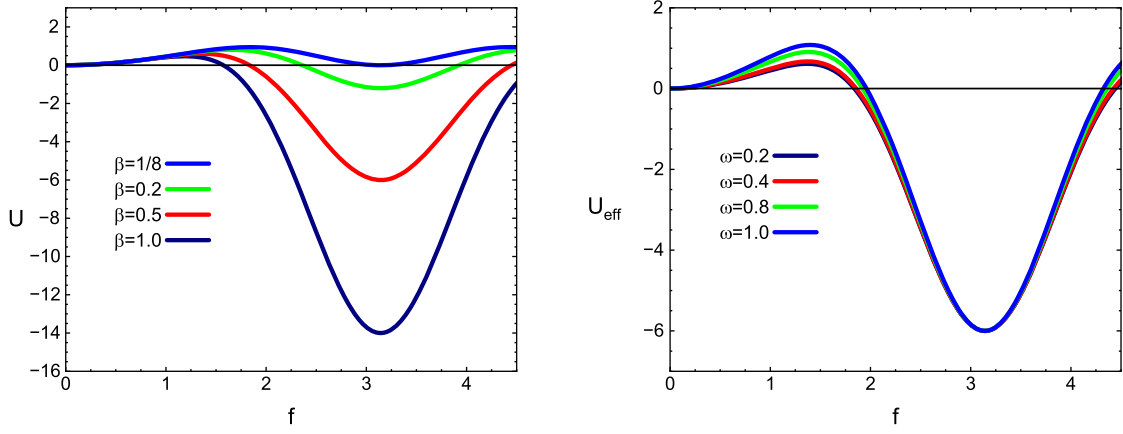


FIG. 1. The potential (3) for different values of the parameter β , $\mu^2 = 1$ (left) and the effective potential $U_{\text{eff}} = U + \frac{1}{2}\omega^2 \sin^2 f$ for $e = 0$ and different values of the frequency ω at $\beta = 0.5$ and $\mu^2 = 1$ (right), vs the value of the radial function f .

Stable soliton solutions of the model (1) do not exist as the rescaled parameter β becomes smaller than $1/8$.

The model (1) is invariant with respect to the local Abelian gauge transformations

$$(\phi^1 + i\phi^2) \rightarrow e^{ie\zeta}(\phi^1 + i\phi^2), \quad A_\mu \rightarrow A_\mu + \partial_\mu \zeta, \quad (4)$$

where ζ is a real function of the coordinates. The corresponding Noether current can be written as follows:

$$j^\mu = \phi^1 D^\mu \phi^2 - \phi^2 D^\mu \phi^1. \quad (5)$$

Variation of the Lagrangian (1) with respect to the fields A_μ and ϕ_a leads to the dynamical equations

$$\partial_\mu F^{\mu\nu} = e j^\nu, \quad D^\mu D_\mu \phi^a + \phi^a (D^\mu \phi^b D_\mu \phi^b) + \frac{\delta U}{\delta \phi} \phi^a = 0, \quad (6)$$

where we take into account the dynamical constraint imposed on the $O(3)$ field. The vacuum boundary conditions are $\phi_\infty^a = (0, 0, 1)$, $D_\mu \phi^a = 0$, and $F_{\mu\nu} = 0$. In the static gauge, where the fields have no explicit dependence on time, the asymptotic boundary conditions on the gauge potential are

$$A_0(\infty) = V, \quad A_i(\infty) = 0, \quad (7)$$

where V is a real constant. Note that the asymptotic value of the electric potential $A_0(\infty)$ can be adjusted via the residual $U(1)$ transformations (4), choosing $\zeta = -Vt$. In such a stationary gauge, $A_0(\infty) = 0$ and two components of the scalar triplet acquire an explicit harmonic time dependence with frequency $\omega = eV$.

The scalar and electromagnetic components of the energy-momentum tensor are

$$T_\phi^{\mu\nu} = D^\mu \phi^a D^\nu \phi^a - \eta^{\mu\nu} \left(\frac{1}{2} D^\rho \phi^a D_\rho \phi^a + U \right), \quad (8)$$

$$T_{em}^{\mu\nu} = F^{\mu\rho} F_\rho^\nu - \frac{1}{4} \eta^{\mu\nu} F^{\rho\sigma} F_{\rho\sigma}, \quad (9)$$

respectively. Here $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ is the Minkowski metric.

B. Scale transformations and virial identity

Let us establish that the model (1) with two-component potential (3) supports finite energy localized solutions. The total energy functional of the system can be written as

$$E = \int d^3x (T_\phi^{00} + T_{em}^{00}) = E_0 + E_2 + E_4, \quad (10)$$

where

$$E_0 = \int d^3x U(\phi), \quad E_2 = \frac{1}{2} \int d^3x D_i \phi^a D_i \phi^a, \quad E_4 = \frac{1}{2} \int d^3x (E_i^2 + B_i^2), \quad (11)$$

and $E_i = F_{0i}$ and $B_i = \frac{1}{2} \epsilon_{ijk} F_{jk}$ are the electric and magnetic fields, respectively. For the potential (3) E_0 , which represents a difference of two positive-definite terms, $E_0 = E_0^{(1)} - E_0^{(2)}$, where both $E_0^{(1)}$ and $E_0^{(2)}$ are positive definite.

The critical points of the total energy functional of the $O(3)$ sigma model (10) must satisfy the arguments of Derrick's theorem. Note that the corresponding scale transformation $x_i \rightarrow x'_i = \lambda^{-1} x_i$ does not affect the field $\phi^a \rightarrow \phi^a$ because of the sigma model constraint, and the gauge potential transforms as $A_\mu(x_i) \rightarrow A'_\mu(x'_i) = \lambda A_\mu(x_i)$. Hence,

$$E(\lambda) = \lambda^{-1}E_2 + \lambda E_4 + \lambda^{-3}E_0^{(1)} - \lambda^{-3}E_0^{(2)} \quad (12)$$

and $\partial_\lambda^2 E(\lambda=1) = 2[E_2 + 6(E_0^{(1)} - E_0^{(2)})] \geq 0$. The corresponding virial identity follows from the condition $\partial_\lambda E|_{(\lambda=1)} = 0$; it gives

$$E_2 + 3E_0^{(1)} = E_4 + E_0^{(2)}, \quad (13)$$

which secures the existence of soliton solutions even in the absence of an electromagnetic field [31,53]. It should be noted that electromagnetic interaction in three spatial dimensions alone cannot stabilize a soliton. On the other hand, there are ungauged nontopological solitons in a model with a potential like (3) [30,31,53].

C. Ansätze and boundary conditions

Imposing spherical symmetry, we employ for the scalar field the usual $O(3)$ Ansatz, which is parametrized in terms of the radial function $f(r)$ and the harmonic time dependence

$$\phi^a = [\sin f(r) \cos \omega t, \sin f(r) \sin \omega t, \cos f(r)]. \quad (14)$$

For the gauge potential, we consider a purely electrostatic Ansatz

$$A_\mu dx^\mu = A_0(r) dt. \quad (15)$$

Note that the $U(1)$ invariance of the model (1) admits the static gauge fixing setting $\omega = 0$. However, following our previous works [31,53,57], we employ the gauge fixing $A_0 \rightarrow 0$ as $r \rightarrow \infty$ and retain the dependence of the solutions on the angular frequency ω .

Substituting (14) and (15) into (1), we arrive at the reduced Lagrangian

$$L_{\text{eff}} = -\frac{1}{2}(A'_0)^2 - \frac{1}{2}(\omega + eA_0)^2 \sin^2 f + \frac{1}{2}(f')^2 - \mu^2(1 - \cos f) + \beta(1 - \cos f)^4.$$

The corresponding field equations become

$$\begin{aligned} f'' + \frac{2f'}{r} + (\omega + eA_0)^2 \sin f \cos f &= [\mu^2 - 4\beta(1 - \cos f)^3] \sin f, \\ A''_0 + \frac{2A'_0}{r} - e(\omega + eA_0) \sin^2 f &= 0, \end{aligned} \quad (16)$$

where a prime denotes the radial derivative.

As for the usual Q balls, there are two important characteristics of the $U(1)$ gauged solitons of the $O(3)$ sigma model, the mass of the configurations

$$M = \int d^3x T^{00} = 4\pi \int r^2 dr \left(\frac{1}{2}f'^2 + \frac{1}{2}A_0'^2 + \frac{1}{2}(\omega + eA_0)^2 \sin^2 f + \mu^2(1 - \cos f) - \beta(1 - \cos f)^4 \right) \quad (17)$$

and the Noether charge

$$Q = \int d^3x j^0 = 4\pi \int r^2 dr \sin^2 f (\omega + eA_0). \quad (18)$$

It must be emphasized that, unlike ordinary Q balls in a complex scalar field theory, the Noether charge Q (18) cannot be interpreted as the particle number since the norm of the $O(3)$ field is fixed, and only two components of the scalar triplet contribute to the charge. On the other hand, the mass of the configuration (17) includes the contributions of all components of the scalar field. Therefore, one can expect a different type of dependence $M(Q)$ for solitons of the $O(3)$ sigma model and in the usual scalar theory.

It follows from the system of equations (16) that, as the fields approach the vacuum values on the spacial asymptotics,

$$f(r) \rightarrow 0 + h(r) + \dots, \quad A_0(r) \rightarrow 0 + a(r) + \dots,$$

the linearized asymptotic equations describe perturbative excitations around the vacuum,

$$h'' - (\mu^2 - \omega^2)h = 0, \quad a'' = 0. \quad (19)$$

Notably, the contribution of the term, which is proportional to β , decays faster than the usual mass term; a localized configuration of the $O(3)$ field with an exponentially decaying tail may exist if $\omega \leq \mu$ where μ is the mass of scalar perturbations. On the other hand, the gauge field remains massless.

III. MECHANICAL PROPERTIES OF NONTOPOLOGICAL $O(3)$ SOLITONS

It was suggested by Polyakov [44,45] that solitonlike configurations may be considered by analogy with an elastic medium. Explicitly, the spacial components of the stress-energy tensor (9) can be associated with the

distribution of the pressure anisotropy (shear forces) $s(r)$ and the elastic pressure $p(r)$ as

$$T_{ij} = \left(\frac{x_i x_j}{r^2} - \frac{1}{3} \delta_{ij} \right) s(r) + p(r) \delta_{ij}, \quad i, j = 1, 2, 3. \quad (20)$$

Using the spherically symmetric *Ansätze* (14) and (15) and the expression for the energy-momentum tensor (9), we obtain

$$\begin{aligned} p(r) &= -\frac{1}{6} f'^2 + \frac{1}{6} A_0^2 + \frac{1}{2} (\omega + eA_0)^2 \sin^2 f \\ &\quad - \mu^2 (1 - \cos f) + \beta (1 - \cos f)^4, \\ s(r) &= -A_0^2 + f'^2. \end{aligned} \quad (21)$$

Conservation of the energy-momentum tensor yields the “equilibrium equation” [46,51]

$$d(r) = p'(r) + \frac{2}{3} s'(r) + \frac{2}{r} s(r) = 0 \quad (22)$$

or

$$p'(r) = -\frac{2}{3r^3} \frac{d}{dr} [r^3 s^2(r)]. \quad (23)$$

Another criterion of stability of a soliton is the von Laue condition [45,58,59], related to the repulsive and attractive force balance in the interior of a soliton,

$$\int_0^\infty r^2 dr p(r) = 0. \quad (24)$$

Indeed, integrating this equation by parts over r and imposing a finite upper integration limit R , we obtain

$$\int_0^R r^2 dr p(r) = \left[\frac{r^3}{3} p(r) \right]_0^R - \int_0^R dr \frac{r^3}{3} p'(r). \quad (25)$$

Using the relation (23), we obtain

$$\int_0^R r^2 dr p(r) = \left[\frac{r^3}{3} \left(p(r) + \frac{2}{3} s(r) \right) \right]_0^R. \quad (26)$$

Since the asymptotic behavior of the solutions of the system (16) secures that the pressure and shear force densities decay at large distances faster than $1/r^3$, the von Laue condition is satisfied. Consequently, the pressure function $p(r)$ must possess at least one zero. This is a necessary (though not sufficient) condition of stability; it can also be reformulated as a virial relation connecting integrals of components of the energy-momentum tensor.

The expression in brackets in (26) corresponds to the distribution of the normal component of the net force acting on an infinitesimal area element $dA \hat{e}_r$ at a distance r [45,60], $dF_j = T_{jk} dA_k$, and the normal force must be directed outward,

$$c(r) = \frac{dF_r}{dA_r} = p(r) + \frac{2}{3} s(r) > 0. \quad (27)$$

Further, by analogy with related studies of the internal structure of hadrons, one can introduce one more mechanical characteristic, the so-called D-term (or the Druck term), which is related to the distribution of the internal forces inside the configuration [44]. It can be expressed in terms of the pressure and shear force distributions as

$$D = -\frac{M}{3} \int d^3x r^2 s(r) = \frac{5}{4} M \int d^3x r^2 p(r). \quad (28)$$

Indeed, there is yet another relation between the energy density $\epsilon(r) = T_{00}(r)$, the pressure $p(r)$, shear forces $s(r)$, and the charge density $j^0(r)$,

$$\epsilon + p = (\omega + eA_0) j^0 + \frac{1}{3} s + A_0^2. \quad (29)$$

Multiplying this relation by r^2 and integrating over d^3x , one can express D in terms of other properties,

$$D = \frac{5}{9} (\omega M Q \langle r_Q^2 \rangle - M^2 \langle r_E^2 \rangle), \quad (30)$$

where the mean square radii of the energy and charge densities are

$$\langle r_E^2 \rangle = \frac{\int T_{00} r^2 d^3x}{\int T_{00} d^3x}, \quad \langle r_Q^2 \rangle = \frac{\int j_0 r^2 d^3x}{\int j_0 d^3x}, \quad (31)$$

respectively. It was suggested that the condition for the stability of the configuration is a negative value of the D-term [45].

Finally, localized configurations usually satisfy the energy conditions. They follow from the assumption that the energy density should be positive everywhere in space. Explicitly, the following energy conditions must hold:

$$\text{weak energy condition: } \epsilon = T_{\mu\nu} X^\mu X^\nu \geq 0, \quad (32)$$

$$\text{strong energy condition: } \rho = \left(T_{\mu\nu} - \frac{T}{2} \eta_{\mu\nu} \right) X^\mu X^\nu \geq 0, \quad (33)$$

where $X^\mu = (1, 0, 0, 0)$ is a timelike vector field. As we will see, both conditions can be violated for the soliton solutions of the model (1).

IV. NUMERICAL RESULTS

The system (16) can be solved numerically. The corresponding boundary conditions are found by considering the asymptotic expansions of the equations on the boundary of the domain of integration together with the assumption of regularity. Explicitly, for the spherically symmetric configurations, we impose $A_0 = f = 0$ at spacial infinity and $\partial_r A_0 = \partial_r f = 0$ at the origin.

Numerical calculations are performed on an equidistant one-dimensional grid, employing the compact radial coordinate $x = r/(C + r) \in [0:1]$ where C is an arbitrary constant used to adjust the grid according to the contraction of the solutions. In our numerical calculations, we have made use of a sixth-order finite difference scheme, where the system of equations is discretized on a grid with a typical size of about 200 points in the radial direction. The emerging system of nonlinear algebraic equations has been solved using the Newton-Raphson scheme, and typical errors are of order 10^{-8} .

Let us briefly review the basic properties of the spherically symmetric solutions of the model (1) discussed previously in [31].

In Fig. 2 we present radial profiles of the scalar and gauge fields for a set of values of the parameter β and the gauge coupling e . For a fixed value of e , the size of the configuration rapidly increases as β decreases toward the critical value $\beta_c = 1/8$. An increase of the gauge coupling e leads to the opposite effect, as seen in Fig. 2, left plot. The central value of the profile function of the scalar field approaches antivacuum $\phi^3 = (0, 0 - 1)$; it decreases as β grows, the short-range attractive force becomes stronger, and the core of the soliton shrinks.

The electric potential A_0 is a constant in the interior of the soliton, as displayed in Fig. 2, right plot. The electric field of the gauged configuration vanishes in the interior. Outside the core of the soliton, it possesses a long-range Coulomb asymptotic tail sourced by the charge $Q_e = eQ$.

Soliton solutions of the model (1) exist for all nonzero values of the frequency $\omega \in [0, 1]$ where the upper critical value corresponds to the mass of free charged quanta of scalar excitations.

It is worth emphasizing the following important difference between the solutions under consideration and ordinary Q balls in the theory of a complex scalar field. In the latter case, Q balls arise smoothly from perturbative excitations about the vacuum, as the angular frequency is decreasing below the mass threshold, and the scalar quanta condense into a nontopological soliton. The situation is different for the solitonic solutions of the $O(3)$ sigma model with potential (3); they are disconnected from the vacuum excitations for all the range of values of the frequency ω and the parameter β .

In Fig. 3 we present the total mass M of the solutions vs the angular frequency ω (upper plots) and Noether charge Q (bottom plots) for a set of values of the parameter β and fixed gauge coupling e and vice versa.

The gauged solitons form a branch, which extends forward as ω increases. The limiting fundamental solution at $\omega = 0$ is unchanged; it corresponds to the nontopological soliton stabilized by potential (3).

The mass of the configuration increases monotonically with increasing frequency. In the case of ungauged solitons ($e = 0$) it diverges as the angular frequency approaches the upper critical value, as seen in Fig. 3, upper plots. We observe that the critical behavior of the electrically charged solitons is different. They possess finite energy and charge for all ranges of values of the angular frequency and parameter β , see Fig. 3, upper plots. Increasing the gauge coupling e leads to a decrease of the mass, while, at fixed charge Q , the mass M increases with increasing coupling e . This behavior can be attributed to the simultaneous enhancement of the electrostatic repulsion and the attractive contribution associated with the structure of the potential (3).

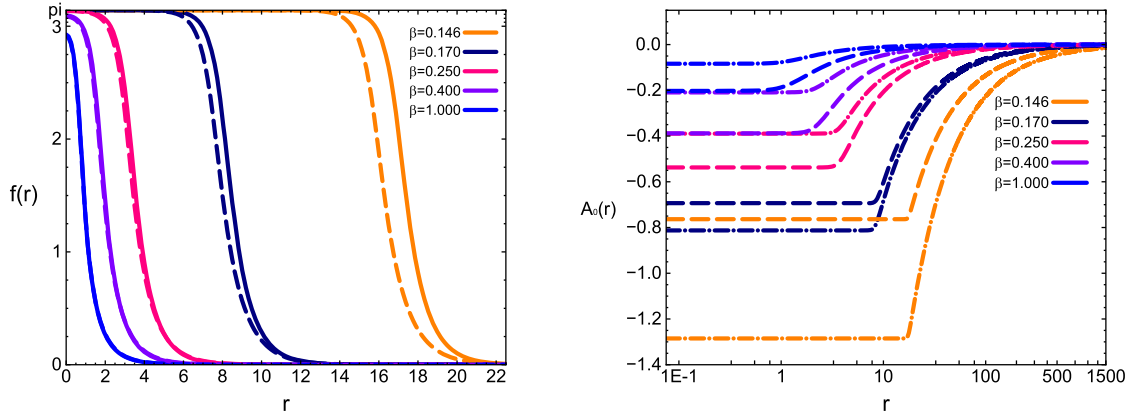


FIG. 2. Profiles of the radial profile functions of the soliton $f(r)$, $A_0(r)$ for different values of the parameter β at $\omega = 0.5$ and $e = 0$ (solid lines), $e = 0.2$ (dash-dotted lines), and $e = 0.6$ (dashed lines). (A logarithmic scale is used for the function A_0).

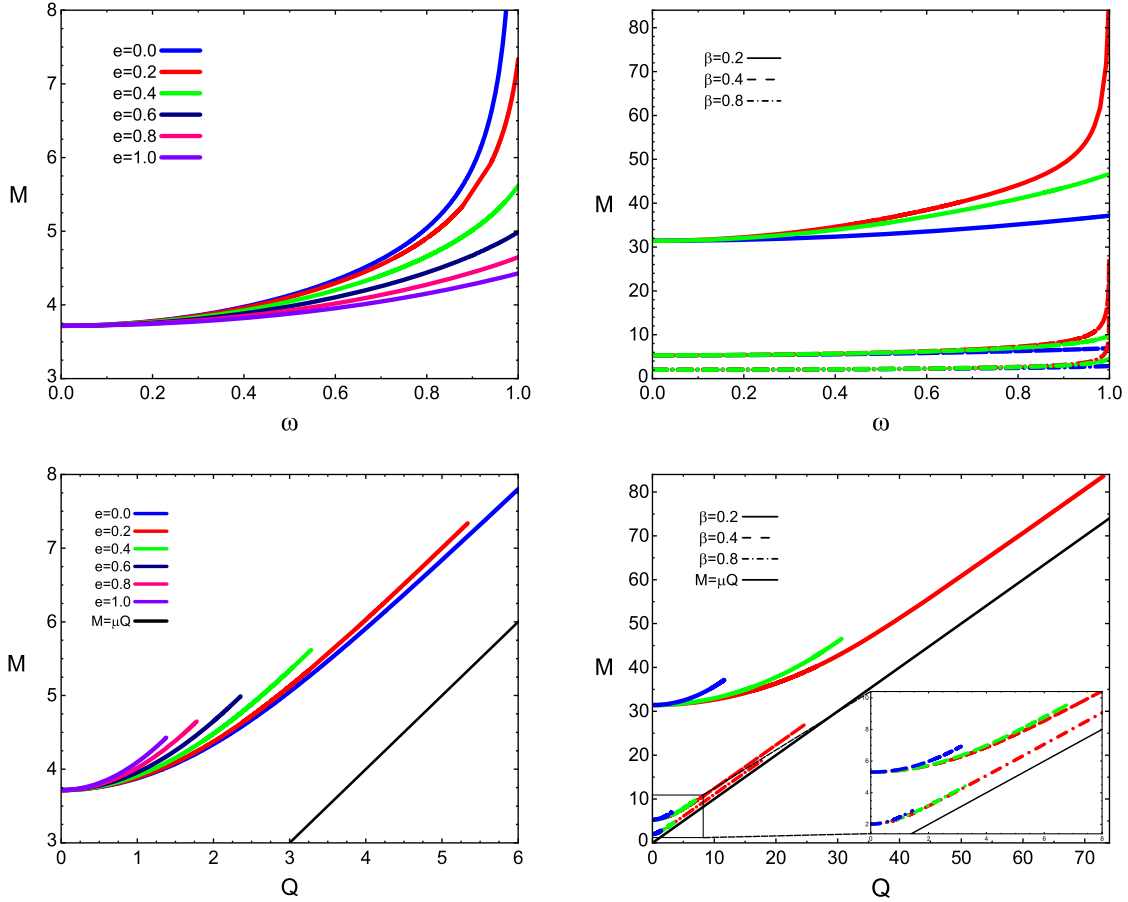


FIG. 3. Spherically symmetric $U(1)$ gauged $O(3)$ solitons: The mass M vs the frequency ω (upper row) and vs the Noether charge Q (bottom row) for some set of values of the gauge coupling e for $\beta = 0.5$ (left) and for some set of values of the parameter β (right) for $e = 0$ (red line), $e = 0.2$ (green line), $e = 0.6$ (blue line).

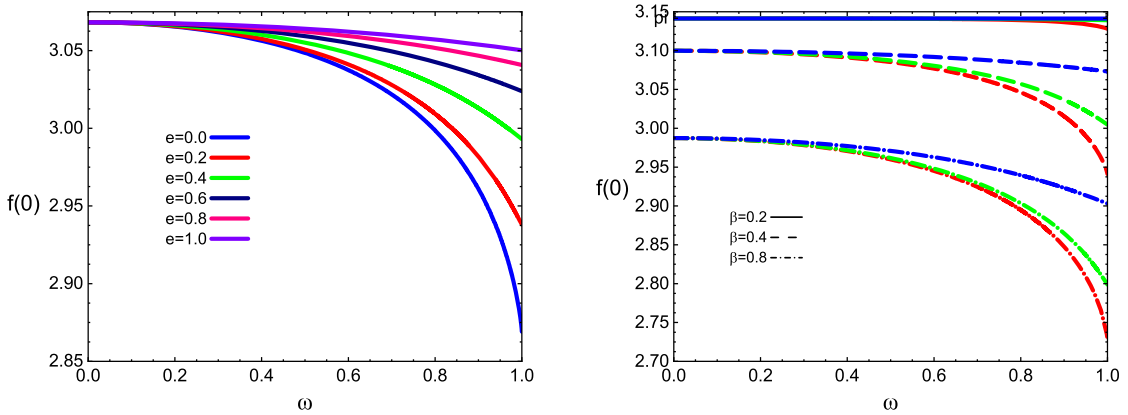


FIG. 4. The central value of the scalar profile function $f(0)$ vs the frequency ω for some set of values of e at $\beta = 0.5$ (left) and for some set of values of β at $e = 0$ (red line), $e = 0.2$ (green line), and $e = 0.6$ (blue line) (right).

The parameter β has a crucial impact on both the solutions and the mass. As β approaches the critical value $\beta_c = \frac{1}{8}$, the mass of the configuration rapidly increases. On the other hand, for a fixed value of e , the mass-charge relation approaches the linear form $M = \mu Q$, as β increases, see Fig. 3, bottom right plot.

As can be seen from Fig. 4, the central value of the scalar profile function $f(0)$ decreases as the frequency ω increases all the way up to the mass threshold. The decrease rate is slower for electrically charged solitons, also the value of the profile function at the center of a soliton decreases faster for larger values of β .

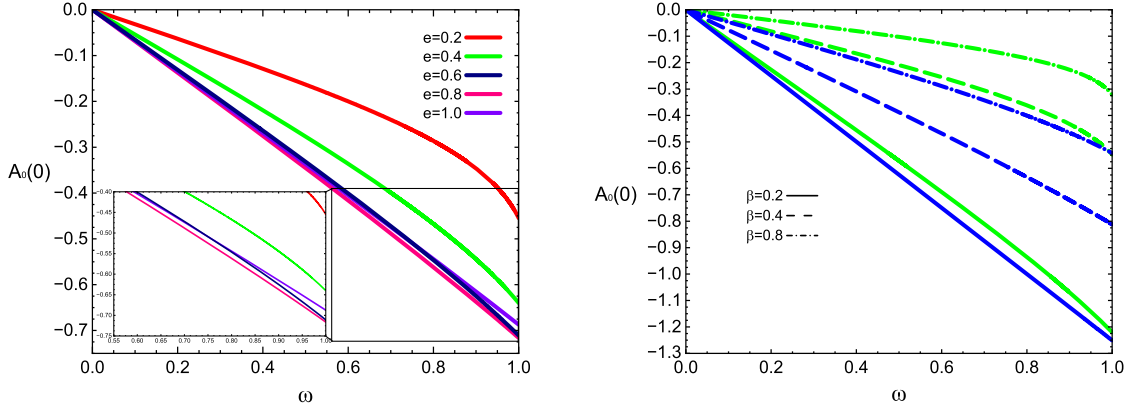


FIG. 5. The central value of electric potential $A_0(0)$ vs the frequency ω for some set of values of e at $\beta = 0.5$ (left) and for some set of values of β at $e = 0.2$ (green line) and $e = 0.6$ (blue line) (right).

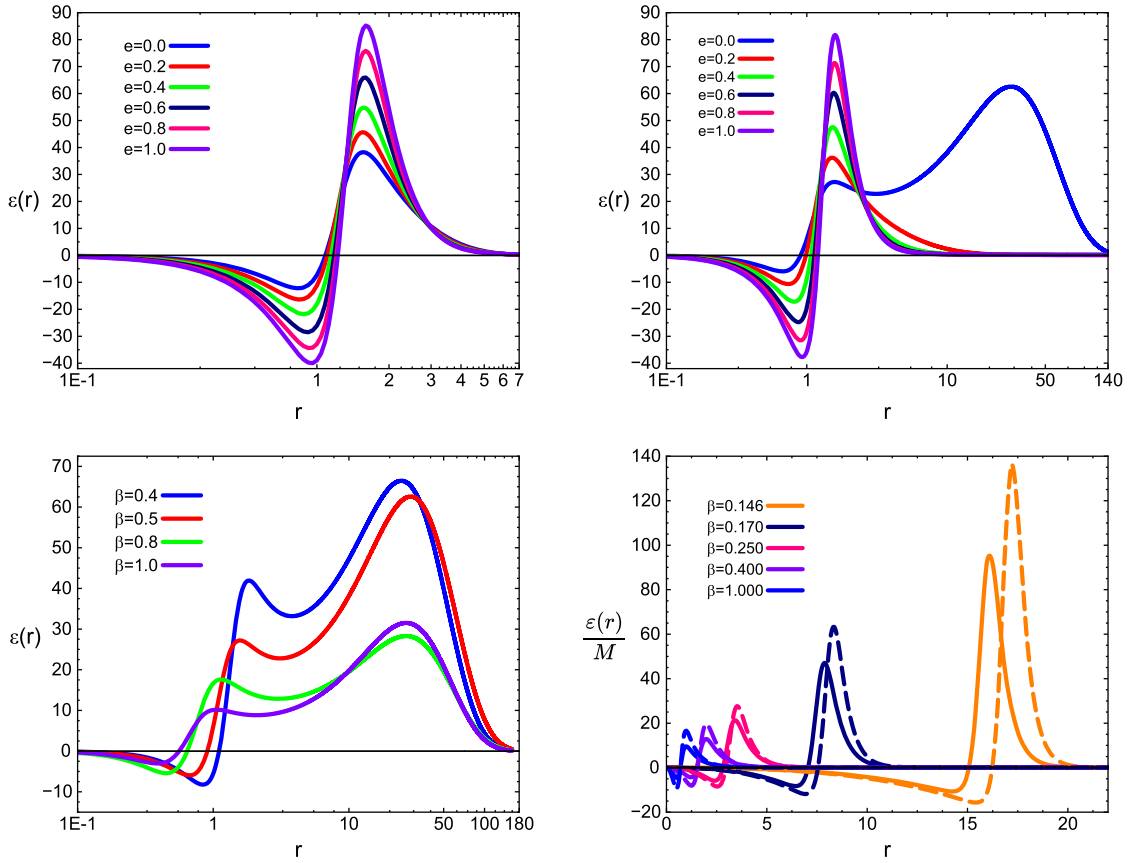


FIG. 6. Upper row: Radial profiles of the energy density distributions $\varepsilon(r)$ are displayed for some set of values of e for $\beta = 0.5$ at $\omega = 0.8$ (upper left) and $\omega = 0.995$ (upper right). Bottom row: Radial profiles of the energy density distributions $\varepsilon(r)$ are displayed for some set of values of β at $e = 0$ and $\omega = 0.995$ (bottom left) and at $e = 0$ (solid line) and $e = 0.6$ (dashed line) for $\omega = 0.5$ (bottom right, normalized by M).

In Fig. 5 we display the central value of electric potential $A_0(0)$ vs frequency ω for some set of values of the gauge coupling e and the parameter β . This quantity indicates the strength of electrostatic repulsion. Clearly, for a given β , it decreases monotonically as ω increases. The increase of β results in slower decay rate. Because of the interplay of two

compensating forces of electrostatic repulsion and scalar interaction, the function $A_0(0)$ exhibits a nontrivial dependence on e ; in particular, for $e > 0.8$, the slope of the $A_0(0)$ vs ω curve stops increasing, see Fig. 5 left plot.

Remarkably, for all solitons, the energy density distribution is not positive everywhere, see Fig. 6, although the

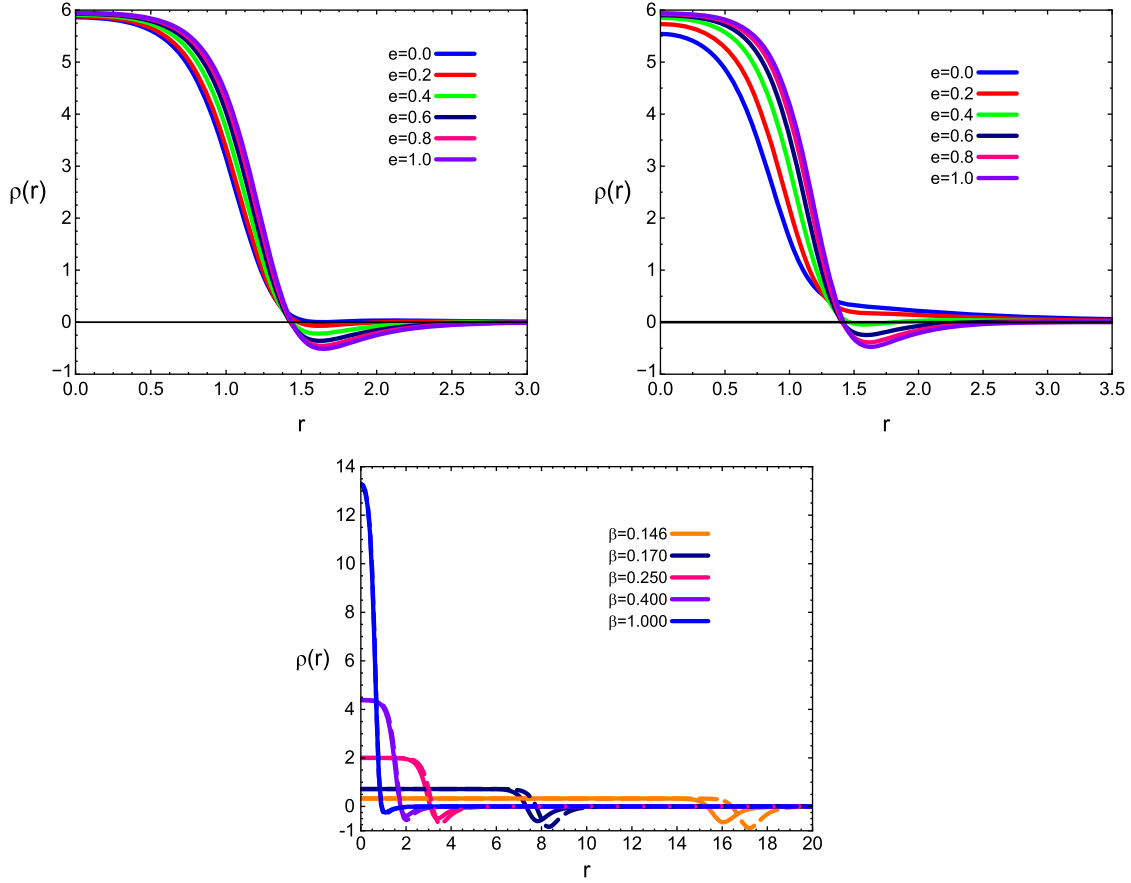


FIG. 7. Profiles of the function $\rho(r)$ with different values of gauge coupling e at $\omega = 0.8$ (upper left) and $\omega = 0.995$ (upper right) for $\beta = 0.5$ and for some set of values of parameter β (bottom) at $e = 0$ (solid line) and $e = 0.6$ (dashed line) for $\omega = 0.5$.

total integrated mass remains positive in any case. The structure of the potential (3) leads to the appearance of a domain with negative energy density in the interior of the soliton. Hence, the weak energy condition (32) is always violated, as expected. For a given value of the parameter β and a fixed frequency ω , an increase in the constant e leads to an increase in both domains of positive and negative energy densities, as displayed in Fig. 6, upper plots. Note that, for ungauged solitons, in the nonrelativistic limit, as the frequency approaches the mass threshold, the function of the radial energy distribution ceases to be monotonic in the positive domain, see Fig. 6. For example, as $\omega = 0.995$ and $e = 0.0$, the function $\varepsilon(r)$ develops a pronounced hump and becomes more extended, approaching zero only at $r = 140$, see Fig. 6, bottom left plot.

In Fig. 7 we exhibit radial profiles of the function $\rho(r)$, which represents the strong energy condition (33). As it turns out, this condition holds for some set of values of parameters of the model. For example, this condition is satisfied for the ungauged configurations at $\beta = 0.5$, see Fig. 7, upper plots. However, it becomes violated as the gauge coupling increases. For $\omega < \omega_c = 0.8$, the strong energy condition is violated for all values of e while

$\beta = 0.5$. The critical value of the frequency increases with increasing e and decreasing β .

Interestingly, the presence of domains with negative energy density and violation of the energy conditions most likely does not lead to the destabilization of solitons.

Indeed, in Fig. 8 we displayed the radial distributions of the pressure function $p(r)$ and the shear forces $s(r)$ (21) and the stability criteria function $c(r)$ (27) for some set of values of the gauge coupling and the parameter β . We can clearly see that for all solutions the pressure function $p(r)$ possesses no more than one radial node, while the shear force $s(r)$ is positive approaching zero both at the origin and on the spacial asymptotic. The pressure is maximal at the center of the soliton, and it approaches zero from below on the spacial asymptotic.

The value of the D-term slightly varies with ω , however, it always remains negative, see Fig. 8, bottom left plot.

Figures 9 and 10 illustrate dependency of the pressure function $p(r)$, the shear forces $s(r)$, and the stability criteria function $c(r)$ on the strength of the gauge coupling. For $\omega = 0.8$ and $\beta = 0.5$ the increase of the gauge coupling slightly increases the minimal negative value of the pressure, and central values of the functions $c(0)$ and

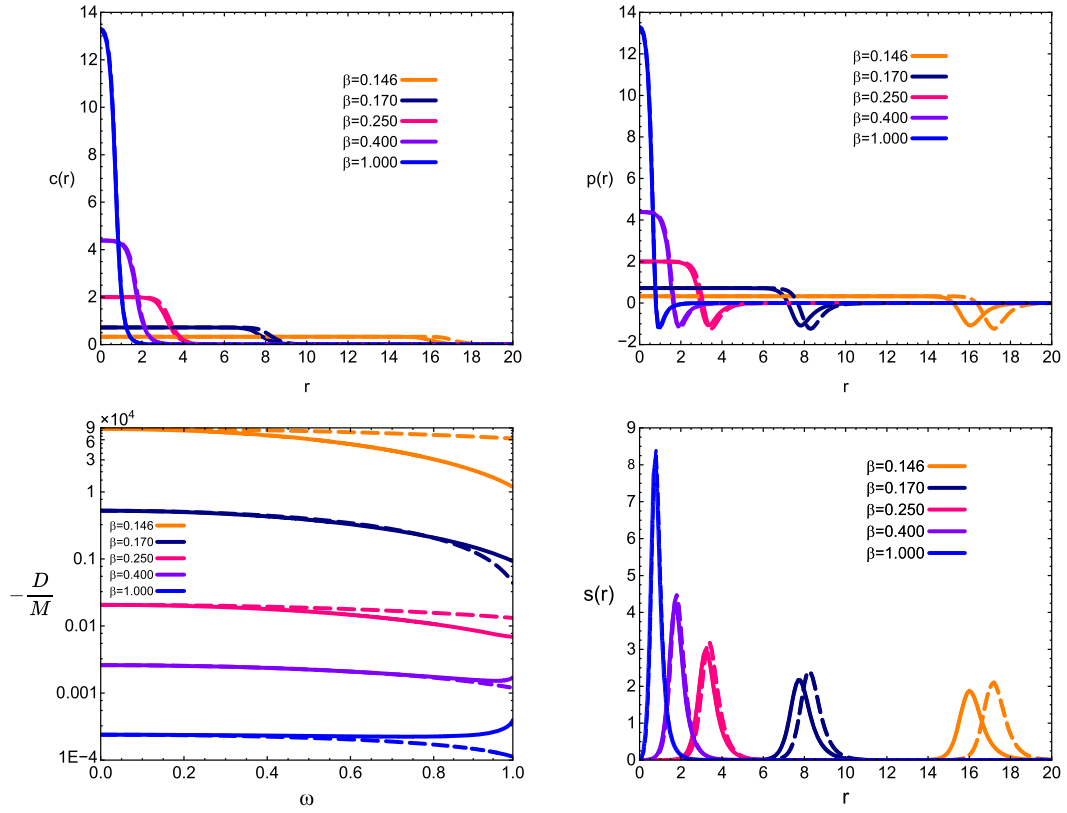


FIG. 8. Radial distributions of functions $c(r)$, $p(r)$, $s(r)$ at $\omega = 0.5$ and $-\frac{D(\omega)}{M}$ for different values of the parameter β and $e = 0$ (solid lines), $e = 0.2$ (dash-dotted lines), and $e = 0.6$ (dashed lines).

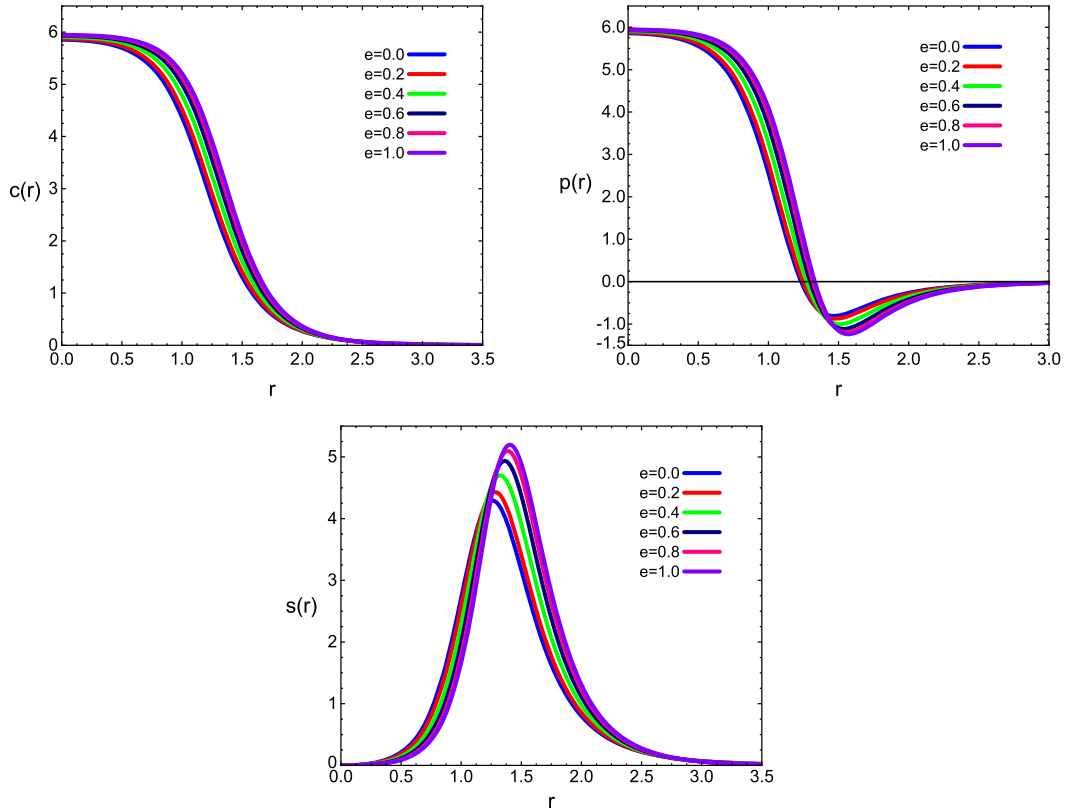


FIG. 9. Profiles of the functions $c(r)$, $p(r)$, $s(r)$ for some set of values of the coupling e at $\omega = 0.8$ and $\beta = 0.5$.

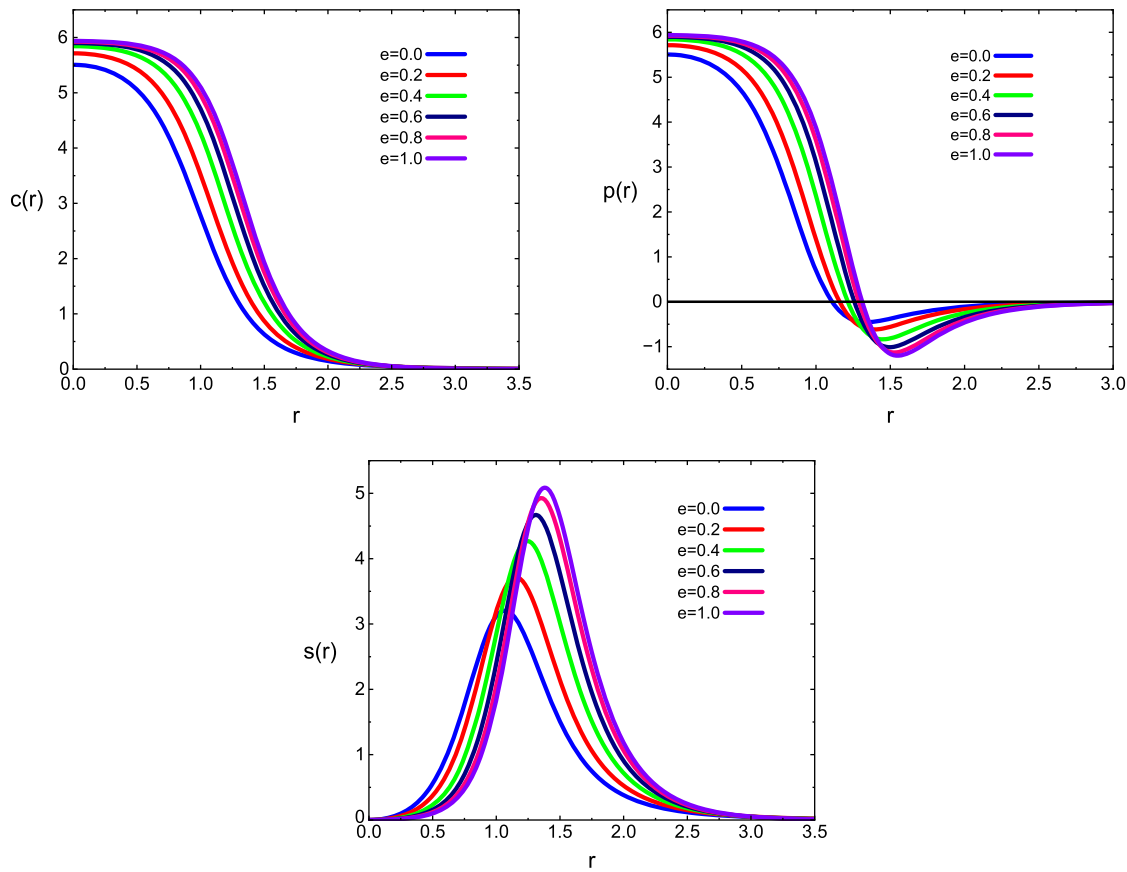


FIG. 10. Profiles of the functions $c(r)$, $p(r)$, $s(r)$ for some set of values of the coupling e at $\omega = 0.995$ and $\beta = 0.5$.

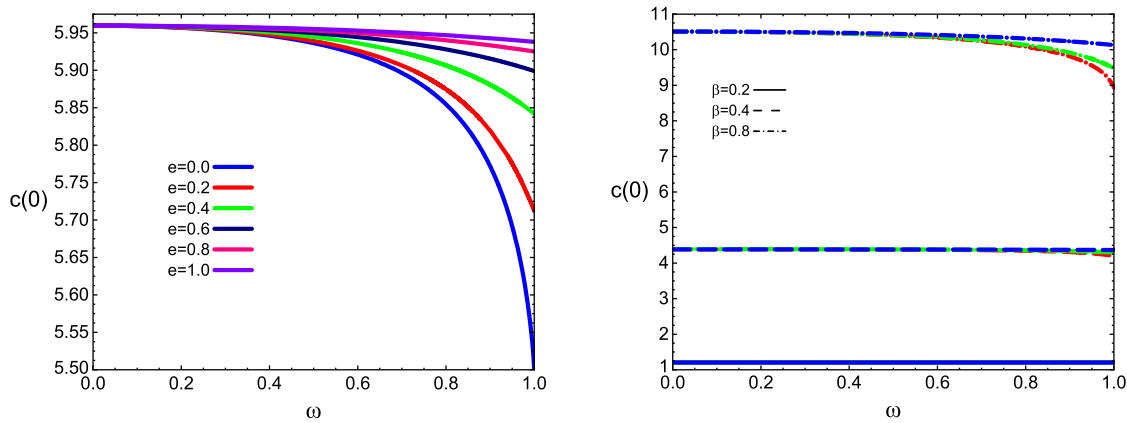


FIG. 11. The central value of the function $c(0)$ vs the frequency ω for some set of values of the coupling e at $\beta = 0.5$ (left) and for some set of values of parameter β (right) at $e = 0$ (red line), $e = 0.2$ (green line), $e = 0.6$ (blue line), respectively.

$p(0)$ marginally increase. This effect becomes more noticeable as the frequency ω approaches the upper limit, see Figs. 10 and 11. However, there are no qualitative changes in the behavior of these solutions; the pressure distribution function $p(r)$ exhibits a single zero for all parameter values, the criterion function $c(r)$ remains positive, and the D-term is always negative. We may conclude that the $U(1)$ gauged soliton solutions are stress stable.

It is interesting to compare mechanical properties of these solitons with those of the well-established Q balls, see [46–48]. First, the radial distributions of the energy density are very different; for the usual Q balls, it is always positive with a nonzero value at the center of the configuration and a pronounced peak on the surface of the soliton. On the other hand, qualitatively, the pressure function $p(r)$ differs little from its analog in the case of Q balls; it is positive inside the

core of the soliton and possess at least one zero. Note that for the $U(1)$ gauged Q balls the pressure function may possess more than one node [48], and the configuration becomes unstable.

For the stable branch of $U(1)$ gauged Q balls both the shear force function $s(r)$ and the criteria function $c(r)$ differ qualitatively little from their analogs in the nonlinear $O(3)$ sigma model. However, there is an unstable branch of the $U(1)$ gauged Q balls, and the corresponding shear force $s(r)$ becomes negative both inside of the core of the configuration and on the spacial asymptotic, while the criteria function $c(r)$ becomes negative in these domains [48]. This pattern is excluded for the $U(1)$ gauged solitons of the nonlinear $O(3)$ sigma model.

Finally, we note that the dependence of the solutions on the parameter β is crucial. As β approaches the critical value $1/8$, the stability criterion function $c(r)$, the energy density ρ , and the pressure distribution $p(r)$ become more spatially extended, while their central values increase. The peak of the shear force radial distribution $s(r)$ then shifts toward larger values of r , and its amplitude decreases.

V. CONCLUSIONS AND OUTLOOK

In this paper, we have considered the problem of classical stability of spherically symmetric nontopological soliton solutions of the nonlinear $O(3)$ sigma model with a symmetry-breaking potential containing a nonpositive term responsible for the additional energy of the short-range interaction. Clearly, the presence of the negative energy density domain at the center of the soliton violates the weak energy condition. Further, coupling to the gauge field yields additional electrostatic repulsion which may also destabilize the configuration.

Our approach to the study of soliton stability is based on a treatment of the $U(1)$ gauged nontopological soliton as an elastic medium and study of corresponding matrix elements of energy-momentum tensor, which contains information about spatial distribution of internal forces acting in the interior of the configuration.

Our results indicate that the von Laue stability condition is always satisfied. Further, we analyze a local stability criteria, and the results of numerical simulations demonstrate that the $U(1)$ gauged nontopological solitons of the nonlinear $O(3)$ sigma model are classically stable for the full range of values of the parameters of the system.

Looking ahead, several promising avenues for further study emerge. A natural next step will be to analyze the internal mechanical properties of the axially symmetric solitonic solutions of the $O(3)$ sigma model [31].

An interesting open question is whether the usual Vakhitov-Kolokolov criterion of stability [52] can be directly applied to the nontopological solitons of the nonlinear $O(3)$ sigma model with a harmonic time dependence. More generally, there is an interesting task to study the full spectrum of linearized perturbations of these solitons.

Finally, we note that apart from the nontopological solitons considered in this paper, the $O(3)$ sigma model in three spacial dimensions also supports topological solitons with a nonzero Hopf charge. Notably, both for Hopfion solutions and for the $O(3)$ nontopological solitons, the field is approaching the antivacuum $\phi^3 = -1$ at the center of configurations. It would be interesting to investigate the possibility of the existence of topological soliton solutions in a model with a potential of the type (3) and without terms with higher-order derivatives, like the Skyrme term or Dzyaloshinskii-Moriya interaction.

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DATA AVAILABILITY

The data are not publicly available. The data are available from the authors upon reasonable request.

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