New method for exact results on quasinormal modes of black holes

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We develop a new method for writing simple exact equations characterizing gravity solutions among which are black holes and in particular quasinormal modes. More precisely, we derive the full system of functional and thermodynamic Bethe ansatz nonlinear integral equations of quantum integrability. In particular, we prove that the quasinormal modes verify different equivalent exact quantization conditions and identify them with Bethe roots. We numerically solve the integral equation and compare the results with other methods. Eventually, we can definitely certify its simplicity, accuracy, and effectiveness. Furthermore, this method connects different unexpected fields and paves the way for innovative ways of investigations in gravity and gauge theories.

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I. INTRODUCTION, PLAN AND GOALS

Thanks to the extraordinary discovery of gravitational waves in 2015 [1], a new window for our understanding of the Universe has opened. In fact, it has become possible to make progress also in fundamental physics, testing general relativity (GR) in extreme regimes and in particular to discriminate between GR black holes (BHs) and exotic compact objects or fuzzballs appearing in modified theories of gravity or string Theory. this is possible importantly by analysing the quasinormal modes (QNMs) that characterize the linear perturbations of the metric and the fields, for instance in the ringdown phase of BHs merging [2–4]. Computing QNMs numerically, as well as finding new exact analytic characterizations for them, is an increasingly important task. In this direction, a significant improvement has been realized very recently as QNMs have been identified by exact quantization conditions on *dual* periods of some $\mathcal{N}=2$ supersymmetric gauge, i.e. deformed Seiberg-Witten (SW), theories [5–7]. On the latter we have in fact some exact control and then this surprising SW-QNM correspondence—so dubbed in [8]—has already allowed us to use Nekrasov functions to find many new theoretical and computational results for BHs and other spacetime geometries (cf. for instance [9–11]). An explanation of this correspondence has been constructed in a rather general case [11] by exploiting another correspondence between $\mathcal{N}=2$ gauge theory and conformal field theory [12]. However, we are going to show that it is possible to explain the SW-QNM correspondence by analyzing closely the ordinary differential equations (ODEs) describing the perturbations in gravitational physics. We are able to do this on the basis of our previous works [13,14], where we have connected the $\mathcal{N}=2$ gauge theories to quantum integrable theories, in particular the gauge periods to the Baxter's Q and T functions. To this aim we have started from the ODEs characterizing the periods and developed further the elegant ODE/IM correspondence between ODEs and integrable models (IM) [15–18].

Here we prove that QNMs are nothing but the zeros of the Q function (Bethe roots) and then find an entirely new set of functional and integral (nonlinear) equations for them (in particular quantization conditions). In specific, we derive a new quantization condition involving the solution of a celebrated nonlinear integral equation, the thermodynamic Bethe ansatz (TBA) ones. As a result, this way turns out to be a very simple and powerful way to compute QNMs. In the light of these findings, we understand and develop here the lines of generality in our construction [13,14] to put it into correspondence with gravitational physics.

In this letter, after introducing QNMs in Sec. II, in Sec. III we explain the integrability/gauge/gravity correspondence in the simplest case of the D3-brane spacetime, corresponding to our previous treatment of $SU(2)N_f=0$ gauge theory via the self-dual Liouville theory [13]. To show the generality of the integrability/gravity correspondence, in Sec. IV we develop the same theory for the

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spacetime given by the intersection of D3-branes, a generalization of extremal Reissner-Nordström (RN) BHs, corresponding to $SU(2)N_f = 2$ gauge theory [8,19] and a new IM (which reproduces known IMs in some limits, cf. below). In Sec. V we highlight the reasons for applicability of our method and conclude that different features of many integrability/gauge/gravity cases can be treated in future perspective [20].

II. BRIEF OVERVIEW ON QNMS

We recall the definition and use of quasinormal modes following [21]. A linear perturbation of a spherically symmetric BH (e.g. Schwarzschild) is a solution $\Phi(t,x)$ of some linear separable partial differential equation (PDE) derived from the equations for the fields and metric. Its Laplace transform $\hat{\Psi}(s,x)$ ($\Re s > 0$) satisfies a second order ODE with nonhomogeneous term $\mathcal{I}(s,x) = -[s\Phi(t,x) + \partial_t\Phi(t,x)]_{t=0}$ whose corresponding homogeneous equation is

$$\left\{-\frac{\partial^2}{\partial x^2} + U(x) + s^2\right\}\Psi(s,x) = 0, \tag{1}$$

with a given potential $U(x)$. As usual, x is the so-called *tortoise* coordinate, such that the BH horizon is put at $x \to -\infty$ and spacetime infinity at $x \to +\infty$. Moreover, also rotating Kerr black holes obey a separable PDE which still reduces to a couple of second order ODEs of the same kind. The bounded solutions at $x \to \pm\infty$ behave, respectively, as

$$\Psi_\pm(s,x) \sim e^{\mp sx}. \tag{2}$$

In terms of these Green functions we can write the solution of the original nonhomogenous ODE $\hat{\Psi}(s,x)$ and then take its inverse Laplace transform to obtain the perturbation in the form [21]

$$\Phi(t,x) = \sum_n e^{s_n t}\mathrm{Res}\left(\frac{1}{W(s)}\right)\Big|_{s_n} \times \int_{-\infty}^{\infty}\Psi_-(s_n,x_<)\Psi_+(s_n,x_>)\mathcal{I}(s_n,x')dx' + \ldots, \tag{3}$$

where, outside the main signal, the dots represent the prompt and late time contributions, $x_< = \min(x',x), x_> = \max(x',x)$, $W(s) = W[\Psi_+,\Psi_-]$ is the Wronskian of the regular solutions (2) and s_n are its zeroes

$$W(s_n) = 0. \tag{4}$$

Crucially, this Wronskian and ODE (1) themselves play an important an role and can be characterized by integrability physics and methods, as we are going to show in the next sections. Then the perturbation (3) is a series over the residues, which then depend only on the proportionality factors of the zeroes of W. Besides, condition (4) means that at these special points the two solutions (2) (in general linear independent) become proportional. Therefore, by setting $s_n = -i\omega_n$ we recover the usual definition of QNMs as the frequencies of plane wave solutions both incoming at the horizon and outgoing at infinity. Rigorously speaking, they have an imaginary part $\Im\omega_n < 0$, so that the perturbation they describe is damped to zero as $t \to \infty$ [21].

III. AN EXAMPLE: THE D3-BRANE

The D3-brane is described by the line element

$$ds^2 = H(r)^{-\frac{1}{2}}(-dt^2 + d\mathbf{x}^2) + H(r)^{\frac{1}{2}}(dr^2 + r^2 d\Omega_5^2), \tag{5}$$

where $\mathbf{x}$ are the longitudinal coordinates, $H(r) = 1 + L^4/r^4$ and $d\Omega_5^2$ denotes the metric of the transverse round S^5-sphere [3]. The ODE which describes the scalar field perturbation of the D3-brane is [3,22]

$$\frac{d^2\phi}{dr^2} + \left[\omega^2\left(1 + \frac{L^4}{r^4}\right) - \frac{(l+2)^2 - \frac{1}{4}}{r^2}\right]\phi = 0. \tag{6}$$

Upon the change of variables

$$r = Le^{\frac{y}{2}} \quad \omega L = -2ie^{\theta} \quad P = \frac{1}{2}(l+2), \tag{7}$$

the equation reduces to the generalized Mathieu equation

$$-\frac{d^2}{dy^2}\psi + \left[e^{2\theta}(e^y + e^{-y}) + P^2\right]\psi = 0. \tag{8}$$

It corresponds to the conformal self-dual Liouville theory with momentum P and rapidity θ [13,23]. Instead, to obtain the $\mathcal{N} = 2$ $SU(2)$ pure ($N_f = 0$) gauge theory with omega background in the Nekrasov Shatashvili limit $\epsilon_2 \to 0$, $\epsilon_1 = \hbar \neq 0$, we need to use the change of parameters $\omega L = -2i\frac{\Lambda_0}{\hbar}, \frac{1}{8}(l+2)^2 = \frac{u}{\hbar^2}$, where u is the Coulomb branch modulus and Λ_0 the instanton coupling [13]. Here we will just state the necessary results of the ODE/IM correspondence procedure for the Liouville field theory and refer to [13] for details and the proper 2D statistical interpetation. In fact, we want to explain ODE/IM correspondence for a more general case in Sec. III. In general, the starting point of the ODE/IM method is the definition of a basis of two regular solutions $\psi_{+,0}, \psi_{-,0}$ at $y \to \pm\infty$, respectively, and the Q function as their Wronskian,

$$Q(\theta,P) = W[\psi_{+,0},\psi_{-,0}](\theta,P). \tag{9}$$

Of course, $\psi_{+,0}(y) = \psi_{-,0}(-y)$ and cf. the next section for physical interpretation. Then we derived the fundamental QQ functional relation

$$Q(\theta+i\pi/2)Q(\theta-i\pi/2)=1+Q(\theta)^2, \tag{10}$$

from which the rest of the ODE/IM construction follows. Here and in the following we can omit the dependence on P as it stays fixed. Crucially, the QNMs condition (4) translates into

$$Q(\theta_n)=0, \tag{11}$$

namely the zeros of the Baxter's Q function which are the Bethe roots [24]. Now, we prove that condition (11) is equivalent to the quantization condition of the dual gauge period

$$A_D(a,\Lambda_{0,n},\hbar)=\pi n\hbar, \qquad n\in\mathbb{N}, \tag{12}$$

as conjectured by [5]. For this relevant relation holds

$$Q(a,\Lambda_0,\hbar)=i\frac{\sin\frac{A_D(a,\Lambda_0,\hbar)}{\hbar}}{\sin\frac{2\pi a}{\hbar}}, \tag{13}$$

involving the period a and its instanton dual $A_D=\partial\mathcal{F}/\partial a$ [7]: it can be proven by connecting the ODE/IM and Floquet basis where e^{A_D} shows up as governing the asymptotics [or by computing the lhs with TBA (15), cf. also [25]]. Reference [5] proposed the computation of A_D for (12) from the expansion of the prepotential $\mathcal{F}$ in powers (number of instantons) of Λ_0^4 [26,27]; the period a is related to the moduli parameter u (or P) through the Matone's relation [28,29]. In this manner, only the first instanton contributions are easily accessible and summing them up (naively) is accurate as long as $|\Lambda_0|/\hbar\ll 1$. Thus, this makes it hard to access QNMs values $|\Lambda_{0,n}|/\hbar\gg 1$.

As an improvement, we found this general procedure to an exact result, which resums all instantons. Let us define the $Y(\theta,P)=Q^2(\theta,P)$ function and derive from (10) the Y system

$$Y(\theta+i\pi/2)Y(\theta-i\pi/2)=(1+Y(\theta))^2. \tag{14}$$

Eventually, we solve it explicitly (*up to quadratures*) via a TBA integral equation for the pseudoenergy $\varepsilon(\theta)=-\ln Y(\theta)$,

$$\varepsilon(\theta)=\frac{16\sqrt{\pi^3}}{\Gamma(\frac{1}{4})^2}e^{\theta}-2\int_{-\infty}^{\infty}\frac{\ln\left[1+\exp\{-\varepsilon(\theta')\}\right]}{\cosh(\theta-\theta')}\frac{d\theta'}{2\pi}. \tag{15}$$

In this P does not appear explicitly, but fixes the solution by its asymptotic linear behavior $\varepsilon(\theta,P)\simeq+8P\theta$, $P>0$, at $\theta\to-\infty$. Eventually, the QQ system (10) characterizes the QNMs as $Y(\theta_n-i\pi/2)=-1$, i.e. the *TBA quantization condition*

$$\varepsilon(\theta_n-i\pi/2)=-i\pi(2n+1), \qquad n\in\mathbb{N} \tag{16}$$

TABLE I. QNMs of (6) through (15) and other methods (with $L=1$).

n	l	TBA	Leaver	WKB $l\to\infty$
0	1	*2.09176 − 0.50175i*	*2.09176 − 0.50181i*	2.121 − 0.500*i*
0	2	*2.80629 − 0.50098i*	*2.80629 − 0.50099i*	2.828 − 0.500*i*
0	3	*3.51783 − 0.50063i*	*3.51783 − 0.50063i*	3.535 − 0.500*i*
0	4	*4.22790 − 0.50044i*	*4.22790 − 0.50044i*	4.243 − 0.500*i*
0	5	*4.93711 − 0.50032i*	*4.93711 − 0.50032i*	4.950 − 0.500*i*
1	2	*2.52815 − 1.50966i*	*2.52879 − 1.51065i*	2.828 − 1.500*i*
1	4	*4.04748 − 1.50407i*	*4.04761 − 1.50446i*	4.243 − 1.500*i*
1	6	*5.51170 − 1.50225i*	*5.51175 − 1.50246i*	5.657 − 1.500*i*
1	8	*6.95539 − 1.50143i*	*6.95542 − 1.50156i*	7.071 − 1.500*i*
1	10	*8.38908 − 1.50100i*	*8.38910 − 1.50107i*	8.485 − 1.500*i*

which can be easily implemented by using the TBA (15) as Tables I and II show. In these tables and the subsequent ones, the italic digits highlight output values which resulted identical by truncation (not rounding). These values match very well those obtained by the method of continued fractions by Leaver and reach the $l\to\infty$ and $n\to\infty$ Wentzel-Kramers-Brillouin approximation (WKB) exact values, respectively [3,30]. Moreover the exponential map (7) allows one to obtain all overtones $\omega_n\propto -ie^{\theta_n}\in\mathbb{C}$ with increasing $|\Im\omega_{n+1}|>|\Im\omega_n|$ within $|\Im\theta|<\pi/2$. Although we encountered some numerical instability for large n, we recovered effectively the $n\to\infty$ value [31].

Within our setup of functional and integral equations for entire functions in θ (integrability), we can find other quantization conditions on the roots θ_n (QNMs). For instance, the TQ relation [13]

$$T(\theta)Q(\theta)=Q(\theta-i\pi/2)+Q(\theta+i\pi/2) \tag{17}$$

means $Q(\theta_n-i\pi/2)+Q(\theta_n+i\pi/2)=0$. This and the QQ relation (10) actually fixes $Q(\theta_n+i\pi/2)Q(\theta_n-i\pi/2)=1$ and then

$$Q(\theta_n\pm i\pi/2)=\pm i \tag{18}$$

are fixed, too. Again (10) around θ_n forces $Q(\theta+i\pi/2)=i\pm Q(\theta)+\cdots$ and $Q(\theta-i\pi/2)=-i\pm Q(\theta)+\cdots$ up to

TABLE II. QNMs of (6) (with $L=1$) through (15) and the large n asymptotic $\omega_n\simeq -1.85407(n+1/2)i$.

n	l	TBA	WKB $n\to\infty$
5	25	18.4914 − 5.50747*i*	−10.1974*i*
10	25	16.8121 − 10.5541*i*	−19.4677*i*
20	25	7.98806 − 21.0398*i*	−38.0084*i*
40	25	10^{-10} − 69.2803*i*	−75.0898*i*
100	25	10^{-14} − *184.166i*	−*186.334i*
700	25	-10^{-15} − *1298.47i*	−*1298.78i*
1600	25	-10^{-16} − *2967.31i*	−*2967.44i*
3000	25	-10^{-16} − *5565.08i*	−*5565.14i*

smaller corrections (dots). Therefore, (17) imposes

$$T(\theta_n) = \pm 2. \tag{19}$$

On equating the invariant trace of the monodromy matrix in the Floquet and ODE/IM bases (the coefficients in the latter involve the T function), we obtain a nice equality involving the period a ($= \nu$, Floquet index) [13,20]

$$T(\theta) = 2\cos 2\pi a. \tag{20}$$

In conclusion, condition (11) means that also the period a is quantized

$$a(\theta_n) = \frac{n}{2}, \qquad n \in \mathbb{N}. \tag{21}$$

This is exactly the condition used by [3]. Yet, here we have fixed the general limits of its validity as relying on specific forms of the TQ and QQ systems (17) and (10), respectively: it does not work in general, but we will see in the next section the specific conditions for its validity.

IV. GENERAL THEORY: INTERSECTING D3-BRANES AND BEYOND

To illustrate how our method develops in general, we consider the spacetime given by the intersection of four stacks of D3-branes in type IIB supergravity. These geometries are characterized by four different charges $\mathcal{Q}_i$ which, if all equal, lead to an extremal RN BH. In isotropic coordinates the line element writes

$$ds^2 = -f(r)dt^2 + f(r)^{-1}[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)], \tag{22}$$

with $f(r) = \prod_{i=1}^4 (1 + \mathcal{Q}_i/r)^{-\frac{1}{2}}$. The ODE describing the scalar perturbation is, with $\Sigma_k = \sum_{i_1<\ldots<i_k}^4 \mathcal{Q}_{i_1}\cdots\mathcal{Q}_{i_k}$ [8,19],

$$\frac{d^2\phi}{dr^2} + \left[-\frac{(l+\frac{1}{2})^2 - \frac{1}{4}}{r^2} + \omega^2\left(1 + \sum_{k=1}^4 \frac{\Sigma_k}{r^k}\right)\right]\phi = 0. \tag{23}$$

Changing variables as $r = \sqrt{4}\Sigma_4 e^y$ and

$$\omega\sqrt{4}\Sigma_4 = -ie^\theta M_j = \frac{1}{2}\frac{\Sigma_{2j-1}}{\sqrt{4}\Sigma_4{}^{2j-1}}e^\theta$$
$$P^2 = \left(l+\frac{1}{2}\right)^2 - \omega^2\Sigma_2, \tag{24}$$

($j = 1, 2$) the ODE takes the integrability form

$$-\frac{d^2}{dy^2}\psi + \left[e^{2\theta}(e^{2y} + e^{-2y}) + 2e^\theta(M_1 e^y + M_2 e^{-y}) + P^2\right]\psi = 0. \tag{25}$$

This theory may have its own 2D statistical field theory interpretation and also describes $SU(2)N_f = 2 = (1,1)$ gauge theory with flavor masses $m_j = \hbar M_j$, modulus $P^2 = u/\hbar^2$ and instanton coupling $\Lambda_2 = 4\hbar e^\theta$ [8,32]. In fact, in the $m_2 \to \infty$ limit the ODE represents $N_f = 1$ which links to the perturbed hairpin IM [33] (and further goes to $N_f = 0$). An advantage of ODE is that these limits are simple on it though cumbersome on the IM.

Now, we develop the ODE/IM procedure for this general model by extending the procedure of [13]. The regular solutions at $y \to \pm\infty$ are determined by

$$\psi_{\pm,0}(y) \simeq 2^{-\frac{1}{2}-M_\pm} e^{-\left(\frac{1}{2}+M_\pm\right)\theta \mp \left(\frac{1}{2}+M_\pm\right)y} e^{-e^{\theta\pm y}}, \qquad y \to \pm\infty, \tag{26}$$

with $M_+ = M_1$ and $M_- = M_2$. Equation (25) enjoys the discrete symmetries

$$\Omega_\pm:\ \theta \to \theta + i\pi/2, \qquad y \to y \pm i\pi/2,$$
$$M_1 \to \mp M_1, \qquad M_2 \to \pm M_2, \tag{27}$$

which can be formulated in the brane dictionary (24) as $\Sigma_1 \to \pm i\Sigma_1, \Sigma_2 \to -\Sigma_2, \Sigma_3 \to \mp i\Sigma_3, \Sigma_4 \to \Sigma_4$. Thanks to these symmetries, one can define other independent solutions $\psi_{-,k} = \Omega_-^k\psi_{-,0}$, $\psi_{+,k} = \Omega_+^k\psi_{+,0}$, and has these invariance properties $\Omega_+^k\psi_{-,0} = \psi_{-,0}$, $\Omega_-^k\psi_{+,0} = \psi_{+,0}$. These solutions are normalized so that their Wronskians are $W[\psi_{-,k+1},\psi_{-,k}] = -i\exp\{(-1)^k i\pi M_2\}$, $W[\psi_{+,k+1},\psi_{+,k}] = i\exp\{(-1)^k i\pi M_1\}$. One can now define a Baxter's Q function as the Wronskian

$$Q_{+,+}(\theta, P) = W[\psi_{+,0},\psi_{-,0}](\theta, P, M_1, M_2). \tag{28}$$

[We will use the notation $Q_{\pm,\pm} = Q(\theta, \pm M_1, \pm M_2)$, $Q_{\pm,\mp} = Q(\theta, \pm M_1, \mp M_2)$.] By the properties of Wronskians, we can write the linear connection relations as

$$ie^{i\pi M_1}\psi_{-,0} = Q_{-,+}(\theta + i\pi/2)\psi_{+,0} - Q_{+,+}(\theta)\psi_{+,1}$$
$$ie^{i\pi M_1}\psi_{-,1} = Q_{-,-}(\theta + i\pi)\psi_{+,0} - Q_{+,-}(\theta + i\pi/2)\psi_{+,1} \tag{29}$$

and taking their Wronskian we obtain the QQ system

$$Q_{+,-}\left(\theta + \frac{i\pi}{2}\right)Q_{-,+}\left(\theta - \frac{i\pi}{2}\right) = e^{-i\pi(M_1 - M_2)} + Q_{-,-}(\theta)Q_{+,+}(\theta). \tag{30}$$

From this, as above, one may define the Y functions $Y_{+,\pm}(\theta) = \exp\{i\pi(M_1 \mp M_2)\}Q_{+,\pm}(\theta)Q_{-,\mp}(\theta)$, $Y_{-,\pm}(\theta) = \exp\{i\pi(-M_1 \mp M_2)\}Q_{-,\pm}(\theta)Q_{+,\mp}(\theta)$ and then derive the Y system [in integrability variables (28)]

$$Y_{+,-}\left(\theta+\frac{i\pi}{2}\right)Y_{-,+}\left(\theta-\frac{i\pi}{2}\right)=[1+Y_{+,+}(\theta)]\times[1+Y_{-,-}(\theta)]. \quad (31)$$

In gravity variables $Y(\theta,\Sigma_1,\Sigma_2,\Sigma_3,\Sigma_4)=Y_{+,+}(\theta)$, it becomes

$$Y\left(\theta+\frac{i\pi}{2},-i\Sigma_1,-\Sigma_2,i\Sigma_3\right)Y\left(\theta-\frac{i\pi}{2},-i\Sigma_1,-\Sigma_2,i\Sigma_3\right)=[1+Y(\theta,\Sigma_1,\Sigma_2,\Sigma_3)][1+Y(\theta,-\Sigma_1,\Sigma_2,-\Sigma_3)] \quad (32)$$

so that it can be inverted into TBA integral equations

$$\varepsilon_\pm(\theta)=\left[f_\pm\mp\frac{i\pi}{2}\left(\frac{\Sigma_1}{\Sigma_4^{1/4}}-\frac{\Sigma_3}{\Sigma_4^{3/4}}\right)\right]e^\theta-\varphi*(\bar L_\pm+\bar L_\mp)(\theta)$$
$$\bar\varepsilon_\pm(\theta)=\left[\bar f_\pm\pm\frac{\pi}{2}\left(\frac{\Sigma_1}{\Sigma_4^{1/4}}+\frac{\Sigma_3}{\Sigma_4^{3/4}}\right)\right]e^\theta-\varphi*(L_\pm+L_\mp)(\theta), \quad (33)$$

where $\varepsilon_\pm(\theta)=-\ln Y(\theta,\pm\Sigma_1,\Sigma_2,\pm\Sigma_3,\Sigma_4)$, $\bar\varepsilon_\pm(\theta)=-\ln Y(\theta,\pm i\Sigma_1,-\Sigma_2,\mp i\Sigma_3,\Sigma_4)$, $L=\ln(1+Y)$, $\varphi(\theta)=(2\pi\cosh\theta)^{-1}$. Importantly, (29) yields $Q_{+,+}(\theta)=-ie^{i\pi M_1}\lim_{y\to+\infty}\psi_{-,0}(y)/\psi_{+,1}(y)$ whose $\theta\to+\infty$ limit gives the forcing terms $f_\pm=c_{+,\pm}+c_{-,\mp}$ as sums of $c_{\pm,\pm}=\int_{-\infty}^{\infty}[\sqrt{2\cosh(2y)\pm\frac{\Sigma_1}{\sqrt{4\Sigma_4}}e^y\pm\frac{\Sigma_3}{\sqrt{4\Sigma_4^3}}e^{-y}+\frac{\Sigma_2}{\sqrt{\Sigma_4}}}-\text{reg.}]dy$, and similarly for the barred ones upon replacing $(\Sigma_1,\Sigma_2,\Sigma_3)\to(i\Sigma_1,-\Sigma_2,-i\Sigma_3)$. These elliptic integrals can be expressed through a triple power series for small parameters. For a numerical solution of TBA (33), we have to numerically input l by fixing the asymptotic condition $\varepsilon_\pm(\theta)\simeq 4P\theta\simeq 4(l+1/2)\theta$, $\theta\to-\infty$, as following from the ODE (25) (precision improves upon adding the subleading constant). These TBA equations generalize those in [33,34] (Perturbed Hairpin IM and equal masses $N_f=2$ gauge theory, respectively), with benefit from the change of variables from integrability to gravity (and gauge) as first noted in [13]. Anew we find the QNMs to be given by the Bethe root conditions

$$Q_{+,+}(\theta_n)=0,\qquad \bar\varepsilon_+(\theta_n-i\pi/2)=-i\pi(2n+1) \quad (34)$$

and show in Tables III–V, IV and V their agreement with continued fraction (Leaver) method and WKB approximation for $l\to\infty$ and $n\to\infty$ [19,30,35]. For $\Sigma_1\neq\Sigma_3$ and $\Sigma_4\neq 1$ the original Leaver method is not applicable, as the recursion involves more than three terms [30] (even though a more involved matrix variant should be plausible [36]). Moreover, Table V shows (even better than Table II) an interestingly smooth transition to the $n\to\infty$ asymptotic behavior. The accuracy is sometimes lower than with the Leaver method, but in principle it could be increased by using higher integral cutoffs and grid resolution (up to infinity). All this reinforces the validity of our new TBA method as an effective, general completion and alternative to standard methods.

From ODE/IM we have evidence that generalizations of formula (13) can be derived (some are already available for the limit $N_f=1$ [37]). In this way we expect that in general the integrability Bethe roots condition (34), which we have shown to follow straightforwardly from BHs physics, corresponds in gauge theory to the quantization of the A_D period, as conjectured in [5].

The ODE/IM construction goes further because the presence of the irregular singularity of (25) at $y\to\pm\infty$

TABLE III. QNMs of (23) through (33) and other methods, with $\Sigma_1=\Sigma_3=0.2$, $\Sigma_2=0.4$, $\Sigma_4=1$.

n	l	TBA	Leaver	WKB $l\to\infty$
0	2	*1.47799 − 0.36814i*	*1.47789 − 0.36824i*	1.494 − 0.3660*i*
0	4	*2.68035 − 0.36664i*	*2.68035 − 0.36664i*	2.689 − 0.3660*i*
0	5	*3.27959 − 0.36641i*	*3.27959 − 0.36641i*	3.287 − 0.3660*i*
0	6	*3.87833 − 0.36629i*	*3.87833 − 0.36629i*	3.884 − 0.3660*i*
0	8	*5.07501 − 0.36615i*	*5.07501 − 0.36615i*	5.080 − 0.3660*i*
1	2	*1.21708 − 1.12174i*	*1.22004 − 1.12585i*	1.494 − 1.0979*i*
1	4	*2.54243 − 1.10434i*	*2.54246 − 1.10513i*	2.689 − 1.0979*i*
1	6	*3.78385 − 1.10090i*	*3.78380 − 1.10122i*	3.885 − 1.0979*i*
1	8	*5.00305 − 1.09963i*	*5.00300 − 1.09981i*	5.080 − 1.0979*i*

TABLE IV. QNMs of (23) through (33), with $\Sigma_1=0.1$, $\Sigma_2=0.2$, $\Sigma_3=0.3$, $\Sigma_4=1$, compared with the large l asymptotic.

n	l	TBA	WKB $l\to\infty$
0	2	*1.53080 − 0.39676i*	1.551 − 0.3946*i*
0	4	*2.78076 − 0.39525i*	*2.792 − 0.3946i*
0	8	*5.26781 − 0.39477i*	*5.274 − 0.3946i*
0	16	*10.2380 − 0.39472i*	*10.238 − 0.3946i*
0	32	*20.1738 − 0.39473i*	*20.165 − 0.3946i*
0	64	*40.0373 − 0.39473i*	*40.019 − 0.3946i*
0	128	*79.7581 − 0.39473i*	*79.729 − 0.3946i*
0	256	*159.185 − 0.39473i*	*159.15 − 0.3946i*

TABLE V. QNMs of (23) through (33), with $\Sigma_1=0.1$, $\Sigma_2=0.2$, $\Sigma_3=0.3$, $\Sigma_4=1$, compared with the large n asymptotic $\omega_n\simeq-1.43627(n+1/2)i$.

n	l	TBA	WKB $n\to\infty$
10	30	17.6290 − 8.32576	−15.0809*i*
20	30	13.4594 − 16.5027*i*	−29.4436*i*
30	30	0.19177 − 24.0114*i*	−43.8063*i*
50	30	0.02194 − 66.3169*i*	−72.5317*i*
160	30	0.00236 − *228.118i*	−230.522*i*
300	30	0.00054 − *430.662i*	−*431.600i*
1000	30	0.00005 − *1436.71i*	−*1436.99i*
2000	30	0.00001 − *2873.12i*	−*2873.26i*

allows us to define T functions (Stokes coefficients)

$$T_{+,+}(\theta) = -iW[\psi_{-,-1}, \psi_{-,1}], \tilde{T}_{+,+}(\theta) = iW[\psi_{+,-1}, \psi_{+,1}]. \tag{35}$$

Then, similarly to (29), the TQ relations follow upon expressing $\psi_{\pm,1}$ as linear combination of $\psi_{\pm,0}$ and $\psi_{\pm,-1}$,

$$\begin{aligned} T_{+,+}(\theta)Q_{+,+}(\theta) &= e^{i\pi M_2}Q_{+,-}\left(\theta - \frac{i\pi}{2}\right) \\ &\quad + e^{-i\pi M_2}Q_{+,-}\left(\theta + \frac{i\pi}{2}\right), \\ \tilde{T}_{+,+}(\theta)Q_{+,+}(\theta) &= e^{i\pi M_1}Q_{-,+}\left(\theta - \frac{i\pi}{2}\right) \\ &\quad + e^{-i\pi M_1}Q_{-,+}\left(\theta + \frac{i\pi}{2}\right). \end{aligned} \tag{36}$$

As above (20), we equate the invariant trace of the monodromy matrix in the Floquet and ODE/IM bases (the latter involving the Ts) and obtain equalities showing now the masses M_1 and M_2, too [20],

$$\begin{aligned} 2\cos 2\pi\nu + 2\cos 2\pi M_1 &= \tilde{T}_{+,+}(\theta)\tilde{T}_{-,+}\left(\theta + i\frac{\pi}{2}\right), \\ 2\cos 2\pi\nu + 2\cos 2\pi M_2 &= T_{+,+}(\theta)T_{+,-}\left(\theta + i\frac{\pi}{2}\right). \end{aligned} \tag{37}$$

This generalizes a formula conjectured in [33] (and the same method should give that guessed in [14]). Besides, also in this case the period $a = \nu$, the Floquet index.

Only for *equal masses* $M_1 = M_2 \equiv M$ we can make considerations on TQ and QQ systems (36), (30) like in Sec. III and conclude that

$$T_{+,+}(\theta_n)T_{-,-}(\theta_n) = 4, \tag{38}$$

which generalizes (19). Then, we use $T_{+,-}(\theta + i\frac{\pi}{2}) = T_{+,+}(\theta)$, $\tilde{T}_{-,+}(\theta + i\frac{\pi}{2}) = \tilde{T}_{+,+}(\theta)$ and (38) in (37) (invariant upon changing sign of M) and conclude

$$[\cos 2\pi\nu + \cos 2\pi M]_{\theta=\theta_n} = \pm 2. \tag{39}$$

Still for equal masses, there is the extra symmetry $\psi_{+,0}(y) = \psi_{-,0}(-y)$ as in the previous $N_f = 0$ case. It exchanges infinity ($y = \infty$) and the (analog) horizon ($y = -\infty$), leaving the photon-sphere ($y = 0$) fixed as in [35], and imposes the identity $\tilde{T}_{+,+}(\theta) = T_{+,+}(\theta)$.

V. PERSPECTIVES

From the general treatment in the previous section it is manifest that our method applies to many other theories, both to the angular and the radial problem [5]. We can list, for instance, the $SU(2)N_f = (2,0)$ gauge theory and the associated gravity counterparts, like D1D5 fuzzballs and CCLP 5D BHs, as well as $SU(2)N_f = 3$ and the asymptotically flat Kerr BH [8]. Instead of only irregular singularities as above, in these cases the ODE shows regular ones as well: so that the associated integrability structure may share some similarity with the original correspondence of conformal minimal models [15]. Another simple but tricky case happens when all the singularities are regular, e.g. $SU(2)N_f = 4$ or asymptotically AdS BHs, since then the ODE has only regular singularities and no Stokes sectors. Actually, our procedure should still apply as in [38] for the supersymmetric XXZ spin chain. Yet, more recently these regular equations were discovered to appear as ODEs in the spectral parameter as derived from a "standard" (irregular) ODE [18]: this bispectrality map is to be followed. Moreover, second or higher order BH perturbations are governed by the same homogeneous ODEs (the difference residing in the nonhomogeneous term) [39]. In any case, the TBA method seems in many cases convenient with respect to other methods, like Leaver or SW, which cannot always be effectively applied. The latter for instance needs a clever numerical resummation procedure [5]. Instead, when the PDE is not separable (such as in the case of Kerr-Newman BH with nonscalar perturbations), one should pursue a possible extension of the present (connection coefficients and monodromy) theory to PDE, which is very important in itself beyond applications.

Eventually, much of the BH theory seems to go in parallel to the ODE/IM construction and its 2D statistical field theory interpretation, beyond the determination of QNMs: as an example, the *absorption coefficient* seems a ratio of Qs. In this respect it is relevant to understand the role of ODE/IM excitations [40,41] which preserve the form of the functional equations, but change the integral ones.

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DATA AVAILABILITY

No data were created or analyzed in this study.

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