

Quantum estimation in QED scattering

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We tackle the issue of estimating dynamical parameters in quantum electrodynamics. We numerically compute the quantum Fisher information matrix (QFIM) of physical parameters in electron-muon and Compton scattering at tree level. In particular, we consider the estimation of center-of-mass three-momentum magnitude and polar scattering angle through measurements on the internal degrees of freedom (helicity or polarization) of the scattered particles. Computations are carried out for pure and maximally mixed initial states. The QFIM values are then used to compute the quantum Cramér-Rao lower bounds on the estimations at hand. Further, we compare such ultimate bounds to the classical Fisher information of local polarization or helicity degrees of freedom.

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I. INTRODUCTION

Making inferences based on experimental measurements is a fundamental step of the scientific method [1]. One example of such an inference is the estimation of one or multiple parameters on which a measurement may depend [2]. This is referred to as the parameter estimation problem and is approached using the methods of estimation theory [3]. In the context of quantum measurements, it is the framework of quantum metrology, or quantum estimation theory (QET), that offers methods to establish the precision limits constraining quantum measurements, as well as, in principle, schemes that achieve these limits and potentially surpass the performance of classical systems [4]. A key concept in QET is the quantum Fisher information (QFI) of a quantum state, calculated with respect to the parameters to be estimated [3]. The QFI allows one to place a fundamental precision bound on parameter estimation, known as the Cramér-Rao lower bound.

QFI calculations have been applied in a variety of contexts, including condensed matter physics [5,6], cosmology [7–9], and optics [10–12]. The QFI has also been used to show a connection between classical fluid dynamics and relativistic quantum mechanics [13].

The growing interest in the connection between quantum information and high-energy physics (HEP) has also

led to investigations of the QFI in HEP. Quantum information metrics [14–16] and the QFI in particular [17–23], have been studied in the context of the anti-de Sitter/conformal field theory (AdS/CFT) correspondence too. The QFI of an Unruh-DeWitt detector has also been investigated in Ref. [24]. Reference [25] calculated the classical Fisher information matrix (CFIM) of the probability distribution associated with measuring a particular field configuration in a massless Klein-Gordon quantum field theory. The CFIM in Ref. [25] was calculated with respect to the coordinate labels of a flat spacetime background. Furthermore, Ref. [26] calculated the quantum and classical Fisher information with respect to the Dirac CP-violating phase in the context of neutrino oscillation detection. However, there is no known study examining the QFI of excitations of the quantum field in the context of HEP scattering. This is the central issue addressed by the present paper, as we consider QFI in QED scattering at the tree level. Hence, while Ref. [25] demonstrated that a massless Klein-Gordon field carries information about a spacetime coordinate label, we show that the Dirac and gauge fields in QED carry information about a momentum label. This paper is also a development of previous investigations considering the quantum informational properties of QED scattering. In this specific regard, Refs. [27,28] investigated entanglement between the helicity degrees of freedom in tree-level QED, in particular focusing on the conditions for maximal entanglement, whilst Ref. [29] investigated such quantum correlations over a wider range of scattering energies, for pure and mixed scattering states. In recent years, further developments have been made investigating entanglement in scattering processes in QED [30–32], noncommutative QED [33], and quantum chromodynamics [34].

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The tree-level scattering processes we shall consider in this paper are electron-muon and Compton scattering, which will be prototypical of QED processes.

The specific estimation we will be considering is better explained by employing the services of Alice and Bob, the archetypal actors of experiments in the quantum information tradition. In our scenario, Alice prepares two initial particles with known center-of-mass energy and lets them scatter off each other. Hence, she selects the outgoing scattering angle with infinite precision, say through an infinitely narrow collimator. The two-particle quantum state she thus prepares and sends off (in opposite directions) is characterized by the parameters p , the conserved total three-momentum magnitude in the center of mass frame, and θ , the selected scattering angle. Bob receives this state, but has no knowledge of the two parameters. In order to reconstruct them, he can only access internal degrees of freedom (polarization for photons, helicity for massive particles). We assume such an access to be global, that is, Bob can perform global measurements on the internal two-qubit state he receives from Alice (but then also consider a local estimation scheme). We therefore consider a multiparameter quantum estimation of the external degrees of freedom of the scattering process, namely the three-momentum magnitude and the scattering angle, based on such internal degrees of freedom quantum measurements. To this aim, we calculate the quantum Fisher information matrix (QFIM) of the output two-qubit state after scattering, whose inverse provides a lower bound on the covariance matrix associated with the estimation of the external parameters considered, by virtue of the quantum Cramér-Rao bound. Thus, we place ultimate, rigorous bounds on the estimation procedure. Besides, we also provide a specific example of estimation, based on diagonal helicity or polarization measurements, and contrast it with the general bounds.

Albeit the scenario we just described is not, *per se*, directly relevant in practice [35], it does lend an exact operational meaning to the bounds we will derive. On a fundamental level, our investigation addresses the hitherto-untackled question of how much information about the external degrees of freedom can be gleaned by accessing the internal degrees of freedom of a quantum field. As we shall discuss in our conclusions, our approach may be further developed and extended to improve estimation in cases with direct practical impact.

II. SETTING AND PRELIMINARIES

In order to provide the relevant set-up for this paper, Sec. II A introduces the problem of multiparameter quantum estimation and the concepts of classical and quantum Fisher information. Section II D states the estimation process that we consider. Section II B provides the derivation of the postscattering, four-momentum-filtered quantum state density matrix, whose QFIM is then calculated. Finally, Sec. II C provides the optimal positive operator-valued

measure (POVM) for the measurement of pure quantum states.

A. Fisher information

1. Classical Fisher information

In estimation theory we consider a vector of physically observable random variables $\mathbf{X} = (X_1, X_2, \dots, X_n)$ [36] and define $\mathbf{x} = (x_1, x_2, \dots, x_n)$ to be a vector of values that $\mathbf{X}$ take after a single measurement is performed. This measurement process can then be repeated, giving us a set of different $\mathbf{x}$ values. The probability that $\mathbf{X} = \mathbf{x}$ for a given measurement is described by the joint probability density function (PDF) $p(\mathbf{x}) = p(\mathbf{x}|\boldsymbol{\sigma})$, which depends on the vector of parameters $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \dots, \sigma_m)$. Within this framework, the task of accurately estimating $\boldsymbol{\sigma}$, given a set of $\mathbf{x}$ measurements, is rephrased as the task of finding a suitable estimator $\hat{\boldsymbol{\sigma}}(\mathbf{x})$, which is close to the true $\boldsymbol{\sigma}$ [37]. $\hat{\boldsymbol{\sigma}}(\mathbf{x})$ is defined as a mapping,

$$\hat{\boldsymbol{\sigma}}: \mathcal{X} \rightarrow \boldsymbol{\Sigma}, \quad (1)$$

where $\mathcal{X}$ is the space of observations $\mathbf{x}$, and $\boldsymbol{\Sigma}$ is the space of parameter values $\boldsymbol{\sigma}$ [38].

If the measurement performed on $\mathbf{X}$ is classical, then the study of $\hat{\boldsymbol{\sigma}}(\mathbf{x})$ is subject to classical estimation theory (CET). In this paper we consider the properties of unbiased estimators, which are defined to have an expectation value equal to the true parameter value [36]:

$$\mathbb{E}[\hat{\boldsymbol{\sigma}}(\mathbf{x})] = \boldsymbol{\sigma}. \quad (2)$$

Without making any further assumptions about the properties of $\hat{\boldsymbol{\sigma}}(\mathbf{x})$, we are interested in lower bounding the precision of the estimation $\hat{\boldsymbol{\sigma}}(\mathbf{x})$ performs. For a classical measurement, this precision lower bound is placed on the variance of $\hat{\boldsymbol{\sigma}}(\mathbf{x})$ [i.e., $\text{Var}[\hat{\boldsymbol{\sigma}}(\mathbf{x})]$] and is defined as the classical Cramér-Rao lower bound (CCRLB). In order to define the CCRLB we introduce the CFIM $\mathbf{I}_{\boldsymbol{\sigma}, \mathbf{x}}^{(C)}$. The following review of the CFIM is adapted from [39]. The CFIM is a positive semidefinite, symmetric $m \times m$ matrix, with its (i, j) element given by

$$(\mathbf{I}_{\boldsymbol{\sigma}, \mathbf{x}}^{(C)})_{i,j} = \text{Cov}[L_{\sigma_i}^{(C)}, L_{\sigma_j}^{(C)}]. \quad (3)$$

Here $L_{\sigma_i}^{(C)} = \frac{\partial \log p(\mathbf{x}|\boldsymbol{\sigma})}{\partial \sigma_i}$ is defined as the score function, with its derivative evaluated at the true value of σ_i [40,41], while the covariance $\text{Cov}[X, Y]$ of two random variables X and Y is defined as

$$\text{Cov}[X, Y] = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])]. \quad (4)$$

Accordingly, $\text{Cov}[X, X] = \text{Var}[X]$. In the single-parameter estimation case, where $\boldsymbol{\sigma} = \sigma$, the CFIM $\mathbf{I}_{\boldsymbol{\sigma}, \mathbf{x}}^{(C)}$ reduces to the

classical Fisher information (CFI) $I_{\sigma, \mathbf{x}}^{(C)}$ given by

$$\begin{aligned} I_{\sigma, \mathbf{x}}^{(C)} &= \text{Var}[L_{\sigma}^{(C)}(\mathbf{x})] \\ &= E[(L_{\sigma}^{(C)}(\mathbf{x}))^2], \end{aligned} \quad (5)$$

where $L_{\sigma}^{(C)}(\mathbf{x}) = \frac{\partial \log p(\mathbf{x}|\sigma)}{\partial \sigma}$. In order to intuitively understand the motivation behind the definition of $I_{\sigma, \mathbf{x}}^{(C)}$ as a quantifier of information about the value of σ , it is useful to consider a scenario where we collect N different measurements $\mathbf{x}^{(i)}$ for $i \in \{1, \dots, N\}$ and then plot N different $\log p(\mathbf{x}^{(i)}|\sigma)$ - σ curves, where $\log p(\mathbf{x}^{(i)}|\sigma)$ is defined as the log-likelihood [42] of the respective curve. In these plots, the PDF $p(\mathbf{x}^{(i)}|\sigma)$ tells us the probability of the parameter σ being equal to any given value, for a fixed measurement result $\mathbf{x}^{(i)}$. The next step is to compute the score function $L_{\sigma}^{(C)}(\mathbf{x})$ of each curve. $L_{\sigma}^{(C)}(\mathbf{x})$ can be thought of as a piece of information about the shape of the curve, or as an “instruction” about where to find its local minimum. If the N score functions $L_{\sigma}^{(C)}(\mathbf{x})$ have a large variance of values around a given σ , then we have a distribution of sharp $\log p(\mathbf{x}^{(i)}|\sigma)$ - σ curves, meaning that $p(\mathbf{x}|\sigma)$ is sensitive to small changes in σ , hence $p(\mathbf{x}|\sigma)$ contains substantial information about σ . This corresponds to a large Fisher information value $I_{\sigma, \mathbf{x}}^{(C)}$ around a given σ . Conversely, a small variance of score functions at a given σ corresponds to a large spread of broad curves, i.e., $p(\mathbf{x}|\sigma)$ contains less information about σ , which is quantified by a smaller $I_{\sigma, \mathbf{x}}^{(C)}$.

Having motivated the definition of $I_{\sigma, \mathbf{x}}^{(C)}$, we now derive the CCRLB in a single-parameter estimation scenario. The CCRLB is ultimately of a geometric origin, as it arises from the Cauchy–Schwarz (CS) inequality. The CS inequality for two real-valued random variables X and Y , with an inner product defined by $\langle X, Y \rangle := E[XY]$, is given by $E[XY]^2 \leq E[X^2]E[Y^2]$, which leads to $\text{Cov}[X, Y]^2 \leq \text{Var}[X]\text{Var}[Y]$. Upon substituting $X \rightarrow \hat{\sigma}(\mathbf{x})$ and $Y \rightarrow L_{\sigma}^{(C)}(\mathbf{x})$, the covariance inequality yields

$$\text{Cov}[\hat{\sigma}(\mathbf{x}), L_{\sigma}^{(C)}(\mathbf{x})]^2 \leq \text{Var}[\hat{\sigma}(\mathbf{x})]I_{\sigma, \mathbf{x}}^{(C)}. \quad (6)$$

Using the property $E[L_{\sigma}^{(C)}(\mathbf{x})] = 0$ and the expectation value of an unbiased estimator (2) leads to

$$\text{Cov}[\hat{\sigma}, L_{\sigma}^{(C)}] = E[\hat{\sigma}L_{\sigma}^{(C)}] = E\left[\hat{\sigma} \cdot \frac{\partial p}{\partial \sigma} \frac{1}{p}\right]. \quad (7)$$

In the case of a continuous random variable $\mathbf{x}$, this yields

$$\text{Cov}[\hat{\sigma}, L_{\sigma}^{(C)}] = \int \hat{\sigma}(\mathbf{x}) \cdot \frac{\partial p(\mathbf{x}|\sigma)}{\partial \sigma} d\mathbf{x} = 1, \quad (8)$$

where we allowed the interchange of derivative and integral and then applied the unbiased estimator condition (2). Hence, by substituting (8) into (6) one obtains the expression of the CCRLB [41]:

$$\text{Var}[\hat{\sigma}(\mathbf{x})] \geq \frac{1}{I_{\sigma, \mathbf{x}}^{(C)}}. \quad (9)$$

The Fisher information is by definition additive over repeated, independent samplings of the distribution, so that for N such samplings one has the CCRLB $\text{Var}[\hat{\sigma}(\mathbf{x})] \geq 1/(NI_{\sigma, \mathbf{x}}^{(C)})$. The Cramér-Rao bound sets the ultimate limit to the precision achievable on the estimation of a parameter (at least in terms of its estimation variance).

The multiparameter case can be treated along the same lines, although there the CCRLB takes the form of a matrix inequality relating the inverse of the $n \times n$ matrix $\mathbf{I}_{\sigma, \mathbf{x}}^{(C)}$ to the covariance matrix $\text{Cov}[\hat{\sigma}(\mathbf{x})]$ of the multiparameter estimator $\hat{\sigma}(\mathbf{x})$, as [41]

$$\text{Cov}[\hat{\sigma}(\mathbf{x})] \geq [\mathbf{I}_{\sigma, \mathbf{x}}^{(C)}]^{-1}. \quad (10)$$

It is very instructive to consider a two-parameter instance of this estimation:

$$\begin{pmatrix} \text{Var}[\hat{x}] & \text{Cov}[\hat{x}, \hat{y}] \\ \text{Cov}[\hat{x}, \hat{y}] & \text{Var}[\hat{y}] \end{pmatrix} \geq \frac{1}{(\mathbf{I}_{xx}\mathbf{I}_{yy} - \mathbf{I}_{xy}^2)} \begin{pmatrix} \mathbf{I}_{yy} & -\mathbf{I}_{xy} \\ -\mathbf{I}_{xy} & \mathbf{I}_{xx} \end{pmatrix}, \quad (11)$$

where we write $\mathbf{I}_{\sigma, \mathbf{x}}^{(C)} = \mathbf{I}_{\sigma}$ for simplicity and we set $\sigma_1 = x$ and $\sigma_2 = y$. First off, notice that, as one should expect, in the absence of correlations between the two variables (i.e., for $\text{Cov}[\hat{x}, \hat{y}] = 0$), one recovers two independent single-parameter CCRLB's in terms of the respective Fisher informations I_{xx} and I_{yy} . Further, notice that the bound is generally worse than in the single-parameter case—i.e., the fundamental lower bounds on the individual variances, on the matrix's main diagonal, are larger than the single-parameter bounds. The single-parameter bounds are also recovered if the Fisher information on the other parameter diverges—i.e., letting $I_{yy} \rightarrow \infty$ yields

$$\text{Var}[\hat{x}]_{\text{fixed } y} \geq \frac{1}{I_{xx}}. \quad (12)$$

2. Quantum Fisher information

Let us now move on to *quantum* estimation theory, by considering the case of a quantum state $\rho(\sigma)$, rather than a classical probability distribution, depending on a parameter σ . The key realization here is that, once a POVM, i.e., a specific quantum measurement, is set, the Born rule extracts a classical probability distribution from a quantum state, and one can apply the classical analysis presented

above, whereby the variance of a parameter σ is minimized by the (achievable) CCRLB. The optimization of the CCRLB over all possible POVMs then gives the quantum Cramér-Rao lower bound (QCRLB), in terms of the quantum Fisher information. The latter is therefore just the classical Fisher information optimized over all possible POVMs.

The following description of QET is adapted from [3]. Again, let us treat first the single-parameter estimation case. In QET, we describe our system by the density operator rather than the PDF $p(\mathbf{x}|\sigma)$ [37]. The two are related by the Born rule $p(\mathbf{x}|\sigma)d\mathbf{x} = \text{Tr}[d\Pi_{\mathbf{x}}\rho(\sigma)]$, where $d\Pi_{\mathbf{x}}$ are POVM elements satisfying $\int d\Pi_{\mathbf{x}} = 1$.

In QET, the (classical) score function $L_{\sigma}^{(C)}(\mathbf{x})$ is generalized to the symmetric logarithmic derivative (SLD) quantum operator L_{σ} . Since, in the classical case, $L_{\sigma}^{(C)}(\mathbf{x})p(\mathbf{x}|\sigma) = \partial_{\sigma}p(\mathbf{x}|\sigma)$, in QET one sets

$$\frac{1}{2}(L_{\sigma}\rho(\sigma) + \rho(\sigma)L_{\sigma}) = \partial_{\sigma}\rho(\sigma). \quad (13)$$

Then, the QFI of state $\rho(\sigma)$ with respect to the parameter σ is defined by

$$\mathbf{I}(\sigma) = \text{Tr}[\rho(\sigma)(L_{\sigma})^2], \quad (14)$$

which is the quantum generalization of the classical Fisher information expression in Eq. (5), in the sense that it maximizes the classical Fisher information over all POVMs. For N independent quantum measurements of $\mathbf{X}$ on $\rho(\sigma)$, the variance of any estimator $\hat{\sigma}$ is in fact constrained by the QCRLB:

$$\text{Var}[\hat{\sigma}(\mathbf{x})] \geq \frac{1}{N\mathbf{I}_{\sigma}}. \quad (15)$$

Analogously to its classical counterpart, in the multi-parameter estimation case, the QFI generalizes to the QFIM $\mathbf{I}(\sigma)$. For m parameters $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_m)$ the QFIM is a positive semidefinite, symmetric $m \times m$ matrix, with its (i, j) element given by

$$(\mathbf{I}_{\sigma})_{i,j} = \text{Tr}\left[\rho(\sigma) \frac{L_{\sigma_i}L_{\sigma_j} + L_{\sigma_j}L_{\sigma_i}}{2}\right]. \quad (16)$$

This expression is equivalent to [43]

$$(\mathbf{I}_{\sigma})_{i,j} = 2 \sum_{k,l} \frac{\text{Re}[\langle k|\partial_i\rho(\sigma)|l\rangle\langle l|\partial_j\rho(\sigma)|k\rangle]}{\lambda_k + \lambda_l}, \quad (17)$$

where $\partial_j\rho(\sigma) = \frac{\partial\rho(\sigma)}{\partial\sigma_j}$ and $\rho(\sigma) = \sum_k \lambda_k(\sigma)|k\rangle\langle k|$ is the eigendecomposition of the density matrix $\rho(\sigma)$. Here $\{|k\rangle\}$ and $\{\lambda_k\}$ are, respectively, the sets of eigenstates and

eigenvalues of $\rho(\sigma)$. For a pure state $\rho(\sigma) = \rho^2(\sigma) = |\psi\rangle\langle\psi|$ Eq. (17) reduces to [43]

$$(\mathbf{I}_{\sigma})_{i,j} = 4\text{Re}[\langle\partial_i\psi|\partial_j\psi\rangle - \langle\partial_i\psi|\psi\rangle\langle\psi|\partial_j\psi\rangle], \quad (18)$$

where $|\partial_j\psi\rangle = \frac{\partial|\psi\rangle}{\partial\sigma_j}$. In the multiparameter estimation scenario we consider here, the QCRLB bounds the covariance matrix $\text{Cov}[\hat{\sigma}(\mathbf{x})]$ as

$$\text{Cov}[\hat{\sigma}(\mathbf{x})] \geq \frac{1}{N}\mathbf{I}_{\sigma}^{-1}, \quad (19)$$

where $\mathbf{I}_{\sigma}^{-1}$ is the inverse of the QFIM.

It is worthwhile adding that one of the (not necessarily unique) POVMs that achieves the QCRLB is that whose elements are single-rank projectors on the eigenvalues of the SLD operator. In principle, this provides a systematic method to identify an optimal measurement (although it may well be impossible to implement in practice).

Notice that the QFIM acts as a Riemannian metric on the space of quantum states, equivalent to the Bures distance between quantum states over the manifold spanned by the parameters under consideration (except for values at which the state's rank varies [44]). So, by computing the QFIM, we explore a property of the space of quantum states with respect to parameters encoded through the particular quantum field dynamics we consider.

B. QED scattering and filtering

Here, we present the steps required to prepare and describe the state of a quantum field whose external degrees of freedom are to be estimated based on a measurement of its internal degrees of freedom. This state preparation process consists of QED scattering and followed by a momentum measurement.

We consider the scattering of two asymptotically free particles for a two-particle initial state $\rho_{-\infty}$ with given center-of-mass momentum magnitude and maximally mixed in the internal (spin) degrees of freedom:

$$\rho_{-\infty} = \sum_{\lambda} \frac{1}{4V^2} |\mathbf{p}, \lambda\rangle\langle\mathbf{p}, \lambda|, \quad (20)$$

where $E_{\mathbf{p}} = \sqrt{|\mathbf{p}|^2 + m^2}$ and $V = \int_{\mathbb{R}^3} d\mathbf{x}$. Here we use the short-hand $|\mathbf{p}, \lambda\rangle = a_{\mathbf{p}, \lambda_1}^{\dagger} b_{-\mathbf{p}, \lambda_2}^{\dagger} |0\rangle$, where a and b are (bosonic or fermionic) field operators for three-momentum $\mathbf{p}$, while $\lambda \equiv (\lambda_1, \lambda_2) \in \{LL, LR, RL, RR\}$. L labels a left-handed helicity eigenstate for a fermion or a left-handed circular polarization state for a photon, while R labels right-handedness in a similar way.

As already stated, we will consider a scenario where an observer measures the momentum $\mathbf{q}$ of a scattered particle (which, clearly, also fixes the momentum of the other one in the center-of-mass frame) on the final,

scattered state $\rho_{+\infty}$, by applying the POVM element $d\Pi_{\mathbf{q}} = \frac{d^6\mathbf{q}}{4(2\pi)^6} \sum_{\eta} |\mathbf{q}, \eta\rangle \langle \mathbf{q}, \eta| = dK_{\mathbf{q}}^\dagger dK_{\mathbf{q}}$. This idealized (infinitely sharp in momentum) filtering process projects $\rho_{+\infty}$ onto a two-qubit state of the internal degrees of freedom that depends on the initial and final momenta $\mathbf{p}$ and $\mathbf{q}$. Our aim is to consider the quantum estimation of these dynamical parameters based on measurements on the internal degrees of freedom.

If S is the QED S-matrix operator, i.e., the unitary relating the initial state to the final state after scattering, the resulting state after the momentum filtering process is

$$\begin{aligned} \rho_{\text{out}} &= \frac{dK_{\mathbf{q}} S \rho_{-\infty} S^\dagger dK_{\mathbf{q}}^\dagger}{\text{Tr}[dK_{\mathbf{q}} S \rho_{-\infty} S^\dagger dK_{\mathbf{q}}^\dagger]} \\ &= \frac{\sum_{\lambda, \eta, \eta'} |\mathbf{q}, \eta\rangle \langle \mathbf{q}, \eta| S |\mathbf{p}, \lambda\rangle \langle \mathbf{p}, \lambda| S^\dagger |\mathbf{q}, \eta'\rangle \langle \mathbf{q}, \eta'|}{\sum_{\lambda, \nu} \langle \mathbf{q}, \nu| S |\mathbf{p}, \lambda\rangle \langle \mathbf{p}, \lambda| S^\dagger |\mathbf{q}, \nu\rangle} \\ &= \sum_{\eta, \eta'} \rho_{\eta, \eta'} |\mathbf{q}, \eta\rangle \langle \mathbf{q}, \eta'|, \end{aligned} \quad (21)$$

where $\text{Tr}[\cdot]$ is the trace operator on the two-particle helicity or polarization Hilbert space, and ρ_{out} can be represented by a 4×4 matrix in the same space. Equation (21) shows that the entries of the two-qubit density matrix $\rho_{\eta, \eta'}$ can be written in terms of the S-matrix elements $\mathcal{M}_{\mathbf{p}\lambda \rightarrow \mathbf{q}\eta}$ that are evaluated through Feynman diagrams in the standard perturbative expansion, as per

$$\rho_{\eta, \eta'} = \frac{\sum_{\lambda} \mathcal{M}_{\mathbf{p}\lambda \rightarrow \mathbf{q}\eta} \mathcal{M}_{\mathbf{p}\lambda \rightarrow \mathbf{q}\eta'}^*}{\sum_{\lambda, \nu} \mathcal{M}_{\mathbf{p}\lambda \rightarrow \mathbf{q}\nu} \mathcal{M}_{\mathbf{p}\lambda \rightarrow \mathbf{q}\nu}^*}. \quad (22)$$

We also consider a scenario where we start with a pure incoming state instead, which is of the form

$$|\psi_{-\infty}\rangle = \frac{1}{V} |\mathbf{p}, \lambda\rangle, \quad (23)$$

with $|\lambda\rangle \in \{LL, LR, RL, RR\}$ and evolves to $|\psi_{\infty}\rangle = S|\mathbf{p}, \lambda\rangle$ after scattering. Then, the scattered and filtered state is

$$|\psi_{\text{out}}\rangle = \frac{1}{\sqrt{\sum_{\nu} |\mathcal{M}_{\mathbf{p}\lambda \rightarrow \mathbf{q}\nu}|^2}} \sum_{\eta} \mathcal{M}_{\mathbf{p}\lambda \rightarrow \mathbf{q}\eta} |\mathbf{q}, \eta\rangle. \quad (24)$$

In this paper, we will evaluate the matrix elements $\mathcal{M}_{\mathbf{p}\lambda \rightarrow \mathbf{q}\eta}$ at tree level.

C. Optimal measurement of pure states

The set of eigenvectors $\{|e_{\pm}\rangle\}$ of L_{σ} (the SLD with respect to parameter σ) form an optimal measurement basis for which the QCRLB is saturated [45]. For a pure quantum state $\rho = \rho^2 = |\psi\rangle \langle \psi|$, the SLD has the simple form

$$\begin{aligned} L_{\sigma} &= 2\partial_{\sigma}(|\psi\rangle \langle \psi|) \\ &= 2(|\partial_{\sigma}\psi\rangle \langle \psi| + |\psi\rangle \langle \partial_{\sigma}\psi|). \end{aligned} \quad (25)$$

Given the pure outgoing state $|\psi\rangle = |\psi_{\text{out}}\rangle$ [Eq. (24)] our aim is to find the optimal measurement basis for the estimation of the external parameters p and θ . For this purpose we let $\sigma \in \{p, \theta\}$ and decompose the state $|\partial_{\sigma}\psi\rangle$ in the orthonormal $\{|\psi\rangle, |\psi^{\perp}\rangle\}$ basis, which satisfies

$$\langle \psi | \psi^{\perp} \rangle = 0. \quad (26)$$

$|\partial_{\sigma}\psi\rangle$ is written as

$$|\partial_{\sigma}\psi\rangle = \frac{\alpha|\psi\rangle + \beta|\psi^{\perp}\rangle}{\sqrt{|\alpha|^2 + |\beta|^2}}, \quad (27)$$

where $\alpha = \langle \psi | \partial_{\sigma}\psi \rangle$, and $\beta = \langle \psi^{\perp} | \partial_{\sigma}\psi \rangle$. Hence, the decomposition of the SLD in the $\{|\psi\rangle, |\psi^{\perp}\rangle\}$ basis is

$$L = \frac{2(\alpha|\psi\rangle + \beta|\psi^{\perp}\rangle)(\langle \psi| + 2\langle \psi^{\perp}|)(\alpha^*\langle \psi| + \beta^*\langle \psi^{\perp}|)}{\sqrt{|\alpha|^2 + |\beta|^2}}, \quad (28)$$

which can be written in matrix form as

$$L = \frac{2}{\sqrt{|\alpha|^2 + |\beta|^2}} \begin{pmatrix} \alpha + \alpha^* & \beta\beta^* \\ \beta & 0 \end{pmatrix}. \quad (29)$$

The eigenvectors of this matrix are

$$|e_{\pm}\rangle = \frac{\text{Re}(\alpha) \pm \sqrt{\text{Re}(\alpha)^2 + |\beta|^2}}{\beta} |\psi\rangle + |\psi^{\perp}\rangle, \quad (30)$$

where

$$|\psi^{\perp}\rangle = \frac{\sqrt{|\alpha|^2 + |\beta|^2} |\partial_{\sigma}\psi\rangle - \alpha|\psi\rangle}{\beta}. \quad (31)$$

We now consider an optimal projective measurement on ρ described by $\Pi_{\pm} = |e_{\pm}\rangle \langle e_{\pm}|$. The probability distribution associated with this measurement is given by the PDF

$$p(e_{\pm}|\sigma) = |\langle e_{\pm} | \psi_{\text{out}} \rangle|^2, \quad (32)$$

where $|e_{\pm}\rangle$ are the eigenvalues of L_{σ} . The CFI

$$\mathcal{I}_{\sigma, e}^{(C)} = \sum_{\pm} \frac{1}{p(e_{\pm}|\sigma)} \cdot \left(\frac{\partial p(e_{\pm}|\sigma)}{\partial \sigma} \right)^2 \quad (33)$$

associated with this PDF is equivalent to the QFI associated with ρ . This is compared with the case where, instead, a projective measurement $\Pi_{\lambda} = |\lambda\rangle \langle \lambda|$ is applied in the $\lambda \in \{LL, LR, RL, RR\}$ helicity/polarization basis, inducing the PDF

$$p(\lambda|\sigma) = |\langle \lambda | \psi_{\text{out}} \rangle|^2, \quad (34)$$

which has an associated CFI given by

$$I_{\sigma,\lambda}^{(C)} = \sum_{\lambda} \frac{1}{p(\lambda|\sigma)} \cdot \left(\frac{\partial p(\lambda|\sigma)}{\partial \sigma} \right)^2. \quad (35)$$

D. Classical Fisher information of local measurements

Beside evaluating the absolute bounds determined by the quantum Fisher information matrix, we shall also examine the estimation performance of completely local schemes, based on projective measurements in the helicity/polarization product basis $\{|LL\rangle, |LR\rangle, |RL\rangle, |RR\rangle\}$. This will be done by considering the *classical* Fisher information with respect to the relevant parameters (p and θ) of the probability distribution obtained through the Born rule from the final scattered state and the local POVM above. We will limit ourselves to single-parameter Fisher informations in this case, which we shall denote with $I_{p,\lambda}^{(C)}(p, \theta)$ and $I_{\theta,\lambda}^{(C)}(p, \theta)$.

Notice that the eigenvectors of Eq. (30) in the pure state case and, more generally, the eigenvectors of the SLD operator which optimize the quantum estimation, are typically entangled, calling for nonlocal, entangled measurements as optimal schemes. Evaluating the quantities $I_{p,\lambda}^{(C)}(p, \theta)$ and $I_{\theta,\lambda}^{(C)}(p, \theta)$ is then a way to assess the efficacy of local—more realistic and less demanding than global—measurement schemes.

III. METHOD

We considered electron-muon ($e^- \mu^- \rightarrow e^- \mu^-$), as well as electron-photon (Compton: $e^- \gamma \rightarrow e^- \gamma$) scattering, for both maximally mixed and pure initial states. We then modeled the scattering and four-momentum filtering of these states, as shown in Sec. II B, treating them at tree level in the COM frame. The three external degrees of freedom considered initially were p , θ and ϕ . For each scattering process, the 16 scattering amplitudes $\mathcal{M}_{p\lambda \rightarrow q\eta}$ (for the four different prescattering and postscattering helicity configurations $\lambda, \eta \in \{LL, LR, RL, RR\}$) were calculated at the lowest order of perturbation (i.e., at tree level). The calculated $\mathcal{M}_{p\lambda \rightarrow q\eta}$ values were then used to find the scattered and four-momentum filtered states, given by Eq. (21) in the mixed state case, and by Eq. (24) in the pure state case. The QFIM terms with respect to p , θ , and ϕ were then calculated using Eq. (17) in the mixed state case, and by Eq. (18) in the pure state case. For pure states in the single-parameter estimation case, the QFI terms were compared to the CFI terms resulting from a helicity/polarization basis (nonoptimal) measurement. All calculations were done numerically over the following range of parameters: $p \in [\Delta p, 5]$ MeV, $\theta \in [\Delta \theta, \pi]$, and $\phi \in [0, 2\pi]$ for electron-muon scattering

and $p \in [\Delta p, 5]$ MeV, $\theta \in [0, \pi]$, and $\phi \in [0, 2\pi]$ for Compton scattering. Numerical steps of $\Delta p = 0.01$ MeV, $\Delta \theta = \frac{\pi}{500}$, and $\Delta \phi = \frac{\pi}{500}$ were chosen. The 5 MeV momentum upper bound was chosen as a sensible cutoff of the high-momentum scattering regime, where we observed less variation in the values of all computed quantities when compared to the low-momentum regime. The reason for restricting the p and θ domains is the occurrence of an infrared (IR) divergence at $p = 0$ in Compton [46] and electron-muon scattering calculations, as well as the t -channel divergence at $\theta = 0$ in electron-muon scattering.

As one should expect by symmetry, no dependence on ϕ remains in the output state, even for cases of definite initial helicity or polarization. We have tested this numerically as a preliminary check on our evaluations and, henceforth, we can just focus on the estimation of p and θ , characterized by the symmetric 2×2 matrix $\mathbf{I}(p, \theta)$. Under such a rank reduction, Eq. (19) can be written down explicitly to obtain the following expression for the QCRLB:

$$\begin{pmatrix} \text{Var}[\hat{p}] & \text{Cov}[\hat{p}, \hat{\theta}] \\ \text{Cov}[\hat{\theta}, \hat{p}] & \text{Var}[\hat{\theta}] \end{pmatrix} \geq \frac{1}{N(\mathbf{I}_{pp} \mathbf{I}_{\theta\theta} - \mathbf{I}_{p\theta}^2)} \begin{pmatrix} \mathbf{I}_{\theta\theta} & -\mathbf{I}_{p\theta} \\ -\mathbf{I}_{p\theta} & \mathbf{I}_{pp} \end{pmatrix}, \quad (36)$$

Note that an experimental design which maximizes the determinant in the denominator above—and, in general, the determinant of the Fisher matrix—is called D-optimal, as this results in a minimized covariance matrix. Using the diagonal terms of the minimized covariance matrix, we computed the signal-to-noise ratio (SNR) of the optimal estimator $\hat{\sigma}$ given an optimal-basis measurement of the scattered two-particle state we consider. We adapt the SNR definition from Ref. [47] for $N = 1$ measurement as

$$\text{SNR}_{\sigma} = \frac{\hat{\sigma}}{\sqrt{\text{Var}[\hat{\sigma}]}} \quad (37)$$

and we let $\sigma = p, \theta$. The single-parameter lower bound shown in Eq. (12), which can be extended to QET, was used to compute the optimal-estimator SNRs ($\text{SNR}_{p, \text{fixed } \theta}$ and $\text{SNR}_{\theta, \text{fixed } p}$) in single-parameter estimation. Note that, in this work, we have evaluated the SNR by using the true value of the parameter σ , rather than an actual estimator $\hat{\sigma}$ (bear in mind, however, that the latter does converge to the former for unbiased estimators). Let us also recall that $\text{Var}[\hat{\sigma}]$ is just a value for a single measurement; dividing its square root by the square root of a sampling rate f_s , yields the error $\Delta_{\hat{\sigma}}$ as a function of the sampling time t_s . This can be written in terms of SNR_{σ} as

$$\Delta_{\hat{\sigma}} = \frac{\hat{\sigma}}{\text{SNR}_{\sigma} \sqrt{f_s t_s}}, \quad (38)$$

which is just a rewriting of Eq. (36) for optimal estimators and measurements of the parameter σ .

IV. RESULTS

In this section, the numerical values of the following terms are presented as plots of p and θ : the three independent QFIM elements $\mathbf{I}_{pp}(p, \theta)$, $\mathbf{I}_{p\theta}(p, \theta)$, and $\mathbf{I}_{\theta\theta}(p, \theta)$, as well as the covariance and SNR values of the optimal estimators $\hat{p}$ and $\hat{\theta}$ for an estimation in the optimal measurement basis $\{|e_{\pm}\rangle\}$ shown in Eq. (30). In the case of single-parameter estimation based on pure-state measurements, we compare these results to the ones resulting from estimation in the local helicity/polarization basis.

We report our findings in full detail in the following sections, and then provide the reader with a summary in Sec. IV C. We work in natural units ($\hbar = c = 1$) throughout.

A. Electron-muon scattering

Here we present the QFIM elements as well as the covariance and SNR values of optimal estimators for the electron-muon scattering process ($e^- \mu^- \rightarrow e^- \mu^-$) with a Feynman diagram shown in Fig. 1. The scattering amplitude for this process is given by [48]

$$i\mathcal{M} = \bar{u}(s_2, p_2)(-ie\gamma^\mu)u(s_1, p_1) \frac{-ig_{\mu\nu}}{(p_2 - p_1)^2} \times \bar{u}(r_2, q_2)(-ie\gamma^\nu)u(r_1, q_1), \quad (39)$$

where u are Dirac spinors in the helicity basis, γ^μ are the Dirac matrices, and $g_{\mu\nu}$ is the Minkowski metric in flat spacetime for $\mu, \nu = 0, 1, 2, 3$.

Results are shown for a scattered electron-muon pair state, given that the two particles were initially in a maximally mixed state (see Sec. IV A 1) or in a pure state (see Sec. IV A 2).

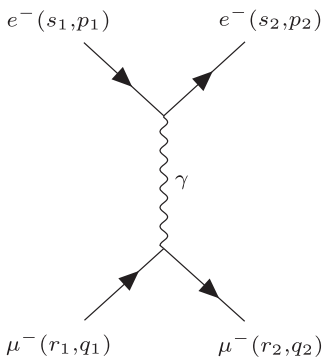


FIG. 1. Feynman diagram of an electron-muon scattering process ($e^- \mu^- \rightarrow e^- \mu^-$).

1. Mixed state

A scattered maximally mixed electron-muon state has three unique QFIM elements $\mathbf{I}_{pp}(p, \theta)$, $\mathbf{I}_{p\theta}(p, \theta)$, and $\mathbf{I}_{\theta\theta}(p, \theta)$ with similar features [see Fig. 2(a)–(c)]. All three QFIM elements have consistently increasing values as p and θ increase, i.e., the electron and muon helicities carry more information about p and θ in a high-momentum, wide-angle scattering process. This means that, by selecting a backscattered electron-muon pair, Alice can send more information about p and θ to Bob. Heuristically, this complies with the loose intuition that the particles interact more strongly in this regime, thus making the final state very sensitive to initial parameters. In the low- p scattering regime there is no θ dependence as the QFIM terms are negligible for all polar angles. For large p the QFIM terms exhibit a much stronger θ dependence than a p dependence, especially $\mathbf{I}_{pp}(p, \theta)$, suggesting that the precisions of any unbiased $\hat{p}$ and $\hat{\theta}$ estimators must also have a weak p -dependence (especially $\hat{p}$). The largest QFI values are observed in the backscattering regime. For any given physical scenario (where the true p and θ are fixed) $\mathbf{I}_{\theta\theta}(p, \theta)$ is larger than $\mathbf{I}_{pp}(p, \theta)$, meaning that in general the electron and muon helicities carry more information about θ than p . For example, for large p and θ , $\mathbf{I}_{\theta\theta}(p, \theta)$ is larger than $\mathbf{I}_{pp}(p, \theta)$ by two orders of magnitude.

We now consider the covariance and SNR values of optimal p and θ estimators for a single ($N = 1$) optimal quantum measurement of the electron-muon state. The results for single-parameter and multiparameter estimation are compared. The values of SNR_p and SNR_θ [Figs. 2(d) and (f), respectively] both reach their maximum in the large θ range, as expected given the QFIM results. A significant difference in the features of the two quantities is that SNR_p reaches a maximum around $p \approx 2.4$ MeV, whereas SNR_θ increases with p in the considered domain. After a single optimal measurement of the mixed-state electron-muon helicities, the maximum $\text{SNR}_p \approx 0.6$ associated with the optimal estimator $\hat{p}$ suggests the standard deviation of $\hat{p}$ is consistently larger than the true value p . Hence, a reliable estimation of p in such a scenario is not possible for $N = 1$ and more measurements would be required. On the other hand, the maximum value of $\text{SNR}_\theta \approx 10$ for large θ suggests that a high-precision estimation of θ is possible even just for one optimal-basis measurement performed by Bob, provided that Alice has chosen a backscattered maximally mixed electron-muon pair. The precision of θ estimation in such a scenario is one order of magnitude larger than the precision of p estimation.

As implied by Eq. (12), which also holds in QET, the QCRLBs of $\text{Var}[\hat{p}]_{\text{fixed } \theta}$ and $\text{Var}[\hat{\theta}]_{\text{fixed } p}$ (and hence the behavior of $\text{SNR}_{p, \text{fixed } \theta}$ and $\text{SNR}_{\theta, \text{fixed } p}$) can be inferred from the behavior of $\mathbf{I}_{pp}(p, \theta)$ and $\mathbf{I}_{\theta\theta}(p, \theta)$, respectively. Hence, these results are not shown explicitly here. No

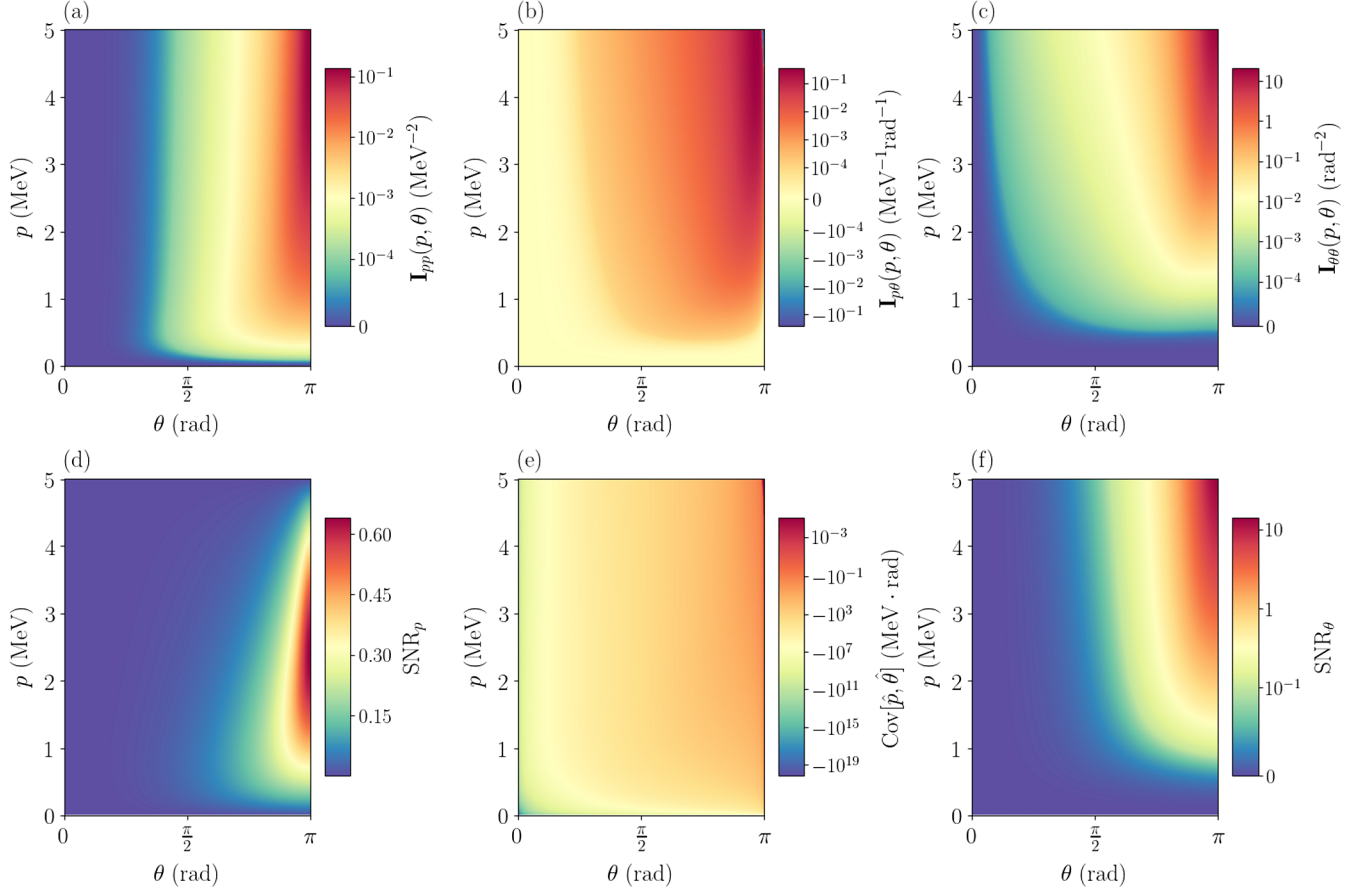


FIG. 2. The Quantum Fisher information matrix elements (a) $\mathbf{I}_{pp}(p, \theta)$, (b) $\mathbf{I}_{p\theta}(p, \theta)$, and (c) $\mathbf{I}_{\theta\theta}(p, \theta)$ shown as functions of the particles' three-momentum magnitude p and the polar scattering angle θ in electron-muon scattering. In the case of multiparameter quantum estimation and for $N = 1$ optimal-basis measurement, the signal-to-noise ratios (d) SNR_p of the optimal estimator $\hat{p}$ and (f) SNR_θ of the optimal estimator $\hat{\theta}$ are shown, as well as the estimators' covariance (e) $\text{Cov}[\hat{p}, \hat{\theta}]$.

significant differences in the behavior of $\text{SNR}_{p, \text{fixed } \theta}$ and $\text{SNR}_{\theta, \text{fixed } p}$ relative to SNR_p and SNR_θ were observed.

Furthermore, as seen in Fig. 2(e), for $N = 1$ measurement the saturated QCRLB of $\text{Cov}[\hat{p}, \hat{\theta}]$ is small and negative for all p and for small θ values. $\text{Cov}[\hat{p}, \hat{\theta}]$ then gradually increases with θ , until it becomes positive for $\theta \approx \pi$. This means that for an optimal-basis measurement, any pair of optimal unbiased $\hat{p}$ and $\hat{\theta}$ estimators must also exhibit a weak negative correlation for small θ , which becomes a stronger negative correlation for larger θ . The two are positively correlated for $\theta \approx \pi$.

2. Pure states

Under a parity transformation $\hat{P}$, the two-particle helicity eigenstates transform as $|LL\rangle \xleftrightarrow{\hat{P}} |RR\rangle$ and $|LR\rangle \xleftrightarrow{\hat{P}} |RL\rangle$. The QED Lagrangian is invariant under parity transformation, so the outgoing (scattered) pure quantum states transform as $|\psi\rangle_{\text{out}, LL} \xleftrightarrow{\hat{P}} |\psi\rangle_{\text{out}, RR}$ and $|\psi\rangle_{\text{out}, LR} \xleftrightarrow{\hat{P}} |\psi\rangle_{\text{out}, RL}$, up to relative phase terms, where $|\psi\rangle_{\text{out}, \lambda}$ is the outgoing state for an incoming state $|\lambda\rangle$. The results shown in this section

demonstrate that all pure-state QFIM elements are invariant under a parity transformation, as the QFIM for $|\psi\rangle_{\text{out}, LL}$ equals the QFIM for $|\psi\rangle_{\text{out}, RR}$ and the QFIM for $|\psi\rangle_{\text{out}, LR}$ equals the one for $|\psi\rangle_{\text{out}, RL}$. As a shorthand we will refer to

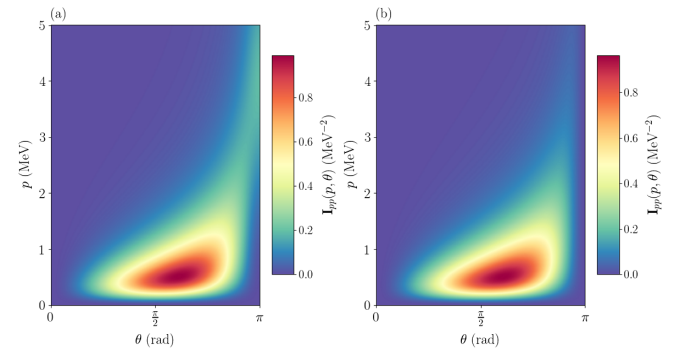


FIG. 3. Quantum Fisher information matrix element $\mathbf{I}_{pp}(p, \theta)$ with respect to the particles' three-momentum magnitude p in electron-muon scattering. Results are shown for same-helicity pure incoming electron-muon states $|LL\rangle$ and $|RR\rangle$ in (a) and opposite-helicity states $|LR\rangle$ and $|RL\rangle$ in (b). These are plotted as functions of p and the polar scattering angle θ .

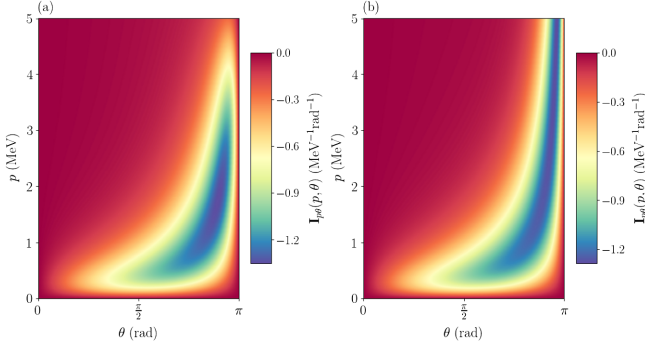


FIG. 4. Quantum Fisher information matrix element $\mathbf{I}_{p\theta}(p, \theta) = \mathbf{I}_{\theta p}(p, \theta)$ with respect to the particles' three-momentum magnitude p and the polar scattering angle θ in electron-muon scattering. Results are shown for same-helicity pure incoming electron-muon states $|LL\rangle$ and $|RR\rangle$ in (a) and opposite-helicity states $|LR\rangle$ and $|RL\rangle$ in (b). These are plotted as functions of p and θ .

the states $|LL\rangle$ and $|RR\rangle$ as “same-helicity” states, whereas $|LR\rangle$ and $|RL\rangle$ are referred to as “opposite-helicity” states.

For pure-state electron-muon scattering, the three QFIM elements do not change significantly for opposite-helicity vs same-helicity states. As shown in Fig. 3, $\mathbf{I}_{pp}(p, \theta)$ has higher values in the low- p region, centered around ($p = 0.52, \theta = 1.91$). Hence for all possible electron-muon pure states, the helicities of the two particles carry significant information about θ only in the low- p scattering regime. The only difference between the $\mathbf{I}_{pp}(p, \theta)$ functions for the two families of scattering electron-muon states is that for same-helicity states, $\mathbf{I}_{pp}(p, \theta)$ has slightly larger values for large p and θ .

For both opposite-helicity and the same-helicity state scattering events $\mathbf{I}_{p\theta}(p, \theta) \geq 0$ has a larger magnitude that is skewed in the $\frac{\pi}{2} < \theta < \pi$ region (see Fig. 4). The only difference between the two pure-state parity scenarios is that for the scattering of same-helicity pure states, a gradual decrease of the $\mathbf{I}_{pp}(p, \theta)$ magnitude is seen again for $p > 4$ MeV, while no such decrease is seen for opposite-helicity pure states in the considered p domain. As seen in Eq. (36), the off-diagonal QFIM terms affect the QCRLB on $\text{Cov}[\hat{p}, \hat{\theta}]$, which has a direct interpretation. As it will be shown, in the case of electron-muon scattering the $\mathbf{I}_{p\theta}(p, \theta)$ values are consistently of a smaller magnitude than $\mathbf{I}_{pp}(p, \theta)$ and $\mathbf{I}_{\theta\theta}(p, \theta)$ and hence the former are dominated by the latter in the values of the $\text{Cov}[\hat{p}, \hat{\theta}]$ lower bound.

As shown in Fig. 5 the $\mathbf{I}_{\theta\theta}(p, \theta)$ QFIM term also has similar features for incoming same-helicity and opposite-helicity electron-muon states. Significant $\mathbf{I}_{\theta\theta}(p, \theta)$ values are only seen in the region where θ approaches π , with larger values for larger p . Hence, after high- p (and hence high-energy) backscattering, the electron and muon helicities carry more information about θ . Even though

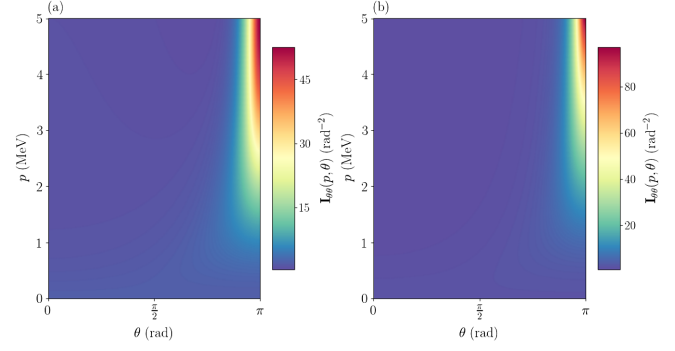


FIG. 5. Quantum Fisher information matrix element $\mathbf{I}_{\theta\theta}(p, \theta)$ with respect to the polar scattering angle θ in electron-muon scattering. Results are shown for same-helicity pure incoming electron-muon states $|LL\rangle$ and $|RR\rangle$ in (a) and opposite-helicity states $|LR\rangle$ and $|RL\rangle$ in (b). These are plotted as functions of the particles' center-of-mass three-momentum p and θ .

$\mathbf{I}_{\theta\theta}(p, \theta)$ reaches higher maxima than the two other unique QFIM elements, a significant $\mathbf{I}_{\theta\theta}(p, \theta)$ value can only be achieved for a limited family of physical scenarios for pure-state electron-muon scattering. Furthermore, the maximum calculated information they can carry is larger if the two helicities were different before scattering. However, opposite-helicity incoming states reach a larger $\mathbf{I}_{\theta\theta}(p, \theta)$ maximum for a smaller domain of p and θ values distributed over larger p and θ compared to the same-helicity state case. Comparing Fig. 5 with Fig. 2(c) shows that $\mathbf{I}_{pp}(p, \theta)$ reaches a higher maximum for a pure, scattered electron-muon state than for a mixed state, albeit over a smaller $p - \theta$ domain.

We also computed the CFI terms $I_{p,\lambda}^{(C)}(p, \theta)$ and $I_{\theta,\lambda}^{(C)}(p, \theta)$ associated with a measurement of a pure, scattered electron-muon state in the electron-muon helicity basis. Invariance under state parity transformation was observed for the CFI terms as well. However, these terms are not shown here as they have the same features as those of $\mathbf{I}_{pp}(p, \theta) = I_{p,e}^{(C)}(p, \theta)$ in Fig. 3 and $\mathbf{I}_{\theta\theta}(p, \theta) = I_{\theta,e}^{(C)}(p, \theta)$ in Fig. 5, respectively. Even though $I_{p,\lambda}^{(C)}(p, \theta) < \mathbf{I}_{pp}(p, \theta)$ as expected, the values of $I_{p,\lambda}^{(C)}(p, \theta)$ and $\mathbf{I}_{pp}(p, \theta)$ are of the same order of magnitude for both same-helicity and opposite-helicity electron-muon states. Therefore, when Bob is estimating p , he does not lose a significant amount of information by measuring Alice's electron-muon state in the helicity basis rather than the optimal measurement basis. This is in contrast with the θ estimation scenario, where the values of $I_{\theta,\lambda}^{(C)}(p, \theta)$ are significantly smaller than those of $\mathbf{I}_{\theta\theta}(p, \theta)$. More precisely, $\mathbf{I}_{\theta\theta}(p, \theta) \in [0.51, 52.59] \text{ rad}^{-2}$ and $I_{\theta,\lambda}^{(C)}(p, \theta) \in [0.50, 31.92] \text{ rad}^{-2}$ for same-helicity states, while $\mathbf{I}_{\theta\theta}(p, \theta) \in [0.91, 97.32] \text{ rad}^{-2}$ and $I_{\theta,\lambda}^{(C)}(p, \theta) \in [0.50, 56.90] \text{ rad}^{-2}$ for opposite-helicity states. Hence, a measurement of a prepared electron-muon state in the

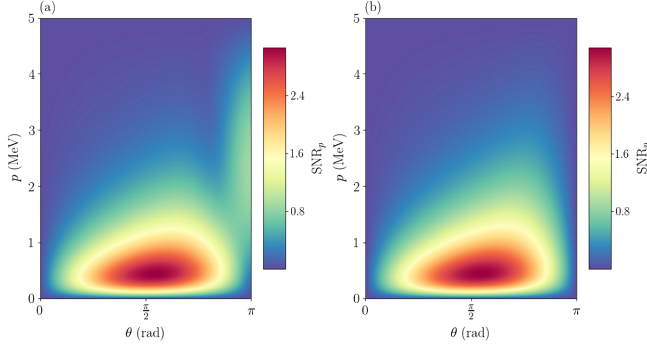


FIG. 6. The signal-to-noise ratio SNR_p of an optimal estimator $\hat{p}$ of the particles' three-momentum magnitude p in electron-muon scattering. $N = 1$ optimal-basis measurement of the electron-muon state is considered in the context of multiparameter estimation. Results are shown for same-helicity pure incoming electron-muon states $|LL\rangle$ and $|RR\rangle$ in (a) and opposite-helicity states $|LR\rangle$ and $|RL\rangle$ in (b). Values are plotted as functions of p and the polar scattering angle θ .

helicity basis of product states, rather than the optimal measurement basis, places a larger limit on the single-parameter estimation of θ than it does on the estimation of p . Additionally, we have evaluated the concurrence (a measure of entanglement easily evaluated for two-qubit states [49]) of the optimal measurement-basis states $|e_+\rangle$ and $|e_-\rangle$ of Eq. (30) and found that (in the low- p scattering regime) each of the two states is less entangled in the helicity degrees of freedom in the case of p estimation than in the case of θ estimation. As $I_{p,\lambda}^{(C)}(p, \theta)$ and $\mathbf{I}_{pp}(p, \theta)$ have significant values for small p , this finding complies with the fact that the CFI associated with standard, local helicity measurements is much closer to the actual QFI for p than for θ .

We now compare the SNRs of optimal $\hat{p}$ and $\hat{\theta}$ estimators in three different scenarios; single vs multiparameter estimation for $N = 1$ optimal measurement, and single-parameter estimation for one helicity/polarization-basis measurement. As shown in Fig. 6, SNR_p has similar values for all electron-muon pure states, with a maximum of $\text{SNR}_p \approx 2.4$ for $\theta \approx \frac{\pi}{2}$ and $p \approx 0.5$ MeV. A distinguishing feature of same-helicity pure electron-muon states is a slightly larger SNR_p for $\theta \approx \pi$ for a shown range of p values. The maximum reached by SNR_p for an optimal $\hat{p}$ and optimal-basis measurement of a pure state is four times larger than the maximum reached by SNR_p in the mixed-state case, suggesting that in the former case a precise estimation of p is possible in the low- p region. However, in both cases $\text{SNR}_p < 1$ for a large domain of p and θ values, making corresponding estimations (based on a single measurement) unreliable. In the single-parameter estimation scenario when θ is fixed to a known value, $\text{SNR}_{p,\text{fixed } \theta}$ has similar features to SNR_p , but reaches a higher maximum of $\text{SNR}_{p,\text{fixed } \theta} \approx 4$. Hence, if Bob knows the true

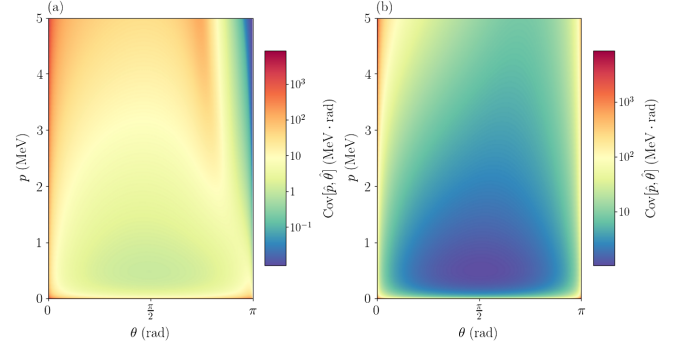


FIG. 7. The lower bound of the covariance $\text{Cov}[\hat{p}, \hat{\theta}]$ of optimal estimators $\hat{p}$ and $\hat{\theta}$ of, respectively, the particles' three-momentum magnitude p and the polar scattering angle θ in electron-muon scattering for $N = 1$ optimal-basis measurement. Results are shown for same-helicity pure incoming electron-muon states $|LL\rangle$ and $|RR\rangle$ in (a) and opposite-helicity states $|LR\rangle$ and $|RL\rangle$ in (b). The variances are plotted as functions of p and the polar scattering angle θ .

value of θ , the precision limit on his estimation of p is improved. In the case of p estimation, no significant differences are observed between the values of $\text{SNR}_{p,\text{fixed } \theta}$ resulting from an optimal-basis vs a helicity basis measurement of the electron-muon state helicities.

$\text{Cov}[\hat{p}, \hat{\theta}]$, with respect to a pure state of a scattering electron-muon pair after $N = 1$ optimal basis measurement, has a QCRLB that is positive for all p and θ , implying a positive correlation between the estimators $\hat{p}$ and $\hat{\theta}$ when p and θ are both unknown (see Fig. 7). This is in contrast with the distribution of $\text{Cov}[\hat{p}, \hat{\theta}]$ for a maximally mixed electron-muon state, which has a negative lower bound for most p and θ values. Therefore, the correlation between $\hat{p}$ and $\hat{\theta}$ has been shown to depend on the purity of the electron-muon state. The maximum value of the QCRLB of $\text{Cov}[\hat{p}, \hat{\theta}]$ in the pure state case is $\approx 10^3$ MeV · rad for both same-helicity and opposite-helicity states. Hence, the scale of $\text{Cov}[\hat{p}, \hat{\theta}]$ for pure-state electron-muon scattering is much larger in the pure-state case than in the mixed-state one, corresponding to a much stronger correlation between the estimators. The shape of the pure-state $p - \theta$ distribution of the optimal-measurement $\text{Cov}[\hat{p}, \hat{\theta}]$ QCRLB depends strongly on whether the helicities of the electron and muon are the same or not, even though the scale of the $\text{Cov}[\hat{p}, \hat{\theta}]$ QCRLB values does not.

As shown in Fig. 8, in multiparameter estimation SNR_θ due to one optimal-basis measurement of a pure electron-muon state shares the feature of mixed-state SNR_θ of reaching its maximum values in the large θ scattering regime. For same-helicity electron-muon states, SNR_θ reaches a maximum of ≈ 20 , corresponding to an estimation precision two times larger than in the equivalent mixed-state case, but applicable in a smaller domain of (p, θ) values. Note that this is the largest SNR reached in all different

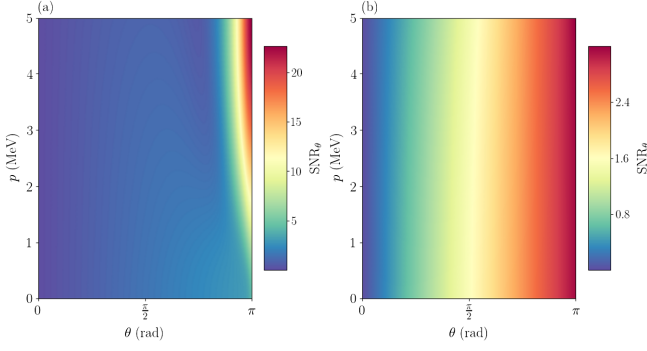


FIG. 8. The signal-to-noise ratio SNR_θ of an optimal estimator $\hat{\theta}$ of the polar scattering angle in electron-muon scattering. $N = 1$ optimal-basis measurement of the electron-muon state is considered in the context of multiparameter estimation. Results are shown for same-helicity pure incoming electron-muon states $|LL\rangle$ and $|RR\rangle$ in (a) and opposite-helicity states $|LR\rangle$ and $|RL\rangle$ in (b). Values are plotted as functions of the particles' three-momentum magnitude p and θ .

electron-muon scattering scenarios considered. On the other hand, in the opposite-helicity case, the maximum achievable SNR_θ is ≈ 2.4 , which is a decrease relative to the mixed-state case. However, in the former scenario, the domain of large SNR_θ extends over a larger domain of θ values. Now we consider the SNR when θ takes a known, fixed value, and the electron-muon state is still measured in the optimal basis. In the case of same-helicity states, no significant changes in the features and the value range of $\text{SNR}_{\theta, \text{fixed } p}$ are observed relative to SNR_θ . However, for different electron-muon helicities, $\text{SNR}_{\theta, \text{fixed } p}$ reaches a maximum of ≈ 24 (one order of magnitude higher than when θ is also estimated). Furthermore, the (p, θ) domain of large $\text{SNR}_{\theta, \text{fixed } p}$ values is restricted to the high- θ scattering regime, in a similar way to the same-helicity case. For single-parameter estimation based on a helicity/polarization-basis measurement of the electron-muon state, the features of $\text{SNR}_{\theta, \text{fixed } p}$ remain identical to those in the optimal-basis measurement case, but the $\text{SNR}_{\theta, \text{fixed } p}$ values reach lower maxima. For large θ we computed $\text{SNR}_{\theta, \text{fixed } p} \approx 16$ for same-helicity states and $\text{SNR}_{\theta, \text{fixed } p} \approx 20$ for opposite-helicity states. Hence, in contrast to the p estimation case, for θ estimation we observe a significant decrease in the precision of optimal estimation when a nonoptimal measurement is performed.

B. Compton scattering

Here we present the QFIM elements and covariance matrix elements' lower bounds in Eq. (36) for Compton scattering ($e^- \gamma \rightarrow e^- \gamma$) with a Feynman diagram shown in Fig. 9. The scattering amplitude for this process is given by [48]

$$\mathcal{M} = \mathcal{M}_s + \mathcal{M}_u, \quad (40)$$

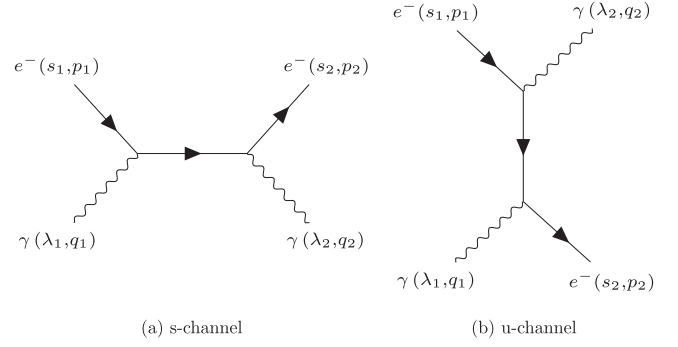


FIG. 9. Feynman diagrams of the s and u channels for Compton scattering ($e^- \gamma \rightarrow e^- \gamma$).

where the s -channel scattering amplitude is given by

$$i\mathcal{M}_s = \bar{u}(s_2, p_2)(-ie\gamma^\mu)\epsilon_\mu^*(\lambda_2, q_2) \frac{\not{p}_1 + \not{q}_1 + m_e}{(p_1 + q_1)^2 - m_e^2} \times (-ie\gamma^\nu)\epsilon_\nu(\lambda_1, q_1)u(s_1, p_1), \quad (41)$$

and the u -channel scattering amplitude is given by

$$i\mathcal{M}_u = \bar{u}(s_2, p_2)(-ie\gamma^\nu)\epsilon_\nu(\lambda_1, q_1) \frac{\not{p}_1 - \not{q}_2 + m_e}{(p_1 - q_2)^2 - m_e^2} \times (-ie\gamma^\mu)\epsilon_\mu^*(\lambda_2, q_2)u(s_1, p_1), \quad (42)$$

where m_e is the mass of an electron. The QFIM element functions are shown for a scattered electron-photon pair state, given that the two particles were initially in a maximally mixed state (see Sec. IV B 1) or in a pure state (see Sec. IV B 2).

1. Mixed states

In this section, we present Compton scattering results for an electron and a photon that are initially in a maximally mixed internal quantum state. The three unique QFIM elements for this scenario are shown in Fig. 10(a)–(c). The QFIM elements have functions of p and θ with similar features. All three elements have small values in the region of small θ . The size of this region increases for larger p . Therefore, for a maximally mixed electron-photon pair, the amount of information the internal degrees of freedom hold about the external degrees of freedom of the scattering is skewed with larger values for larger true θ . The values of $\mathbf{I}_{pp}(p, \theta)$ and $\mathbf{I}_{\theta\theta}(p, \theta)$ both decrease locally at $\frac{\pi}{2}$ for small p . The line of small $\mathbf{I}_{pp}(p, \theta)$ values also skews towards increasing θ as p increases. $\mathbf{I}_{p\theta}(p, \theta)$ shares a similar feature, with the difference that its values go from negative to positive around $\theta = \frac{\pi}{2}$ for small p . The maximum information of p and θ carried by the scattered electron helicity and scattered photon polarization is of the same order of magnitude as 10^2 MeV^{-2} for p and 10^2 rad^{-2} for θ . This is larger than the information maxima that were

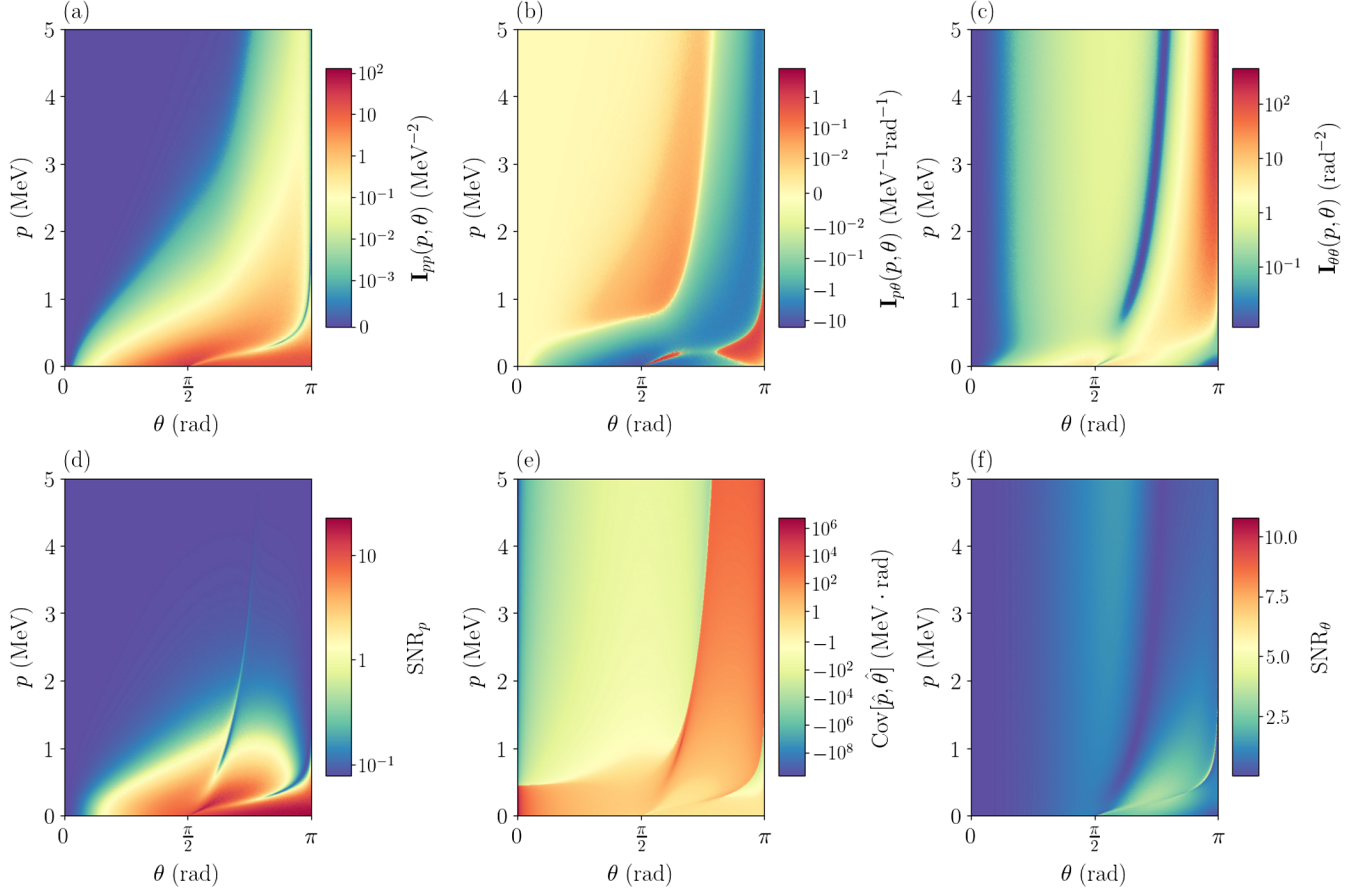


FIG. 10. The quantum Fisher information matrix elements (a) $\mathbf{I}_{pp}(p, \theta)$, (b) $\mathbf{I}_{p\theta}(p, \theta)$, and (c) $\mathbf{I}_{\theta\theta}(p, \theta)$ shown as functions of the particles' three-momentum magnitude p and the polar scattering angle θ in Compton scattering. In the case of multiparameter quantum estimation and for $N = 1$ optimal-basis measurement, the signal-to-noise ratios (d) SNR_p of the optimal estimator $\hat{p}$ and (f) SNR_θ of the optimal estimator $\hat{\theta}$ are shown, as well as the estimators' covariance (e) $\text{Cov}[\hat{p}, \hat{\theta}]$.

observed in mixed-state electron-muon scattering. However, for Compton scattering $\mathbf{I}_{pp}(p, \theta)$ has its largest values in the low- p scattering regime, while $\mathbf{I}_{\theta\theta}(p, \theta)$ is larger in the high- θ , near-backscattering regime. The latter is a feature shared with electron-muon mixed-state scattering.

The covariance and SNR values of the optimal $\hat{p}$ and $\hat{\theta}$ (for one optimal-basis measurement) shown in Fig. 10(d)–(f) exhibit features that are similar to those of the relevant QFIM elements, with special behavior at $\theta = \frac{\pi}{2}$ and in the $\theta \in [\frac{\pi}{2}, \pi]$ domain. SNR_p and SNR_θ reach maxima that are of the same order of magnitude as 10 (more precisely, 10.78 for SNR_p and 22.85 for SNR_θ) and both occur as $\theta \rightarrow \pi$. The difference between the two estimators is that the SNR_p approaches its maximum value as $p \rightarrow 0$, while SNR_θ has its maximum value at $p = 1.52$ for. A SNR value of the same order of magnitude as 10 indicates that the optimal estimators in this scenario exhibit a high degree of precision. However, for most true p and θ values, especially for high- p scattering, $\text{SNR} < 1$ and a reliable estimation based on one optimal measurement is

not possible. Despite the need for multiple measurements, the mixed-state estimation precision for Compton scattering is overall higher than in the electron-muon scattering case, especially in the case of p estimation.

The values of these estimator SNRs are significantly larger in the single-parameter estimation case. The general features of $\text{SNR}_{p, \text{fixed } \theta}$ and $\text{SNR}_{\theta, \text{fixed } p}$ are similar to those of SNR_p and SNR_θ , but the former have larger values for larger p and θ domains than the latter. The maxima reached by $\text{SNR}_{p, \text{fixed } \theta}$ and $\text{SNR}_{\theta, \text{fixed } p}$ are 56.76 and 66.54 respectively, compared to 10.78 for SNR_p and 22.85 for SNR_θ . As expected, fixing one of the parameters improves the precision of the estimation of the other parameter.

As shown in Fig. 10(e), the maximum-magnitude values of the $\text{Cov}[\hat{p}, \hat{\theta}]$ QCRLB are $\text{Cov}[\hat{p}, \hat{\theta}] \sim 10^6 \text{ MeV} \cdot \text{rad}$ in the positive covariance region (for small p and θ and for large p and θ) and $\text{Cov}[\hat{p}, \hat{\theta}] \sim -10^8 \text{ MeV} \cdot \text{rad}$ in the negative covariance region (for large p and small θ). In these true $p - \theta$ regions there is a strong correlation between the values estimated by $\hat{p}$ and $\hat{\theta}$ after an optimal basis measurement. However, for high p in the $\theta \approx \frac{\pi}{2}$ region

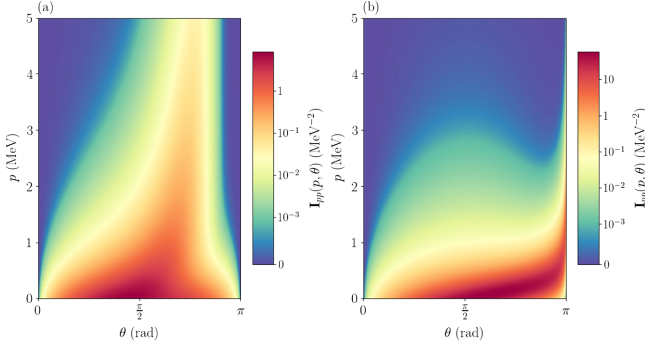


FIG. 11. Quantum Fisher information matrix element $\mathbf{I}_{pp}(p, \theta)$ with respect to the particles' three-momentum magnitude p in Compton scattering. Results are shown for same-helicity pure incoming electron-photon states $|LL\rangle$ and $|RR\rangle$ in (a) and opposite-helicity states $|LR\rangle$ and $|RL\rangle$ in (b) in the helicity-polarisation basis. These are plotted as functions of p and the polar scattering angle θ .

$\text{Cov}[\hat{p}, \hat{\theta}]$ is very small and the correlation between $\hat{p}$ and $\hat{\theta}$ is weak.

2. Pure states

As one should expect, the invariance of QFI and CFI terms under parity transformation $\hat{P}$ is observed for pure-state Compton scattering, as it is for electron-muon scattering. Again, the CFI terms $I_{p,\lambda}^{(C)}(p, \theta)$ and $I_{\theta,\lambda}^{(C)}(p, \theta)$ (associated with a measurement of the pure electron-photon state in the helicity/polarization basis) are not shown as they have the same shape as those of the QFIM terms $\mathbf{I}_{pp}(p, \theta)$ and $\mathbf{I}_{\theta\theta}(p, \theta)$, respectively. As expected, the CFI terms have consistently lower values. Notably, unlike in the electron-muon scattering case, the Fisher information terms for pure-state Compton scattering behave in significantly different ways for same-helicity and opposite-helicity states.

As shown in Fig. 11, the $\mathbf{I}_{pp}(p, \theta)$ terms are both asymmetric about $\theta = \frac{\pi}{2}$ with higher values in the $\theta > \frac{\pi}{2}$ region. $\mathbf{I}_{pp}(p, \theta)$ has a stronger $\theta > \frac{\pi}{2}$ skew for opposite-helicity pure states than for same-helicity ones. The largest calculated $\mathbf{I}_{pp}(p, \theta)$ value is higher for the opposite-helicity scattered states than for same-helicity ones. In both parity cases however, $\mathbf{I}_{pp}(p, \theta)$ values are higher for small p , and decrease for $\theta \approx 0$ and $\theta \approx \pi$. Therefore, the helicity and polarization of the outgoing electron and photon can carry more information about a small p value if the incoming helicity and polarization were different than if they were the same. By comparing Fig. 11 with Fig. 10(a) it is also noted that θ , $\mathbf{I}_{pp}(p, \theta)$ reaches a higher maximum value for a mixed scattered electron-muon state than for a pure state. This is the only QFIM term calculated in this paper for which this property is true. Furthermore, similarly to the electron-muon pure state scattering case, the values of

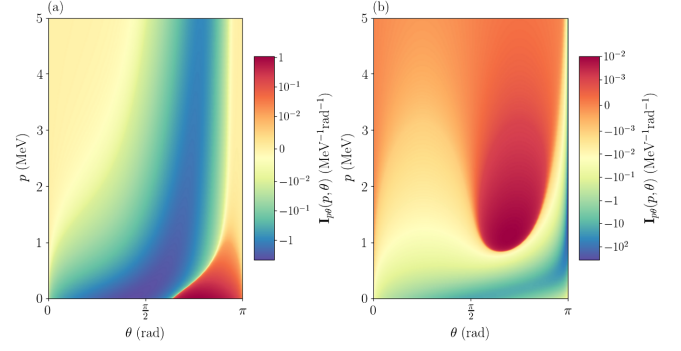


FIG. 12. Quantum Fisher information matrix element $\mathbf{I}_{p\theta}(p, \theta) = \mathbf{I}_{\theta p}(p, \theta)$ with respect to the particles' three-momentum magnitude p and the polar scattering angle θ in Compton scattering. Results are shown for same-helicity pure incoming electron-photon states $|LL\rangle$ and $|RR\rangle$ in (a) and opposite-helicity states $|LR\rangle$ and $|RL\rangle$ in (b) in the helicity-polarization basis. These are plotted as functions of p and θ .

$I_{p,\lambda}^{(C)}(p, \theta)$ are of the same order of magnitude as those of $\mathbf{I}_{pp}(p, \theta)$.

The two pure-state $\mathbf{I}_{p\theta}(p, \theta)$ functions are shown in Fig. 12. As already remarked, the physical interpretation of $\mathbf{I}_{p\theta}(p, \theta)$ is understood by considering its effect on $\text{Cov}[\hat{p}, \hat{\theta}]$.

$\mathbf{I}_{\theta\theta}(p, \theta)$ has significantly different values for same-helicity vs opposite-helicity pure electron-photon states (see Fig. 13). For same-helicity states, large $\mathbf{I}_{\theta\theta}(p, \theta)$ values only occur for a small domain of p and θ values, where $\mathbf{I}_{\theta\theta}(p, \theta)$ increases as p and θ do, reaching a maximum of $\mathbf{I}_{\theta\theta}(p, \theta) \sim 20 \text{ rad}^{-2}$ in the considered domain. In contrast, for opposite-helicity states $\mathbf{I}_{\theta\theta}(p, \theta) \sim 10^4 \text{ rad}^{-2}$ for $\theta \rightarrow \pi$. This is the largest order of magnitude value of a QFIM term calculated in this paper. The increase of $\mathbf{I}_{\theta\theta}(p, \theta)$ for Compton backscattering events is a feature shared between opposite-helicity pure electron-photon states and mixed states, but does not hold for same-helicity states. Furthermore, for same-helicity states the values of $\mathbf{I}_{\theta\theta}(p, \theta)$ and $I_{\theta,\lambda}^{(C)}(p, \theta)$ are of the same order of magnitude. However, for opposite-helicity states $I_{\theta,\lambda}^{(C)}(p, \theta)$ reaches a maximum value that is two orders of magnitude smaller than the one reached by $\mathbf{I}_{\theta\theta}(p, \theta)$. More precisely, $I_{\theta,\lambda}^{(C)}(p, \theta) \in [0.00, 569.78] \text{ rad}^{-2}$ and $\mathbf{I}_{\theta\theta}(p, \theta) \in [0.00, 12021.89] \text{ rad}^{-2}$ for opposite-helicity states. This result is different from the electron-muon scattering result, where lower Fisher information was observed when measuring both same-helicity and opposite-helicity states in the helicity/polarization basis.

We now analyze the behavior of optimal $\hat{p}$ and $\hat{\theta}$ estimators in single and multiparameter estimation for one optimal measurement, as well as in single-parameter estimation for one helicity/polarization-basis measurement. As shown in Fig. 14, given an optimal-basis measurement of

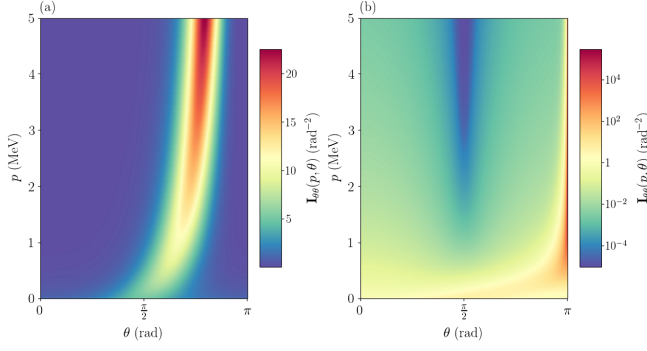


FIG. 13. Quantum Fisher information matrix element $\mathbf{I}_{\theta\theta}(p, \theta)$ with respect to the polar scattering angle θ in Compton scattering. Results are shown for same-helicity pure incoming electron-photon states $|LL\rangle$ and $|RR\rangle$ in (a) and opposite-helicity states $|LR\rangle$ and $|RL\rangle$ in (b) in the helicity-polarization basis. These are plotted as functions of the particles' center-of-mass three-momentum p and θ .

a pure electron-photon state, a high-precision estimate of p is possible only for low- p Compton scattering. Large SNR_p values skew towards the $\theta \in [\frac{\pi}{2}, \pi]$ domain, more so for opposite-helicity electron-photon states. The maximum reached by SNR_p is 18.29 for opposite-helicity pure states, which is larger than the maximum of 11.02 for same-helicity states. In the single-parameter estimation regime when θ is known, $\text{SNR}_{p, \text{fixed } \theta}$ reaches higher values for a larger domain of θ values, while other features, such as the low precision for high p scattering, remain the same as in the multiparameter estimation case. The increase in $\text{SNR}_{p, \text{fixed } \theta}$ relative to SNR_p is more significant for opposite-helicity electron-photon states, where $\text{SNR}_{p, \text{fixed } \theta}$ reaches a maximum of 37.41. In the same-helicity case the maximum $\text{SNR}_{p, \text{fixed } \theta}$ value is 14.34. When the pure electron-photon state is measured in the helicity/polarization

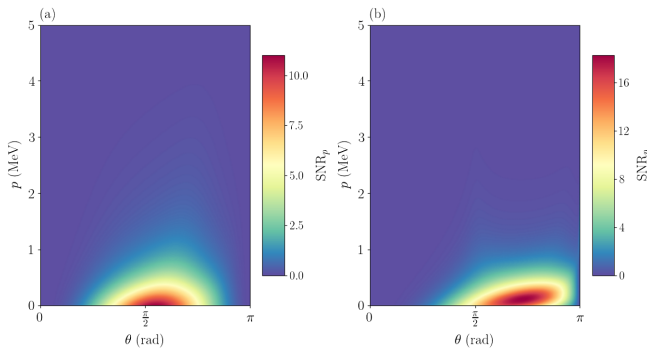


FIG. 14. The signal-to-noise ratio SNR_p of an optimal estimator $\hat{p}$ of the particles' three-momentum magnitude p in Compton scattering. $N = 1$ optimal-basis measurement of the electron-muon state is considered in the context of multiparameter estimation. Results are shown for same-helicity pure incoming electron-photon states $|LL\rangle$ and $|RR\rangle$ in (a) and opposite-helicity states $|LR\rangle$ and $|RL\rangle$ in (b). Values are plotted as functions of p and the polar scattering angle θ .

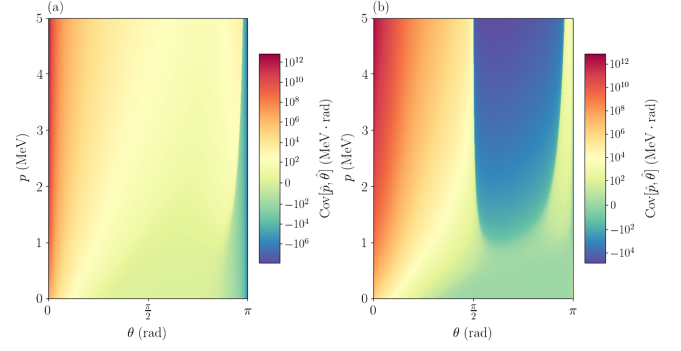


FIG. 15. The lower bound of the covariance $\text{Cov}[\hat{p}, \hat{\theta}]$ of the optimal estimators $\hat{p}$ and $\hat{\theta}$ of, respectively, the particles' three-momentum magnitude p and polar scattering angle θ in Compton scattering for $N = 1$ optimal-basis measurement. Results are shown for same-helicity pure incoming electron-photon states $|LL\rangle$ and $|RR\rangle$ in (a) and opposite-helicity states $|LR\rangle$ and $|RL\rangle$ in (b). The variances are plotted as functions of p and the polar scattering angle θ .

basis instead, the resulting $\text{SNR}_{p, \text{fixed } \theta}$ shares the features of $\text{SNR}_{p, \text{fixed } \theta}$ associated with an optimal-basis measurement. The values of the two SNRs are of the same order of magnitude, as expected due to the similarity between $I_{p\lambda}^{(C)}(p, \theta)$ and $\mathbf{I}_{pp}(p, \theta)$. Hence, performing a single (non-optimal) helicity/polarization-basis measurement leads to no significant decrease in the achievable p estimation precision relative to the optimal measurement case. This is the same behavior as the one observed in electron-muon scattering.

As shown in Fig. 15, the largest positive $\text{Cov}[\hat{p}, \hat{\theta}]$ QCRLB values in the considered $p - \theta$ domain are observed for small θ and increase with p . This corresponds to a strong correlation between the $\hat{p}$ and $\hat{\theta}$ estimators in the small θ scattering regime. The $\text{Cov}[\hat{p}, \hat{\theta}]$ QCRLB functions also have values close to 0 in the small p domain for θ near $\frac{\pi}{2}$. A large negative covariance is observed for $\theta \rightarrow \pi$ in the same-helicity state case, and for the domain $\theta \in [\frac{\pi}{2}, \pi]$ in the opposite-helicity case. The strength of this negative estimator correlation increases with p .

In Fig. 16 we show the SNR_θ of an optimal $\hat{\theta}$ estimator for a single optimal-basis measurement of a pure, Compton-scattered electron-photon state. A significant difference is observed between the SNR_θ values for same-helicity and opposite-helicity electron-photon states. For same-helicity, significant θ estimation precision is achievable for the low- p , $\theta \in [\frac{\pi}{2}, \pi]$ domain of high SNR_θ values. SNR_θ reaches a maximum of ~ 4 . For opposite-helicity states however, $\text{SNR}_\theta \sim 10^3$ is observed for high- p Compton backscattering, i.e., in the limit $\theta \rightarrow \pi$. This is the highest optimal estimation precision (for just one measurement) observed in this paper. The relevant physical scenario is one where Alice prepares an electron

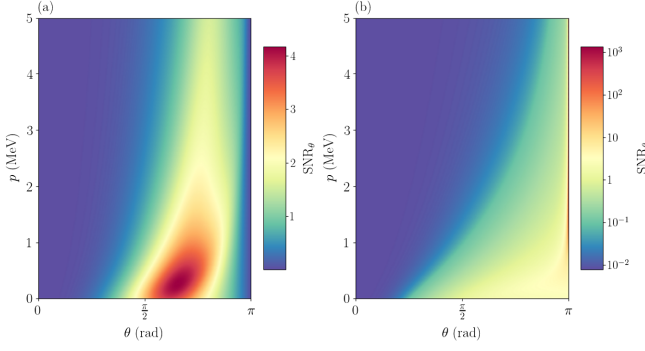


FIG. 16. The signal-to-noise ratio SNR_θ of an optimal estimator $\hat{\theta}$ of the polar scattering angle in Compton scattering. $N = 1$ optimal-basis measurement of the electron-photon state is considered in the context of multiparameter estimation. Results are shown for same-helicity pure incoming electron-photon states $|LL\rangle$ and $|RR\rangle$ in (a) and opposite-helicity states $|LR\rangle$ and $|RL\rangle$ in (b). Values are plotted as functions of the particles' three-momentum magnitude p and θ .

and a photon with opposite helicity and circular polarization values, respectively, then lets them scatter and selects a backscattered two-particle state to send to Bob. Measuring the received state in the optimal basis, Bob can map his measurement result to an estimate of the scattering angle θ . Just for one such measurement, he would be able to estimate θ to a very high precision.

We now consider single-parameter θ estimation in the scenario where p is fixed but an optimal-basis measurement of the scattered electron-photon state is still performed. $\text{SNR}_{\theta, \text{fixed } p}$ share common features with SNR_θ for both same-helicity and opposite-helicity states. It is only for same-helicity states that large $\text{SNR}_{\theta, \text{fixed } p}$ occur for large p (rather than for small p as is the case for SNR_θ). The maximum value of $\text{SNR}_{\theta, \text{fixed } p}$ in the same-helicity case is ≈ 10 . When a measurement is taken in the helicity/polarization basis instead, a decrease in the values of $\text{SNR}_{\theta, \text{fixed } p}$ is observed, even though the features of the functions remain unchanged relative to the optimal-basis measurement, single-parameter estimation case. For a helicity/polarization basis measurement, the maximum of $\text{SNR}_{\theta, \text{fixed } p}$ (reached as $\theta \rightarrow \pi$) is $\sim 10^2$. As for electron-muon scattering, we observe a significant dependence of the precision of θ estimation on the measurement basis chosen.

C. Summary of results

Let us summarize here the salient features of our study:

- (i) For electron-muon scattering and mixed initial states, optimal sensitivity to and estimation of the external parameters is obtained for backward scattering at high energies: if Alice intended to send Bob information about external parameters through the internal degrees of freedom, she would do well to select backscattered particles;
- (ii) The behavior above is common to the estimation of the scattering angle with initial pure product states, whilst the estimation of p with pure initial states display a more sophisticated structure, with optimal angle between $\pi/2$ and π and better resolution at lower energies (around the electron mass);
- (iii) Final measurements in the local helicity basis are close to optimal to estimate the center-of-mass momentum but they are largely suboptimal in determining the scattering angle θ , which would profit from measurements in an entangled basis;
- (iv) Compton scattering estimation has a more structured response to initial energies, achieving optimal estimation below 1 MeV and for a wider range of angles, although wider angles are still optimal;
- (v) Also, in Compton scattering the performance of same-helicity and opposite-helicity initial pure states is very distinct, with the latter preferring lower energies and wider angles, and the former maintaining higher sensitivity for a wider range of energies;
- (vi) For Compton scattering, the highest achievable precision we computed is the one for θ estimation based on an optimal-basis measurement of a back-scattered electron-photon pair, given that the particles initially have opposite helicity and circular polarization eigenvalues.

V. CONCLUSION AND OUTLOOK

We considered asymptotically-free fermions and photons that interact at the tree level, scatter, and then become asymptotically-free again before being measured. We then considered the quantum states of the scattered fermions and photons in, respectively, the helicity-basis and polarization-basis Hilbert spaces, and computed the QFIM of these states with respect to the external parameters of the scattering (namely, the particles' momentum magnitude and the polar scattering angle), which are continuous. In this proof-of-concept study we showed that the internal degrees of freedom of scattered fermions and photons carry information about their external degrees of freedom, and hence, placed precision limits on any unbiased estimation of the latter. We implicitly used quantum field operators and the formalism of QED in order to model the evolution of these particles states.

The results of this investigation are relevant to practical QED scattering scenarios where precise estimation and/or measurement of kinematic parameters is required. A key example is positron emission tomography (PET). In PET imaging, a patient is injected with a radioactive isotope that undergoes positron emission decay. Emitted positrons then combine and annihilate with electrons in the given medium, producing pairs of 511 keV photons as a result [50]. The direction of propagation of these photons is measured using scintillators. If the photons' polarization states were accessible too, say through Compton polarimetry [51], the

calculation we propose would provide bounds on the additional information available about the photons' 3-momentum magnitude and scattering angle (which could in turn lower false coincidences and improve contrast at given dosage).

Another interesting continuation of this investigation would be (following the work done by [17]) to consider the QFIM of a quantum state with respect to the coupling constant of the quantum field itself. This would describe how deformations of the quantum field Lagrangian affect the geometry of the relevant Hilbert space, and would hence reveal a fundamental feature of the quantum field theory under consideration.

An alternative line of development would be to apply the method put forward in this paper to other quantum fields. Ref. [34] showed that two gluons in a pure state are entangled in their polarizations as a result of three and four-gluon vertex interactions, given that they have opposite

polarizations before the interaction. The scattered state derived in Ref. [34] is a function parametrized by the polar scattering angle and the gluons' momenta in the center-of-mass frame. This means that the QFIM of this state can be found using the method presented here.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

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