

D-shell mixing in light baryons and its effect on the orbital motion

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The standard description of the nucleon in the nonrelativistic quark model is an $1S, L = 0$ state without orbital motion. Yet, there are several indications from phenomenology that an admixture of states with nonzero orbital motion may be substantial. In this paper we focus on the “second shell” of the nucleon excitations (D-shell), for which we give a modern description of the wave functions. We follow it by investigating what we call a “maximal mixing” scenario, assuming a hypothetical long-range tensor force. We give the explicit wave functions for all states, before and after mixing, and reassess many predictions such as the magnetic moments, the standard and transitional form-factors from the nucleon to N^* . Unexpectedly, in this scenario we can reproduce the long-puzzling features of the Roper resonance $N^*(1440)$. But even in this extreme case, the admixture of the $1D, L = 2$ state to a nucleon remains significantly smaller than expected from phenomenology.

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I. INTRODUCTION

This paper is a continuation of our efforts to address some open issues in hadronic spectroscopy. In our previous paper [1] we focused on the first shell (negative parity) $L = 1$ baryon resonances. Yet, besides some technical progress, we basically confirmed the classic conclusion by [2] that the quasilocal spin-spin and tensor forces can describe the first shell states, while the spin-orbit forces need to be canceled by a long-distance negative contribution following from the Thomas term due to the QCD string.

We then derived explicitly the orbital-color-flavor-spin wave functions with Fermi statistics, using a novel method based on representations of the permutation group S_3 . In the subsequent paper [3] we generalized this approach to larger systems, with five, six, and even 12 quarks.

A natural step is an extension of this construction to the second shell (positive parity). For that we define the basic antisymmetric states, and use them to evaluate various observables: matrix elements (magnetic moments), transitional form factors, and so on. For a nucleon admixture to the tensor state, the main observables are various form factors.

As we hope to demonstrate, the spectroscopy of the second shell baryonic resonances is not settled yet, with some open questions. Large splittings between these states are challenging to perturbative (or in fact any short-range) spin-dependent potentials. One of 50-year-old puzzle for the quark model is the large downward shift of the famed Roper resonance $N^*(1440)$. Also the issue of admixture amplitude of the nucleon to the second shell states remains unsettled. Using different tensor potentials, we will explore the case of “a maximal mixing scenario,” induced by (so far hypothetical) a long-range tensor force.

Before outlining the main structure of this paper, we will make a few more general remarks on the quark model and phenomenology. Our previous attempts to bridge the standard hadron spectroscopy in the rest frame, with the light-cone description, are still facing open questions.

One issue is the famous “nucleon spin problem”: empirical structure functions (at low normalization scale) still suggest that orbital motion contributes about half of the nucleon spin. Generally, the orbital motion can originate from different sources. One—investigated in this paper—is via admixture to the second shell states with $L = 2$. It may also come via the nucleon admixture to $qqq + q\bar{q}$ pentaquark states. This mechanism, as well as other issues related to pentaquarks, is to be studied in our next papers.

This paper is structured as follows. We start with few short introductory subsections, containing historic references, general information and, briefly, the status of the issues to be discussed in the main part of the paper. In Sec. **IA** we remind the reader of the structure of the second shell baryons. In Sec. **IB** we recall the main observables and puzzles associated with those. The nucleon spin problem is

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widely known and discussed, but for completeness we will recall it briefly in Sec. IC, with emphasis on the orbital part. Many technical details—e.g., explicit wave functions—are delegated to Appendixes, while the main text focuses on the physics and the calculation results.

A. Spectroscopy of the second shell baryons

The positive parity excited states of the baryons have orbital momentum up to $L = 2$ and spin up to $S = 3/2$, so their total angular momentum can be $J = 1/2 \dots 7/2$. In this paper we focus on light u, d quark states. Furthermore, since we will be mostly interested in their admixture to the nucleon, we will focus on a subset of N^* states with the same angular momentum and isospin, namely $I = 1/2$, $J = 1/2$.

QCD-based quark models of baryons, including constituent quark masses, confining potentials, and perturbative spin-spin and tensor forces, have been developed in the 1970s. Isgur and Karl in their classic papers, on the first [4] and second [2] shell, laid the foundation. Their model described rather well the spectrum of the excited baryons known at that time, and predicted many more which (nearly all) were found later. For recent comprehensive reviews of the quark model of baryons see, e.g., [5,6].

By revisiting the baryon spectroscopy again, there is no need for some simplifying assumptions made in those works. Instead of assuming Gaussian wave functions, one can calculate those for realistic confining forces in the hyperdistance approximation. The wave function satisfying Fermi statistics can be constructed explicitly, via representations of the permutation group. The calculations can be carried using modern analytic computer tools, such as Wolfram Mathematica. The spin-isospin space of “monoms” $2^3 \times 2^3 = 64$ (for nonstrange baryons) is used in spin-tensor form. All pertinent operators of spin-dependent forces can be simply represented as matrices in this space, via the “KroneckerProduct” command, giving a very straightforward shape to all calculations. In this work we will carry the Mathematica and the standard group representation analyses for cross-check.

The main motivation for the present work was to check explicitly whether the observed splittings of states in the second shell yield strong mixing, and in particular derive some estimates of the tensor mixing with the ground state (proton). In other papers of our series we derived the quark-quark central and spin-dependent forces using two “vacuum structure” models, the Instanton Liquid Model (ILM) originating from [7], and more recently the Dense Liquid Model including instanton-antiinstanton molecules [1,8]. Both models turn out to produce forces that are rather short-range, good for their applications to quarkonia but not very useful for the second shell baryons, with their wave functions (WFs) vanishing at small distances. To address this issue, we will introduce certain hypothetical long-distance

tensor forces, which we will use to evaluate some observables, and assess their phenomenological limitations.

B. Magnetic moments and form-factors

The central observable related to hadronic spins is their magnetic moments. And indeed, by the 1960s it was shown that the magnetic moments are reasonably described by nonrelativistic qqq models *without* any orbital admixture. If admixture of $L \neq 0$ states, either in qqq or pentaquark $qqqq\bar{q}$ sector is there, some mechanism of cancelations between the spin and orbital contributions in the composition of the magnetic moments is needed.

The next sensitive observables are “transitional” matrix elements between on-shell states of pertinent operators (e.g., electromagnetic currents) with a finite momentum transfer, or transitional form factors. Starting from the diagonal nucleon-nucleon form factors, these are the electric and magnetic Sachs form factors. The hard (photon) current contains either quark charges (Dirac) or quark spins (Pauli).

For a while, it was accepted that the electric-to-magnetic ratio of the form factors is independent of the momentum transfer Q . This supported the idea that the spin and radial parts of the nucleon wave function may factorize. Also, it was expected that the neutron electric form factor G_E^n be zero. And yet, experiments at Jefferson Laboratory (JLAB) [9–11] have shown that both predictions are not supported by experiment. The ratio G_E^p/G_M^p decreases with Q , with even a possible sign flip at the upper end of the available measured range or $Q^2 \leq 8.5 \text{ GeV}^2$. The electric form factor of the neutron is nonzero $G_E^n \neq 0$.

(Let us note that the JLAB data fall in the range of $Q^2 \leq 9 \text{ GeV}^2$, or $Q/2 \approx 1.5 \text{ GeV}$. These experiments are neither at small Q where a soft chiral cloud is expected to dominate, nor at large Q where hard perturbative predictions are expected to work. This is the intermediate or “semihard” region, where form factors have to be calculated nonperturbatively, see discussion in [12].)

How the mixing between states may affect the form factors, has been discussed in the literature. Specifically, the mixing of the ground state nucleons (proton and neutron) with states of the second shell was addressed in [13]. In this work, some schematic model was shown to reproduce the JLAB data on the electric and magnetic, proton, and neutron form-factors. Yet for this to happen it requires a rather strong and specific mixing. As the reader will see from our analysis below, we disagree with the selection of the states in this work, dominant in the admixture, and its mechanism.

When evaluating the form factors, we encountered a number of issues that we would like to mention here. At large momentum transfer $Q \gg M$, relativistic effects are important, so it is natural to boost wave functions, from the c.m. to the light-cone. Examples for the nucleon and delta are given in [14], where the form-factors are also analyzed.

At small Q , we may naively use the Fourier transforms of the nonrelativistic c.m. wave functions, as done, e.g., in atomic physics. Yet it does not work: one reason is that constituent quarks, unlike electrons, are not pointlike objects. The radii of the baryons are affected by their “chiral clouds.”

Yet at intermediate Q the nonrelativistic wave functions can still be used, provided a “simplified boost” (e.g., in [13])

$$\psi(\vec{p}_\perp, p_l) \rightarrow \frac{1}{\gamma} \psi(\vec{p}_\perp, p_l/\gamma)$$

is made, via compression of the longitudinal sizes by a relativistic gamma factor $1/\gamma = \sqrt{1-v^2}$. Below, we will discuss to what extent this prescription can be used to describe the form factors in the intermediate “semi-hard” range.

Furthermore, the *transitional form factors* from the nucleon to N^* and Δ resonances provide a large set of observables, sensitive to their internal structure. In particular, the Roper resonance $N^*(1440)$ famously shows a transitional form factor in clear disagreement with a “minimal scenario,” ascribing it to the radial 2S-excitation. This fact is another strong motivation for the present work.

C. The nucleon spin/orbital problem

The spin structure of the nucleon is a widely discussed issue, which does not need an extensive introduction. There is no shortage of in-depth reviews, e.g., [15]. The so called spin sum rule is

$$\frac{1}{2} = \frac{1}{2} \Delta\Sigma + L_q + L_g + \Delta G \quad (1)$$

where the first (last) term is the integrated quark (gluon) parton distribution functions of a polarized nucleon, e.g.,

$$\Delta\Sigma = \sum_q \int dx \Delta q(x). \quad (2)$$

The naive expectation that the first term is dominant was shattered by experimental data, and more recently by first-principle lattice simulations. Only about *half* of the nucleon spin is coming from the quark spins [16] (and references therein), while the other half must come from the other contributions.

Looking at this issue in more detail, one should clarify its Q^2 dependence. At large Q^2 it is defined by perturbative evolution, and in the semihard regime by the magnitude of the twist-3 power effects. (We expect to complete a separate paper on this issue.) Most specialized reviews on the spin issue, use either the standard lattice normalization point $\mu = 2$ GeV, or an even higher scale related to the actual momentum transfers in experiments. A detailed discussion

of the data evolution to the low normalization point has been made, e.g., by [17].

The lattice results suggest the three contributions $\Delta\Sigma, L_q, L_g$ be similar, or about $\sim 1/3$ each [18].

Yet our series of papers devoted to hadronic spectroscopy describe it at a “low scale” $\mu \leq 1/\rho \sim 0.6$ GeV where there are no perturbative gluons. The only glue is semiclassical, in form of “instantons” or their bosonized zero modes. The description of hadrons at low resolution however, requires the inclusion of chiral field clouds, $\vec{\pi}, \sigma$. In the light-front formulation, these effects are captured by higher Fock contributions, e.g., as $qqq + \bar{q}q$, as was done, e.g., by Miller in [19] and subsequent works.

At high resolution, where perturbative QCD description should hold, the electromagnetic form factors should be related by chirality flips. If the latter are due to quark masses, they should be very strongly suppressed. However, as pointed out in [20], the needed chirality flip can be due to insertions of the orbital momentum operators. This idea shows again that the magnitude of orbital motion in the nucleon is an important issue.

Furthermore, in this paper we do not discuss the contributions of the sea quarks. In this case, the orbital contribution is about $L_q \approx 0.6$ (see, e.g., our paper [16]).

In summary, if the quark orbital angular momentum L_q provides a significant contribution to the nucleon spin, it is in direct contradiction with the nonrelativistic quark models where nucleons are assigned to the S-shell ($L = 0$) states.

II. THEORY AND PHENOMENOLOGY OF THE SECOND SHELL BARYONS

A. Generalities

The two pillars of hadronic spectroscopy are two non-perturbative phenomena characterizing the QCD vacuum. The first is the spontaneous breaking of chiral $SU(N_f)_c$ symmetry, leading to moving *constituent quarks* with effective masses ~ 350 MeV [much larger than the so called *Lagrangian or current* masses proportional to the Higgs vacuum expectation value (VEV)]. The second is *electric color confinement* producing a *confining potential*, linear at large distances σr . Both are believed to be induced by topological solitons: instantons for chiral breaking, and magnetic monopoles for confinement (for extensive pedagogical text see, e.g., [21] and references therein).

The nonrelativistic quark model was formulated in the 1960s, with light mesons and baryons, and complemented in the 1970s by heavy quarkonia. During the last decades the field of hadronic spectroscopy has been revived, with the discovery of multi-quark states, mostly by B factories, as well as LHCb and other LHC detectors.

Three quark systems are traditionally described using two Jacobi coordinates $\vec{\rho}, \vec{\lambda}$ (or momenta), with either location of the c.m. (or total momentum) put to zero,

see the definitions in Appendix A. If all three quarks have the same masses (light u, d , or sss or heavy ccc, bbb) then the ground state wave function is approximated by a spherically symmetric 6D wave function $\psi(Y)$ depending on the radial “hyperdistance” variable

$$Y^2 = \vec{\rho}^2 + \vec{\lambda}^2 = R^2. \quad (3)$$

The hyperspherical (or hypercentral) approximation originates from nuclear physics in the 1960s, where it was successfully used for the description of few-body $A = 3, 4, \dots$ nuclei. If quark masses are the same, the kinetic energy is proportional to a pertinent Laplacian. While the binary potentials are not 6D spherically symmetric, their nonspherical components were evaluated and found to be small. Solving only the 6D radial Schrödinger equation is much simpler than the use of alternative approaches (like the Faddeev equation or the expansion in a large set of basis functions). By now it is considered as a standard method, used by multiple authors, which we will also follow here. We have already used it for baryons of the first shell [1]. The actual calculations for the radial wave functions of the second shell are described in Appendix B.

The classic papers [4] have considered the negative parity baryons from the P shell ($L = 1$), and found that using two of them and disregarding the spin-orbit, they achieve a good description of the observed spectrum. Recently we repeated this analysis in [1] using explicitly the asymmetric wave functions and modern data, and came to the same conclusion: the matrix elements of the spin-orbit V_{SL} from our fit are zero inside the error bars.

This paper focuses on the second, positive parity D shell of nucleon resonances. The classic Isgur-Karl [2] discusses this shell, in a model with only spin-spin and tensor forces, where only one common parameter—basically the gluon exchange coupling α_s fitted from the $N - \Delta$ splitting—was used. Following them, we will also ignore the spin-orbit forces, and focus on the spin-spin and tensor ones.

Yet, as we will show below, there is much room for improvement. Before we come to our calculation, let us add some general comments on that. The spread of the second shell resonances is large: their masses spread between 1400 and 2100 MeV. If in the zeroth order (without spin-dependent forces) one starts from the oscillator states (as done by Isgur-Karl subsequent works) with degenerate masses, this spread is attributed to spin forces. If these matrix elements are indeed large, ~ 300 MeV, they should completely mix the second shell states. Furthermore, assuming that the matrix element between the nucleon and the “tensor” ($L = 2$) state is of similar magnitude, one would expect the nucleon to strongly participate in such a mixing. These are effects we are going to study in detail below.

B. The radial “hyperspherical” equation

The explicit calculation is detailed in the Appendixes. While most of the analysis is rather standard, we decided to detail it, including the explicit wave functions of the basis states.

In Appendix A we define the Jacobi coordinates used, the hyperdistance Y (we use $Y = R$ interchangeably to remain consistent with the notations in our earlier papers) and the six-dimensional Laplace-Beltrami operator. The latter includes the four angles of two 3D vectors $\vec{\rho}, \vec{\lambda}$, complemented by the fifth angle χ . Its eigenvalues follow from the radial hyper-spherical equation, with “realistic” confinement, which is solved numerically using *Mathematica*. The procedure and resulting radial wave functions are given in Appendix B.

We will mainly focus on the $J^P = 1/2^+$ nucleons, starting from the nucleon itself, the 2S state, and three more states of “the second shell” referred to as “scalar,” “vector,” and “tensor” (for the details see the Appendix). The radial hyperspherical equation (we will refer to it as “the zeroth order results”) gives their masses as

$$H_0 = \text{diag}(1070, 1665, 1956, 1956, 2048) \text{ (MeV)}. \quad (4)$$

The results are by no means a degenerate shell of the oscillator basis, as assumed in many works. In fact this spectrum is much closer to the experimental ones (from particle data tables)

$$(938, 1440, 1710, 1880, 2100) \text{ (MeV)}. \quad (5)$$

The largest deviation (about -200 MeV) is seen for the second state, the famous “Roper resonance puzzle” debated in the literature for 60 years since its discovery. For the other states, the shifts are smaller. One may therefore ask which spin-related forces can mix these states in such fashion: we will provide an example of those below. Other room for improvements compared to literature we found is that the non-Gaussian shape of the wave functions significantly impact the resulting form factors.

C. Orbital-spin-isospin structure of the wave functions

For baryons the antisymmetric color wave functions factorize, and so the next step is to combine multiple parts of the orbital-spin-isospin wave function into a unique combinations consistent with Fermi statistics (that is being symmetric under quark permutations), see the next subsection. Given that, one then construct the matrix elements of the spin-dependent potentials, i.e., spin-spin, tensor, and spin-orbit V_{SS}, V_T, V_{SL} , respectively.

The explicit form of the orbital-spin-flavor wave functions is rather technical, and is summarized in the Appendixes. Here we will formulate an alternative to the traditional representations of the $SU(6)$ spin-flavor group,

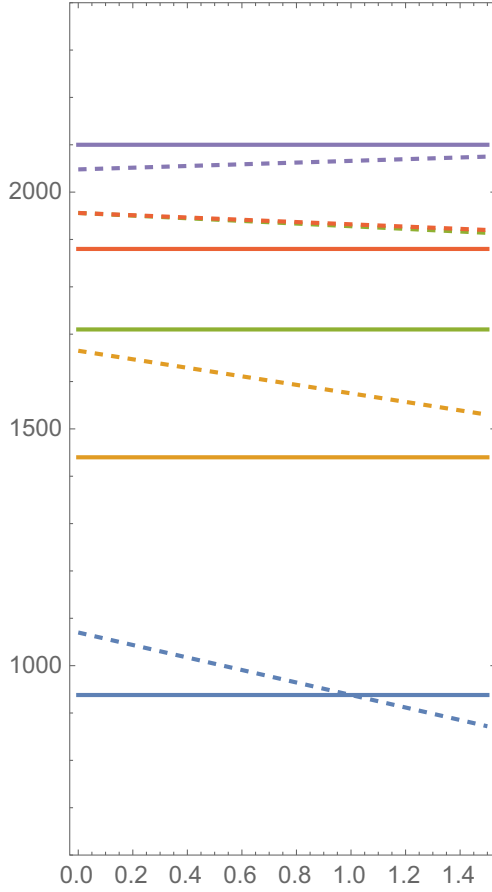


FIG. 1. Five horizontal colored lines indicate the observed masses of the nucleon and other four $J^P = 1/2^+$ resonances, in MeV, from the particle data tables. The horizontal coordinates indicate the strength of the spin-spin forces, and the dashed line shows the corresponding masses from the diagonalized Hamiltonian containing only V_{SS} .

as was done by Isgur-Karl and all subsequent literature. (One may find a bit more detail on this construction in the Appendix of [5]. We have not found where they are given explicitly, to check both their permutation symmetry and pertinent J value.)

Before going further, let us discuss the possible mixing by spin-dependent forces. The spin-spin V_{SS} does not change the orbital momentum, so it will only have diagonal matrix elements. The tensor V_T contains an $L = 2$ part, so it has nonzero matrix elements between all the $L = 0$ and the ψ^C $L = 2$ states, particularly for the nucleon. (Our mixing matrix has many zeroes, which we do not see in the reported Isgur-Karl mixing matrix. It is not puzzling in general as the definitions of states are different.)

Anyway, we will reconstruct these states with a focus on Fermi statistics, in other words asymmetry under all quark permutations. The orbital-flavor-spin wavefunctions for the D-shell were rederived in [1] from the representations of the S_3 symmetry group generators, deriving them explicitly as a tensor product of four objects

coordinate $\times$ coordinate $\times$ flavor $\times$ spin.

For completeness and as a check, the group theoretical construction is also summarized in the Appendixes.

D. Spin-dependent forces

The general expression for interaction of quark pair i, j is used in the form

$$H_{sd}^{ij} = V_{SS}^{ij} \vec{S}_i \cdot \vec{S}_j + V_T^{ij} [(\vec{S}_i \cdot \vec{n}_{ij})(\vec{S}_j \cdot \vec{n}_{ij}) - (\vec{S}_i \cdot \vec{S}_j)/3] \quad (6)$$

where $\hat{n}_{ij}$ is the unit vector along the line connecting two quarks. Following the Isgur-Karl study and subsequent studies (including ours), we neglect the spin-orbit contribution. Since our wave functions are properly symmetrized under permutations, it is sufficient to calculate their matrix elements for the $i = 1, j = 2$ pair, and then multiply the result by 3, the number of pairs.

Using the explicit wave function described in the preceding section and the Appendix, we get the spin-dependent Hamiltonian as the following simple matrix

$$H_{SS} = -\frac{3}{4} \begin{pmatrix} \langle V_{SS}^{1S1S} \rangle & 0 & 0 & 0 \\ 0 & \langle V_{SS}^{2S2S} \rangle & 0 & 0 \\ 0 & 0 & \langle V_{SS}^{3S3S} \rangle & 0 \\ 0 & 0 & 0 & \langle V_{SS}^{1P1P} \rangle & 0 \\ 0 & 0 & 0 & 0 & -\langle V_{SS}^{1D1D} \rangle \end{pmatrix}. \quad (7)$$

The first state is the nucleon (1S), the second is the (2S) state, and the remaining three are scalar (3S), vector (1P), and tensor (1D) states of the second shell, as defined above. Although it is not shown explicitly, the matrix elements involve coordinate-dependent potentials, with the pertinent wave functions of the states. Note that for the 1-2 quark pair the inter-distance is $r_{12} = \sqrt{2}\rho$. It does not depend on the λ coordinate, while the radial wave functions are 6D spherical and depend on the hyperdistance $Y = R = \sqrt{\rho^2 + \lambda^2}$. Note that since the coefficient of V_{SS} depends only on the total spin of the state, therefore only the last—tensor $S = 3/2$ —state has a contribution with a positive sign.

In Fig. 1 we show separately the effects of the spin-spin contributions as a function of their strengths. The results follow from the diagonalization of the Hamiltonian $H_0 + H_{SS}$, using the radial hyperspherical equation described earlier. The spin-spin forces originate from both perturbative and nonperturbative effects, and have been studied on the lattice. Some details are given in the Appendix. Although their origins are different, their contributions are dominated by small distances, $r_{12} \sim 1 \text{ GeV}^{-1}$, and are stronger in the nucleon than in the second shell states. If V_{SS} is normalized to some magnitude such that at

$x = 1$ (the splitting gives the physical nucleon mass), the near-local spin-spin forces cannot describe the splittings in the second shell.

E. The tensor operator

The pairwise tensor interactions are given by

$$\mathbb{V}_T(1, 2, 3) = \sum_{i < j=1}^3 V_T^{ij} T^{i,j} \quad (8)$$

$$T^{i,j} = (S_i \cdot \hat{r}_{ij} S_j \cdot \hat{r}_{ij} - S_i \cdot S_j / 3). \quad (9)$$

They split the even-parity states of the second shell, and can induce mixture with other shells, e.g., the nucleon. This operator can be averaged over five orbit-spin-isospin wave

functions derived and given explicitly in Appendixes. Most of the values in the ensuing 5×5 matrix are zero, except when the “tensor” state is involved.

The 12 relative distance in the tensor is

$$r^{12} = \sqrt{2}\rho = Y\sqrt{2}\cos(\chi),$$

which means that the integration over four angles (two solid angles) can be performed explicitly

$$\int \frac{d\phi_\rho d\theta_\rho d\phi_\lambda d\theta_\lambda}{(4\pi)^2} \sin(\theta_\rho) * \sin(\theta_\lambda) \dots$$

with the results

$$\begin{aligned} \langle P, 1S | T^{12}(Y \cos(\chi)) | VL2 \rangle_{\phi_\rho, \theta_\rho, \phi_\lambda, \theta_\lambda} &= \cos(\chi)^2 / (3\sqrt{5}), \\ \langle 2S | T^{12}(Y \cos(\chi)) | VL2 \rangle_{\phi_\rho, \theta_\rho, \phi_\lambda, \theta_\lambda} &= \cos(\chi)^2 / (3\sqrt{5}), \\ \langle VL0 | T^{12}(Y \cos(\chi)) | VL2 \rangle_{\phi_\rho, \theta_\rho, \phi_\lambda, \theta_\lambda} &= 1/9 \sqrt{2/5} \cos(\chi)^2 (2 - 5 \cos(2\chi)), \\ \langle VL1 | T^{12}(Y \cos(\chi)) | VL2 \rangle_{\phi_\rho, \theta_\rho, \phi_\lambda, \theta_\lambda} &= -\frac{\sin(2\chi)^2}{9\sqrt{10}}, \\ \langle VL2 | T^{12}(Y \cos(\chi)) | VL2 \rangle_{\phi_\rho, \theta_\rho, \phi_\lambda, \theta_\lambda} &= (2/45) \cos(\chi)^2 (-5 + 2 \cos(2\chi)). \end{aligned} \quad (10)$$

The right-hand side still needs to be averaged over two remaining variables Y, χ with the appropriate radial wave functions $\langle \psi^{I*}(Y) V_T^{12}(\chi) \psi^J(Y) \rangle$, with the χ measure

$$\int \frac{d\chi \cos(\chi)^2 \sin(\chi)^2}{(\pi/16)} \dots$$

We will consider two limiting cases of the tensor forces, the perturbative “short-range” tensor (second derivative of Coulomb potential) and a hypothetical “long-range” tensor

$$V_T^{\text{short range}} \sim \frac{1}{\cos^3(\chi)}, \quad V_T^{\text{long range}} \sim \cos(\chi) \quad (11)$$

to be motivated below. In the latter case the averaging over χ is straightforward with the result

$$\begin{aligned} \langle P, 1S | V_T^{12} | VL2 \rangle_{Y,\chi} &= \frac{128}{315\sqrt{5}\pi} \langle P, 1S | V_T^{12} | VL2 \rangle_Y, \\ \langle 2S | V_T^{12} | VL2 \rangle_{Y,\chi} &= \frac{128}{315\sqrt{5}\pi} \langle 2S | V_T^{12} | VL2 \rangle_Y, \\ \langle VL0 | V_T^{12} | VL2 \rangle_{Y,\chi} &= \frac{128\sqrt{2/5}}{2835\pi} \langle VL0 | V_T^{12} | VL2 \rangle_Y, \\ \langle VL1 | V_T^{12} | VL2 \rangle_{Y,\chi} &= -\frac{256\sqrt{2/5}}{2835\pi} \langle VL1 | V_T^{12} | VL2 \rangle_Y, \\ \langle VL2 | V_T^{12} | VL2 \rangle_{Y,\chi} &= -\frac{3328}{14175\pi} \langle VL2 | V_T^{12} | VL2 \rangle_Y \end{aligned} \quad (12)$$

where in the right-hand side only averaging of the radial wave functions remains. The short-range case needs regularization, which is discussed in the Appendix.

The tensor potential $V_T(r)$ can also be expressed as a correlator of Wilson lines appended by two magnetic field strengths [22]. To our knowledge, it has not yet been studied on the lattice. Forces due to instantons or instanton molecules, by their nature, should only apply at sufficient small distances $r_{12} \sim (1-2)\rho_{\text{inst}} \sim (1-2)1/3$ fm, smaller than sizes of the second shell nucleon resonances in question. So, the potential V_T at large distances remains

mostly unknown from first principles, besides phenomenological flavor dependent meson exchange models.

F. Hypothetical long range forces

Most of the studies of the tensor interaction in literature have assumed its Coulomb-like perturbative form $V_T \sim \alpha_s/r^3$, and [in spite of the use of a large coupling constant $\alpha_s \sim O(1)$] have concluded that the effects are small. If we add the instanton-induced contribution, the results are still small in light of their short range character. Yet, looking at the spin-orbit interactions, it was necessary to require that their short-range and long-range contributions cancel out in the first shell. If so, it is then natural to assume that in the second shell, the long-range effects are dominant.

A sketch of the long-range quark-quark interactions is shown in Fig. 2. It is known that they should be dominated by the QCD string. The black dots in the figure refer to certain operators describing the quark interactions with the string endpoints. Most studies describe the QCD string as a Nambu-Goto string, with a simple scalar coupling to the quarks. However, generically, the end points should carry both mass and spin couplings [23]

$$S = \sigma_T \int_{-T/2}^{T/2} d\tau d\sigma \sqrt{h} + \int_{-T/2}^{T/2} d\tau \left(M \sqrt{\dot{X}^2} - \frac{1}{2} \sigma_{\mu\nu} \frac{\dot{X}^\mu \ddot{X}^\nu}{\dot{X}^2} \right) \quad (13)$$

with the volume element given by the determinant

$$h = \det(\eta_{\mu\nu} \partial_a X^\mu \partial_b X^\nu). \quad (14)$$

The last boundary term is the spin factor that enforces the spin-orbit (Thomas precessionlike) interaction at the end-point of the world lines [24,25]. Indeed, if the string endpoints have velocity and acceleration in different directions (e.g., if those are rotating in circles, as sketched in Fig. 2), then spin-dependent forces arise. While the mechanism by which these forces are generated is known, we suggest that a more accurate account for the quantum string motion should lead to additional spin-spin and tensor contributions. After all, the QCD string is thick with curvature corrections for intermediate lengths still unclear.

Here we propose that the standard long range string confining potential $V_{\text{conf}} \sim r$ produced by the classical electric flux tube, may be extended to include spin-dependent coupling operators correlating the string end-points in a quantum string. So far, we have not worked out such operators, yet we will provide a preliminary study of what such operators may lead to. The results will be compared with phenomenological observations. As an example, let us consider what will happen with the second shell states, if the tensor potential possesses a long-range component, say

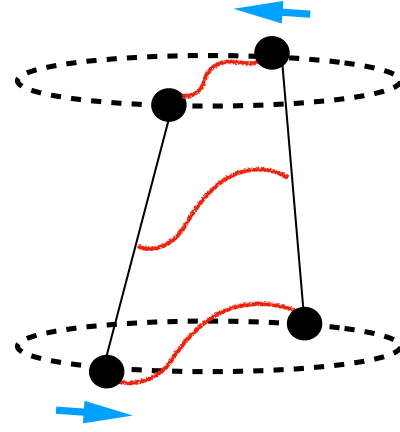


FIG. 2. Sketch of two rotating quarks connected by a QCD string. Time is running upward, so a string is shown at three time slices. The quark spin is connected to the magnetic field related to the velocity-acceleration term, at the string ends.

$$V_T^{\text{long range}}(r) = \alpha_T r \quad (15)$$

with some strength α_T . Note that (15) is comparable to the long range spin-orbit induced potential [23] (and references therein), and much stronger than the string induced van der Waals potential in [26]. In Fig. 3 we show the results of the lowest lying $J^P = \frac{1}{2}^+$ nucleon resonances versus α_T . The dashed lines are the masses evaluated using solely the hyperdistance equation without mixing, while the dots are the results induced by (15) through mixing between the S-D shells versus α_T . The suggested novel “long-range tensor forces” indeed create significant shifts of the low lying masses by S-D mixing.

G. Roper “collectivization” by the tensor force

Using a hypothetical “long range tensor potential” we do indeed find a strong mixing and motion of the lowest lying $J^P = \frac{1}{2}^+$ states as shown in Fig. 3. The strongest shift is the downward motion of the second state. Starting at $\alpha_T = 0$ as a $2S$ hyper-radial state, it crosses the horizontal line at 1440 MeV, corresponding to the particle data group (PDG) name of the so called “Roper resonance” discovered 60 years ago. (Note that modern pole mass fit of it gives even lower value, of 1370 MeV. We still call it 1440 following tradition.)

The history of the Roper state has been covered, e.g., in a separate review [27], where many models are discussed. Apart of unexpectedly small mass, the Roper resonance has an unusual $N \rightarrow N^*(1440)$ form factor, difficult to describe by quark-based models. Nevertheless the authors of the review still conclude that the Roper is basically a “radial excitation” of the nucleon, or the $2S$ state.

In our novel scenario with a long range tensor force, the low mass of the Roper resonance is naturally explained by strong mixing in the S-D shell states. In terms of the five basis states we use, its wave function at $\alpha_T = 0.25$ is

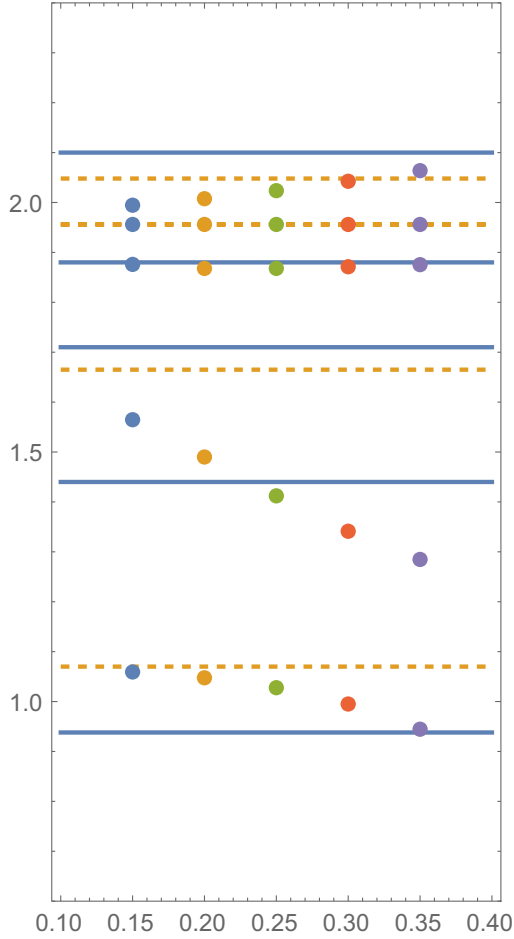


FIG. 3. Horizontal solid lines are the observed masses of the nucleon and 4 $J^P = 1/2^+$ resonances from the particle data tables in GeV. The horizontal dashed lines are masses calculated from the radial hyperdistance equation before mixing. The horizontal coordinate indicates the strength α_T of the hypothetical tensor potential defined in (14). The points are the calculated masses by diagonalization of the Hamiltonian in the presence of the long range tensor interaction.

$$|\mathbf{R}\rangle \approx \{0.28, -0.68, -0.06, 0.12, 0.66\}. \quad (16)$$

So, the 2S contribution is the 1D “tensor” contribution, about 50% each. Even the first state 1S (nucleon) is represented by a significant probability of about $0.28^2 = 7.8\%$.

Our explanation of the unusual feature of the Roper resonance is thus a strong mixing in the second shell, induced by a hypothetical long range tensor interaction to be measured using lattice simulations. This mixing is reminiscent of a phenomenon known in nuclear physics as “a collectivization” of a particular state, producing “giant resonances.” This idea was put forth in [28]. In short, consider an $N \times N$ Hamiltonian of the schematic form

$$H = - \begin{vmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \dots & & & \\ 1 & 1 & \dots & 1 \end{vmatrix} \quad (17)$$

with all matrix elements being the same. One of its eigenvalues is $-N$, while all the others are zero. Its eigenvector $\frac{1}{\sqrt{N}}(1, 1, \dots, 1)$ is a coherent (“collective”) mixture of all basis states. This description of the Roper resonance as a collectivized state, is in line with the Skyrmin description of the Roper resonance as a breathing mode of the solitonic pion field [29]. While in our scenario the Roper is not yet quite “giant,” the probability to remain in the original 2S state is about $1/3$, practically the same as for other states in the mix. So, we suggest the Roper resembles a “giantlike resonance” of the second shell, although with a not-so-large number of intermixed states $N \sim 3$.

H. Nucleon admixtures with the second shell states

The main physics message from Fig. 3, is that the mixing between the 1S nucleon state and the second shell states, remains rather limited. For the tensor coupling $\alpha_T = 0.25$ (which gives correct Roper mass) the wave function of the nucleon in our S-D basis, is

$$|N\rangle \approx \{0.95, 0.11, 0.014, -0.029, -0.28\}. \quad (18)$$

The probability for the nucleon to remain in the original 1S state is 90%, while the probability to be in the tensor second shell state is $0.28^2 \approx 7.8\%$. Such value should be taken as an upper limit, corresponding to a scenario with a “good Roper.”

Could this be the case in reality? Let us compare this result, with similar phenomena in nuclear physics. The most famous example is the deuteron, known to possess a finite quadrupole moment, due to a similar admixture between the main S wave ($L=0$) and the D wave ($L=2$). While we cannot compare the quadrupole moments, we can compare the admixture. The standard form of such an admixture is

$$\cos(\omega)\psi_{L=0} + \sin(\omega)\psi_{L=2}, \quad (19)$$

which amounts to $\sin(\omega) \approx 0.2$, or 4% in terms of the admixture probability. This admixture amplitude is only a factor of 2 smaller than our estimate for a nucleon. It is reasonable, since the tensor force is relativistic $O(v^2/c^2)$ effect, and the deuteron (with much smaller binding of only 2.2 MeV and 3 times larger mass of constituent M_N) is less relativistic than the constituent quarks in the nucleon.

A recent study including the role of D-wave admixture in He^4 has been carried out by one of us [30]. In this study, the potentials without a tensor force were compared to those with a realistic Argonne v14 potential, involving 14 operators in the nuclear force. The admixture induced by the tensor operators was found to be in the few percent range as well.

To summarize: Even an assumed very strong long-range tensor force between constituent quarks, can produce only a moderate D-wave admixture with orbital momentum in the nucleon state. This admixture at the few percent level is insufficient to explain several of the puzzles we mentioned in the Introduction.

III. MAGNETIC MOMENTS

The magnetic moment operator in atomic physics has a long history, going back to the definition $\vec{\mu} = 2\vec{S} + \vec{L}$. While it is not directed along $\vec{J}$, its average is, with a cosine of the relative angle known as the so called Lande factor, a function of J , L , S .

The baryon magnetic moments receive contributions from the intrinsic spin of each quark and their orbital motion,

$$\mu_X = \sum_{i=1}^3 Q_i (g_S S_i^z + g_L L_i^z) \mu_Q \quad (20)$$

with Q_i the i th quark flavor and

$$\mu_Q = \frac{e\hbar}{2m_Q} = \frac{m_N}{m_Q} \mu_N.$$

The “constituent quark magnetic moment” is expressed in terms of the nuclear Bohr magneton, thus the mass ratio. For spin- $\frac{1}{2}$ the Lande factors are $g_S = 2g_L = 2$. In terms of the Jacobi orbital angular momenta

$$\vec{l}_\rho = \vec{\rho} \times \vec{p}_\rho \quad \vec{l}_\lambda = \vec{\lambda} \times \vec{p}_\lambda$$

and the i th-flavor magnetic moment $\mu_i = Q_i \mu_Q$.

Let us focus on the 1-2 quark pair. The Jacobi coordinate $\vec{\rho} = (\vec{r}_1 - \vec{r}_2)/\sqrt{2}$, and the orbital combinations

$$\vec{L}_{1,2} = \vec{r}_{1,2} \times \frac{1}{i} \frac{\partial}{\partial \vec{r}_{1,2}}$$

are the same, so each gives half of the orbital momentum. Their contribution to the magnetic moment is

$$\mu_{12}^L = \frac{e_1}{2e} L_1 + \frac{e_2}{2e} L_2 \quad (21)$$

where charges can be expressed via isospin Pauli matrices as $e_i/e = (1/6 + \tau_3/2)$.

Equation (20) can be recast as

$$\mu_X = g_S \sum_{i=1}^3 \mu_i S_i^z + g_L \left(\frac{1}{2} (\mu_1 + \mu_2) l_\rho^z + \frac{1}{6} (\mu_1 + \mu_2 + 4\mu_3) l_\lambda^z + \frac{1}{2\sqrt{3}} (\mu_1 - \mu_2) (\rho \times p_\lambda + \lambda \times p_\rho)^z \right). \quad (22)$$

Its average values for the five S-D basis states denoted generically by X , normalized to that of the proton, are

$$\frac{\mu_X}{\mu_p} = \left(1, 1, \frac{1}{3}, -\frac{13}{108}, -\frac{49}{72} \right). \quad (23)$$

The signs follow the direction of the spin in all cases. Furthermore, even in the maximally mixed case discussed in this paper, the correction to the proton magnetic moment is only

$$-(49/72)0.078 \approx -0.05,$$

which is perhaps smaller than the current uncertainty of the “quark magneton” value $e/2M_q$.

IV. ELECTRIC, MAGNETIC, AND TRANSITIONAL FORM FACTORS

In the nonrelativistic limit, the form factors are evaluated using the wave functions. More specifically, in the commonly used Breit (brick-wall) frame, the elastic collision is a transition between a hadron i with momentum $\vec{Q}/2$ to a hadron j with momentum $-\vec{Q}/2$ along the z axis, caused by the action of the electromagnetic current operator,

$$\langle j | J_\mu^{\text{em}} | i \rangle.$$

The first step in the evaluation of the form factors is the calculation of the Fourier transform of the wave functions, and the second step is their convolution. As noted earlier, in the nonrelativistic limit, we will use the Lorentz contracted Fourier transforms, as we already mentioned in the Introduction, as detailed in Sec. IV.

In Fig. 4 (upper) we show the form factors calculated using the 1S (blue dots) and 1D (orange dots) wave functions. The comparison is to the empirical dipole fit to the proton magnetic form factor (dashed line). The 1D state is much softer than the 1S state as expected, since the wave function of the state is finite at the origin, while that of the 1D is quadratically suppressed by the centrifugal potential.

Another important feature is that the 1S form factor calculated from the hypercentral wavefunction (blue dots) does not agree with the data (the dashed line), even at small Q^2 . However, their behavior at larger $Q^2 > 0.5 \text{ GeV}^2$ are in fact similar. The fact that the hypercentral wave function

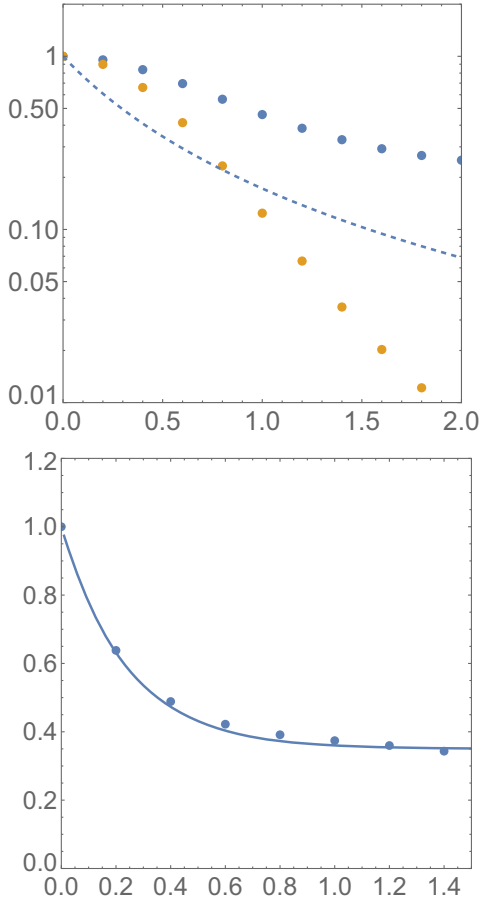


FIG. 4. Upper plot shows the $(1/2)^+$ form factor using the 1S (blue dots) and 1D (orange dots) versus $Q^2(\text{GeV}^2)$. The comparison is to the nucleon dipole fit $FF_{\text{dipole}} = 1/(1 + Q^2/0.71)^2$ (dashed blue). The lower plot shows the “constituent quark form factor” versus $Q^2(\text{GeV}^2)$. The points are the ratio of the calculated form factor (points) to the fitted experimental curve (dashed) in the upper plot. The line is $0.35 + 0.65 \exp(-Q^2 R_{\text{cloud}}^2/6)$ with radius $R_{\text{cloud}} = 1 \text{ fm} \approx 5 \text{ GeV}^{-1}$.

strongly underestimates the nucleon radius, has been noticed by many authors and is well known, see, e.g., [31]. Indeed, the standard radial wave functions are expected to give the spatial distributions of the constituent quark *centers*, not the distributions of charge (or mass). The “constituent quarks” are not point like objects but have certain spatial distribution and form factors by themselves.

To make this argument more quantitative, assume that the experimental form factor is the *product* of two form factors, the global wave function described by the hyper-radius, times the constituent quark one,

$$FF_N(Q^2) = FF_{\text{WF}}(Q^2) \cdot FF_{\text{constituent quark}}(Q^2). \quad (24)$$

In the lower part of Fig. 4, we show the last form factor in (24). A pointlike contribution from the constituent quark is present, which is about $\sim 1/3$ of the total form factor.

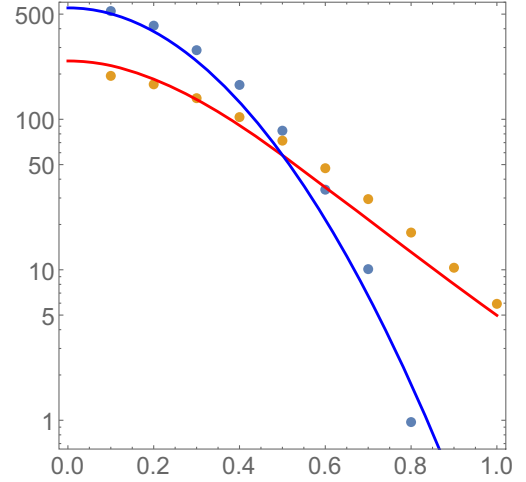


FIG. 5. Red and blue points are for Fourier transform of the 1S and 1D radial wave functions versus momentum $p(\text{GeV})$. The solid lines are the analytical fits in (26).

At small momentum transfer it can be represented by a Gaussian $e^{-Q^2 R^2/6}$ with a large radius $R \approx 5 \text{ GeV}^{-1} \approx 1 \text{ fm}$. Such a form factor is well described by the (instanton-based) semiclassical theory of chiral symmetry breaking. To make a nonzero quark condensate, virtual quarks bound to an instanton (a zero mode) also have to reach the next (anti)instanton in an ensemble, located at distances $R \sim n_{\text{inst}}^{1/4} \sim 1 \text{ fm}$. In other words, a constituent quark carries a “chiral cloud.”

In the remaining part of this section, we will briefly detail parts of the calculation entering the discussion above. More specifically, the radial (hyperdistance) wave functions of their six-dimensional Fourier transforms were all evaluated numerically. For analytical parametrizations it is useful to recall the exact Fourier transforms for Gaussian and exponential wave functions

$$\begin{aligned} \exp\left(-\frac{Y^2}{2\Lambda^2}\right) &\rightarrow \exp\left(-\frac{P^2\Lambda^2}{2}\right) \\ \exp(-Y \cdot A) &\rightarrow \frac{1}{(1 + P^2/A^2)^{7/2}}. \end{aligned} \quad (25)$$

With this in mind, the numerical form factors shown in Fig. 5 for the 1S state (blue dots) and 1D state (orange dots) are well fitted by

$$\begin{aligned} \psi_{1S}(p) &\approx \frac{4\pi 1.6}{(0.7^2 + p^2)^{7/2}}, \\ \psi_{1D}(p) &\approx 550 \cdot \exp(-9 * p^2). \end{aligned} \quad (26)$$

As noted earlier, the Fourier transform of the 1D state is softer than the 1S state, and not dominated by small distances for this range of momenta.

The form factor in the Breit frame is a convolution of the wave functions corresponding to total momenta $\vec{Q}/2$ and $-\vec{Q}/2$. The boost amounts to dividing the longitudinal momenta by the gamma factor

$$1/\gamma = \sqrt{1 - v^2} = \sqrt{1/(1 + Q^2/4M^2)}$$

so the “Lorentz-compressed” wave functions are $\phi = \psi(\vec{p}_\perp, p_{\text{long}}/\gamma)$. The form factor includes scatterings on all three quarks, labeled by $i = 1, 2, 3$ and weighted by their charges e_i ,

$$G_E(Q) = \sum_i \frac{e_i}{e\gamma} \int d^3 p_\rho d^3 p_\lambda (2\pi)^6 \times \phi(p_\rho, p_\lambda) \phi(\vec{p}_\rho + \nu_\rho \vec{Q}, \vec{p}_\lambda + \nu_\lambda \vec{Q}) \quad (27)$$

for the proton $2/3, 2/3, -1/3$, with ν_ρ, ν_λ referring to the weights of the respective Jacobi coordinates as fraction of the total momentum $\vec{Q}$.

We now turn to the transitional form factors, and the effect of level mixing. Since in this work we have focused on five basis states, we have calculated all 5×5 transitional form factors $\langle i|J|j \rangle$, where the in-out states consist of the “mixed” state following from a direct diagonalization of the Hamiltonian. The chief example is the transitional electromagnetic form factor between the mixed nucleon state **N** and the mixed Roper state **R**. Using the “strong mixing” value $\alpha_T = 0.25$ of the tensor coupling, we obtain

$$\begin{aligned} \langle \mathbf{N}|J|\mathbf{R} \rangle &\approx -0.62\langle 1|J|2 \rangle + 0.55\langle 1|J|5 \rangle \\ &+ 0.27\langle 1|J|1 \rangle + 0.26\langle 2|J|5 \rangle - 0.18\langle 5|J|5 \rangle. \end{aligned} \quad (28)$$

Note that the original transition $\langle 1|J|2 \rangle$ from 1S to 2S state has the largest weight, but is not dominant. In fact the five contributions are comparable in strength (additional terms with smaller coefficients were dropped). Since all the terms have different shapes, the configuration mixing leads to rather different shapes of the transitional form factor.

V. SUMMARY AND DISCUSSION

The main questions discussed in this paper are:

- (i) Can the observed splitting of the second shell baryons be understood by known (near-local) spin-dependent potentials?
- (ii) Can a mixture of the nucleon with those states be quantified?
- (iii) Can this mixing be reconciled with other hints of a possible quark orbital motion inside the nucleon?

The short answers to those questions are:

- (i) No. The second shell baryons have large sizes, so the effects of near-local forces are small. In particular—in

agreement with earlier studies—it is impossible to use them to explain the puzzling properties of the Roper resonance.

- (ii) A hypothetical long-range tensor force does enhance the orbital mixing of SD-states and provides a satisfactory description of the Roper resonance, as a strong mixture of all second-shell states.
- (iii) Yet even this long range tensor force does not affect the nucleon composition much, which remains dominantly the original 1S state. For the largest mixing, the “tensor” amplitude is ~ 0.28 , corresponding to a small probability $\sim 7.8\%$ in the nucleon. This mixing yields small corrections to the magnetic moments $\sim 5\%$. Both changes should be regarded as maximal, so the tensor mixing cannot explain the large orbital contribution reported from partonic data.

The traditional relativistic spin-dependent contributions (following atomic and nuclear physics) produce spin-spin, tensor, and spin-orbit potentials V_{SS}, V_T, V_{SL} . The perturbative one-gluon exchange gives local $[V_{SS} \sim \delta^3(r)]$ or very short-range $[V_{SS} \sim 1/r^3]$ potentials. The nonperturbative and inhomogeneous field fluctuations we have studied in several of our previous papers, induced by instantons or instanton-anti-instanton molecules, complement the perturbative results and improve the agreement with quarkonia spectroscopy and lattice studies. Yet, by their very nature, they are also rather short range. We investigated these interactions using heavy quarkonia [1], with overall agreement with earlier works. More specifically, the V_{SS}, V_T are in agreement with phenomenological matrix elements, except the spin-orbit V_{SL} which needs to be balanced by a long-distance negative contribution (tied to the Thomas contribution from the confining part). Yet, as we found in this work, all short-range effects do *not* reproduce the splittings of the second shell baryons.

While long range nonperturbative potentials from first principles are not known (future lattice simulations would be welcome), we have posited the possible existence of a “long-range tensor force” perhaps from a thick and confining quantum string. The result is a low mass Roper resonance, and a tensor contribution to the nucleon state. This SD mixing is found to be similar in magnitude to that observed in the deuteron and He-4 states in nuclear physics.

Finally, in this mixing scenario, we have analyzed the nucleon and the nucleon-to-Roper transitional form factors. The description using hyper-spherical wave functions yields form factors (and even rms nucleon radii) that are different from the data. While we have not analyzed this difference in detail, we have shown that the difference can be explained by the fact that hadrons are composed of constituent quarks, which are *not* just pointlike but have intrinsic form factors of their own. This situation is reminiscent of the low-energy electromagnetic form factors of nuclei, as composites of nucleons.

ACKNOWLEDGMENTS

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: COORDINATES

Throughout we made use of the Jacobi coordinates $\vec{\rho}, \vec{\lambda}$ traditionally used for baryons. They are represented by a hyperdistance $Y = R$ and five angles $\theta_\lambda, \phi_\lambda, \theta_\rho, \phi_\rho, \chi$ defined as follows

$$\begin{aligned} x^a = & Y(\cos(\chi)\cos(\theta_\lambda), \cos(\chi)\sin(\theta_\lambda)\sin(\phi_\lambda), \\ & \times \cos(\chi)\sin(\theta_\lambda)\cos(\phi_\lambda), \sin(\chi)\cos(\theta_\rho), \\ & \times \sin(\chi)\sin(\theta_\rho)\sin(\phi_\rho), \sin(\chi)\sin(\theta_\rho)\cos(\phi_\rho)) \end{aligned} \quad (\text{A1})$$

with two 3D solid angle variables ϕ, θ , plus the fifth angle $\lambda = Y\cos(\chi), \rho = Y\sin(\chi)$. The volume element is

$$\sqrt{g} = Y^5 \cos(\chi)^2 \sin(\theta_\lambda) \sin(\theta_\rho) \sin(\chi)^2 \quad (\text{A2})$$

and the line element, which defines the metric tensor, is

$$\begin{aligned} dl^2 = & dY^2 + Y^2(\cos(\chi)^2 d\theta_\lambda^2 + d\chi^2 \\ & + \cos(\chi)^2 d\phi_\lambda^2 \sin(\theta_\lambda)^2 + d\theta_\rho^2 \sin(\chi)^2 \\ & + d\phi_\rho^2 \sin(\theta_\rho)^2 \sin(\chi)^2). \end{aligned} \quad (\text{A3})$$

The corresponding Laplace-Beltrami operator is

$$\begin{aligned} -\frac{1}{\sqrt{g}} \frac{\partial}{\partial m} \sqrt{g} g^{mn} \frac{\partial}{\partial n} = & \frac{\partial^2}{\partial Y^2} + \frac{5}{Y} \frac{\partial}{\partial Y} + \frac{1}{Y^2} \left(\sec(\chi)^2 \frac{\partial^2}{\partial \theta_\lambda^2} + \text{Csc}[\theta_\lambda]^2 \text{Sec}[\chi]^2 \frac{\partial^2}{\partial \phi_\lambda^2} + \text{Csc}[\chi]^2 \frac{\partial^2}{\partial \theta_\rho^2} + \text{Cot}[\theta_\rho] (\text{Csc}[\chi]^2) \frac{\partial}{\partial \theta_\rho} \right. \\ & \left. + \text{Csc}[\theta_\rho]^2 \text{Csc}[\chi]^2 \frac{\partial^2}{\partial \phi_\rho^2} + 2\text{Cos}[2\chi] \text{Csc}[\chi] \text{Sec}[\chi] \frac{\partial}{\partial \chi} + \frac{\partial^2}{\partial \chi^2} \right). \end{aligned} \quad (\text{A4})$$

APPENDIX B: HYPERSPHERICAL WAVE FUNCTIONS

The hyperspherical radial Schrödinger equation differs from the usual one only by: (i) a six-dimensional Laplacian in the kinetic energy, and (ii) a 6D solid-angle average binary interaction potential

$$H\psi(Y) = -\frac{1}{2M_q} (\psi(Y)'' + \frac{5}{Y} \psi(Y)') + (3M_q + \bar{V})\psi(Y) \quad (\text{B1})$$

with

$$\bar{V} = 3 \left(\frac{32 \cdot \sqrt{2} Y 0.113}{30\pi} \right) - \frac{16 \cdot \sqrt{2} 0.64}{3\pi Y} \frac{1}{2}. \quad (\text{B2})$$

Here $Y = R$ is the hyperdistance and $M_q = 0.35$ GeV the constituent mass. We solve this equation for the 1S, 2S, 1D states as shown in Fig. 6, with the normalization

$$\int |\psi(Y)|^2 Y^5 dY = 1. \quad (\text{B3})$$

The eigenvalues are the masses in the lowest approximation (without spin-dependent forces). If the 1S mass is taken to be the spin-average N, Δ mass, then the corresponding three masses are

$$M_{1S}^0 = 1.07 \text{ GeV}, \quad M_{2S}^0 = 1.67 \text{ GeV}, \quad M_{1D}^0 = 1.82 \text{ GeV}. \quad (\text{B4})$$

Note that the last two numbers would be different in the oscillator model used in [4] and many subsequent works. Their difference ~ 150 MeV is not negligible and comparable to shifts due to spin-spin and tensor forces.

In the transition form factor, the dominant contribution stems from small sizes. In Fig. 6 we have chosen to normalize the wave functions to the same value at the origin. Out of the two approximations—Gaussian used in [13] and our exponential fit—we see that the latter is much closer to the numerical solution of the radial equation. (This

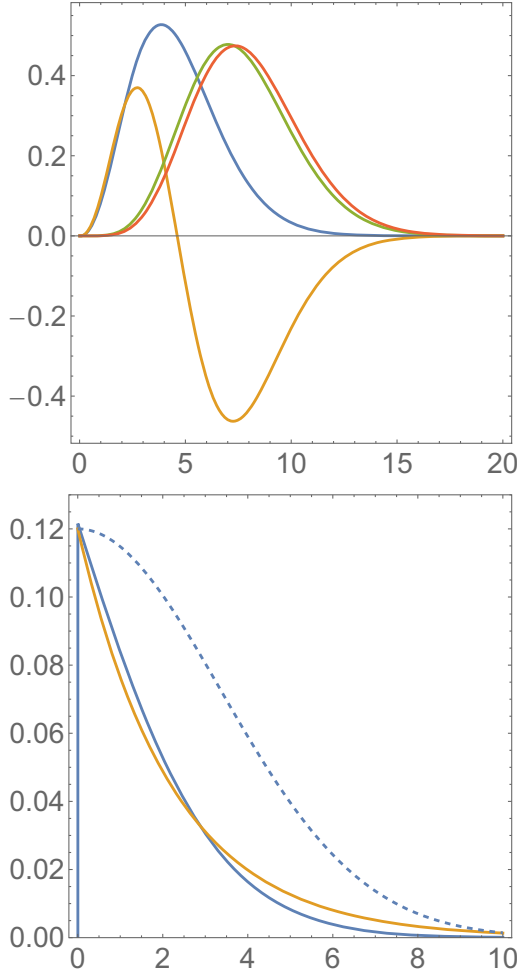


FIG. 6. Upper: Hyperspherical wave functions plotted as $\psi(Y)Y^{5/2}$ versus the hyperdistance $Y(\text{GeV}^{-1})$. The blue and orange curves are the 1S, 2S states respectively, with zero angular Laplacian. The green and red curves correspond to the values 16 and 19, respectively. Lower: Solid blue curve is the same numerical solution for the 1S state as in the upper figure, but now showing the 1S wave function $\psi(Y)$ only. It is finite at the origin and about exponential for small Y . The red curve is the exponential approximation mentioned in the text. The dashed curve shows the Gaussian approximation used by [13]. All are normalized to the same wave function value at the origin.

is to be expected, since the potential at small distances is Coulomb-like rather than oscillatorlike.)

APPENDIX C: SYMMETRIC ORBITAL-SPIN-FLAVOR WFS FROM REPRESENTATION OF THE PERMUTATION GROUPS S_3

The key constraint on the construction of the nucleon wave functions is Fermi statistics. The full wave function is antisymmetric under any pair exchange. For example, under permutation of particle 1 and 2 P_{12} , which amounts to $\rho \rightarrow -\rho, \lambda \rightarrow \lambda$, hence the first angular combination is odd and the other even. Therefore the flavor-spin WFs for the ρ, λ contributions must be different.

The explicit construction of the ρ, λ parts of the wave functions can be done in two ways. One, developed in our previous paper [1], is based on the representation of the permutation group S_3 . Here we only need to add few things relevant for the second shell. The second derivation based on Young tableaux is discussed in the next Appendix. Both derivations were developed for consistency checks.

We now recall the main ideas. The construction of the spin and isospin combinations of the same form as the ρ, λ coordinates allows the building of the representation of the permutation group S_3 , e.g.,

$$\begin{aligned} S_\rho &= (\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow)/\sqrt{2}, \\ S_\lambda &= (\downarrow\uparrow\uparrow + \uparrow\downarrow\uparrow - 2\uparrow\uparrow\downarrow)/\sqrt{6}. \end{aligned} \quad (\text{C1})$$

The isospin states I_ρ, I_λ (or equivalently F_ρ, F_λ as used in our previous work and the Young Tableaux below) are defined similarly.

Note that the permutation P_{12} changes the sign of S_ρ, I_ρ but not S_λ, I_λ , e.g.,

$$\begin{aligned} [P_2 = (12)] \begin{pmatrix} M^\rho \\ M^\lambda \end{pmatrix} &= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} M^\rho \\ M^\lambda \end{pmatrix}, \\ [P_4 = (23)] \begin{pmatrix} M^\rho \\ M^\lambda \end{pmatrix} &= \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} M^\rho \\ M^\lambda \end{pmatrix}, \end{aligned} \quad (\text{C2})$$

which clearly shows that the combination $(S_\rho I_\rho + S_\lambda I_\lambda)$ is permutation invariant.

In general, we will consider a number of products of N such objects, and will seek to find the representation of the N -th tensor power of the ρ, λ mixing matrices under permutations.

The cases discussed in this paper start with a nucleon in the ground state, with spin and isospin terms of $N = 2$. In this case, we only have the symmetric combination $S_\rho I_\rho + S_\lambda I_\lambda$, as is well known. In the first (negative parity) shell case $N = 3$, as the coordinates are added to the mix. In this case, there is only one symmetric combination

$$V_{ABC} = A_\rho(B_\rho C_\lambda + B_\lambda C_\rho) + A_\lambda(B_\rho C_\rho - B_\lambda C_\lambda). \quad (\text{C3})$$

The second shell has two coordinates, spin and isospin so in general it requires four objects $N = 4$, or the construction of all combinations $A_\alpha B_\beta C_\gamma D_\delta$ with binary indices ρ, λ . Those should be symmetric under any quark permutation. Selecting the permutations $1 \leftrightarrow 2$ and $2 \leftrightarrow 3$, as described by the 16-by-16 dimensional matrices

$$P_{12}^4 = P_{12} \otimes P_{12} \otimes P_{12} \otimes P_{12},$$

$$P_{23}^4 = P_{23} \otimes P_{23} \otimes P_{23} \otimes P_{23},$$

and diagonalizing them, yields the number of symmetric and antisymmetric combinations. It turns out there are three such combinations as given in (A11) of [1],

$$V_1 = A_\lambda B_\lambda C_\lambda D_\lambda + A_\rho B_\lambda C_\lambda D_\rho + A_\lambda B_\rho C_\rho D_\lambda + A_\rho B_\rho C_\rho D_\rho, \quad (C4)$$

$$V_2 = A_\lambda B_\lambda C_\rho D_\rho - A_\rho B_\lambda C_\lambda D_\rho - A_\lambda B_\rho C_\rho D_\lambda + A_\rho B_\rho C_\lambda D_\lambda, \quad (C5)$$

$$V_3 = (A_\rho B_\lambda - A_\lambda B_\rho)(C_\rho D_\lambda - C_\lambda D_\rho). \quad (C6)$$

In total, we discuss five nucleon states, all with $J = 1/2, I = 1/2$: the (1S) nucleon, the second (2S) state, two scalar $L = 0$ states formed from the V_1, V_2 combinations defined above, the “vector” $L = 1$ formed from V_3 (including the vector product of $[\vec{\rho} \times \vec{\lambda}]$), and finally the “tensor” $L = 2, S = 3/2$ combination. Using the standard Clebsch-Gordon coefficients, we first combine two coordinates into states with fixed $L = 0, 1, 2$, and then combine them with spin $S = 1/2$ or $S = 3/2$, as appropriate to get states with the desired $J = 1/2$. We use the spin-tensors in the spin-isospin monom space (64-dimensional) with analytic manipulations using *Mathematica*. The expressions in terms of monomons are too long to be listed here. They are orthogonal to each other and normalized to 1, after integration over all angles. Acting on them by the angular part of the Laplacian given above, we find that the eigenvalues for those six states are $[0, 0, 0, 16, 19]$, respectively. Acting on them by spin-dependent forces leads to the matrices discussed in the main text.

APPENDIX D: ALTERNATIVE DERIVATION OF THE ORBITAL-SPIN-FLAVOR WFS

In this appendix we present the alternative derivation based on Young tableaux. The idea, used already in [4], is to construct mixed symmetry wave functions $M^{\rho,\lambda}$, out of Jacobi ρ -like and λ -like blocks that have pure S or A permutation properties, both for spin and isospin parts. Throughout this appendix, we will adhere to the notations used in [4] and also followed in our recent analysis of the P states [1]. More specifically, for the hyperdistance $Y \rightarrow R$, and the integration measure will be given in terms of $d\vec{\rho}d\vec{\lambda}$ without carrying explicitly the relative angular integration over χ . Relative signs in transition matrix elements reflect on the overall sign undermination of the wave function.

Construction of representations of S_3 is done like for any other groups, e.g., familiar generalization from spinor representation of $O(3)$ to spin-1, spin-3/2, etc. The tensor product of two generic representations X_a and X_b with different symmetries under S_3 , is a sum pf representation X_{ab} each with symmetries being $S, A, M^{\rho,\lambda}$. It is obvious that

$$\begin{aligned} S_a \otimes S_b &= S_{ab}, \\ A_a \otimes A_b &= S_{ab}, \\ S_a \otimes A_b &= A_{ab} \end{aligned} \quad (D1)$$

but the product of the mixed representations $M^{\rho,\lambda}$ viewed as primitive 2_M doublets is more subtle and needs more detailed studies.

The product of two mixed representations $M^{\rho,\lambda}$ of S_3 yields the sum over the irreducible representations of S_3

$$\begin{array}{|c|c|} \hline & \\ \hline \end{array} \otimes \begin{array}{|c|c|} \hline & \\ \hline \end{array} \sim \begin{array}{|c|} \hline \\ \hline \end{array} \otimes \begin{array}{|c|} \hline \\ \hline \end{array} = \begin{array}{|c|} \hline \\ \hline \end{array} \oplus \begin{array}{|c|c|} \hline & \\ \hline \end{array} \quad (D2)$$

corresponding to the 1_A singlet antisymmetric, and to the 3_S triplet symmetric representations, respectively.

The singlet 1_A antisymmetric representation is

$$A_{ab} = \frac{1}{\sqrt{2}} M_a^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} M_b = \frac{1}{\sqrt{2}} (M_a^\rho M_b^\lambda - M_a^\lambda M_b^\rho) \quad (D3)$$

while one of the triplet 3_S is, e.g.,

$$M_{ab}^\rho = \frac{1}{\sqrt{2}} M_a^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} M_b = \frac{1}{\sqrt{2}} (M_a^\rho M_b^\lambda + M_a^\lambda M_b^\rho). \quad (D4)$$

The remaining part of the triplet 3_S with projection $\pm$, is regrouped in the manifestly orthogonal combinations

$$S_{ab} = \frac{1}{\sqrt{2}} M_a^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} M_b = \frac{1}{\sqrt{2}} (M_a^\rho M_b^\rho + M_a^\lambda M_b^\lambda), \quad (D5)$$

$$M_{ab}^\lambda = \frac{1}{\sqrt{2}} M_a^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} M_b = \frac{1}{\sqrt{2}} (M_a^\rho M_b^\rho - M_a^\lambda M_b^\lambda). \quad (D6)$$

S_{ab} is invariant under all the six rotations, hence all the six permutations. It is manifestly symmetric. The combinations $M_{ab}^{\rho,\lambda}$ can be checked to transform as a doublet under all permutations, e.g.,

$$[P_4 = (23)] \begin{pmatrix} M_{ab}^\rho \\ M_{ab}^\lambda \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} M_{ab}^\rho \\ M_{ab}^\lambda \end{pmatrix}. \quad (D7)$$

1. Nucleon even parity states: Radial excitations

The lowest nucleon state with even parity carries symmetric spin-flavor combination

$$S_{ab} = \frac{1}{\sqrt{2}} (S_a^\rho F_b^\rho + S_a^\lambda F_b^\lambda) \quad (D8)$$

hence (spectroscopic notation $|LSJm\rangle_{X_R^P}$)

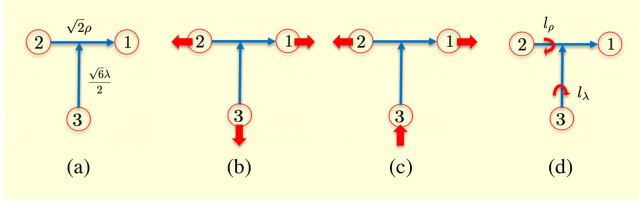


FIG. 7. Three particle states for the nucleon in Jacobi coordinates: (a) the ground state; (b) the radial monopole excitation; (c) the radial tripole excitation; (d) the orbitally excited states with internal $l_{\rho, \lambda} \neq 0$.

$$\left| 0^{+} \frac{1}{2} \frac{1}{2} m \right\rangle_{p_s^{+}} = \varphi_{1S}(R) \frac{1}{\sqrt{2}} \left(S_{\frac{1}{2}m}^{\rho} F_{\frac{1}{2}}^{\rho} + S_{\frac{1}{2}m}^{\lambda} F_{\frac{1}{2}}^{\lambda} \right) \quad (\text{D9})$$

where the wave function $\varphi_{1S}(R)$ depends on the 6d hyperdistance $R = Y$, with the squared norm

$$\|0^{+}\|^2 = \int d\vec{\rho} d\vec{\lambda} |\varphi_{1S}(R)|^2. \quad (\text{D10})$$

The radially excited states with zero total orbital angular momentum $L = l_{\rho} + l_{\lambda} = 0$, and null $l_{\rho} = l_{\lambda} = 0$, but a total angular momentum $J = L + S = \frac{1}{2}$, are illustrated in Fig. 7(b),(c). These states are distinguishable by their transformation under the permutation group. More specifically, Fig. 7(b) corresponds to the symmetric (S) and radially excited state

$$\varphi_{2S}^S(R) = \frac{1}{\sqrt{2}} (R^2 - \overline{R}^2) \varphi_{1S}(R) \quad (\text{D11})$$

with a single node, which is orthogonal to ground state $\varphi_{1S}(R)$.

The states in Fig. 7(c) are the radially excited scalar states with ρ, λ -mixed symmetry ($M^{\rho, \lambda}$),

$$\begin{aligned} \varphi_{3S}^{\lambda}(\rho, \lambda) &= \frac{1}{\sqrt{2}} (\rho \cdot \rho - \lambda \cdot \lambda) \varphi_{3S}(R), \\ \varphi_{3S}^{\rho}(\rho, \lambda) &= \frac{1}{\sqrt{2}} (\rho \cdot \lambda + \lambda \cdot \rho) \varphi_{3S}(R). \end{aligned} \quad (\text{D12})$$

The lowest symmetric (S) and radially excited state of the nucleon follows from (D11)

$$\left| 0^{+*} \frac{1}{2} \frac{1}{2} m \right\rangle_{p_s^{+*}} = \varphi_{2S}^S \frac{1}{\sqrt{2}} \left(S_{\frac{1}{2}m}^{\rho} F_{\frac{1}{2}}^{\rho} + S_{\frac{1}{2}m}^{\lambda} F_{\frac{1}{2}}^{\lambda} \right), \quad (\text{D13})$$

which is orthogonal to 0^{+} with the squared norm

$$\|0^{+*}\|^2 = \int d\vec{\rho} d\vec{\lambda} \frac{1}{2} (R^2 - \overline{R}^2)^2 |\varphi_{1S}(R)|^2. \quad (\text{D14})$$

The lowest mixed symmetry and radially excited state of the nucleon follows from (D12)

$$\begin{aligned} \left| 0^{+} \frac{1}{2} \frac{1}{2} m \right\rangle_{p_M^{**+}} &= \frac{1}{2} \left(F_{\frac{1}{2}}^{\rho} \left(S_{\frac{1}{2}m}^{\rho} \varphi_{3S}^{\lambda} + S_{\frac{1}{2}m}^{\lambda} \varphi_{3S}^{\rho} \right) \right. \\ &\quad \left. + F_{\frac{1}{2}}^{\lambda} \left(S_{\frac{1}{2}m}^{\rho} \varphi_{3S}^{\rho} - S_{\frac{1}{2}m}^{\lambda} \varphi_{3S}^{\lambda} \right) \right), \end{aligned} \quad (\text{D15})$$

with the first bracket carrying M^{ρ} symmetry and the second bracket carrying M^{λ} symmetry. Equation (D15) is orthogonal to both p^{+}, p^{+*} with squared norm

$$\|0^{+**}\|^2 = \int d\vec{\rho} d\vec{\lambda} \frac{1}{4} (\rho^4 + \lambda^4 - 2\rho^2 \lambda^2 + 4(\vec{\lambda} \cdot \vec{\rho})^2) |\varphi_{3S}(R)|^2. \quad (\text{D16})$$

2. Nucleon even parity states: Orbital excitations

The orbitally excited states of the nucleon are P, D states with $L = 1, 2$ respectively, which carry $l_{\rho, \lambda} \neq 0$ as illustrated in Fig. 7(d). For the S state with $L = 1$ or with $l_{\rho} = 1$ and $l_{\lambda} = 0$ and vice versa, the spatial wave function with positive parity 1^{+} , is antisymmetric under the permutation group (A)

$$\begin{aligned} \varphi_{11}^A(\rho, \lambda) &= \frac{1}{\sqrt{2}} (\rho_{+} \sqrt{2} \lambda_{z} - \sqrt{2} \rho_{z} \lambda_{+}) \varphi_{1P}(R), \\ \varphi_{10}^A(\rho, \lambda) &= \frac{1}{\sqrt{2}} (\sqrt{2} \rho_{z} \sqrt{2} \lambda_{z} - \sqrt{2} \rho_{z} \sqrt{2} \lambda_{z}) \varphi_{1P}(R) = 0, \\ \varphi_{1-1}^A(\rho, \lambda) &= \frac{1}{\sqrt{2}} (\rho_{-} \sqrt{2} \lambda_{z} - \sqrt{2} \rho_{z} \lambda_{-}) \varphi_{1P}(R). \end{aligned} \quad (\text{D17})$$

In contrast, for the D state with $L = 2$ or with $l_{\rho, \lambda} = 0, 1, 2$, the spatial wavefunctions with positive parity 2^{+} and maximum weight, split into symmetric (S) and mixed symmetry ($M^{\lambda, \rho}$) states are tensor ρ, λ combinations

$$\begin{aligned} \varphi_{22}^S(\rho, \lambda) &= \frac{1}{\sqrt{2}} (\rho_{+}^2 + \lambda_{+}^2) \varphi_{1D} = \sqrt{\frac{16\pi}{15}} (\rho^2 Y_2^2(\hat{\rho}) + \lambda^2 Y_2^2(\hat{\lambda})) \varphi_{1D}, \\ \varphi_{22}^{\lambda}(\rho, \lambda) &= \frac{1}{\sqrt{2}} (\rho_{+}^2 - \lambda_{+}^2) \varphi_{1D} = \sqrt{\frac{16\pi}{15}} (\rho^2 Y_2^2(\hat{\rho}) - \lambda^2 Y_2^2(\hat{\lambda})) \varphi_{1D}, \\ \varphi_{22}^{\rho}(\rho, \lambda) &= \frac{1}{\sqrt{2}} (\rho_{+} \lambda_{+} + \lambda_{+} \rho_{+}) \varphi_{1D} = \frac{4\pi}{3\sqrt{2}} (Y_1^1(\hat{\rho}) Y_1^1(\hat{\lambda}) + Y_1^1(\hat{\lambda}) Y_1^1(\hat{\rho})) \rho \lambda \varphi_{1D}(R). \end{aligned} \quad (\text{D18})$$

The P-excited states of the nucleon with $L^\pi = 1^+$ and positive parity follow from (D17)

$$\left| 1^+ \frac{1}{2} \frac{1}{2} m \right\rangle_{p_A^+} = C_{1m_L \frac{1}{2} m_S}^{\frac{1}{2} m} \varphi_{1m_L}^A \frac{1}{\sqrt{2}} \left(S_{\frac{1}{2} m_S}^\lambda F_{\frac{1}{2}}^\rho - S_{\frac{1}{2} m_S}^\rho F_{\frac{1}{2}}^\lambda \right). \quad (\text{D19})$$

They are orthogonal to the S states, with the squared norm

$$\begin{aligned} \|1^+\|^2 &= \left\langle 1^+ \frac{1}{2} \frac{1}{2} m \left| 1^+ \frac{1}{2} \frac{1}{2} m' \right. \right\rangle \\ &= \delta_{mm'} \int d\vec{\rho} d\vec{\lambda} \frac{1}{3} \rho^2 \lambda^2 |\varphi_{1P}(R)|^2. \end{aligned} \quad (\text{D20})$$

The D-excited states of the nucleon with $L^\pi = 2^+$ and positive parity follow from (D18)

$$\left| 2^+ \frac{1}{2} \frac{1}{2} m \right\rangle_{p_S^+} = C_{2m_L \frac{1}{2} m_S}^{\frac{1}{2} m} S_{\frac{1}{2} m_S}^S \frac{1}{\sqrt{2}} \left(F_{\frac{1}{2}}^\rho \varphi_{2m_L}^\rho + F_{\frac{1}{2}}^\lambda \varphi_{2m_L}^\lambda \right). \quad (\text{D21})$$

They are orthogonal to the S, P states, with the squared norm

$$\begin{aligned} \|2^+\|^2 &= \left\langle 2^+ \frac{1}{2} \frac{1}{2} m \left| 2^+ \frac{1}{2} \frac{1}{2} m' \right. \right\rangle \\ &= \delta_{mm'} \int d\vec{\rho} d\vec{\lambda} \left(\frac{1}{15} (\rho^4 + \lambda^4) + \frac{2}{9} \rho^2 \lambda^2 \right) |\varphi_{1D}(R)|^2. \end{aligned} \quad (\text{D22})$$

To summarize: the positive parity nucleon state will be assumed to be an admixture of the five states

$$\left[\left| 0^+ \frac{1}{2} \frac{1}{2} m \right\rangle_{p_S^+}, \left| 0^+ \frac{1}{2} \frac{1}{2} m \right\rangle_{p_S^{*+}}, \left| 0^+ \frac{1}{2} \frac{1}{2} m \right\rangle_{p_M^{*++}}, \left| 1^+ \frac{1}{2} \frac{1}{2} m \right\rangle_{p_A^+}, \left| 2^+ \frac{1}{2} \frac{1}{2} m \right\rangle_{p_S^+} \right], \quad (\text{D23})$$

where the ground state p_S^+ is the dominant contribution.

3. Spin-spin interaction

The pair spin-spin interactions

$$\mathbb{V}_{SS}(1, 2, 3) = \sum_{i < j=1}^3 V_{SS}(i, j) S_i \cdot S_j \quad (\text{D24})$$

split the degeneracy of the even-parity states of the nucleon in (D23). Since the states have good permutation symmetry, it is enough to evaluate the 12-pair contribution

$$V_{SS}(\sqrt{2}\rho) S_1 \cdot S_2 \quad (\text{D25})$$

and multiply the net result by three. We now detail some of the spin-spin matrix elements between the states (D23), and summarize the final results.

In the nucleon ground state p_S^+ we have

$$\begin{aligned} \left\langle 0^+ \frac{1}{2} \frac{1}{2} m \left| \mathbb{V}_{SS} \right| 0^+ \frac{1}{2} \frac{1}{2} m \right\rangle &= \frac{3}{2} (S_{\frac{1}{2} m}^\rho S_1 \cdot S_2 S_{\frac{1}{2} m}^\rho + S_{\frac{1}{2} m}^\lambda S_1 \cdot S_2 S_{\frac{1}{2} m}^\lambda) \int d\vec{\rho} d\vec{\lambda} V_{SS}(\sqrt{2}\rho) |\varphi_{1S}(R)|^2 \\ &= \frac{3}{2} \left(-\frac{3}{4} + \frac{1}{4} \right) \int d\vec{\rho} d\vec{\lambda} V_{SS}(\sqrt{2}\rho) |\varphi_{1S}(R)|^2 \equiv -\frac{3}{4} \langle V_{1S1S} \rangle \end{aligned} \quad (\text{D26})$$

with the overall factor of three counting the number of pairs. Similarly in the nucleon D-state p_S^+ , we note that

$$S_1 \cdot S_2 \left| 2^+ \frac{1}{2} \frac{1}{2} m \right\rangle_{p_S^+} = \frac{1}{4} \left| 2^+ \frac{1}{2} \frac{1}{2} m \right\rangle_{p_S^+} \quad (\text{D27})$$

hence

$$\left\langle 2^+ \frac{1}{2} \frac{1}{2} m \left| \mathbb{V}_{SS} \right| 2^+ \frac{1}{2} \frac{1}{2} m \right\rangle = \frac{3}{4} \langle V_{1S1S} \rangle. \quad (\text{D28})$$

The final result for the spin-spin interaction in the positive

parity nucleon basis-set (D23) is diagonal, and in agreement with (7)

$$\mathbb{V}_{SS} = -\frac{3}{4} \begin{pmatrix} \langle V_{1S1S} \rangle & 0 & 0 & 0 & 0 \\ 0 & \langle V_{2S2S} \rangle & 0 & 0 & 0 \\ 0 & 0 & \langle V_{3S3S} \rangle & 0 & 0 \\ 0 & 0 & 0 & \langle V_{1L1L} \rangle & 0 \\ 0 & 0 & 0 & 0 & -\langle V_{1D1D} \rangle \end{pmatrix}. \quad (\text{D29})$$

4. Tensor interaction

Permutation symmetry reduces the tensor matrix elements to the evaluation of the 12-pair contribution

$$V_T(\sqrt{2}\rho)(3S_1 \cdot \hat{\rho} S_2 \cdot \hat{\rho} - S_1 \cdot S_2) \quad (D30)$$

$$\mathbb{V}_{TS}[\alpha, \beta] = \sqrt{\frac{\pi}{5}} \begin{pmatrix} +\frac{3}{2_s} & +\frac{1}{2_s} & +\frac{1}{2_a} & +\frac{1}{2_p} & -\frac{1}{2_p} & -\frac{1}{2_a} & -\frac{1}{2_s} & -\frac{3}{2_s} \\ +\frac{3}{2_s} & Y_2^0 & \sqrt{2}Y_2^{-1} & Y_2^{-1} & 0 & 0 & 2Y_2^{-2} & \sqrt{2}Y_2^{-2} & 0 \\ +\frac{1}{2_s} & -\sqrt{2}Y_2^1 & -Y_2^0 & -Y_2^0 & 0 & 0 & -\sqrt{3}Y_2^{-1} & 0 & \sqrt{2}Y_2^{-2} \\ +\frac{1}{2_a} & -Y_2^1 & -\sqrt{2}Y_2^0 & 0 & 0 & 0 & 0 & -\sqrt{3}Y_2^{-1} & -2Y_2^{-2} \\ +\frac{1}{2_p} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2_p} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2_a} & 2Y_2^2 & \sqrt{3}Y_2^1 & 0 & 0 & 0 & 0 & \sqrt{2}Y_2^0 & Y_2^{-1} \\ -\frac{1}{2_s} & \sqrt{2}Y_2^2 & 0 & \sqrt{3}Y_2^1 & 0 & 0 & \sqrt{2}Y_2^0 & -Y_2^0 & -\sqrt{2}Y_2^{-1} \\ -\frac{3}{2_s} & 0 & \sqrt{2}Y_2^2 & -2Y_2^2 & 0 & 0 & -Y_2^1 & \sqrt{2}Y_2^1 & Y_2^0 \end{pmatrix} \quad (D32)$$

with all spherical harmonics valued with $\hat{\rho}$. Throughout the calculations of the tensor transition matrix elements to follow and to avoid the multiplication of notations, we will use the same labels for the *normalized* radial wave functions, e.g.,

$$\varphi_{1P}/||1^+|| \rightarrow \varphi_{1P}, \quad \varphi_{1D}/||2^+|| \rightarrow \varphi_{1D} \quad \text{etc.}$$

a. 1S1D, 2S1D

The tensor induced S-D mixing is given by the transition matrix element

$$\begin{aligned} & \left\langle 0^+ \frac{111}{222} \right| \mathbb{V}_T \left| 2^+ \frac{111}{222} \right\rangle \\ &= 3C_{2m_L \frac{3}{2} m_S}^{\frac{11}{22}} \int d\vec{\rho} d\vec{\lambda} (\varphi_{1S} \varphi_{2m_L}^\lambda V_T(\sqrt{2}\rho)) \frac{1}{2} S_{\frac{11}{22}}^\lambda \mathbb{V}_{TS}^\rho S_{\frac{3}{2} m_S}. \end{aligned} \quad (D33)$$

The λ -tensor wave function selected by the nonvanishing Clebsches,

$$\begin{aligned} C_{22 \frac{3}{2} - \frac{3}{2}}^{\frac{11}{22}} &= +\sqrt{\frac{4}{10}}, & C_{21 \frac{3}{2} - \frac{1}{2}}^{\frac{11}{22}} &= -\sqrt{\frac{3}{10}}, \\ C_{20 \frac{31}{22}}^{\frac{11}{22}} &= +\sqrt{\frac{2}{10}}, & C_{2-1 \frac{33}{22}}^{\frac{11}{22}} &= -\sqrt{\frac{1}{10}}, \end{aligned} \quad (D34)$$

with the net result multiplied by three. In the spin basis set

$$[\alpha] = S_{\frac{3}{2} m_S = \pm \frac{3}{2}, \pm \frac{1}{2}}^S, \quad S_{\frac{1}{2} m_\rho = \pm \frac{1}{2}}^\rho, \quad S_{\frac{1}{2} m_\lambda = \pm \frac{1}{2}}^\lambda \quad (D31)$$

the entries $[\alpha, \beta]$ of the 8×8 spin-tensor part in (D30) are captured by the block-form matrix

can be recast in spherical harmonics

$$\begin{aligned} \varphi_{2-1}^\lambda &= +\sqrt{\frac{8\pi}{15}} (\rho^2 Y_2^{-1}(\hat{\rho}) - \lambda^2 Y_2^{-1}(\hat{\lambda})) \varphi_{1D}(R), \\ \varphi_{20}^\lambda &= +\sqrt{\frac{8\pi}{15}} (\rho^2 Y_2^0(\hat{\rho}) - \lambda^2 Y_2^0(\hat{\lambda})) \varphi_{1D}(R), \\ \varphi_{21}^\lambda &= +\sqrt{\frac{8\pi}{15}} (\rho^2 Y_2^1(\hat{\rho}) - \lambda^2 Y_2^1(\hat{\lambda})) \varphi_{1D}(R), \\ \varphi_{22}^\lambda &= +\sqrt{\frac{16\pi}{15}} (\rho^2 Y_2^2(\hat{\rho}) - \lambda^2 Y_2^2(\hat{\lambda})) \varphi_{1D}(R). \end{aligned} \quad (D35)$$

Inserting (D34)–(D35) into the transition matrix (D33), using the pertinent spin-tensor entries in (D32), we obtain

$$\left\langle 0^+ \frac{111}{222} \right| \mathbb{V}_T \left| 2^+ \frac{111}{222} \right\rangle = -\frac{3}{10\sqrt{15}} \langle V_{1S1D} \rangle \quad (D36)$$

with

$$\langle V_{1S1D} \rangle = \int d\vec{\rho} d\vec{\lambda} \rho^2 \varphi_{1S}(R) \varphi_{1D}(R) V_T(\sqrt{2}\rho). \quad (D37)$$

A similar result holds for the 2S1D mixing, with the replacement $\varphi_{1S} \rightarrow \varphi_{2S}$,

$$\left\langle 0^{+*} \frac{111}{222} \right| \mathbb{V}_T \left| 2^+ \frac{111}{222} \right\rangle = -\frac{3}{10\sqrt{15}} \langle V_{2S1D} \rangle \quad (D38)$$

with

$$\langle V_{2S1D} \rangle = \int d\vec{\rho} d\vec{\lambda} \rho^2 \varphi_{2S}(R) \varphi_{1D}(R) V_T(\sqrt{2}\rho). \quad (\text{D39})$$

b. 3S1D

For the 3S-1D mixing, we have

$$\begin{aligned} \left\langle 0^{+**} \frac{1}{2} \frac{1}{2} m \left| \mathbb{V}_T \right| 2^+ \frac{1}{2} \frac{1}{2} m \right\rangle &= 3 C_{2m_L \frac{3}{2} m_S}^{\frac{1}{2} m} \int d\vec{\rho} d\vec{\lambda} \left(\frac{1}{2} (F_{\frac{1}{2}}^\rho (S_{\frac{1}{2}m}^\rho \varphi_{3S}^\lambda + S_{\frac{1}{2}m}^\lambda \varphi_{3S}^\rho) + F_{\frac{1}{2}}^\lambda (S_{\frac{1}{2}m}^\rho \varphi_{3S}^\rho - S_{\frac{1}{2}m}^\lambda \varphi_{3S}^\lambda))^* V_T^\rho S_{\frac{3}{2}m_S}^S \right. \\ &\quad \left. \times \frac{1}{\sqrt{2}} (F_{\frac{1}{2}}^\rho \varphi_{2m_L}^\rho + F_{\frac{1}{2}}^\lambda \varphi_{2m_L}^\lambda) \right), \end{aligned} \quad (\text{D40})$$

which reduces to

$$\left\langle 0^{+**} \frac{1}{2} \frac{1}{2} m \left| \mathbb{V}_T \right| 2^+ \frac{1}{2} \frac{1}{2} m \right\rangle = \frac{3}{2\sqrt{2}} C_{2m_L \frac{3}{2} m_S}^{\frac{1}{2} m} \int d\vec{\rho} d\vec{\lambda} \left(\varphi_{3S}^{\rho*} \varphi_{2m_L}^\rho - \varphi_{3S}^{\lambda*} \varphi_{2m_L}^\lambda \right) S_{\frac{1}{2}m}^\lambda V_T^\rho S_{\frac{3}{2}m_S}^S. \quad (\text{D41})$$

The ρ , λ -tensor wave functions selected by the nonvanishing Clebsches (D34), can be recast in spherical harmonics using (D35) and

$$\begin{aligned} \varphi_{2-1}^\rho &= + \frac{4\pi\sqrt{2}}{3} (Y_1^{-1}(\hat{\rho}) Y_1^0(\hat{\lambda}) + Y_1^{-1}(\hat{\lambda}) Y_1^0(\hat{\rho})) \rho \lambda \varphi_{1D}(R), \\ \varphi_{20}^\rho &= + \frac{4\pi\sqrt{2}}{3} (Y_1^0(\hat{\rho}) Y_1^0(\hat{\lambda}) + Y_1^0(\hat{\lambda}) Y_1^0(\hat{\rho})) \rho \lambda \varphi_{1D}(R), \\ \varphi_{21}^\rho &= + \frac{4\pi\sqrt{2}}{3} (Y_1^1(\hat{\rho}) Y_1^0(\hat{\lambda}) + Y_1^1(\hat{\lambda}) Y_1^0(\hat{\rho})) \rho \lambda \varphi_{1D}(R), \\ \varphi_{22}^\rho &= + \frac{4\pi\sqrt{2}}{3} (Y_1^1(\hat{\rho}) Y_1^1(\hat{\lambda}) + Y_1^1(\hat{\lambda}) Y_1^1(\hat{\rho})) \rho \lambda \varphi_{1D}(R) \end{aligned} \quad (\text{D42})$$

with the 3S-radial wave functions given in (D12), i.e.,

$$\begin{aligned} \varphi_{3S}^\lambda(\rho, \lambda) &= \frac{1}{\sqrt{2}} (\rho^2 - \lambda^2) \varphi_{3S}(R), \\ \varphi_{3S}^\rho(\rho, \lambda) &= \frac{1}{\sqrt{2}} (\rho^i \lambda^i + \lambda^i \rho^i) \varphi_{3S}(R) = \frac{4\pi\sqrt{2}}{3} (Y_1^{1*}(\hat{\rho}) Y_1^1(\hat{\lambda}) + Y_1^{1*}(\hat{\lambda}) Y_1^1(\hat{\rho}) + Y_1^0(\hat{\rho}) Y_1^0(\hat{\lambda})) \rho \lambda \varphi_{3S}(R). \end{aligned} \quad (\text{D43})$$

The spin-tensor matrix elements in (D41) can be read off (D32), with the result

$$S_{\frac{11}{22}}^\lambda V_T^\rho S_{\frac{3}{2}m_S}^S = -\delta_{m_S+m_L, \frac{1}{2}} \sqrt{\frac{(2+m_L)\pi}{5}} Y_2^{-m_L} \quad (\text{D44})$$

for the selected $m_L = 2, 1, 0, -1$ and fixed $m = \frac{1}{2}$. Inserting (D34)–(D44) into the transition matrix (D41), we obtain

$$\begin{aligned} \left\langle 0^{+**} \frac{1}{2} \frac{1}{2} m \left| \mathbb{V}_T \right| 2^+ \frac{1}{2} \frac{1}{2} m \right\rangle &= -\frac{1}{5\sqrt{15}} (2 + \sqrt{2}) \int d\vec{\rho} d\vec{\lambda} \rho^2 \lambda^2 \varphi_{3S}(R) \varphi_{1D}(R) \\ &\quad + \frac{1}{6\sqrt{15}} \int d\vec{\rho} d\vec{\lambda} (\rho^2 - \lambda^2) \rho^2 \varphi_{3S}(R) \varphi_{1D}(R) \end{aligned} \quad (\text{D45})$$

with the first contribution stemming from the $\rho\rho$ term, and the second contribution from the $\lambda\lambda$ term.

c. 1P1D

A similar analysis yields the tensor P-D matrix element

$$\left\langle 1^+ \frac{1}{2} \frac{1}{2} \frac{1}{2} \middle| \mathbb{V}_T \middle| 2^+ \frac{1}{2} \frac{1}{2} \frac{1}{2} \right\rangle = \frac{3}{2} C_{1m_L \frac{1}{2} m_S}^{\frac{1}{2} \frac{1}{2}} C_{2\bar{m}_L \frac{3}{2} \bar{m}_S}^{\frac{1}{2} \frac{1}{2}} \int d\vec{\rho} d\vec{\lambda} \left(\varphi_{1m_L}^{A*} \varphi_{2\bar{m}_L}^\rho S_{\frac{1}{2} m_S}^\lambda V_T^{\rho S} S_{\frac{3}{2} \bar{m}_S}^S - \varphi_{1m_L}^{A*} \varphi_{2\bar{m}_L}^\rho S_{\frac{1}{2} m_S}^\lambda V_T^{\rho S} S_{\frac{3}{2} \bar{m}_S}^S \right). \quad (\text{D46})$$

From (D32) it follows that all entries in $S_{\frac{1}{2} m_S}^\rho V_T^{\rho S} S_{\frac{3}{2} \bar{m}_S}^S = 0$, and (D46) simplifies

$$\left\langle 1^+ \frac{1}{2} \frac{1}{2} \frac{1}{2} \middle| \mathbb{V}_T \middle| 2^+ \frac{1}{2} \frac{1}{2} \frac{1}{2} \right\rangle = \frac{3}{2} C_{1m_L \frac{1}{2} m_S}^{\frac{1}{2} \frac{1}{2}} C_{2\bar{m}_L \frac{3}{2} \bar{m}_S}^{\frac{1}{2} \frac{1}{2}} \int d\vec{\rho} d\vec{\lambda} \varphi_{1m_L}^{A*} \varphi_{2\bar{m}_L}^\rho S_{\frac{1}{2} m_S}^\lambda V_T^{\rho S} S_{\frac{3}{2} \bar{m}_S}^S. \quad (\text{D47})$$

The 1^+ -antisymmetric wave functions in (D17) can be recast in spherical harmonics

$$\begin{aligned} \varphi_{11}^A &= \sqrt{\frac{4\pi}{3}} (Y_1^1(\hat{\rho}) Y_1^0(\hat{\lambda}) - Y_1^0(\hat{\rho}) Y_1^1(\hat{\lambda})) \rho \lambda \varphi_{1P}(R), \\ \varphi_{10}^A &= \sqrt{\frac{4\pi}{3}} (Y_1^0(\hat{\rho}) Y_1^0(\hat{\lambda}) - Y_1^0(\hat{\rho}) Y_1^0(\hat{\lambda})) \rho \lambda \varphi_{1P}(R) = 0, \\ \varphi_{1-1}^A &= \sqrt{\frac{4\pi}{3}} (Y_1^{-1}(\hat{\rho}) Y_1^0(\hat{\lambda}) - Y_1^0(\hat{\rho}) Y_1^{-1}(\hat{\lambda})) \rho \lambda \varphi_{1P}(R). \end{aligned} \quad (\text{D48})$$

The reduction of (D47) using the explicit wave functions (D42), (D48) is tedious and will not be detailed here.

d. 1D1D

Finally, the tensor D-D matrix element

$$\begin{aligned} \left\langle 2^+ \frac{1}{2} \frac{1}{2} m \middle| \mathbb{V}_T \middle| 2^+ \frac{1}{2} \frac{1}{2} m \right\rangle &= 3 C_{2m_L \frac{3}{2} m_S}^{\frac{1}{2} m} C_{2\bar{m}_L \frac{3}{2} \bar{m}_S}^{\frac{1}{2} m} \int d\vec{\rho} d\vec{\lambda} \frac{1}{\sqrt{2}} \left(F_{\frac{1}{2}}^\rho \varphi_{2m_L}^\rho + F_{\frac{1}{2}}^\lambda \varphi_{2m_L}^\lambda \right)^* S_{\frac{3}{2} m_S}^S V_T^{\rho S} S_{\frac{3}{2} \bar{m}_S}^S \frac{1}{\sqrt{2}} \left(F_{\frac{1}{2}}^\rho \varphi_{2\bar{m}_L}^\rho + F_{\frac{1}{2}}^\lambda \varphi_{2\bar{m}_L}^\lambda \right) \\ &= \frac{3}{2} C_{2m_L \frac{3}{2} m_S}^{\frac{1}{2} m} C_{2\bar{m}_L \frac{3}{2} \bar{m}_S}^{\frac{1}{2} m} \int d\vec{\rho} d\vec{\lambda} \left(\varphi_{2m_L}^{\rho*} \varphi_{2\bar{m}_L}^\rho + \varphi_{2m_L}^{\lambda*} \varphi_{2\bar{m}_L}^\lambda \right) S_{\frac{3}{2} m_S}^S V_T^{\rho S} S_{\frac{3}{2} \bar{m}_S}^S \end{aligned} \quad (\text{D49})$$

simplifies by angular integration. The reduction of (D49) is also tedious, and for illustration of the procedure we present only the contribution stemming from the diagonal spin-tensor entries in (D49). Using the spin-tensor entries in (D32) we have

$$S_{\frac{3}{2} m_S}^S V_T^{\rho S} S_{\frac{3}{2} \bar{m}_S}^S = (-1)^{\frac{5}{2} + |m_S|} \sqrt{\frac{\pi}{5}} Y_2^0(\hat{\rho}), \quad (\text{D50})$$

which for $m = \frac{1}{2}$ gives

$$\frac{3}{2} \sqrt{\frac{\pi}{5}} \left(|C_{22\frac{3}{2}\frac{1}{2}}^{\frac{1}{2}}|^2 \int d\vec{\rho} d\vec{\lambda} |\varphi_{22}^\lambda|^2 Y_2^0(\hat{\rho}) - |C_{21\frac{3}{2}\frac{1}{2}}^{\frac{1}{2}}|^2 \int d\vec{\rho} d\vec{\lambda} |\varphi_{21}^\lambda|^2 Y_2^0(\hat{\rho}) \right). \quad (\text{D51})$$

Since the mixed symmetry wave functions (D18) can be recast in spherical harmonics

$$\varphi_{22}^\lambda = \sqrt{\frac{16\pi}{15}} (\rho^2 Y_2^2(\hat{\rho}) - \lambda^2 Y_2^2(\hat{\lambda})) \varphi_{1D} \quad \varphi_{21}^\lambda = \sqrt{\frac{8\pi}{15}} (\rho^2 Y_2^1(\hat{\rho}) - \lambda^2 Y_2^1(\hat{\lambda})) \varphi_{1D} \quad (\text{D52})$$

and using the identities

$$\int d\vec{\rho} Y_2^{2*}(\hat{\rho}) Y_2^0(\hat{\rho}) Y_2^2(\hat{\rho}) = 2 \int d\vec{\rho} Y_2^{1*}(\hat{\rho}) Y_2^0(\hat{\rho}) Y_2^1(\hat{\rho}) = \frac{2}{7} \sqrt{\frac{5}{4\pi}} \quad (\text{D53})$$

the spin-diagonal contribution in (D49) is

$$\begin{aligned} & \left(\left| C_{22\frac{3}{2}-\frac{3}{2}}^{\frac{11}{2}} \right|^2 - \frac{1}{4} \left| C_{21\frac{3}{2}-\frac{1}{2}}^{\frac{11}{2}} \right|^2 \right) \frac{2}{35} \langle V_{1D1D} \rangle \\ &= \left(\sqrt{\frac{1}{5}} - \frac{1}{4} \sqrt{\frac{3}{10}} \right) \frac{2}{35} \langle V_{1D1D} \rangle = \frac{1}{140} \langle V_{1D1D} \rangle \quad (\text{D54}) \end{aligned}$$

with

$$\langle V_{1D1D} \rangle = \int d\vec{\rho} d\vec{\lambda} \rho^4 \phi_{1D} \phi_{1D} V_T(\sqrt{2}\rho). \quad (\text{D55})$$

The additional off-diagonal spin contributions in (D49) follow from the same reasoning and will not be presented.

5. Details of V_{SS}, V_T matrix elements

The perturbative (one gluon exchange) expressions for the spin-dependent potentials are

$$\begin{aligned} V_{SS}^{12} &= \frac{2\alpha_s}{3m^2} \frac{8\pi}{3} \delta^3(\sqrt{2}\vec{\rho}), \\ V_T^{12} &= \frac{2\alpha_s}{3m^2} \frac{1}{(\sqrt{2}\rho)^3} \end{aligned} \quad (\text{D56})$$

where the inter-quark distance is $r_{12} = (\sqrt{2}\rho)$. The matrix elements of the spin and tensor operators (D56) over the wave functions include integration over all six coordinates $\vec{\rho}, \vec{\lambda}$.

For the perturbative spin-spin term, we use the angular measure $d^3\rho d^2\lambda d\lambda \sin(\theta_\lambda) d\theta_\lambda d\phi_\lambda$, with the result

$$\begin{aligned} \langle A | V_{SS} | B \rangle &= 3 \frac{2\alpha_s}{3m^2} \frac{8\pi}{3} \frac{1}{2^{3/2}} \\ &\times \int \psi_A^*(\lambda) (\vec{S}^1 \cdot \vec{S}^2) \psi_B(\lambda) \lambda^2 d\lambda \sin(\theta_\lambda) d\theta_\lambda d\phi_\lambda. \end{aligned} \quad (\text{D57})$$

The perturbative tensor force needs to be regulated

$$1/r^3 \rightarrow 1/(r^2 + \delta^2)^{3/2}.$$

The regulator δ is fixed by physical considerations. First, for $\delta < 1/m_q$ we loose the range of validity of the non-relativistic approximation. The resulting “zitterbewegung” can in principle be remedied by the Dirac equation. Second, as we discussed in the section on form factors, the constituent quark is not a pointlike object. Its form factor well agrees with that from the ILM, which relates δ to the mean instanton size $\rho_{\text{inst}} \approx 0.3 \text{ fm} \sim 1.4 \text{ GeV}^{-1}$. While in principle there remains some dependence on the regulator size, this dependence is in practice quite weak. In Fig. 8 we show the transition matrix element of the perturbative tensor force between 1S and 1D state

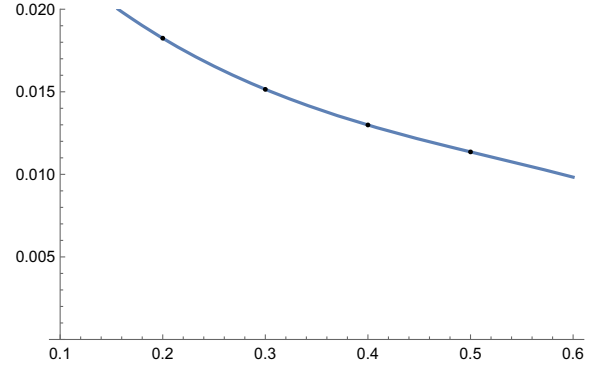


FIG. 8. Dependence of the integral (D58) on the regulator δ (in GeV^{-1}).

$$T(\delta) = \int \frac{\psi_{1S}^*(Y) \psi_{1D}(Y)}{(\rho^2 + \lambda^2)^5 (\rho^2 + \delta^2)^{3/2}} \rho^2 d\rho \lambda^2 d\lambda \quad (\text{D58})$$

with $Y = \sqrt{\rho^2 + \lambda^2}$. The tensor operator producing this mixing has a fixed structure, but its magnitude in the existing theory papers has relied mostly on the perturbative derivation. Lacking a full nonperturbative derivation from first principles, we can rely on various observables to quantify its magnitude.

APPENDIX E: EXPLICIT ORBIT-SPIN-FLAVOR WAVE FUNCTIONS IN THE “MONOM REPRESENTATION”

Let us start with the standard spin-isospin wave function for the nucleon

$$|1S\rangle = \psi_{1S}(Y) (S_\rho I_\rho + S_\lambda I_\lambda) / \sqrt{2} \quad (\text{E1})$$

in terms of the spin and isospin operators defined in (C1). The 2S state is the same, with the appropriate radial wave function. In the main part of the paper, we have used the decomposition into spin-flavor or 64 “monoms,” writing them as tables in Mathematica with indices defined in the order

$$\begin{aligned} & \text{Sum}[\dots \{s1, 1, 2\}, \{s2, 1, 2\} \{s3, 1, 2\} \{f1, 1, 2\} \\ & \{f2, 1, 2\} \{f3, 1, 2\}], \end{aligned}$$

the nucleon 1S and 2S states have the same angle-independent wave functions shown in Fig. 9.

Here only nine out of 64 possible “monom” terms are nonzero, so it may appear unnecessary to write it in this way. However, with the orbital parts containing two powers of coordinates, this reporting is very useful, as we now detail. Indeed, the construction of the actual wave functions can start with the permutation invariant combinations $V_{ABC}, V1, V2, V3$ as defined in Appendix C. This proceeds through two steps: (i) we combine a pair of standard

[illegible]

FIG. 9. 1S (proton) and 2S wave function in “flattened” spin-flavor tensor form.

$Y_{1,m}(\theta, \phi)$ functions into three combinations with $L = 0, 1, 2$ (scalar, vector, and tensor) using Clebsch-Gordon coefficients; (ii) we combine them with spin operators (using another set of Clebsch-Gordon coefficients) to obtain states with total angular momentum $J = 1/2$.

More specifically, let us start with the state with the largest angular momentum $L = 2$. The $J = 1/2$ can only be formed if the total spin is $S = 3/2$, which are permutation symmetric operators

$$\text{spin}[3/2, 3/2] = \uparrow\uparrow\uparrow, \quad q\text{spin}[3/2, 1/2] = (\downarrow\uparrow\uparrow + \uparrow\downarrow\uparrow + \uparrow\uparrow\downarrow)/\sqrt{3} \quad (\text{E2})$$

and $\text{spin}[3/2, -1/2]$, $\text{spin}[3/2, 3/2]$ defined by a reflection. Using the Clebsch-Gordon coefficients, we now define the wave function of the $L = 2, S = 3/2, J = 1/2$ “tensor” state as

$$\begin{aligned}
\text{“tensor”} = & \frac{1}{\sqrt{10}} \left[-(-2\text{spin}[3/2, -(3/2)]\psi[2, -2, \lambda\lambda] + \sqrt{3}\text{spin}[3/2, -(1/2)]\psi[2, -1, \lambda\lambda] \right. \\
& - \sqrt{2}\text{spin}[3/2, +(1/2)]\psi[2, 0, \lambda\lambda] + \text{spin}[3/2, +(3/2)]\psi[2, 1, \lambda\lambda] \\
& + 2\text{spin}[3/2, 3/ - 2]\psi[2, -2, \rho\rho] - \sqrt{3}\text{spin}[3/2, -(1/2)]\psi[2, -1, \rho\rho] \\
& + \sqrt{2}\text{spin}[3/2, +(1/2)]\psi[2, 0, \rho\rho] - \text{spin}[3/2, +(3/2)]\psi[2, 1, \rho\rho] * I_\lambda \\
& + (-2\text{spin}[3/2, -(3/2)]\psi[2, -2, \rho\lambda] + \sqrt{3}\text{spin}[3/2, -1/2]\psi[2, -1, \rho\lambda] \\
& - \sqrt{2}\text{spin}[3/2, +(1/2)]\psi[2, 0, \rho\lambda] + \text{spin}[3/2, +(3/2)]\psi[2, 1\rho\lambda] \\
& - 2\text{spin}[3/2, -(3/2)]\psi[2, -2, \lambda\rho] + \sqrt{3}\text{spin}[3/2, -(1/2)]\psi[2, -1, \lambda\rho] \\
& \left. - \sqrt{2}\text{spin}[3/2, +(1/2)]\psi[2, 0, \lambda\rho] + \text{spin}[3/2, +(3/2)]\psi[2, 1, \lambda\rho] * I_\rho \right]. \quad (\text{E3})
\end{aligned}$$

Here the coordinate parts of the wave functions are two spin-1 harmonics $Y_{1m}(\theta_i, \phi_i)$, combined into $L = 2$, with the remaining pair of indices indicating their angular part. For example $\psi[2, -2, \lambda\lambda]$ means $L = 2, L_z = -2$ combination and both angles are $\theta_\lambda, \phi_\lambda$. In this explicit form λ, ρ means the length of the corresponding vectors, with their angles shown explicitly. (Of course, λ, ρ can be expressed via the fifth angle χ , but we have not done so here for

brevity of notations.) Their (flattened) spin-tensor form is given in Fig. 10 below.

The other cases start with the invariants $V1, V2, V3$ made of four ρ, λ type objects. Not all of them may contribute. In particular, since $A_\alpha B_\beta$ are just coordinates $\vec{\rho}, \vec{\lambda}$, it is convenient to start with $L = 1$ corresponding to the antisymmetric combination $[\vec{\rho} \times \vec{\lambda}]$ so that only $V3$ contributes. The positive parity $L = 1$ state is shown in Fig. 11.

The $L = 1, S = 1/2, J = 1/2$ “vector” state is

$$\begin{aligned} \text{“vector”} = & ((\psi[1, -1, \rho\lambda] - \psi[1, -1, \lambda\rho]) * (\text{spin}[\rho, -1/2] * I_\lambda - \text{spin}[\lambda, -1/2] * I_\rho) * \text{Sqrt}[2/3] \\ & - (\psi[1, 0, \rho\lambda] - \psi[1, 0, \lambda\rho]) * (\text{spin}[\rho, 1/2] * I_\lambda - \text{spin}[\lambda, 1/2] * I_\rho) * 1/\text{Sqrt}[3]) \end{aligned} \quad (\text{E4})$$

where the $\text{spin}[\rho, \pm 1/2]$ or $\text{spin}[\lambda, \pm 1/2]$ are spin 1/2 operators of rho or lambda kind. I_ρ and I_λ are similar isospin operators, all with up (+1/2) components as we discuss protonlike states.

$$\begin{aligned}
& \left\{ 0, \frac{\sqrt{\frac{3}{5}} \left(e^{-i\phi\lambda} \lambda^2 \cos[\theta\lambda] \times \sin[\theta\lambda] - e^{-i\phi\rho} \rho^2 \cos[\theta\rho] \times \sin[\theta\rho] \right)}{4\pi}, \right. \\
& \frac{1}{8\sqrt{5}\pi} e^{-i(\phi\lambda+\phi\rho)} \left(-\lambda \cos[\theta\lambda] \left(\sqrt{3} e^{i\phi\rho} \lambda \sin[\theta\lambda] - 3 e^{i\phi\lambda} \rho \sin[\theta\rho] \right) + \right. \\
& \quad \left. \rho \cos[\theta\rho] \left(3 e^{i\phi\rho} \lambda \sin[\theta\lambda] + \sqrt{3} e^{i\phi\lambda} \rho \sin[\theta\rho] \right) \right), 0, \\
& \frac{1}{8\sqrt{5}\pi} e^{-i(\phi\lambda+\phi\rho)} \left(-\lambda \cos[\theta\lambda] \left(\sqrt{3} e^{i\phi\rho} \lambda \sin[\theta\lambda] + 3 e^{i\phi\lambda} \rho \sin[\theta\rho] \right) + \right. \\
& \quad \left. \rho \cos[\theta\rho] \left(-3 e^{i\phi\rho} \lambda \sin[\theta\lambda] + \sqrt{3} e^{i\phi\lambda} \rho \sin[\theta\rho] \right) \right), \\
& 0, 0, 0, 0, \frac{-\lambda^2 + \rho^2 - 3\lambda^2 \cos[2\theta\lambda] + 3\rho^2 \cos[2\theta\rho]}{8\sqrt{15}\pi}, \frac{1}{48\sqrt{5}\pi} \\
& \left(\sqrt{3} \lambda^2 (1 + 3\cos[2\theta\lambda]) - 24\lambda\rho \cos[\theta\lambda] \times \cos[\theta\rho] - \right. \\
& \quad \left. \sqrt{3} \rho^2 (1 + 3\cos[2\theta\rho]) + 6 e^{-i(\phi\lambda+\phi\rho)} \left(e^{2i\phi\lambda} + e^{2i\phi\rho} \right) \lambda \rho \sin[\theta\lambda] \times \sin[\theta\rho] \right), \\
& 0, \frac{1}{48\sqrt{5}\pi} \left(\sqrt{3} \lambda^2 (1 + 3\cos[2\theta\lambda]) + 24\lambda\rho \cos[\theta\lambda] \times \cos[\theta\rho] - \right. \\
& \quad \left. \sqrt{3} \rho^2 (1 + 3\cos[2\theta\rho]) - 6 e^{-i(\phi\lambda+\phi\rho)} \left(e^{2i\phi\lambda} + e^{2i\phi\rho} \right) \lambda \rho \sin[\theta\lambda] \times \sin[\theta\rho] \right), \\
& 0, 0, 0, 0, \frac{-\lambda^2 + \rho^2 - 3\lambda^2 \cos[2\theta\lambda] + 3\rho^2 \cos[2\theta\rho]}{8\sqrt{15}\pi}, \frac{1}{48\sqrt{5}\pi} \\
& \left(\sqrt{3} \lambda^2 (1 + 3\cos[2\theta\lambda]) - 24\lambda\rho \cos[\theta\lambda] \times \cos[\theta\rho] - \sqrt{3} \rho^2 (1 + 3\cos[2\theta\rho]) + \right. \\
& \quad \left. 6 e^{-i(\phi\lambda+\phi\rho)} \left(e^{2i\phi\lambda} + e^{2i\phi\rho} \right) \lambda \rho \sin[\theta\lambda] \times \sin[\theta\rho] \right), 0, \frac{1}{48\sqrt{5}\pi} \\
& \left(\sqrt{3} \lambda^2 (1 + 3\cos[2\theta\lambda]) + 24\lambda\rho \cos[\theta\lambda] \times \cos[\theta\rho] - \sqrt{3} \rho^2 (1 + 3\cos[2\theta\rho]) - \right. \\
& \quad \left. 6 e^{-i(\phi\lambda+\phi\rho)} \left(e^{2i\phi\lambda} + e^{2i\phi\rho} \right) \lambda \rho \sin[\theta\lambda] \times \sin[\theta\rho] \right), 0, 0, 0, 0, \\
& \frac{\sqrt{\frac{3}{5}} \left(-e^{i\phi\lambda} \lambda^2 \cos[\theta\lambda] \times \sin[\theta\lambda] + e^{i\phi\rho} \rho^2 \cos[\theta\rho] \times \sin[\theta\rho] \right)}{4\pi}, \frac{1}{8\sqrt{5}\pi} \\
& \left(\lambda \cos[\theta\lambda] \left(\sqrt{3} e^{i\phi\lambda} \lambda \sin[\theta\lambda] - 3 e^{i\phi\rho} \rho \sin[\theta\rho] \right) - \right. \\
& \quad \left. \rho \cos[\theta\rho] \left(3 e^{i\phi\lambda} \lambda \sin[\theta\lambda] + \sqrt{3} e^{i\phi\rho} \rho \sin[\theta\rho] \right) \right), \\
& 0, \frac{1}{8\sqrt{5}\pi} \left(\lambda \cos[\theta\lambda] \left(\sqrt{3} e^{i\phi\lambda} \lambda \sin[\theta\lambda] + 3 e^{i\phi\rho} \rho \sin[\theta\rho] \right) + \right. \\
& \quad \left. \rho \cos[\theta\rho] \left(3 e^{i\phi\lambda} \lambda \sin[\theta\lambda] - \sqrt{3} e^{i\phi\rho} \rho \sin[\theta\rho] \right) \right), \\
& 0, 0, 0, 0, \frac{-\lambda^2 + \rho^2 - 3\lambda^2 \cos[2\theta\lambda] + 3\rho^2 \cos[2\theta\rho]}{8\sqrt{15}\pi}, \frac{1}{48\sqrt{5}\pi} \\
& \left(\sqrt{3} \lambda^2 (1 + 3\cos[2\theta\lambda]) - 24\lambda\rho \cos[\theta\lambda] \times \cos[\theta\rho] - \sqrt{3} \rho^2 (1 + 3\cos[2\theta\rho]) + \right. \\
& \quad \left. 6 e^{-i(\phi\lambda+\phi\rho)} \left(e^{2i\phi\lambda} + e^{2i\phi\rho} \right) \lambda \rho \sin[\theta\lambda] \times \sin[\theta\rho] \right), 0, \frac{1}{48\sqrt{5}\pi} \\
& \left(\sqrt{3} \lambda^2 (1 + 3\cos[2\theta\lambda]) + 24\lambda\rho \cos[\theta\lambda] \times \cos[\theta\rho] - \sqrt{3} \rho^2 (1 + 3\cos[2\theta\rho]) - \right. \\
& \quad \left. 6 e^{-i(\phi\lambda+\phi\rho)} \left(e^{2i\phi\lambda} + e^{2i\phi\rho} \right) \lambda \rho \sin[\theta\lambda] \times \sin[\theta\rho] \right), 0, 0, 0, 0, \\
& \frac{\sqrt{\frac{3}{5}} \left(-e^{i\phi\lambda} \lambda^2 \cos[\theta\lambda] \times \sin[\theta\lambda] + e^{i\phi\rho} \rho^2 \cos[\theta\rho] \times \sin[\theta\rho] \right)}{4\pi}, \frac{1}{8\sqrt{5}\pi} \\
& \left(\lambda \cos[\theta\lambda] \left(\sqrt{3} e^{i\phi\lambda} \lambda \sin[\theta\lambda] - 3 e^{i\phi\rho} \rho \sin[\theta\rho] \right) - \right. \\
& \quad \left. \rho \cos[\theta\rho] \left(3 e^{i\phi\lambda} \lambda \sin[\theta\lambda] + \sqrt{3} e^{i\phi\rho} \rho \sin[\theta\rho] \right) \right), \\
& 0, \frac{1}{8\sqrt{5}\pi} \left(\lambda \cos[\theta\lambda] \left(\sqrt{3} e^{i\phi\lambda} \lambda \sin[\theta\lambda] + 3 e^{i\phi\rho} \rho \sin[\theta\rho] \right) + \right. \\
& \quad \left. \rho \cos[\theta\rho] \left(3 e^{i\phi\lambda} \lambda \sin[\theta\lambda] - \sqrt{3} e^{i\phi\rho} \rho \sin[\theta\rho] \right) \right)
\end{aligned}$$

FIG. 10. Orbital-spin-flavor wave function $VL2$, or the “tensor” state.

$$\begin{aligned}
& \rho \cos[\theta\rho] \left(3 e^{i\phi\lambda} \lambda \sin[\theta\lambda] - \sqrt{3} e^{i\phi\rho} \rho \sin[\theta\rho] \right), \theta, \theta, \theta, \theta, \\
& \frac{\sqrt{\frac{3}{5}} \left(-e^{i\phi\lambda} \lambda^2 \cos[\theta\lambda] \times \sin[\theta\lambda] + e^{i\phi\rho} \rho^2 \cos[\theta\rho] \times \sin[\theta\rho] \right)}{4\pi}, \frac{1}{8\sqrt{5}\pi} \\
& \left(\lambda \cos[\theta\lambda] \left(\sqrt{3} e^{i\phi\lambda} \lambda \sin[\theta\lambda] - 3 e^{i\phi\rho} \rho \sin[\theta\rho] \right) - \right. \\
& \left. \rho \cos[\theta\rho] \left(3 e^{i\phi\lambda} \lambda \sin[\theta\lambda] + \sqrt{3} e^{i\phi\rho} \rho \sin[\theta\rho] \right) \right), \\
& 0, \frac{1}{8\sqrt{5}\pi} \left(\lambda \cos[\theta\lambda] \left(\sqrt{3} e^{i\phi\lambda} \lambda \sin[\theta\lambda] + 3 e^{i\phi\rho} \rho \sin[\theta\rho] \right) + \right. \\
& \left. \rho \cos[\theta\rho] \left(3 e^{i\phi\lambda} \lambda \sin[\theta\lambda] - \sqrt{3} e^{i\phi\rho} \rho \sin[\theta\rho] \right) \right), \\
& 0, \theta, \theta, \theta, \frac{\sqrt{\frac{3}{5}} \left(-e^{2i\phi\lambda} \lambda^2 \sin[\theta\lambda]^2 + e^{2i\phi\rho} \rho^2 \sin[\theta\rho]^2 \right)}{4\pi}, \\
& \frac{\sqrt{3} e^{2i\phi\lambda} \lambda^2 \sin[\theta\lambda]^2 - 6 e^{i(\phi\lambda+\phi\rho)} \lambda \rho \sin[\theta\lambda] \times \sin[\theta\rho] - \sqrt{3} e^{2i\phi\rho} \rho^2 \sin[\theta\rho]^2}{8\sqrt{5}\pi}, \theta, \\
& \frac{\sqrt{3} e^{2i\phi\lambda} \lambda^2 \sin[\theta\lambda]^2 + 6 e^{i(\phi\lambda+\phi\rho)} \lambda \rho \sin[\theta\lambda] \times \sin[\theta\rho] - \sqrt{3} e^{2i\phi\rho} \rho^2 \sin[\theta\rho]^2}{8\sqrt{5}\pi}, \theta, \theta, \theta \}
\end{aligned}$$

FIG. 10. (Continued)

The last state $VL0$ is made out of two “scalar” combinations of $L = 0$ possible states out of $V1, V2$. After they are formed, it turns out that they are not orthogonal to each other, and after orthogonalization we found that one combination has no angles, and is just another copy of the $2S$ wave function. The physical one takes the form

shown in Fig. 12. While these functions are not yet normalized, they are orthogonal to each other. Multiple tests were performed, among them with the operators S_z and L_z , the third components of the spin and orbital momentum. Using five states in the order we perform averages with them, $|J\rangle = \{1S, 2S, VL0, VL1, VL2\}$; we

$$\begin{aligned}
& \left\{ 0, \theta, \theta, \theta, \theta, \theta, \theta, \theta, \theta, \theta, \theta, \frac{e^{-i(\phi\lambda+\phi\rho)} \left(-e^{2i\phi\lambda} + e^{2i\phi\rho} \right) \lambda \rho \sin[\theta\lambda] \times \sin[\theta\rho]}{4\sqrt{2}\pi}, \right. \\
& 0, \frac{e^{-i(\phi\lambda+\phi\rho)} \left(e^{2i\phi\lambda} - e^{2i\phi\rho} \right) \lambda \rho \sin[\theta\lambda] \times \sin[\theta\rho]}{4\sqrt{2}\pi}, \theta, \theta, \\
& 0, \theta, \frac{e^{-i(\phi\lambda+\phi\rho)} \left(e^{2i\phi\lambda} - e^{2i\phi\rho} \right) \lambda \rho \sin[\theta\lambda] \times \sin[\theta\rho]}{4\sqrt{2}\pi}, \theta, \theta, \\
& \frac{e^{-i(\phi\lambda+\phi\rho)} \left(-e^{2i\phi\lambda} + e^{2i\phi\rho} \right) \lambda \rho \sin[\theta\lambda] \times \sin[\theta\rho]}{4\sqrt{2}\pi}, \theta, \theta, \theta, \\
& 0, \frac{\lambda \rho \left(e^{i\phi\lambda} \cos[\theta\rho] \times \sin[\theta\lambda] - e^{i\phi\rho} \cos[\theta\lambda] \times \sin[\theta\rho] \right)}{2\sqrt{2}\pi}, \\
& \frac{\lambda \rho \left(-e^{i\phi\lambda} \cos[\theta\rho] \times \sin[\theta\lambda] + e^{i\phi\rho} \cos[\theta\lambda] \times \sin[\theta\rho] \right)}{2\sqrt{2}\pi}, \theta, \theta, \\
& 0, \theta, \theta, \theta, \frac{e^{-i(\phi\lambda+\phi\rho)} \left(-e^{2i\phi\lambda} + e^{2i\phi\rho} \right) \lambda \rho \sin[\theta\lambda] \times \sin[\theta\rho]}{4\sqrt{2}\pi}, \\
& \frac{e^{-i(\phi\lambda+\phi\rho)} \left(e^{2i\phi\lambda} - e^{2i\phi\rho} \right) \lambda \rho \sin[\theta\lambda] \times \sin[\theta\rho]}{4\sqrt{2}\pi}, \theta, \theta, \theta, \theta, \theta, \\
& 0, \frac{\lambda \rho \left(-e^{i\phi\lambda} \cos[\theta\rho] \times \sin[\theta\lambda] + e^{i\phi\rho} \cos[\theta\lambda] \times \sin[\theta\rho] \right)}{2\sqrt{2}\pi}, \theta, \theta, \\
& \frac{\lambda \rho \left(e^{i\phi\lambda} \cos[\theta\rho] \times \sin[\theta\lambda] - e^{i\phi\rho} \cos[\theta\lambda] \times \sin[\theta\rho] \right)}{2\sqrt{2}\pi}, \theta, \theta, \theta, \\
& 0, \theta, \frac{\lambda \rho \left(e^{i\phi\lambda} \cos[\theta\rho] \times \sin[\theta\lambda] - e^{i\phi\rho} \cos[\theta\lambda] \times \sin[\theta\rho] \right)}{2\sqrt{2}\pi}, \theta, \\
& \frac{\lambda \rho \left(-e^{i\phi\lambda} \cos[\theta\rho] \times \sin[\theta\lambda] + e^{i\phi\rho} \cos[\theta\lambda] \times \sin[\theta\rho] \right)}{2\sqrt{2}\pi}, \theta, \theta, \theta, \theta, \theta, \theta, \theta, \theta, \theta, \theta \}
\end{aligned}$$

FIG. 11. Orbital-spin-flavor wave function $VL1$.

$$\begin{aligned}
& \left\{ 0, 0, 0, 0, 0, 0, 0, 0, 0, \frac{\lambda^2 - \rho^2}{2\sqrt{3}\pi}, \right. \\
& \frac{\sqrt{3}(-\lambda^2 + \rho^2) - 6\lambda\rho \cos[\theta\lambda] \times \cos[\theta\rho] - 3e^{-i(\phi\lambda + \phi\rho)} (e^{2i\phi\lambda} + e^{2i\phi\rho}) \lambda\rho \sin[\theta\lambda] \times \sin[\theta\rho]}{12\pi}, \\
& 0, \\
& \frac{\sqrt{3}(-\lambda^2 + \rho^2) + 6\lambda\rho \cos[\theta\lambda] \times \cos[\theta\rho] + 3e^{-i(\phi\lambda + \phi\rho)} (e^{2i\phi\lambda} + e^{2i\phi\rho}) \lambda\rho \sin[\theta\lambda] \times \sin[\theta\rho]}{12\pi}, \\
& 0, 0, 0, 0, \\
& \frac{\sqrt{3}(-\lambda^2 + \rho^2) - 6\lambda\rho \cos[\theta\lambda] \times \cos[\theta\rho] - 3e^{-i(\phi\lambda + \phi\rho)} (e^{2i\phi\lambda} + e^{2i\phi\rho}) \lambda\rho \sin[\theta\lambda] \times \sin[\theta\rho]}{12\pi}, \\
& \frac{\sqrt{3}(-\lambda^2 + \rho^2) + 6\lambda\rho \cos[\theta\lambda] \times \cos[\theta\rho] + 3e^{-i(\phi\lambda + \phi\rho)} (e^{2i\phi\lambda} + e^{2i\phi\rho}) \lambda\rho \sin[\theta\lambda] \times \sin[\theta\rho]}{12\pi}, \\
& 0, \frac{\lambda^2 - \rho^2}{2\sqrt{3}\pi}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \\
& \frac{\sqrt{3}(-\lambda^2 + \rho^2) + 6\lambda\rho \cos[\theta\lambda] \times \cos[\theta\rho] + 3e^{-i(\phi\lambda + \phi\rho)} (e^{2i\phi\lambda} + e^{2i\phi\rho}) \lambda\rho \sin[\theta\lambda] \times \sin[\theta\rho]}{12\pi}, \\
& \frac{\lambda^2 - \rho^2}{2\sqrt{3}\pi}, 0, \\
& \frac{\sqrt{3}(-\lambda^2 + \rho^2) - 6\lambda\rho \cos[\theta\lambda] \times \cos[\theta\rho] - 3e^{-i(\phi\lambda + \phi\rho)} (e^{2i\phi\lambda} + e^{2i\phi\rho}) \lambda\rho \sin[\theta\lambda] \times \sin[\theta\rho]}{12\pi}, \\
& \left. 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \right\}
\end{aligned}$$

FIG. 12. Orbital-spin-flavor wave function $VL0$, or the “scalar” state.

have the following

$$\langle S_z \rangle = \{1/2, 1/2, 1/2, -(1/6), -(1/2)\}, \quad (E5)$$

$$\langle L_z \rangle = \{0, 0, 0, 2/3, 1\}, \quad (E6)$$

which adds correctly to $\langle J_z \rangle = 1/2$ for each of them. The same test carries to the isospin operators $I_z = 1/2$, so the results describe correctly the proton and its excitations with the same global quantum number $J^P = (1/2)^+$.

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