

Dark matter in the scale-invariant 3-3-1-1 model

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We propose a novel scale-invariant model with the 3-3-1-1 gauge symmetry featuring a universal seesaw mechanism for all fermion masses, which, through the inclusion of additional vectorlike quarks, provides a partial explanation for the observed fermion mass hierarchies. A discrete remnant of the gauge group, the matter parity (P_M), stabilizes a fermionic dark matter candidate, and the scalar sector includes two triplets (minimal for 3-3-1 breaking) and two scalar singlets. We identify the lightest P_M -odd fermion, f_d , as a viable dark matter candidate. Our analysis shows that f_d satisfies the observed relic density constraint within the mass range $220 \text{ GeV} \lesssim m_{f_d} \lesssim 555 \text{ GeV}$, primarily due to resonant annihilation via the new scalar H_2 . While this mass range depends on the symmetry-breaking scale v_χ , which has a lower bound of $v_\chi \gtrsim 3.6 \text{ TeV}$ from LEP constraints on the ρ_0 parameter, we adopt a more conservative lower bound of $v_\chi > 10 \text{ TeV}$. This choice is made to ensure that the Z' boson mass remains above approximately 4 TeV, and is motivated by recent LHC results and future projections for Z' boson searches, which provide more stringent constraints than previous bounds or those from the ρ_0 parameter. Spin-independent (SI) interactions dominate the direct detection phenomenology of f_d . We calculate the SI elastic scattering cross section and find that parameter points satisfying the relic density constraint are consistent with current experimental limits from LZ and PandaX-4T for certain parameter choices, particularly depending on the α_{12} angle. Some regions of the viable parameter space lie below the neutrino floor. Prospects for detection by future experiments like XLZD and PandaX-xT are also presented and discussed.

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I. INTRODUCTION

The enormous success of the Standard Model (SM) has still left unanswered quite a few questions, such as neutrino

mass, dark matter (DM), and the number of fermion families. The last question finds an answer in the family of 3-3-1 models, which have been extensively studied in the literature [1–10]. One of the most notable features of these models is that the number of fermion families must be a multiple of three due to the nontrivial nature of anomaly cancellation. Additionally, these models exhibit a rich structure enabling the incorporation of new particles that can address other open problems of the SM, such as the dark matter puzzle, the strong CP problem, and the origin of neutrino masses [11–17].

Motivated by the potential of 3-3-1 models to address these shortcomings, we propose a model with dynamical symmetry breaking based on the gauge group

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$SU(3)_C \otimes SU(3)_L \otimes U(1)_X \otimes U(1)_N$ that extends the 3-3-1 gauge group by an additional $U(1)$ symmetry [12–16]. The $SU(3)_C$ represents the color symmetry of the Standard Model, $SU(3)_L$ extends the left-handed chirality symmetry, and $U(1)_X$ and $U(1)_N$ are a combination of charge and $B - L$. Building on this symmetry framework, this work presents a new model that, while sharing the same gauge symmetry as the framework of [18], departs significantly from it in both scalar and fermionic field content, as well as in phenomenological structure. The extended gauge sector leaves remnant symmetries after the spontaneous symmetry breaking, which allow us to economically implement the universal seesaw mechanism and to stabilize dark matter.

One of the driving forces behind the proposed model is the reduction in the number of scalar fields in high representations of the gauge group and the free parameters in the scalar potential, especially those with mass dimension, when compared to the original 3-3-1 models. These parameters tend to make the models less predictive and introduce scalar complexity that, in some cases, becomes excessive compared to other extensions of the SM, such as the two-Higgs-doublet model (see [19]) and the minimal supersymmetric standard model (see [20]). For instance, in the original 3-3-1 models, the scalar sector includes an enlarged set of nontrivial $SU(3)_L$ multiplets of scalar fields, such as three triplets [1,4–10] and, in some cases, three triplets plus a sextet [3].

In order to achieve a very compact and more predictive scalar sector, we follow the same strategy as in [18]. Specifically, we introduce the minimal number of triplets required to dynamically break the gauge symmetry to $SU(3)_C \otimes U(1)_Q$ via the Coleman-Weinberg mechanism [21] in the Gildener-Weinberg approximation [22]. The minimal number of triplets needed to achieve this is two. However, as first discussed in [11], despite these two triplets breaking the gauge symmetry, accidental chiral symmetries typically remain. These symmetries are usually associated with generators that are combinations of the diagonal generators of $SU(3)_L$ and a Peccei-Quinn-like symmetry, as shown in [23].

As it happens, this symmetry is also chiral in the quark sector (and occasionally in the lepton sector), leaving some particles massless. This situation is inconsistent with experimental observations. Therefore, the accidental symmetry must be broken to generate mass terms for these particles. To address this issue, we introduce new vectorlike quarks and leptons.

At first glance, this might seem like a mere translation of the complexity from the scalar sector to the fermion sector. However, there are several advantages to this approach. The first is that these new fermions can be used to implement a universal seesaw mechanism [24], which partially addresses the hierarchy of fermion masses—an unresolved issue in the Standard Model. In our construction, the

seesaw truly deserves the qualifier “universal”: a single set of vectorlike partners simultaneously generates predictive mass matrices for quarks, charged leptons, and neutrinos—whereas earlier 3-3-1 approaches handled each sector in isolation or relied on *ad hoc* extensions.

Moreover, these new fermions acquire masses at an energy scale related to scale invariance, which in this model is significantly higher than the electroweak and 3-3-1 scales due to the requirements of the seesaw mechanism. As a result, the number of physical degrees of freedom at the TeV scale becomes smaller and more manageable, making the model more feasible for phenomenological studies.

As a benefit of the extended gauge group, the matter parity that stabilizes DM arises as a discrete remnant symmetry from the symmetry breaking [12,14,18,25–30]. While we have several potential dark matter candidates, only a dark Majorana fermion f_d can satisfy the direct detection constraints. Its stability is protected by a residual $\mathbb{Z}_2$ symmetry that naturally remains unbroken due to the representation content and gauge charge assignments, without the need to impose any additional discrete symmetry. Furthermore, radiative corrections involving f_d contribute significantly to the stabilization of the scalar potential, establishing a nontrivial connection between the dark matter sector and vacuum dynamics.

We systematically explore the key aspects of the proposed model. In Sec. II, we define the field representations, charge assignments, and symmetry-breaking pattern, emphasizing the role of the scalar triplets η and χ in inducing a distinct vacuum structure. Section III is devoted to the scalar sector, where we analyze the full scalar potential and express the quartic couplings in terms of parameters such as scalar masses, VEVs, and mixing angles, which provide a more physically transparent and intuitive parametrization. In Sec. IV, we derive perturbative unitarity constraints at tree level, which prove to be highly constraining: several regions of parameter space, especially those with large scalar mixings, are excluded due to violations of unitarity bounds.

Section V provides a complete analysis of the gauge boson spectrum. We derive analytical expressions for the masses of the charged and neutral gauge bosons and perform the diagonalization of the neutral gauge boson mass matrix, which has a particularly nontrivial structure. This allows for the identification of the physical states and the isolation of the heavy neutral boson Z'' , whose presence is essential to ensuring vacuum stability. Section VI is devoted to the generation of fermion masses via a universal seesaw mechanism, elegantly yielding the observed hierarchical pattern across all fermionic sectors.

In Sec. VII, we examine phenomenological constraints from lepton flavor-violating processes. Section VIII is devoted to the dark matter sector, demonstrating that the model successfully reproduces the observed relic density

and features parameter regions within the projected sensitivity of future experiments. Our main results and conclusions are summarized in Sec. IX.

II. THE SCALE-INVARIANT 3-3-1-1 MODEL

Although we follow the same general strategy as in [18], there are some important differences in both the symmetry structure and the field content. To be more specific, we choose to introduce the η triplet [instead of $\rho = (\rho_1^+, \rho_2^0, \rho_3^+)^T \sim (1, 3, 2/3)$] and the χ triplet, which transform under the $SU(3)_C \otimes SU(3)_L \otimes U(1)_X \otimes U(1)_N$ group as $(1, 3, -1/3, 1/3)$ and $(1, 3, -1/3, -2/3)$, respectively.

In other words,

$$\begin{aligned}\eta &= \left(\eta_1^0 \equiv \frac{v_\eta + S_\eta + iA_\eta}{\sqrt{2}}, \eta_2^-, \eta_3^0 \right)^T, \\ \chi &= \left(\chi_1^0, \chi_2^-, \chi_3^0 \equiv \frac{v_\chi + S_\chi + iA_\chi}{\sqrt{2}} \right)^T.\end{aligned}\quad (1)$$

Note that η and χ transform differently under the $U(1)_N$ symmetry. This is an important point because we require that η_1^0 and χ_3^0 acquire vacuum expectation values (VEVs) in order to break the gauge symmetry down to $SU(3)_C \otimes U(1)_Q$. If these fields transformed identically under all the gauge groups, a gauge rotation could be performed to set one of the VEVs to zero.

As usual, we also embed the SM quarks and leptons in the representations

$$\begin{aligned}L_{iL} &= (\nu_i, e_i, f_i)_L^T, & Q_{aL} &= (d_a, -u_a, D_a)_L^T, \\ Q_{3L} &= (u_3, d_3, U_3)_L^T,\end{aligned}\quad (2)$$

where $a = 1, 2$, $i = 1, 2, 3$, and L_{iL} , Q_{aL} , and Q_{3L} transform under the gauge group $SU(3)_C \otimes SU(3)_L \otimes U(1)_X \otimes U(1)_N$ as $(1, 3, -1/3, -2/3)$, $(3, \bar{3}, 0, 0)$, and $(3, 3, 1/3, 2/3)$, respectively. As previously mentioned, the two scalar triplets η and χ in Eq. (1) are not enough to generate masses for all the quarks and leptons. Thus, we introduce vectorlike fermions $\mathcal{F}_{iL,R}$, $\mathcal{K}_{aL,R}$, and $\mathcal{K}_{3L,R}$

$$\begin{aligned}\mathcal{F}_{iL,R} &= (E_i, -f'_i, E'_i)_{L,R}^T, \\ \mathcal{K}_{aL,R} &= (\mathcal{U}_a, \mathcal{D}_a, \mathcal{U}'_a)_{L,R}^T, \\ \mathcal{K}_{3L,R} &= (\mathcal{D}_3, -\mathcal{U}_3, \mathcal{D}'_3)_{L,R}^T,\end{aligned}\quad (3)$$

in order to allow terms such as $\overline{L_{iL}} \eta^* \mathcal{F}_{jR}$, $\overline{\mathcal{F}_{iL}} \chi^* e_{jR}$, $\overline{Q_{aL}} \eta \mathcal{K}_{bR}$, and $\overline{Q_{3L}} \eta^* \mathcal{K}_{3R}$ which ensure that all the SM fermions become massive, the fields $\mathcal{F}_{iL,R}$, $\mathcal{K}_{aL,R}$, and $\mathcal{K}_{3L,R}$ have to belong to $(1, \bar{3}, -2/3, -1/3)$, $(3, 3, 1/3, -1/3)$ and $(3, \bar{3}, 0, 1)$ representation of the 3-3-1-1 gauge group.

However, it is still necessary to introduce a new real singlet scalar,

$$\varphi = \frac{v_\varphi + S_\varphi}{\sqrt{2}} \sim (1, 1, 0, 0), \quad (4)$$

to give masses for the $\mathcal{F}_{iL,R}$, $\mathcal{K}_{aL,R}$, and $\mathcal{K}_{3L,R}$ through terms such as $\varphi \overline{\mathcal{F}_{iL}} \mathcal{F}_{jR}$, $\varphi \overline{\mathcal{K}_{aL}} \mathcal{K}_{bR}$, and $\varphi \overline{\mathcal{K}_{3L}} \mathcal{K}_{3R}$.

Although the lepton triplet L_{iL} in Eq. (2) contains ν_{iL} , the gauge symmetries do not allow neutrino masses as in the Standard Model. To address this, we introduce right-handed neutrinos $\nu_{iR} \sim (1, 1, 0, -1)$, motivated by current oscillation experiments, which require at least two neutrinos to be massive and necessarily nondegenerate. Together with the introduction of a complex scalar field

$$\sigma = \frac{v_\sigma + S_\sigma + iA_\sigma}{\sqrt{2}} \sim (1, 1, 0, 2), \quad (5)$$

this allows the implementation of a type-I seesaw mechanism [31–35]. Note that σ is the only field responsible for breaking the $U(1)_N$ symmetry and for giving mass to the neutral gauge boson associated with that symmetry.

Regarding gauge symmetries, this model shares most of its features with the rest of the 3-3-1 models. Specifically, the choice of the field representations under the $SU(3)_C \otimes SU(3)_L \otimes U(1)_X$ symmetry is determined by anomaly cancellation, the allowance of tree-level mass generation, and ensuring that the SM fermions have the correct electric charges. It is also important to note that the $U(1)_X$ charges are chosen such that no quarks possess exotic electric charges. The electric charge operator Q is defined as

$$Q = T_3 - \frac{1}{\sqrt{3}} T_8 + X \mathbf{1}_{3 \times 3}, \quad (6)$$

where T_3 and T_8 are the diagonal generators of $SU(3)_L$.

Moreover, the representation of the fields under $U(1)_N$ is chosen such that the SM fermions have the conventional $B - L$ charges, and a discrete symmetry

$$P_M = (-1)^{3(B-L)+2s}, \quad (7)$$

where s is the spin of the fields, remains after spontaneous symmetry breaking [12,14,18,25–30]. Here, $B - L$ is defined as

$$B - L = -\frac{2}{\sqrt{3}} T_8 + N \mathbf{1}_{3 \times 3}, \quad (8)$$

For example, in the particular case of the $U(1)_N$ charge for the L_{iL} triplet, it should be chosen as $-2/3$. A summary of

TABLE I. Summary of fermion and scalar fields, their representations under the gauge group $SU(3)_C \otimes SU(3)_L \otimes U(1)_X \otimes U(1)_N$, electric charges (Q), $B - L$ charges, and their transformations under the dark parity P_M symmetry.

Fields	3-3-1-1	Q	B - L	P_M -Parity
L_{iL}	$(1, 3, -1/3, -2/3)$	$(0, -1, 0)^T$	$(-1, -1, 0)^T$	$(+, +, -)^T$
e_{iR}	$(1, 1, -1, -1)$	-1	-1	+
Q_{aL}	$(3, \bar{3}, 0, 0)$	$(-1/3, 2/3, -1/3)^T$	$(1/3, 1/3, -2/3)^T$	$(+, +, -)^T$
Q_{3L}	$(3, 3, 1/3, 2/3)$	$(2/3, -1/3, 2/3)^T$	$(1/3, 1/3, 4/3)^T$	$(+, +, -)^T$
u_{iR}	$(3, 1, 2/3, 1/3)$	$2/3$	$1/3$	+
d_{iR}	$(3, 1, -1/3, 1/3)$	$-1/3$	$1/3$	+
ν_{iR}	$(1, 1, 0, -1)$	0	-1	+
f_{iR}	$(1, 1, 0, 0)$	0	0	-
$\mathcal{F}_{iL,R}$	$(1, \bar{3}, -2/3, -1/3)$	$(-1, 0, -1)^T$	$(0, 0, -1)^T$	$(-, -, +)^T$
U_{3R}	$(3, 1, 2/3, 4/3)$	$2/3$	$4/3$	-
D_{aR}	$(3, 1, -1/3, -2/3)$	$-1/3$	$-2/3$	-
$\mathcal{K}_{aL,R}$	$(3, 3, 1/3, -1/3)$	$(2/3, -1/3, 2/3)^T$	$(-2/3, -2/3, 1/3)^T$	$(-, -, +)^T$
$\mathcal{K}_{3L,R}$	$(3, \bar{3}, 0, 1)$	$(-1/3, 2/3, -1/3)^T$	$(4/3, 4/3, 1/3)^T$	$(-, -, +)^T$
η	$(1, 3, -1/3, 1/3)$	$(0, -1, 0)^T$	$(0, 0, 1)^T$	$(+, +, -)^T$
χ	$(1, 3, -1/3, -2/3)$	$(0, -1, 0)^T$	$(-1, -1, 0)^T$	$(-, -, +)^T$
σ	$(1, 1, 0, 2)$	0	2	+
φ	$(1, 1, 0, 0)$	0	0	+

all the fields and their representations under the gauge group is provided in Table I.

The P_M symmetry stabilizes several neutral fields that can, in principle, serve as candidates for dark matter. Finally, we note that a neutral fermion charged under the P_M symmetry, $f_{iR} \sim (1, 1, 0, 0)$, is introduced to also implement a seesaw mechanism in the DM leptonic sector.

III. SCALAR SECTOR

In this section, we study the scalar sector at tree level, as it plays a crucial role in understanding dynamical symmetry breaking. Additionally, it allows us to determine the mass eigenstates in different sectors, which we primarily classify as CP -even and CP -odd. In this model, the CP -odd sector is also P_M -odd. The P_M -odd sector refers to scalars that are nontrivially charged under the P_M symmetry, which, in principle, stabilizes the dark matter sector. Due to our consideration that all quartic coupling constants and vacuum expectation values are real, these sectors do not mix with each other. The charged scalar sector in this model is rather straightforward, as all charged fields, $(\eta_2^-, \chi_2^-, \eta_2^+, \chi_2^+)$, exhibit a mass matrix that is identically zero. Consequently, these fields correspond to Nambu-Goldstone bosons, which are subsequently absorbed by the respective charged gauge fields.

A. Scalar potential

Let us consider the most general renormalizable, scale-invariant scalar potential that preserves the gauge symmetry of the model

$$\begin{aligned}
 V_0 = & \lambda_\chi (\chi^\dagger \chi)^2 + \lambda_\eta (\eta^\dagger \eta)^2 + \lambda_{\eta\chi} (\chi^\dagger \chi) (\eta^\dagger \eta) \\
 & + \lambda'_{\eta\chi} (\chi^\dagger \eta) (\eta^\dagger \chi) + \lambda_{\chi\varphi} (\chi^\dagger \chi) \varphi^2 \\
 & + \lambda_{\chi\sigma} (\chi^\dagger \chi) (\sigma^* \sigma) + \lambda_{\eta\varphi} (\eta^\dagger \eta) \varphi^2 \\
 & + \lambda_{\eta\sigma} (\eta^\dagger \eta) (\sigma^* \sigma) + \lambda_\varphi \varphi^4 + \lambda_\sigma (\sigma^* \sigma)^2 \\
 & + \lambda_{\varphi\sigma} (\sigma^* \sigma) \varphi^2.
 \end{aligned} \tag{9}$$

Since the potential V_0 is biquadratic in the fields, it can be conveniently expressed as

$$V_0 = (\Phi^{\circ 2})^T \Lambda(\theta) \Phi^{\circ 2}, \tag{10}$$

where the vector of field norms $(\Phi^{\circ 2})^T \equiv (|\eta|^2, |\chi|^2, |\sigma|^2, |\varphi|^2) \geq 0$, the symmetric matrix

$$\Lambda(\theta) = \begin{pmatrix} \lambda_\eta & \frac{1}{2}(\lambda_{\eta\chi} + \lambda'_{\eta\chi}\theta) & \frac{1}{2}\lambda_{\eta\sigma} & \frac{1}{2}\lambda_{\eta\varphi} \\ \star & \lambda_\chi & \frac{1}{2}\lambda_{\chi\sigma} & \frac{1}{2}\lambda_{\chi\varphi} \\ \star & \star & \lambda_\sigma & \frac{1}{2}\lambda_{\varphi\sigma} \\ \star & \star & \star & \lambda_\varphi \end{pmatrix} \tag{11}$$

contains quartic couplings, and the orbit parameter $\theta \equiv (\chi^\dagger \eta)(\eta^\dagger \chi) / (|\eta|^2 |\chi|^2) \in [0, 1]$ due to the Cauchy-Schwartz inequality. Notice here that we are using the Hadamard product, defined as the element-wise product, e.g., $[(A)^{\circ 2}]_{ij} = a_{ij}^2$ for a matrix A with elements a_{ij} .

B. Potential minimum and P_M -even mass matrix

We proceed to calculate the mass eigenvalues and eigenstates of the CP -even sector. To do this, we must

first find the minimum of the scalar potential. Following standard procedures, we expand V_0 in terms of the fields $(\Phi^{\circ 2})^T = \frac{1}{2}(S_{\eta 1}^2, S_{\chi 3}^2, S_{\sigma}^2, S_{\phi}^2)$, which acquire VEVs, while setting all other fields to zero. Additionally, we work within the Gildener-Weinberg approximation [22], which allows us to calculate the effective potential perturbatively along the flat direction, as this method is convenient for models with multiple scalar fields. Moreover, as we will see in Eq. (25), $\lambda'_{\eta\chi}$ is positive, ensuring that, at the minimum of the potential, the orbit parameter θ equals zero. In other words, at the minimum $V_0(\Phi) = (\Phi^{\circ 2})^T \Lambda(\theta=0) \Phi^{\circ 2} \equiv (\Phi^{\circ 2})^T \Lambda \Phi^{\circ 2}$, where, for simplicity, we denote $\Lambda \equiv \Lambda(0)$.

To explicitly determine the flat direction, described by $\Phi = \xi \mathbf{n}$, where ξ represents the radial degree of freedom (the dilaton) and $\mathbf{n}$ is a unit vector, we proceed following [36]. The central idea is to minimize the potential $V_0(\Phi) = (\Phi^{\circ 2})^T \Lambda \Phi^{\circ 2}$, subject to the condition $V_0(\mathbf{n}) = 0$. As shown in detail in Ref. [36], the Hadamard square of the unit vector $\mathbf{n}$ is an eigenvector of the quartic coupling matrix Λ with a null eigenvalue, given by

$$\mathbf{n}^{\circ 2} = \frac{\text{adj}(\Lambda)e}{e^T \text{adj}(\Lambda)e}. \quad (12)$$

Here, $\text{adj}(\Lambda)$ is the adjugate matrix of Λ , defined as $\Lambda \text{adj}(\Lambda) = \det(\Lambda)I$, the vector $e = (1, \dots, 1)^T$ on the respective space has all elements equal to unity, and I is the identity matrix. The condition $V_0(\mathbf{n}) = 0$ is guaranteed by taking $\det(\Lambda) = 0$, as seen by inserting Eq. (12) in Eq. (10).

The 1-loop effective potential along the flat direction is given by (see [18,21,36,37])

$$V^{1\text{-loop}}(\xi \mathbf{n}) = B(\mathbf{n}) \xi^4 \left(\ln \frac{\xi^2}{v_{\xi}^2} - \frac{1}{2} \right), \quad (13)$$

where ξ is the dilaton field, $v_{\xi} \equiv \sqrt{v_{\eta}^2 + v_{\chi}^2 + v_{\sigma}^2 + v_{\phi}^2}$ its VEV, and

$$B(\mathbf{n}) = \frac{1}{64\pi^2 v_{\xi}^4} \left(\sum_S m_S^4 + 3 \sum_V m_V^4 - 4 \sum_F m_F^4 \right) \quad (14)$$

is the sum over all degrees of freedom.

The mass matrix for the CP -even fields that get VEVs can be written as [38]

$$M_{\Phi}^2 = v_{\xi}^2 \nabla_{\mathbf{N}}^T \nabla_{\mathbf{N}} V(\mathbf{N})|_{\mathbf{N}=\mathbf{n}} = 2\Lambda \circ (\mathbf{n}\mathbf{n}^T) v_{\xi}^2, \quad (15)$$

where $\mathbf{N}$ is a unit vector in the Φ direction. Solving Eq. (15) for the quartic coupling matrix Λ , we get

$$\Lambda = \frac{1}{2v_{\xi}^2} M_{\Phi}^2 \circ (\mathbf{n}\mathbf{n}^T)^{\circ -1}, \quad (16)$$

where $(\mathbf{n}\mathbf{n}^T)^{\circ -1}$ means the element-wise inverse, i.e., $[(\mathbf{n}\mathbf{n}^T)^{\circ -1}]_{ij} = 1/(n_i n_j)$ in Hadamard notation.

Equation (16) is particularly useful as it enables us to express the quartic couplings in terms of the tree-level scalar masses, the VEVs, and mixing angles. To illustrate this, let us define the diagonal mass matrix $M_{\text{CP-even}}^2$ for the CP -even fields that acquire VEVs as $M_{\text{CP-even}}^2 = \text{diag}(m_{H_1}^2, m_{H_2}^2, m_{H_3}^2, m_{H_4}^2)$, which is related to M_{Φ}^2 by

$$M_{\Phi}^2 = O M_{\text{CP-even}}^2 O^T, \quad (17)$$

where the orthogonal matrix O is given by [39,40]

$$O = O_{34} O_{24} O_{14} O_{23} O_{13} O_{12}. \quad (18)$$

Each O_{ij} defines a rotation in the ij -plane; we denote the mixing angle in O_{ij} as α_{ij} and also abbreviate $s_{ij} \equiv \sin \alpha_{ij}$ and $c_{ij} \equiv \cos \alpha_{ij}$.

Considering $m_{H_4} = m_{\xi} = 0$ as the dilaton mass at tree level, we have the last column of O as the flat direction

$$\mathbf{n} = (s_{14} \quad c_{14}s_{24} \quad c_{14}c_{24}s_{34} \quad c_{14}c_{24}c_{34})^T. \quad (19)$$

Thus, the quartic couplings can be expressed in terms of the VEVs, CP -even scalar masses, and mixing angles using Eq. (16). Here, we note that the angles in Eq. (19), α_{14} , α_{24} , α_{34} , can be easily written in terms of the VEVs as

$$\begin{aligned} \alpha_{14} &= \arctan \left(\frac{v_{\eta}}{\sqrt{v_{\chi}^2 + v_{\sigma}^2 + v_{\phi}^2}} \right), \\ \alpha_{24} &= \arctan \left(\frac{v_{\chi}}{\sqrt{v_{\sigma}^2 + v_{\phi}^2}} \right), \\ \alpha_{34} &= \arctan \left(\frac{v_{\sigma}}{v_{\phi}} \right). \end{aligned} \quad (20)$$

The remaining angles in Eq. (18), α_{12} , α_{13} , and α_{23} , cannot be expressed only in terms of the VEVs. However, they are strongly constrained by the perturbative unitarity conditions and the hierarchy in the VEVs, $v_{\eta} \ll v_{\chi} \ll v_{\sigma}, v_{\phi}$ arising from the imposition of the seesaw mechanism throughout the fermion sector. In this model, it is natural to identify v_{η} with the Standard Model VEV, $v_{\text{SM}} = 246.22$ GeV, since the $W^{\pm}$ mass, as shown in Eq. (37), is entirely determined by v_{η} . Thus, we identify the mass eigenstate H_1 as the SM-like Higgs with mass $m_{H_1} = 125.11$ GeV. The H_2 eigenstate acquires a mass at the 331-breaking scale, v_{χ} . Finally, the H_3 eigenstate acquires a mass at the scale of v_{σ} and v_{ϕ} , leading to the hierarchy $m_{H_1} \ll m_{H_2} \ll m_{H_3}$. The dilaton H_4 obtains a mass at one-loop level: it is given by

$$m_{H_4}^2 = 8B(\mathbf{n})v_\xi^2. \quad (21)$$

For the SM-like Higgs $H_1 \approx S_{\eta 1}$, it is required that $\alpha_{12} \approx 0$. Additionally, to keep the couplings within the perturbative regime, the angles α_{13} and α_{23} must be small. To derive useful expressions for the CP -even eigenstates, we set these angles to zero. Under this approximation, we have

$$\begin{aligned} H_1 &= \frac{(v_\chi^2 + v_\sigma^2 + v_\phi^2)S_{\eta 1} - v_\eta(v_\chi S_{\chi 3} + v_\phi S_\phi) - v_\eta v_\sigma S_\sigma}{\sqrt{(v_\sigma^2 + v_\phi^2)(v_\eta^2 + v_\sigma^2 + v_\phi^2)}} \approx S_{\eta 1}, \\ H_2 &= \frac{(v_\sigma^2 + v_\phi^2)S_{\chi 3} - v_\chi(v_\sigma S_\sigma + v_\phi S_\phi)}{\sqrt{(v_\eta^2 + v_\sigma^2 + v_\phi^2)(v_\eta^2 + v_\chi^2 + v_\sigma^2 + v_\phi^2)}} \approx S_{\chi 3}, \\ H_3 &= \frac{v_\phi S_\sigma - v_\sigma S_\phi}{\sqrt{(v_\sigma^2 + v_\phi^2)}}, \\ H_4 &= \frac{v_\eta S_{\eta 1} + v_\chi S_{\chi 3} + v_\sigma S_\sigma + v_\phi S_\phi}{\sqrt{(v_\eta^2 + v_\chi^2 + v_\sigma^2 + v_\phi^2)}} \approx \frac{v_\sigma S_\sigma + v_\phi S_\phi}{\sqrt{(v_\sigma^2 + v_\phi^2)}}. \end{aligned} \quad (22)$$

Actually, we need to have small nonzero values for the angles α_{12} , α_{13} , and α_{23} to allow nonzero couplings of the fermion DM candidate to scalar field in order to ensure the correct relic density.

C. P_M -odd sector

Finally, we focus our attention on the calculation of the P_M scalar sector. In the basis, $B_d = (\eta_3, \chi_1^*)^T$, the P_M -odd mass matrix is

$$M_d^2 = \begin{pmatrix} \frac{1}{2}\lambda'_{\eta\chi}v_\chi^2 & \frac{1}{2}\lambda'_{\eta\chi}v_\eta v_\chi \\ \frac{1}{2}\lambda'_{\eta\chi}v_\eta v_\chi & \frac{1}{2}\lambda'_{\eta\chi}v_\eta^2 \end{pmatrix}. \quad (23)$$

The matrix M_d is diagonalized by the orthogonal transformation $O_d^T M_d^2 O_d = \text{diag}(m_{G_d}, m_{S_d})$, where

$$O_d = \begin{pmatrix} -\frac{v_\eta}{\sqrt{v_\chi^2 + v_\eta^2}} & \frac{v_\chi}{\sqrt{v_\chi^2 + v_\eta^2}} \\ \frac{v_\chi}{\sqrt{v_\chi^2 + v_\eta^2}} & \frac{v_\eta}{\sqrt{v_\chi^2 + v_\eta^2}} \end{pmatrix}. \quad (24)$$

Its eigenvalues are $m_{G_d}^2 = 0$ for a dark Goldstone and $m_{S_d}^2 = \frac{1}{2}(v_\eta^2 + v_\chi^2)\lambda'_{\eta\chi}$ for a physical dark mass eigenstate S_d . We can now find the $\lambda'_{\eta\chi}$ parameter

$$\lambda'_{\eta\chi} = \frac{2m_{S_d}^2}{(v_\eta^2 + v_\chi^2)}. \quad (25)$$

The dark scalar mass eigenstates are given by

$$\begin{aligned} G_d &= -\frac{v_\eta}{\sqrt{v_\chi^2 + v_\eta^2}}\eta_3 + \frac{v_\chi}{\sqrt{v_\chi^2 + v_\eta^2}}\chi_1^*, \\ S_d &= \frac{v_\chi}{\sqrt{v_\chi^2 + v_\eta^2}}\eta_3 + \frac{v_\eta}{\sqrt{v_\chi^2 + v_\eta^2}}\chi_1^*. \end{aligned} \quad (26)$$

Note that the scalar S_d can, in principle, act as a dark matter candidate.

IV. PERTURBATIVE UNITARITY

Perturbative unitarity constraints arise from the requirement that the scattering matrix be unitary, ensuring the conservation of probability in particle interactions, and are explored by analyzing the tree-level two-to-two scattering amplitudes dominated by quartic interactions in the high-energy limit $\sqrt{s} \rightarrow \infty$. In this context, the eigenvalues $\mathcal{T}^{ba}$ of the scattering matrix must satisfy the condition $\Re(\mathcal{T}_0^{ba}) \leq \frac{1}{2}$ for the theory to remain perturbatively unitary.

At the zeroth partial wave order, in the high-energy limit, the matrix element can be expressed as [41]

$$\mathcal{T}_0^{ba} = \frac{1}{16\pi} \frac{1}{\sqrt{2^{\delta_{S_1, S_2}} 2^{\delta_{S_3, S_4}}}} \frac{\partial V_0}{\partial S_1 \partial S_2 \partial S_3^* \partial S_4^*} \Big|_n \leq \frac{1}{2}, \quad (27)$$

where a corresponds to the initial state $S_1 + S_2$ and b to the analogous final state $S_3 + S_4$ for the two-to-two scattering process.

By calculating the scattering matrix for all possible two-to-two initial and final states, including Goldstone bosons, we derive constraints on the quartic couplings. These constraints are given by

$$\begin{aligned}
|\lambda_\eta| &\leq 4\pi, \quad |\lambda_{\eta\sigma}| \leq 8\pi, \quad |\lambda_{\eta\varphi}| \leq 8\pi, \quad |\lambda_\sigma| \leq 4\pi, \quad |\lambda_\chi| \leq 4\pi, \\
|\lambda_\eta - \sqrt{\lambda_\eta^2 - 2\lambda_\eta\lambda_\chi + \lambda_{\eta\chi}^2 + \lambda_\chi^2 + \lambda_\chi}| &\leq 8\pi, \\
|\lambda_\eta + \sqrt{\lambda_\eta^2 - 2\lambda_\eta\lambda_\chi + \lambda_{\eta\chi}^2 + \lambda_\chi^2 + \lambda_\chi}| &\leq 8\pi, \\
|\lambda_{\eta\chi} - \lambda'_{\eta\chi}| &\leq 8\pi, \quad |\lambda_{\chi\sigma}| \leq 8\pi, \quad |\lambda_{\chi\varphi}| \leq 8\pi, \\
|\lambda_{\eta\chi}| &\leq 8\pi, \quad |\lambda'_{\eta\chi} + \lambda_{\eta\chi}| \leq 8\pi, \quad |3\lambda'_{\eta\chi} + \lambda_{\eta\chi}| \leq 8\pi.
\end{aligned} \tag{28}$$

Additionally, the four solutions λ_i to the following quartic equation must satisfy $|\lambda_i| \leq 4\pi$

$$\lambda^4 + \lambda^3 A + \lambda^2 B + \lambda C + D = 0, \tag{29}$$

with coefficients explicitly given as

$$A = -\frac{1}{2}(8\lambda_\eta + 4\lambda_\sigma + 3\lambda_\varphi + 8\lambda_\chi), \tag{30}$$

$$\begin{aligned}
B = \frac{1}{8} &(-64\lambda_\eta\lambda_\sigma - 48\lambda_\eta\lambda_\varphi - 128\lambda_\eta\lambda_\chi + 6\lambda_{\eta\sigma}^2 + 3\lambda_{\eta\varphi}^2 + 2\lambda_{\chi\eta}^2 + 12\lambda'_{\chi\eta}\lambda_{\chi\eta} - 24\lambda_\sigma\lambda_\varphi - 64\lambda_\sigma\lambda_\chi \\
&- 48\lambda_\varphi\lambda_\chi + \lambda_{\varphi\sigma}^2 + 18\lambda_{\chi\eta}^2 + 6\lambda_{\chi\sigma}^2 + 3\lambda_{\chi\varphi}^2),
\end{aligned} \tag{31}$$

$$\begin{aligned}
C = \frac{1}{8} &\left(9\lambda_{\eta\sigma}^2\lambda_\varphi - 96\lambda_\eta\lambda_\sigma\lambda_\varphi + \lambda_{\chi\eta}^2(4\lambda_\sigma + 3\lambda_\varphi) + 4\lambda_\eta\lambda_{\varphi\sigma}^2 + 24\lambda_{\eta\sigma}^2\lambda_\chi - 256\lambda_\eta\lambda_\sigma\lambda_\chi - 192\lambda_\eta\lambda_\varphi\lambda_\chi - 96\lambda_\sigma\lambda_\varphi\lambda_\chi + 4\lambda_{\varphi\sigma}^2\lambda_\chi \right. \\
&+ 6\lambda_{\eta\varphi}^2(\lambda_\sigma + 2\lambda_\chi) + 36\lambda_\sigma\lambda_{\chi\eta}^2 + 27\lambda_\varphi\lambda_{\chi\eta}^2 - 18\lambda_{\eta\sigma}\lambda_{\chi\eta}\lambda_{\chi\sigma} + 24\lambda_\eta\lambda_{\chi\sigma}^2 + 9\lambda_\varphi\lambda_{\chi\sigma}^2 - 3\lambda_{\varphi\sigma}\lambda_{\chi\sigma}\lambda_{\chi\varphi} + 12\lambda_\eta\lambda_{\chi\varphi}^2 + 6\lambda_\sigma\lambda_{\chi\varphi}^2 \\
&\left. + 3\lambda'_{\chi\eta}(8\lambda_\sigma\lambda_{\chi\eta} + 6\lambda_\varphi\lambda_{\chi\eta} - 2\lambda_{\eta\sigma}\lambda_{\chi\sigma} - \lambda_{\eta\varphi}\lambda_{\chi\varphi}) - 3\lambda_{\eta\varphi}(\lambda_{\eta\sigma}\lambda_{\varphi\sigma} + 3\lambda_{\chi\eta}\lambda_{\chi\varphi})\right),
\end{aligned} \tag{32}$$

$$\begin{aligned}
D = \frac{1}{32} &\left(\lambda_{\chi\eta}^2(-24\lambda_\sigma\lambda_\varphi + \lambda_{\varphi\sigma}^2) - 96\lambda_{\eta\varphi}^2\lambda_\sigma\lambda_\chi - 144\lambda_{\eta\sigma}^2\lambda_\varphi\lambda_\chi + 1536\lambda_\eta\lambda_\sigma\lambda_\varphi\lambda_\chi + 48\lambda_{\eta\sigma}\lambda_{\eta\varphi}\lambda_{\varphi\sigma}\lambda_\chi - 64\lambda_{\eta\varphi}\lambda_{\varphi\sigma}^2\lambda_\chi - 216\lambda_\sigma\lambda_\varphi\lambda_{\chi\eta}^2 \right. \\
&+ 9\lambda_{\varphi\sigma}^2\lambda_{\chi\eta}^2 + 108\lambda_{\eta\sigma}\lambda_{\varphi\sigma}\lambda_{\chi\eta}\lambda_{\chi\sigma} - 18\lambda_{\eta\varphi}\lambda_{\varphi\sigma}\lambda_{\chi\eta}\lambda_{\chi\sigma} + 9\lambda_{\eta\varphi}^2\lambda_{\chi\sigma}^2 - 144\lambda_\eta\lambda_{\varphi\sigma}\lambda_{\chi\sigma}^2 + 72\lambda_{\eta\varphi}\lambda_{\varphi\sigma}\lambda_{\chi\eta}\lambda_{\chi\varphi} - 18\lambda_{\eta\sigma}\lambda_{\varphi\sigma}\lambda_{\chi\eta}\lambda_{\chi\varphi} \\
&- 18\lambda_{\eta\sigma}\lambda_{\eta\varphi}\lambda_{\chi\sigma}\lambda_{\chi\varphi} + 48\lambda_\eta\lambda_{\varphi\sigma}\lambda_{\chi\sigma}\lambda_{\chi\varphi} + 9\lambda_{\eta\sigma}^2\lambda_{\chi\varphi}^2 - 96\lambda_\eta\lambda_\sigma\lambda_{\chi\varphi}^2 - 6\lambda'_{\chi\eta}(24\lambda_\sigma\lambda_{\varphi\sigma}\lambda_{\chi\eta} - \lambda_{\varphi\sigma}^2\lambda_{\chi\eta} - 6\lambda_{\eta\sigma}\lambda_{\varphi\sigma}\lambda_{\chi\sigma} + \lambda_{\eta\varphi}\lambda_{\varphi\sigma}\lambda_{\chi\sigma} \\
&\left. - 4\lambda_{\eta\varphi}\lambda_{\varphi\sigma}\lambda_{\chi\varphi} + \lambda_{\eta\sigma}\lambda_{\varphi\sigma}\lambda_{\chi\varphi})\right).
\end{aligned} \tag{33}$$

V. GAUGE SECTOR

The gauge sector spectrum arises from the kinetic terms of the scalar bosons, which are given by

$$\mathcal{L} \supset (D_\mu\eta)^\dagger(D^\mu\eta) + (D_\mu\chi)^\dagger(D^\mu\chi) + (D_\mu\sigma)^*(D^\mu\sigma), \tag{34}$$

where the covariant derivative for triplets is defined as

$$D_\mu = \partial_\mu - igW_\mu^a T_a - ig_X X B_\mu - ig_N N N_\mu. \tag{35}$$

Here, $T_a = \frac{\lambda_a}{2}$ are the generators of $SU(3)_L$, with λ_a ($a = 1, \dots, 8$) being the Gell-Mann matrices. The parameters X and N represent the charges of the fields under the $U(1)_X$ and $U(1)_N$ symmetries, respectively. Note that the φ field is a gauge singlet and, therefore, its kinetic term is not included in Eq. (34). The covariant derivative for the σ field only involves the N_μ gauge boson, as σ carries a charge of two under the $U(1)_N$ symmetry.

Since $SU(2)_L \subset SU(3)_L$, the gauge coupling g is the same as in the SM. The gauge bosons associated with the $SU(3)_L$, $U(1)_X$, and $U(1)_N$ symmetries are W_μ^a , B_μ , and N_μ , respectively, with their corresponding gauge couplings denoted as g , g_X , and g_N .

The charged (including the non-Hermitian) gauge bosons are straightforwardly calculated, as they arise from the nondiagonal generators of the $SU(3)_L$ group. The charged gauge eigenstates are given by

$$\begin{aligned}
W_\mu^\pm &= \frac{W_{1\mu} \mp iW_{2\mu}}{\sqrt{2}}, \\
V_\mu^{0(\mp)} &= \frac{W_{4\mu} \mp iW_{5\mu}}{\sqrt{2}}, \\
V_\mu^\pm &= \frac{W_{6\mu} \pm iW_{7\mu}}{\sqrt{2}},
\end{aligned} \tag{36}$$

with masses

$$\begin{aligned} m_{W^\pm}^2 &= \frac{g^2 v_\eta^2}{4}, \\ m_{V^{0(\dagger)}}^2 &= \frac{g^2}{4} (v_\eta^2 + v_\chi^2), \\ m_{V^\pm}^2 &= \frac{g^2 v_\chi^2}{4}. \end{aligned} \quad (37)$$

Note that the fields $V_\mu^\pm$ and $V_\mu^{0(\dagger)}$ are P_M -odd, or dark, under the P_M symmetry defined in Eq. (7). However, since

the neutral gauge boson is heavier than the charged one, i.e., $m_{V^{0(\dagger)}}^2 = m_{W^\pm}^2 + m_{V^\pm}^2$, the dark gauge sector does not provide a suitable DM candidate.

On the other hand, diagonalization of the squared mass matrix of the neutral gauge bosons is more intricate. In the basis

$$\mathbf{B}_{\text{NG}} = (W_{3\mu}, W_{8\mu}, B_\mu, N_\mu),$$

the mass matrix is given by

$$M_{\text{NG}}^2 = \frac{g^2}{4} \begin{pmatrix} v_\eta^2 & \frac{v_\eta^2}{\sqrt{3}} & -\frac{2g_X v_\eta^2}{3g} & \frac{2g_N v_\eta^2}{3g} \\ \frac{v_\eta^2}{\sqrt{3}} & \frac{v_\eta^2 + 4v_\chi^2}{3} & -\frac{2g_X(v_\eta^2 - 2v_\chi^2)}{3\sqrt{3}g} & \frac{2g_N(v_\eta^2 + 4v_\chi^2)}{3\sqrt{3}g} \\ -\frac{2g_X v_\eta^2}{3g} & -\frac{2g_X(v_\eta^2 - 2v_\chi^2)}{3\sqrt{3}g} & \frac{4g_X^2(v_\eta^2 + v_\chi^2)}{9g^2} & -\frac{4g_N g_X(v_\eta^2 - 2v_\chi^2)}{9g^2} \\ \frac{2g_N v_\eta^2}{3g} & \frac{2g_N(v_\eta^2 + 4v_\chi^2)}{3\sqrt{3}g} & -\frac{4g_N g_X(v_\eta^2 - 2v_\chi^2)}{9g^2} & \frac{4g_X^2(v_\eta^2 + 4(9v_\sigma^2 + v_\chi^2))}{9g^2} \end{pmatrix}. \quad (38)$$

For simplicity, we are neglecting the kinetic mixing term between the Abelian fields B_μ and N_μ .

Since there is a hierarchy between the VEVs, at leading order, the mass matrix M_{NG}^2 is diagonalized by the orthogonal transformation matrix U_{NG} , such that

$$U_{\text{NG}}^T M_{\text{NG}}^2 U_{\text{NG}} = \text{diag}(0, m_Z^2, m_{Z'}^2, m_{Z''}^2),$$

where U_{NG} is approximately

$$U_{\text{NG}} \simeq \begin{pmatrix} \sin \theta_W & \cos \theta_W & -\frac{\sqrt{3}g \tan^2 \theta_W v_\eta^2}{2g_X \tan 2\theta_W v_\chi^2} & \frac{g v_\eta^2}{24g_N v_\sigma^2} \\ -\frac{\sin \theta_W}{\sqrt{3}} & \frac{\sin^2 \theta_W}{\sqrt{3} \cos \theta_W} & -\frac{g \tan \theta_W}{g_X} & \frac{g v_\chi^2}{6\sqrt{3}g_N v_\sigma^2} \\ \frac{g \sin \theta_W}{g_X} & -\frac{g \sin^2 \theta_W}{g_X \cos \theta_W} & -\frac{\tan \theta_W}{\sqrt{3}} & \frac{g_X v_\chi^2}{18g_N v_\sigma^2} \\ 0 & -\frac{g \tan^2 \theta_W v_\eta^2}{24 \cos \theta_W g_N v_\sigma^2} & \frac{g_X v_\chi^2}{6\sqrt{3}g_N \tan \theta_W v_\sigma^2} & 1 \end{pmatrix}. \quad (39)$$

Here, the Weinberg angle θ_W is defined as

$$\sin^2 \theta_W = \frac{g_X^2}{g^2 + \frac{4}{3}g_X^2}. \quad (40)$$

Since g can be approximately set to 0.652, based on the $W^\pm$ mass, and using $\sin^2 \theta_W \approx 0.231$ [42], the value of g_X is found to be approximately 0.377.

The U_{NG} matrix allows expressing the gauge symmetry bosons, $W_{3\mu}, W_{8\mu}, B_\mu, N_\mu$, in terms of the physical gauge bosons, $A_\mu, Z_\mu, Z'_\mu, Z''_\mu$. That is,

$$\begin{pmatrix} W_{3\mu} \\ W_{8\mu} \\ B_\mu \\ N_\mu \end{pmatrix} = U_{\text{NG}} \begin{pmatrix} A_\mu \\ Z_\mu \\ Z'_\mu \\ Z''_\mu \end{pmatrix}. \quad (41)$$

Besides the fact that the photon mass is zero, we can obtain the masses of the Z , Z' , and Z'' bosons

$$m_Z^2 \simeq [9g^2 v_\eta^2 v_\chi^2 \cos^2 \theta_W] / [\sin^4 \theta_W (v_\eta^2 (36 + v_\chi^2) + 36v_\chi^2) - 36 \sin^2 \theta_W (v_\eta^2 + 2v_\chi^2) + 9(v_\eta^2 + 4v_\chi^2)], \quad (42)$$

$$m_{Z'}^2 \simeq \frac{g_X^2 v_\chi^2}{3 \tan^2 \theta_W}, \quad (43)$$

$$m_{Z''}^2 \simeq 4g_N^2 v_\sigma^2. \quad (44)$$

The masses $m_{Z'}$ and $m_{Z''}$ were computed at leading order, while m_Z was computed at the next order, where it deviates from its SM counterpart. However, at leading order, m_Z coincides with the SM Z boson mass, as expected. Additionally, the parameter g_N , together with v_σ , determines the mass of the Z'' , as given in Eq. (44). For v_σ on the order of 10^3 TeV and Yukawa couplings of order 1, the potential remains stable [i.e., $B(\mathbf{n}) > 0$, as defined in Eq. (14)] if $g_N \gtrsim 0.598$.

The Z boson properties in our model deviate from the SM predictions due to the (small) mixing between Z and the other massive vector fields. For example, we can obtain a lower bound on v_χ by imposing that the new contribution to the ρ_0 parameter, defined as $\rho_0 \equiv (\frac{M_W}{M_Z \cos \theta_W})^2$, remain

within the experimental limits [43]. This parameter, which is exactly equal to unity in the SM, is modified in our model as

$$\Delta\rho_0 = \rho_0 - 1 \approx \frac{v_\eta^2 \cos^2 2\theta_W \sec^4 \theta_W}{4v_\chi^2}. \quad (45)$$

Using $\sin^2 \theta_W \approx 0.231$ and the allowed experimental deviation $\Delta\rho_0 \lesssim 5.8 \times 10^{-4}$ [43], we find that

$$v_\chi \gtrsim 3571.35 \text{ GeV}, \quad (46)$$

setting the lower bound for the symmetry breaking scale of the 331 model.

VI. YUKAWA SECTOR: SEESAW-SUPPRESSED FERMION MASSES

In this section, we derive the fermion mass spectrum, demonstrating how seesaw mechanisms emerge across all sectors as a result of the VEV hierarchy $v_\eta \ll v_\chi \ll v_\phi, v_\sigma$. The electroweak VEV, v_η , is fixed at $v_{\text{SM}} = 246 \text{ GeV}$, while the 3-3-1 breaking VEV, v_χ , is significantly larger, approximately 3.6 TeV. An even larger VEV, v_ϕ , is required to enable the seesaw mechanism for charged fermions, with v_ϕ typically around 10^6 GeV . Finally, the $B - L$ symmetry-breaking VEV, v_σ , must also be large, satisfying $v_\sigma \gg v_\eta$, to justify the seesaw mechanism for neutrinos. This hierarchical structure plays a crucial role in shaping the Yukawa sector.

These seesaw mechanisms not only provide a partial explanation for the flavor puzzle but also help identify the fermionic dark matter candidate in our model. By incorporating this hierarchy, the model successfully reproduces the observed fermion mass spectrum and offers a natural framework to address open questions in particle physics, such as the flavor puzzle and the nature of dark matter.

A. Lepton spectrum

The renormalizable Yukawa Lagrangian for the lepton sector is expressed as

$$\begin{aligned} -\mathcal{L}_l = & Y_{ij}^\nu \overline{L}_{iL} \eta \nu_{jR} + Y_{ij}^f \overline{L}_{iL} \chi f_{jR} \\ & + \frac{h_{ij}^\nu}{2} \sigma(\nu_{iR})^c \nu_{jR} + \frac{h_{ij}^f}{2} \varphi(f_{iR})^c f_{jR} \\ & + y_{ij}^{\mathcal{F}} \overline{L}_{iL} \eta^* \mathcal{F}_{jR} + y_{ij}^e \overline{\mathcal{F}}_{iL} \chi^* e_{jR} \\ & + h_{ij}^{\mathcal{F}} \varphi \overline{\mathcal{F}}_{iL} \mathcal{F}_{jR} + \text{H.c.} \end{aligned} \quad (47)$$

Following spontaneous symmetry breaking, all leptons acquire mass, as will be demonstrated in the subsequent analysis. It is also noteworthy that all Yukawa couplings in the lepton sector are represented by 3×3 matrices.

For simplicity, we omit explicit indices and work directly with the couplings in matrix form.

Moreover, to ensure compatibility with stringent experimental constraints on CP violation, we assume all Yukawa matrices to be real at leading order. The current 90% confidence level limits, $|d_e| < 4.1 \times 10^{-30} e \cdot \text{cm}$ from the ACME III experiment [44], and $|d_n| < 1.8 \times 10^{-26} e \cdot \text{cm}$ from neutron EDM searches [45], already exclude $\mathcal{O}(1)$ CP phases in many extensions of the Standard Model. In 3-3-1 frameworks, even a single physical phase in the scalar or Yukawa sector typically drives the predicted $d_{e,n}$ close to—or above—these bounds [46–48]. Our choice aligns with the paradigm of *additive CP violation*, where tiny ($\lesssim 10^{-3}$) imaginary perturbations can reproduce CKM/PMNS observables while naturally suppressing EDMs [49]. Since the central aim of this work is to present a classically scale-invariant $SU(3)_C \otimes SU(3)_L \otimes U(1)_X \otimes U(1)_N$ model that addresses the fermion mass hierarchy through a universal seesaw mechanism and provides a naturally fermion dark matter candidate, introducing general complex phases would enlarge the parameter space without altering these core results. A comprehensive study of CP -violating effects, including renormalization group running and a SMEFT analysis of dipole operators, will be presented elsewhere [50].

1. Charged leptons

The charged lepton sector consists of P_M -even fields (e_i and E'_i), which mix to produce the SM charged leptons and three heavy charged leptons, as well as P_M -odd fields (E_i), where $i = 1, 2, 3$.

In the basis $\mathbf{E}_{L,R} = (e_{L,R}, E'_{L,R})$, the mass terms are expressed as $\overline{\mathbf{E}}_L \mathcal{M}_e \mathbf{E}_R$, where the mass matrix $\mathcal{M}_e$ is given by

$$\mathcal{M}_e = \begin{pmatrix} 0 & -y^{\mathcal{F}} v_\eta \\ y^e v_\chi & h^{\mathcal{F}} v_\phi \end{pmatrix}. \quad (48)$$

This mass matrix exhibits a type-I Dirac seesaw structure in the limit $v_\eta \ll v_\chi \ll v_\phi$. To leading order, the matrix $\mathcal{M}_e$ can be block-diagonalized using two unitary matrices, V_L^e and V_R^e . The diagonalization matrices are given by

$$\begin{aligned} V_L^e &= \begin{pmatrix} 1 & -\frac{v_\eta}{v_\phi} y^{\mathcal{F}} (h^{\mathcal{F}})^{-1} \\ \frac{v_\eta}{v_\phi} (h^{\mathcal{F}\dagger})^{-1} y^{\mathcal{F}\dagger} & 1 \end{pmatrix}, \\ V_R^e &= \begin{pmatrix} 1 & \frac{v_\chi}{v_\phi} y^{e\dagger} (h^{\mathcal{F}\dagger})^{-1} \\ \frac{v_\chi}{v_\phi} (h^{\mathcal{F}})^{-1} y^e & 1 \end{pmatrix}, \end{aligned} \quad (49)$$

such that

$$V_L^{e\dagger} \mathcal{M}_e V_R^e = \text{diag}(M_e, M_{E'}), \quad (50)$$

where the mass matrices are given by

$$M_e = \frac{v_\eta v_\chi}{\sqrt{2} v_\varphi} y^\mathcal{F} (h^\mathcal{F})^{-1} y^e, \quad M_{E'} = \frac{v_\varphi}{\sqrt{2}} h^\mathcal{F}. \quad (51)$$

The mass matrix of P_M -odd fields, E_i , is

$$M_E = \frac{v_\varphi}{\sqrt{2}} h^\mathcal{F}, \quad (52)$$

that is they are essentially degenerate with P_M -even fields E'_i .

For simplicity, we assume $y^\mathcal{F} = h^\mathcal{F}$ equal to the identity matrix and y^e to be diagonal. This allows us to express the matrix elements in terms of the masses of the charged leptons e , μ , and τ . The diagonal Yukawa coupling matrix is then given by

$$y^e = \text{diag}(y_1^e, y_2^e, y_3^e). \quad (53)$$

By solving for the matrix elements, we find the following relations:

$$y_1^e = \frac{\sqrt{2} v_\varphi}{v_\eta v_\chi} m_e, \quad y_2^e = \frac{\sqrt{2} v_\varphi}{v_\eta v_\chi} m_\mu, \quad y_3^e = \frac{\sqrt{2} v_\varphi}{v_\eta v_\chi} m_\tau. \quad (54)$$

Considering the masses given in [43], with $v_\eta = v_{\text{SM}}$, $v_\chi = 5 \times 10^3$ GeV, and $v_\varphi = 10^6$ GeV, the Yukawa parameters y^e are of the orders

$$y_1^e \sim 10^{-4}, \quad y_2^e \sim 10^{-2}, \quad y_3^e \sim 1. \quad (55)$$

2. Neutrinos

The neutral lepton spectrum can be divided into the P_M -even (visible) and P_M -odd (dark) sectors. The P_M -even sector corresponds to neutrinos. We define the basis $\mathbf{N}_L = (\nu_L, (\nu_R)^c)$ and write the mass term as $(1/2) \overline{\mathbf{N}}_L \mathcal{M}_\nu (\mathbf{N}_L)^c$, where the mass matrix $\mathcal{M}_\nu$ is given by

$$\mathcal{M}_\nu = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & Y^\nu v_\eta \\ Y^{\nu T} v_\eta & h^\nu v_\sigma \end{pmatrix}. \quad (56)$$

Since $v_\sigma \gg v_\eta$, the matrix $\mathcal{M}_\nu$ exhibits the structure of a type-I seesaw mechanism [51]. To leading order, it can be block-diagonalized by the matrix V^ν , given by

$$V^\nu = \begin{pmatrix} 1 & \frac{v_\eta}{v_\sigma} Y^{\nu*} (h^{\nu*})^{-1} \\ -\frac{v_\eta}{v_\sigma} (h^{\nu T})^{-1} Y^{\nu T} & 1 \end{pmatrix}, \quad (57)$$

such that

$$V^{\nu T} \mathcal{M}_\nu V^\nu = \text{diag}(m_\nu, m_N), \quad (58)$$

where the mass matrices are

$$m_\nu = -\frac{v_\eta^2}{\sqrt{2} v_\sigma} Y^\nu (h^\nu)^{-1} Y^{\nu T}, \quad m_N = \frac{v_\sigma}{\sqrt{2}} h^\nu. \quad (59)$$

For simplicity, we assume h^ν to be diagonal, as sterile neutrino oscillations are not considered in this work. Using the Casas-Ibarra parametrization ([52,53]), and considering the R matrix as the identity matrix, we obtain the solution for Y^ν

$$\begin{aligned} Y^\nu &= -i \frac{\sqrt{\sqrt{2} v_\sigma}}{v_\eta} (m_\nu^{\text{diag}})^{1/2} U_{2\nu}^{-1} \\ &= -i \frac{\sqrt{\sqrt{2} v_\sigma}}{v_\eta} (m_\nu^{\text{diag}})^{1/2} (U_{\text{PMNS}})^T. \end{aligned} \quad (60)$$

In the last equality, we use $U_{2\nu} = U_{\text{PMNS}}^*$ [54], as the charged lepton mass matrix is diagonal under our assumptions. Hence, this result expresses the elements of the Yukawa matrix Y^ν in terms of the neutrino masses $m_{\nu i}$ (for $i = 1, 2, 3$), which correspond to the physical neutrinos ν_{iL} , and the neutrino mixing angles in U_{PMNS} .

3. Dark leptons

The P_M -odd, or dark, neutral leptons are arranged in the basis $\mathbf{F}_L = (f_L, (f_R)^c, (f'_L), (f'_R)^c)$, such that their mass matrix, $(1/2) \overline{\mathbf{F}}_L \mathcal{M}_f (\mathbf{F}_L)^c$, is expressed as

$$\mathcal{M}_f = \begin{pmatrix} 0_{[3 \times 3]} & m_{D[3 \times 9]}^f \\ (m_D^f)^T_{[9 \times 3]} & m_{R[9 \times 9]}^f \end{pmatrix}, \quad (61)$$

where

$$\begin{aligned} m_D^f &= \frac{1}{\sqrt{2}} \begin{pmatrix} Y^f v_\chi & 0 & -y^\mathcal{F} v_\eta \end{pmatrix}, \\ m_R^f &= \frac{1}{\sqrt{2}} \begin{pmatrix} h^f v_\varphi & 0 & 0 \\ 0 & 0 & h^\mathcal{F} v_\varphi \\ 0 & h^{\mathcal{F} T} v_\varphi & 0 \end{pmatrix}. \end{aligned} \quad (62)$$

The structure corresponds to a type-I seesaw, as the eigenvalues of m_R^f are proportional to v_φ , while the entries of m_D^f are at most proportional to v_χ , with $v_\varphi \gg v_\chi \gg v_\eta$. Therefore, the same procedure applied to neutrinos can be repeated. In this case, however, block diagonalization requires two matrices, V_1^f and V_2^f . The transformation is given by

$$V_2^{fT} V_1^{fT} \mathcal{M}_f V_1^f V_2^f = \text{diag}(m_f, m_{f'_1}, m_{f'_2}, m_{f'_3}),$$

where the block matrices on the diagonal are

$$\begin{aligned} m_f &= -\frac{v_\chi^2}{\sqrt{2}v_\phi} Y^f (h^{\mathcal{F}})^{-1} Y^{fT}, \\ m_{f'_1} &= \frac{v_\phi}{\sqrt{2}} h^f, \\ m_{f'_2} &= \frac{v_\phi}{\sqrt{2}} h^{\mathcal{F}}, \\ m_{f'_3} &= -\frac{v_\phi}{\sqrt{2}} h^{\mathcal{F}}. \end{aligned} \quad (63)$$

The matrices V_2^f and V_1^f are explicitly given by

$$\begin{aligned} V_2^f &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \\ V_1^f &= \begin{pmatrix} 1 & \frac{v_\chi}{v_\phi} Y^{f*} (h^{\mathcal{F}*})^{-1} & -\frac{v_\eta}{v_\phi} Y^{\mathcal{F}*} (h^{\mathcal{F}*})^{-1} & 0 \\ -\frac{v_\chi}{v_\phi} (h^{fT})^{-1} Y^{fT} & 1 & 0 & 0 \\ -\frac{v_\eta}{v_\phi} (h^{\mathcal{F}T})^{-1} Y^{\mathcal{F}T} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \end{aligned} \quad (64)$$

As a result, three of the dark neutral leptons acquire seesaw-suppressed masses originating from the m_f matrix and are significantly lighter than the others. The lightest of these fermions serves as the DM candidate.

B. Quark spectrum

We now proceed to apply a similar analysis to the quark sector, detailing the corresponding Yukawa interactions. The general Yukawa Lagrangian for the quarks is written as

$$\begin{aligned} -\mathcal{L}_q &= Y_{aj}^d \overline{Q}_{aL} \eta^* d_{jR} + Y_{ab}^D \overline{Q}_{aL} \chi^* D_{bR} \\ &+ Y_{3j}^u \overline{Q}_{3L} \eta u_{jR} + Y_{33}^U \overline{Q}_{3L} \chi U_{3R} \\ &+ y_{ab}^\mathcal{K} \overline{Q}_{aL} \eta \mathcal{K}_{bR} + y_{33}^\mathcal{K} \overline{Q}_{3L} \eta^* \mathcal{K}_{3R} \\ &+ y_{aj}^u \overline{\mathcal{K}}_{aL} \chi u_{jR} + y_{3i}^d \overline{\mathcal{K}}_{3L} \chi^* d_{iR} \\ &+ h_{ab}^\mathcal{K} \phi \overline{\mathcal{K}}_{aL} \mathcal{K}_{bR} + h_{33}^\mathcal{K} \phi \overline{\mathcal{K}}_{3L} \mathcal{K}_{3R} + \text{H.c.}, \end{aligned} \quad (66)$$

where $a, b = 1, 2$ and $i, j = 1, 2, 3$.

1. Up-type quarks

Starting with the P_M -even up-type quarks, we obtain the following mass matrix when written in the basis $\mathbf{u} = (u_i, \mathcal{U}'_a)$:

$$\mathcal{M}_u = \frac{1}{\sqrt{2}} \begin{pmatrix} 0_{[2 \times 3]} & y_{[2 \times 2]}^\mathcal{K} v_\eta \\ Y_{[1 \times 3]}^u v_\eta & 0_{[1 \times 2]} \\ y_{[2 \times 3]}^u v_\chi & h_{[2 \times 2]}^\mathcal{K} v_\phi \end{pmatrix}. \quad (67)$$

For the sake of clarity, since the quark Yukawa couplings have different dimensions, we explicitly write their dimensions in the mass matrices throughout this section.

The hierarchy between the blocks allows us to employ the seesaw approximation used in the previous section. We do not provide all the details here, as the procedure is quite similar to the one applied in the lepton sector. After performing the block-diagonalization, we obtain the following two matrices:

$$(m_u^2)_{\text{light}} \simeq \frac{v_\eta^2}{2} \begin{pmatrix} \kappa_\chi^2 A [1 - Y^{u\dagger} (Y^u Y^{u\dagger})^{-1} Y^u] A^\dagger & 0_{[2 \times 1]} \\ 0_{[1 \times 2]} & Y^u Y^{u\dagger} \end{pmatrix}, \quad (68)$$

and

$$(M_u^2)_{\text{heavy}} \simeq \frac{v_\phi^2}{2} h^\mathcal{K} h^{\mathcal{K}\dagger}, \quad (69)$$

where $A_{[2 \times 3]} = y^\mathcal{K} h^{\mathcal{K}-1} y^u$ and $\kappa_\chi = v_\chi / v_\phi$.

Consequently, two of the three light up-type quarks acquire seesaw-suppressed masses, proportional to $v_\eta \kappa_\chi$. Meanwhile, the remaining quark has a mass proportional to the electroweak scale, v_η , and is identified as the top quark.

We can determine a constraint on the Y^u matrix related to the top quark. According to [42], the top quark mass is 172.57 ± 0.29 GeV. Thus, we have

$$\sqrt{\frac{v_\eta^2}{2} Y^u (Y^u)^\dagger} = \frac{v_\eta}{\sqrt{2}} \sqrt{Y_1^{u2} + Y_2^{u2} + Y_3^{u2}} \sim 172.57, \quad (70)$$

from which we obtain

$$Y_t \equiv \sqrt{Y_1^{u2} + Y_2^{u2} + Y_3^{u2}} \sim 1. \quad (71)$$

The Yukawa coupling is constrained to be less than 1.67 at 95% confidence level (CL) [55].

Regarding the P_M -odd up-quarks, we choose $\mathbf{U} = (U_3, \mathcal{U}_3, \mathcal{U}_a)$ as the basis. In this framework, the mass matrix for the dark up-type quarks is given by

$$\mathcal{M}_u^{\text{dark}} = \frac{1}{\sqrt{2}} \begin{pmatrix} Y_{33}^U v_\chi & -y_{33}^\mathcal{K} v_\eta & 0_{[1 \times 2]} \\ 0 & h_{33}^\mathcal{K} v_\phi & 0_{[1 \times 2]} \\ 0_{[2 \times 1]} & 0_{[2 \times 1]} & h_{[2 \times 2]}^\mathcal{K} v_\phi \end{pmatrix}. \quad (72)$$

It is clear that $\mathcal{U}_a$ does not mix with the other quarks, and their mass matrix corresponds to the bottom-right block

of $\mathcal{M}_u^{\text{dark}}$. For the remaining two states, there is a small mixing of order $\mathcal{O}(v_\eta/v_\phi)$, and their masses are approximately given by the upper diagonal entries of $\mathcal{M}_u^{\text{dark}}$.

2. Down-type quarks

For the P_M -even down-type quarks, we choose $\mathbf{d} = (d_i, \mathcal{D}_3)$ as the basis and derive the mass matrix as follows:

$$\mathcal{M}_d = \frac{1}{\sqrt{2}} \begin{pmatrix} Y_{[2 \times 3]}^d v_\eta & 0_{[2 \times 1]} \\ 0_{[1 \times 3]} & -y_{33}^\mathcal{K} v_\eta \\ Y_{[1 \times 3]}^d v_\chi & h_{33}^\mathcal{K} v_\phi \end{pmatrix}. \quad (73)$$

Following the seesaw procedure, the diagonalization of $\mathcal{M}_d \mathcal{M}_d^\dagger$ proceeds analogously to the lepton case. The masses are approximately given by

$$(m_d^2)_{\text{light}} \simeq \frac{v_\eta^2}{2} \begin{pmatrix} Y^d Y^{d\dagger} & 0_{[2 \times 1]} \\ 0_{[1 \times 2]} & \kappa_\chi^2 B [1 - Y^{d\dagger} (Y^d Y^{d\dagger})^{-1} Y^d] B^\dagger \end{pmatrix},$$

$$(M_d^2)_{\text{heavy}} \simeq \frac{v_\phi^2}{2} |h_{33}^\mathcal{K}|^2, \quad (74)$$

where $B_{[1 \times 3]} = \frac{y_{33}^\mathcal{K}}{h_{33}^\mathcal{K}} Y^d$. As a result, among the down-type quarks, only one acquires a seesaw-suppressed mass.

Finally, we considered the P_M -odd down-quarks. To analyze this, we fix the basis for the dark down-type quarks as $\mathbf{D} = (D_a, \mathcal{D}_a, \mathcal{D}_3)$. The resulting mass matrix is given by

$$\mathcal{M}_d^{\text{dark}} = \frac{1}{\sqrt{2}} \begin{pmatrix} Y_{[2 \times 2]}^D v_\chi & y_{[2 \times 2]}^\mathcal{K} v_\eta & 0_{[2 \times 1]} \\ 0_{[2 \times 2]} & h_{[2 \times 2]}^\mathcal{K} v_\phi & 0_{[2 \times 1]} \\ 0_{[1 \times 2]} & 0_{[1 \times 2]} & h_{33}^\mathcal{K} v_\phi \end{pmatrix}. \quad (75)$$

The diagonal entries provide the approximate masses. While $\mathcal{D}_3$ remains unmixed, the fields D_a and $\mathcal{D}_a$ exhibit a small mixing suppressed by $\kappa_\eta = v_\eta/v_\phi$.

In summary, this section demonstrated that all fermions acquire mass through a consistent application of the seesaw mechanism, highlighting its universality across the entire fermion sector.

3. LHC phenomenology

The vectorlike quarks $\mathcal{U}_a, \mathcal{U}_3, \mathcal{U}'_3, \mathcal{D}_a, \mathcal{D}_3$, and $\mathcal{D}'_3$ have masses proportional to v_ϕ and are therefore far too heavy to be produced at the LHC. The 331-like exotic quarks U_3 and D_a , however, are relatively light, with masses proportional to v_χ . They are therefore within the reach of LHC searches. The main LHC production channel for the exotic quarks is the gluon-gluon fusion, the exotic quarks being produced in pairs, $gg \rightarrow J\bar{J}$, or in association with an exotic gauge

boson $gg \rightarrow V^\pm J, V^0 J$, where $J = U_3, D_a$. The decay modes of exotic quarks differ from 331 models [56], due to the matter parity P_M . The exotic quarks U_3 and D_a are odd under the matter parity and therefore ultimately decay into dark matter and SM particles. The main decay channels are: $U \rightarrow bV^{+*} \rightarrow b f_d X$ and $D_a \rightarrow (u/c)V^{-*} \rightarrow (u/c) f_d X$, where X are leptonic SM final states. The LHC searches of exotic up- and down-type quarks impose the following tentative bound on their masses: $m_U > 1.6$ TeV and $m_{D_a} > 1.1$ TeV [57,58].

VII. BOUNDS FROM CHARGED LEPTONS

The flavor-violating Yukawa couplings are obtained from Eq. (47). After the block-diagonalization, keeping the leading order, the charged lepton Yukawa couplings can be divided into pure SM lepton couplings

$$\mathcal{L}_{\text{SM}} = \frac{\sqrt{2}}{v_\eta} \bar{e}_L M_e e_R \eta_1^{0*} + \frac{\sqrt{2}}{v_\chi} \bar{e}_L M_e e_R \chi_3^{0*} - \frac{\sqrt{2}}{v_\phi} \bar{e}_L M_e e_R \phi + \text{H.c.}, \quad (76)$$

and couplings containing exotic charged leptons

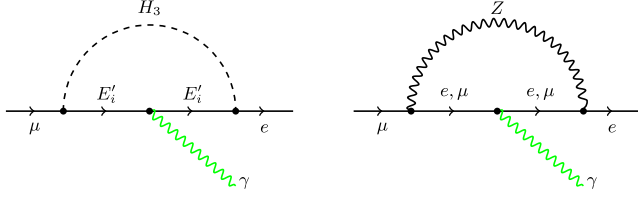
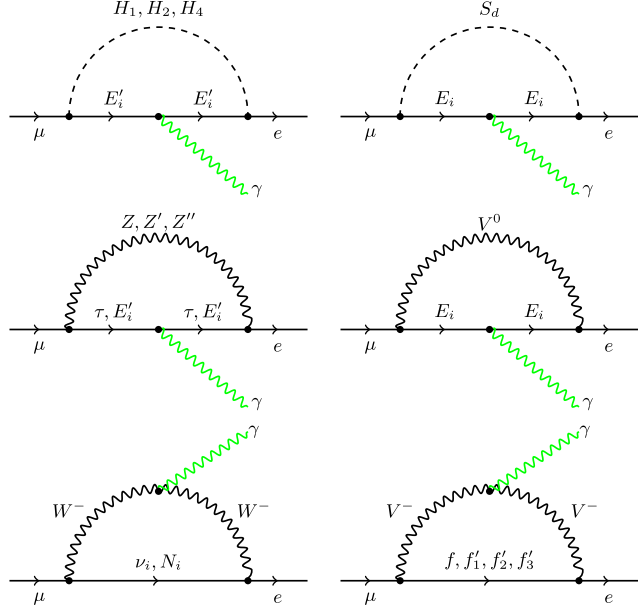
$$\mathcal{L}_{\text{exotic}} = -\bar{e}_L y^F E'_R \eta_1^{0*} + \bar{e}_L y^F E_R \eta_3^{0*} + \bar{E}_L y^e e_R \chi_1^{0*} + \bar{E}'_L y^e e_R \chi_3^{0*} + \bar{E}_L h^F E_R \phi + \bar{E}'_L h^F E'_R \phi + \text{H.c.} \quad (77)$$

The charged lepton mass matrices in Eqs. (52) and (51) are diagonalized in the form $U_L^l M_l U_R^l = M_l^{\text{diag}}$ and physical coupling constants in Eq. (77) become

$$y^{\mathcal{F}} \rightarrow \tilde{y}^{\mathcal{F}} = U_L^{e\dagger} y^{\mathcal{F}} U_R^E, \quad y^e \rightarrow \tilde{y}^e = U_L^{E\dagger} y^e U_R^e, \\ h^{\mathcal{F}} \rightarrow \tilde{h}^{\mathcal{F}} = U_L^{E\dagger} h^{\mathcal{F}} U_R^E. \quad (78)$$

One can see that the Yukawa couplings of SM charged leptons are proportional to their mass matrix and are therefore diagonalized simultaneously with the SM lepton mass matrix. Therefore the neutral scalars do not have flavor-changing neutral currents at the tree level. This also avoids the LHC bounds on the lepton flavor-violating couplings of Higgs. The SM lepton couplings are all suppressed by heavy VEVs, except the first one in Eq. (76). The muon and tau Yukawa couplings have been measured at LHC to be close to the SM values. Therefore the real part of η_1^0 needs to be identified with 125 GeV Higgs. The lepton masses are suppressed by v_χ/v_ϕ . This imposes $v_\chi/v_\phi \gtrsim 10^{-2}$, in order to keep Yukawa coupling smaller than 1.

Even though scalars do not have flavor-changing couplings to a pair of SM charged leptons, the flavor-violating

FIG. 1. The dominant diagrams contributing to $\mu \rightarrow e\gamma$.FIG. 2. Subleading diagrams contributing to $\mu \rightarrow e\gamma$.

effects appear at the loop level due to the mixing between SM leptons and the charged exotic leptons. The most constraining charged lepton flavor-violating (CLFV) process is $\mu \rightarrow e\gamma$, where all neutral scalars and massive gauge bosons mediate it at loop level. The most constraining CLFV diagrams are presented in Fig. 1 and subleading diagrams in Fig. 2. Note that the photon vertex is always flavor conserving. The couplings between scalars, SM charged leptons and E'_i are not suppressed by the mixing between SM leptons and exotic leptons, as can be seen in Eq. (77). The scalar mediated diagrams are, however, heavily suppressed by large masses of E'_i and E_i , as well as the relation

$$\tilde{y}^F (M_{E'}^{\text{diag}})^{-1} \tilde{y}^e = \frac{2}{v_\eta v_\chi} M_e^{\text{diag}}, \quad (79)$$

which originates from the seesaw mechanism. The leading contribution to the scalar diagrams is canceled due to this relation, providing further suppression. The most relevant scalar diagram is mediated by H_3 , corresponding to the branching ratio

$$\text{BR}(\mu \rightarrow e\gamma) = \frac{e^2 \alpha_{13}^2 \alpha_{23}^2 m_\mu^3 m_{H_3}^4}{64(4\pi)^5 \Gamma_\mu^{\text{tot}}} \times \left(\left| \sum_{i=1}^3 \frac{\tilde{y}_{ie}^e \tilde{y}_{\mu i}^F}{m_{E'_i}^3} \right|^2 + \left| \sum_{i=1}^3 \frac{\tilde{y}_{i\mu}^e \tilde{y}_{ei}^F}{m_{E'_i}^3} \right|^2 \right), \quad (80)$$

where Γ_μ^{tot} is the total muon decay width. We can obtain a conservative bound by setting the $m_{H_3}, m_{E'_i} \sim v_\phi$ and the flavor-violating couplings to unity

$$\left(\frac{\text{GeV}}{v_\phi} \right) \alpha_{13} \alpha_{23} < 1.55 \times 10^{-10}, \quad (81)$$

which is satisfied for $v_\phi > 10^6$ GeV and $\alpha_{13}, \alpha_{23} < 10^{-2}$. So even with large, $\mathcal{O}(1)$, flavor-violating couplings, bound from this diagram can be avoided.

The diagrams mediated by the exotic gauge bosons are suppressed by their heavy masses. The couplings involved are also tiny, $\sim v_\eta/v_\phi$, due to them being a result of mixing between SM leptons and exotic ones. These diagrams provide an insignificant contribution to $\mu \rightarrow e\gamma$. The diagrams mediated by W boson are suppressed due to tiny ν_i masses or small mixing between heavy and light neutrinos. The only relevant diagram mediated by gauge bosons is the Z boson mediated one in Fig. 1. It is suppressed only by one flavor-violating coupling and is not mass suppressed due to the Z boson being relatively light. It provides contribution comparable to H_3 mediated diagram. The $\mu \rightarrow e\gamma$ branching ratio corresponding to Z boson mediated diagram is

$$\text{BR}(\mu \rightarrow e\gamma) = \frac{e^2 g^4 m_\mu^5 v_\chi^4 \sin^2 \theta_W \tan^2 \theta_W}{144(4\pi)^5 \Gamma_\mu^{\text{tot}} m_z^4} \left| \frac{3g_3^2}{g_x^2} \Lambda_{\mu e}^* + 3\Lambda_{e\mu} \right|^2, \quad (82)$$

where

$$\Lambda = \tilde{y}^{e\dagger} \left((M_{E'}^{\text{diag}})^{-1} \right)^2 \tilde{y}^e. \quad (83)$$

We can obtain a conservative bound by setting $\Lambda \sim 1/v_\phi^2$

$$\frac{v_\chi}{v_\phi} < 7.7 \times 10^{-3}. \quad (84)$$

Besides $\mu \rightarrow e\gamma$, our setup induces $\mu \rightarrow 3e$ and coherent $\mu - e$ conversion once the heavy states are integrated out. The electromagnetic dipole controlling $\mu \rightarrow e\gamma$ receives contributions from both the $H_3 - E'_i$ loop [Eqs. (80), (81)] and the Z penguin [Eqs. (82), (83)], while the seesaw relation [Eq. (79)] cancels the leading scalar pieces. In the dipole-dominance limit one finds the standard relation

$$\begin{aligned} \text{BR}(\mu \rightarrow 3e) &\simeq \frac{\alpha}{3\pi} \left[\ln \frac{m_\mu^2}{m_e^2} - \frac{11}{4} \right] \text{BR}(\mu \rightarrow e\gamma) \\ &\approx 6.13 \times 10^{-3} \text{BR}(\mu \rightarrow e\gamma), \end{aligned} \quad (85)$$

in agreement with Eq. (117) of Ref. [59]. This implies that current $\mu \rightarrow 3e$ bounds are weaker than those from $\mu \rightarrow e\gamma$. Coherent $\mu - e$ conversion probes the same dipole coefficients together with vector/scalar operators generated by Z, Z', Z'' penguins and neutral-scalar exchange; in the parameter region analyzed here (heavy Z', Z'' and $H_{1,2,4}$, seesaw-suppressed $e - E'$ mixing) the present limits are not more restrictive than $\mu \rightarrow e\gamma$. We stress that the dark-matter benchmarks used in Sec. VIII adopt diagonal Yukawa textures, further suppressing new LFV sources and rendering our benchmark points conservative.

In conclusion the CLFV constraints are easily avoided, even with large flavor-violating couplings. As mentioned in Sec. VIA 1, in our numerics for the dark matter phenomenology we made the simplifying assumptions of making the Yukawa couplings diagonal. In that case the CLFV effects are absent.

VIII. DARK MATTER

As previously mentioned, after the symmetry breaking of the 3-3-1-1 gauge symmetry, the P_M symmetry naturally emerges within the model. This discrete symmetry plays a crucial role in stabilizing the lightest P_M -odd particle, which, being electrically neutral, serves as a compelling DM candidate. This particle could account for the DM relic abundance measured by the Planck satellite [60], given by

$$\Omega_{\text{DM}} h^2 = 0.1200 \pm 0.0012.$$

The model predicts the existence of neutral P_M -odd particles across all sectors: scalar, fermion, and gauge. Specifically, these include the scalar S_d in Eq. (26), the gauge boson V^0 in Eq. (36), and the leptons, which are the eigenstates of the mass matrices defined in Eq. (63). However, as P_M is a $\mathbb{Z}_2$ discrete symmetry, the model does not support a multicomponent DM sector.

The V^0 gauge boson is not a suitable DM candidate because its mass, given by $m_{V^0(f)}^2 = m_{W^\pm}^2 + m_{V^\pm}^2$, allows it to decay into $W^\pm$ and $V^\mp$. Since $V^\pm$ is also a P_M -odd particle, this decay renders V^0 unstable, disqualifying it as a viable DM candidate. As for the scalar candidate, S_d , while it could theoretically provide the correct relic density, it is completely ruled out by current experimental constraints on the spin-independent direct detection cross section, which exclude all regions of its parameter space. Consequently, we will not consider it further in this work.

On the other hand, the dark fermion sector offers a promising candidate for DM. Specifically, the lightest eigenstate of the mass matrix m_f in Eq. (63), which we denote as f_d , emerges as a viable DM candidate. The

remaining dark leptons, i.e., the eigenstates of $m_{f'_1}, m_{f'_2}$, and $m_{f'_3}$, are extremely heavy, with masses proportional to $v_\varphi \sim 10^6$ GeV. Consequently, they do not contribute to the DM phenomenology and are integrated out in the effective Lagrangian.

In fact, not only are the heavy dark leptons integrated out, but all particles with masses on the order of v_σ and v_φ are as well, as they do not play a significant role in the DM analysis. This is because the focus is on DM candidates with masses in the range of 10 GeV to 1.5 TeV, a range that aligns with the sensitivity of current and upcoming accelerator experiments.

Before detailing the parameter scan, an important methodological clarification is in order. Although one could in principle integrate out heavy states sequentially and run the RGEs between the four breaking scales v_η, v_χ, v_σ and v_φ , we deliberately adopt a single one-loop potential evaluated at a single renormalization scale. The justification is twofold

- (i) The fields that gain masses at the highest scales, v_σ (associated with $U(1)_N$ breaking) and v_φ (scale-invariance breaking), lie in the $\mathcal{O}(10^2\text{--}10^3)$ TeV range and therefore decouple completely from TeV phenomenology and from the dark-matter freeze-out epoch. We integrate them out at tree level and keep their leading loop effects through the Gildener-Weinberg formalism.
- (ii) The residual hierarchy $v_\chi/v_\eta \lesssim 10^2$ generates at most

$$\frac{1}{16\pi^2} \ln(v_\chi^2/v_\eta^2) \lesssim 0.06,$$

so even $\mathcal{O}(1)$ couplings shift the light-sector quartics by only a few per cent. This shift in the light-sector quartics, by only a few per cent ($\lesssim 0.06$), is below our scan granularity. It is also modest when compared to the broader theoretical and phenomenological uncertainties inherent in dark matter calculations, which can often significantly exceed the $\sim 1\%$ numerical precision typically targeted by tools like MicrOMEGAs for standard computations [61].

Hence the single-step approach provides accuracy sufficient for the dark-matter analysis, while a full multithreshold RG study—though feasible—would add substantial technical overhead without altering our conclusions and is deferred to future work.

To analyze the phenomenology of DM, we begin by fixing certain parameters of the model. We first focus on the lepton Lagrangian, where the relevant parameters are $Y^\nu, Y^f, h^\nu, h^f, y^{\mathcal{F}}, y^e$, and $h^{\mathcal{F}}$. All these quantities are 3×3 matrices.

Furthermore, throughout the dark matter analysis, we maintain the CP -conserving setup: all Yukawa couplings and scalar potential parameters are real. This ensures that

relic density predictions are free from additional CP -phase dependencies and automatically satisfy current EDM bounds, while remaining robust against the anticipated two-orders-of-magnitude sensitivity improvements from forthcoming molecular-lattice experiments such as BaOH [62].

The matrices $h^{\mathcal{F}}$, h^{ν} , and h^f are required to be of order one to preserve the validity of the seesaw mechanism in both the charged and neutral lepton sectors. For simplicity, we assume h^{ν} and h^f matrices to be equal to the identity matrix and $h^{\mathcal{F}}$ to be 0.5 times the identity matrix. Similarly, the $y^{\mathcal{F}}$ matrix is also taken as the identity matrix. The y^e matrix, in contrast, is diagonal, with its elements specified by Eq. (54), ensuring compatibility with the SM charged leptons. The Y^{ν} matrix is derived using Eq. (60) to reproduce the active neutrino masses and mixings, consistent with neutrino oscillation data [54].

Finally, the Yukawa coupling matrix Y^f , assumed to be diagonal for simplicity, is constrained by $Y^f < \sqrt{4\pi}1$ to maintain the perturbative regime. The diagonal elements of this matrix are directly linked to the masses of the dark leptons, m_f , as specified in Eq. (63).

In a similar manner, we fixed the parameters in the quark sector, which include the matrices Y_{aj}^d , Y_{ab}^D , Y_{3j}^u , Y_{33}^U , $y_{ab}^{\mathcal{K}}$, $y_{33}^{\mathcal{K}}$, y_{aj}^u , y_{3i}^d , $h_{ab}^{\mathcal{K}}$, and $h_{33}^{\mathcal{K}}$, where $a, b = 1, 2$ and $i, j = 1, 2, 3$. Among these, Y^d and Y^u are determined to reproduce the SM quark masses. For simplicity, the remaining matrices are chosen to be equal to the identity. This assumption is consistent with the seesaw mechanism, which is also present in the quark sector. It is important to emphasize that exotic quarks, with masses on the order of v_{χ} , do not significantly influence the DM analysis.

At this point, we have fixed the parameter space in the scalar sector. Equation (16) enables all quartic couplings to be expressed in terms of the scalar masses and rotation angles, as defined in Eq. (18). Additionally, three of these angles have been reformulated in terms of the VEVs, as shown in Eq. (20). Based on the previous discussion, the free parameters used in our DM analysis are: α_{12} , α_{13} , α_{23} , m_{H_2} , m_{H_3} , v_{χ} , v_{σ} , and v_{φ} . Specifically, the ranges and relationships are defined as follows. For the mixing angles, the allowed intervals are

$$\begin{aligned} 10^{-4} &\leq |\alpha_{12}| \leq 10^{-1}, \\ 10^{-4} &\leq |\alpha_{13}| \leq 10^{-3}, \\ 10^{-4} &\leq |\alpha_{23}| \leq 10^{-2}. \end{aligned} \quad (86)$$

The angles, expressed in radians, are, in principle, selected to ensure that all quartic couplings comply with the perturbative unitarity constraints discussed in Sec. IV. However, the maximum value of $\alpha_{12} = 0.32$ is already excluded by direct detection bounds. Therefore, we adopt a more conservative upper limit of 10^{-1} .

The vacuum expectation values are constrained by

$$\begin{aligned} 10 \text{ TeV} &\leq v_{\chi} \leq 20 \text{ TeV}, \\ 10^2 \text{ TeV} &\leq v_{\sigma} \leq 1.5 \times 10^3 \text{ TeV}, \\ 10^3 \text{ TeV} &\leq v_{\varphi} \leq 2.5 \times 10^3 \text{ TeV}. \end{aligned} \quad (87)$$

Here it is important to note that although the minimum v_{χ} value allowed by Eq. (46) is approximately 3.6 TeV, we conservatively adopt a minimum value of 10 TeV, guided by the conservative Z' mass lower bounds established in recent studies [63]. While those studies focused on specific 331 models distinct from ours, their results (exceeding 4 TeV from current LHC data, and projected above 5.8 TeV at HL-LHC, 9.9 TeV at HE-LHC, and 27 TeV at FCC, considering exotic/invisible decays) provide strong support for our conservative choice.

Finally, the masses of the scalar particles are given by

$$100 \text{ GeV} \leq m_{H_2} \leq 2 \text{ TeV}, \quad 10 \text{ TeV} \leq m_{H_3} \leq 100 \text{ TeV}. \quad (88)$$

At this point, an important clarification regarding our choice of scanning variables is in order. Our analysis rigorously preserves classical scale invariance. The scalar quartic couplings λ_i , the fundamental dimensionless inputs of the theory, are *reparametrized* via Eq. (16) in terms of (i) the dimensionless mixing angles α_{ij} , (ii) the physical CP -even scalar masses m_{H_i} , and (iii) the vacuum-expectation values (vevs) v_i . This change of variables is *bijective* within the stable, perturbative region of the potential, hence it is mathematically equivalent to scanning directly over the λ_i .

Crucially, m_{H_i} and v_i are not free inputs: at each sampled point they satisfy the minimization and flat-direction conditions required by the Coleman-Weinberg mechanism. The reparametrization therefore serves only to make the scan outputs more transparent, because α_{ij} , m_{H_i} , and v_i map directly onto observables such as Higgs masses and electroweak vevs. All other dimensionless couplings—Yukawas ($h^{\mathcal{F}}, Y^f, \dots$) and gauge couplings ($g, g_X, \dots$)—are scanned explicitly as fundamental inputs. Hence the procedure remains fully consistent with scale invariance while streamlining the connection to experimental constraints.

With all parameters established, we now proceed to calculate the DM relic density, $\Omega_{\text{DM}} h^2$, resulting from the freeze-out of f_d . To achieve this, we follow the standard procedures outlined in Refs. [64,65]. These procedures involve solving the Boltzmann equation for the DM yield Y , defined as $Y \equiv n/s$ (the ratio of number density to entropy density), which is expressed as

$$\frac{dY}{dx} = -\left(\frac{45}{\pi} G\right)^{-1/2} \frac{g_*^{1/2} m_{f_d}}{x^2} \langle \sigma v \rangle_{\text{ann}} (Y^2 - Y_{\text{eq}}^2). \quad (89)$$

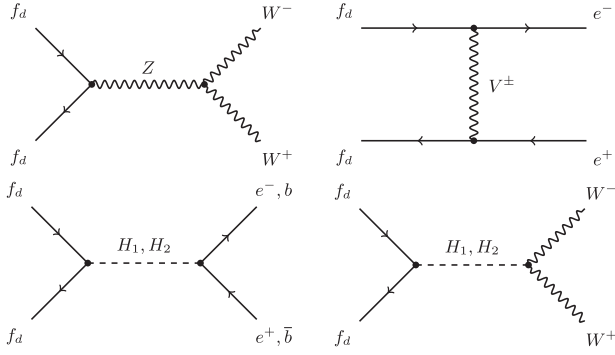


FIG. 3. Dominant annihilation channels for the DM fermion f_d into Standard Model particles include: $f_d \bar{f}_d \rightarrow Z^* \rightarrow W^+ W^-$; $f_d \bar{f}_d \rightarrow e^+ e^-$ (t-channel $V^\pm$ exchange); and $f_d \bar{f}_d \rightarrow H_{1,2} \rightarrow e^+ e^-, b \bar{b}, W^+ W^-$.

Here, $x = m_{f_d}/T$ represents the dimensionless temperature variable, G denotes the gravitational constant, and $Y_{\text{eq}} = n_{\text{eq}}/s$ is the equilibrium yield. The equilibrium number density, n_{eq} , and the effective number of relativistic degrees of freedom, g_* , are defined following the formalism outlined in Ref. [64]. Finally, $\langle \sigma v \rangle_{\text{ann}}$ corresponds to the thermally averaged annihilation cross section, a key quantity in the freeze-out calculation. Once the Boltzmann equation is solved and the DM yield Y is determined at the present time (Y_0), the relic density is computed as

$$\Omega_{\text{DM}} h^2 = 2.82 \times 10^8 \times Y_0 \times \frac{m_{f_d}}{\text{GeV}}. \quad (90)$$

Due to the complexity of the model and the extensive parameter space fixed by Eqs. (86)–(88), solving the Boltzmann equation (89) is computationally demanding. The main difficulty lies in evaluating the thermally averaged annihilation rate $\langle \sigma v \rangle_{\text{ann}}$, which, in our setup, is governed by two leading channels illustrated in Fig. 3: (i) *Vector-portal processes*: $f_d \bar{f}_d \rightarrow Z^* \rightarrow W^+ W^-$ (s-channel) and $f_d \bar{f}_d \rightarrow e^+ e^-$ (t-channel $V^\pm$ exchange); (ii) a *scalar-portal process*, $f_d \bar{f}_d \rightarrow H_{1,2} \rightarrow e^+ e^-, b \bar{b}, W^+ W^-$. These channels dominate the relic abundance by determining the value of $\langle \sigma v \rangle_{\text{ann}}$ that enters the Boltzmann analysis. To address this, we employ the packages Feynrules [66,67], CalcChep [68], and MicrOMEGAs [61], which are widely used tools in phenomenological studies of dark matter. Specifically, we used the MicrOMEGAs package to scan over the DM mass in the range $10 \text{ GeV} \leq m_{f_d} \leq 1500 \text{ GeV}$ and varied the remaining parameters within the ranges defined in Eqs. (86)–(88).

Our results are presented in Fig. 4, where we show the relic density (green region) obtained from a logarithmic random scan over the parameters defined in Eqs. (86)–(88). The horizontal band in this figure represents the measured relic density, $\Omega_{\text{DM}} h^2$, at the 3σ level, as reported by the Planck collaboration [60]. It is evident that the fermionic dark matter candidate in this model satisfies the relic

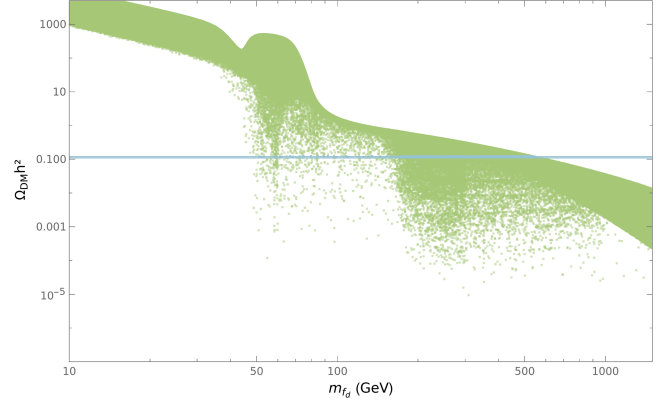


FIG. 4. Relic density $\Omega_{\text{DM}} h^2$ as a function of the fermionic dark matter mass m_{f_d} . The green points represent the results obtained from a logarithmic random scan over the model parameters. The blue horizontal band shows the measured relic density at the 3σ confidence level by the Planck collaboration [60]. The figure illustrates the regions of parameter space satisfying the relic density constraint, particularly the viable mass window $220 \text{ GeV} \lesssim m_{f_d} \lesssim 555 \text{ GeV}$ primarily driven by the H_2 resonance.

density constraint within the range $220 \text{ GeV} \lesssim m_{f_d} \lesssim 555 \text{ GeV}$. For $m_{f_d} < 100 \text{ GeV}$, the relic density exhibits typical resonant behavior, showing the Z-boson resonance around 45 GeV and the Higgs resonance around 62.5 GeV. However, while these resonance regions can yield parameter points satisfying the relic density constraint, none of them manage to simultaneously satisfy the current direct detection limits. This is primarily because, to satisfy direct detection constraints, f_d must be sufficiently decoupled from Standard Model particles. In the explored mass range for f_d , the Z' gauge boson does not play a significant role because, as mentioned, its mass is set above 4 TeV. Specifically, within the range $220 \text{ GeV} \lesssim m_{f_d} \lesssim 555 \text{ GeV}$, the annihilation cross section $\langle \sigma v \rangle_{\text{ann}}$ is significantly enhanced around the H_2 resonance mass, becoming large enough to lower the relic density to the observed value and thus allowing for viable dark matter candidates. For m_{f_d} masses larger than approximately 555 GeV, the relic density falls below the observed value, leading to underabundant dark matter. In this mass region, the dark matter phenomenology in the presented model is predominantly determined by the presence of the new scalar H_2 .

Another important set of constraints on DM candidates comes from contemporary direct detection experiments [69,70], which aim to detect WIMP DM by measuring the kinetic energy transferred to a nucleus during its interaction with a DM particle. These experiments have established upper bounds on the WIMP-nucleus scattering cross section. In general, WIMP-nucleus interactions are classified as either spin-independent (SI) or spin-dependent (SD). Currently, the most stringent limit for spin-independent scattering is set at $36 \text{ GeV}/c^2$, rejecting

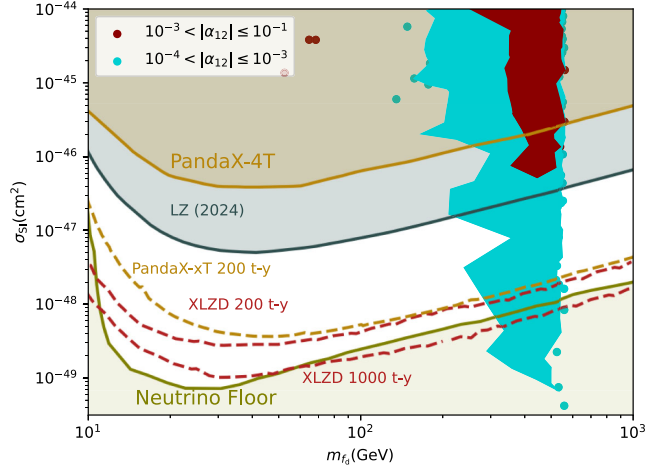


FIG. 5. Spin-independent direct detection results for the dark matter fermion f_d . The figure shows the elastic scattering cross section per nucleon, σ_{SI} , versus dark matter mass m_{f_d} . Parameter points satisfying the relic density constraint are shown, with different regions illustrating results corresponding to varying values of the α_{12} angle, as discussed in the text. Current experimental limits (PandaX-4T [69], LZ [70]), projected future sensitivities (XLZD [73], PandaX-xT [74]; dashed lines), and the neutrino floor [71] are also displayed for comparison.

cross sections above $2.1 \times 10^{-48} \text{ cm}^2$ at the 90% confidence level, as reported by the LUX-ZEPLIN (LZ) collaboration [70]. In this work, we analyse the DM candidate f_d and find that spin-independent (SI) interactions dominate. Using the MicrOMEGAs package, we compute the SI elastic scattering cross section per nucleon, σ_{SI} , for the parameter region that satisfies the relic density constraint. Figure 5 presents the results for this region, showing different areas corresponding to varying values of the α_{12} angle, as this parameter plays a critical role in direct detection within the explored parameter space. It is clear that values of α_{12} greater than 10^{-3} are excluded by current direct detection limits, and a considerable region of the parameter space with α_{12} values in the range $10^{-4} < |\alpha_{12}| \leq 10^{-3}$ is already eliminated by LZ [70]. Meanwhile, other significant region all into the neutrino floor, rendering them inaccessible to current direct detection experiments [71]. Additionally, we present the sensitivity curves for the LZ [72] and PandaX-4T experiments [69], along with the projected sensitivity curves for the near-future XLZD [73] and PandaX-xT [74] experiments, shown as dashed lines. SD interactions are not considered in this analysis, as they are, in this model, several orders of magnitude weaker than the current experimental constraints on SD scattering.

IX. CONCLUSIONS

In this work, we propose a scale-invariant model based on the gauge group $\text{SU}(3)_C \otimes \text{SU}(3)_L \otimes \text{U}(1)_X \otimes \text{U}(1)_N$, with the $\text{U}(1)_N$ symmetry chosen such that a natural P_M

discrete symmetry emerges after the dynamical symmetry breaking. As part of the 3-3-1 family, this construction has the potential to tackle several unresolved challenges of the SM, such as the origin of neutrino masses, the determination of the number of lepton and quark families, and the nature of dark matter.

One of the significant advances of the proposed framework is the reduction in the number of scalar fields in high representations of the gauge group, simplifying the scalar potential while preserving the ability to break the gauge symmetry to $\text{SU}(3)_C \otimes \text{U}(1)_Q$. This optimized scalar sector, combined with the imposition of scale invariance, enhances predictability compared to earlier 3-3-1 constructions, which often suffered from excessive scalar complexity. In Sec. III, we analyse the scalar sector in detail, following the Gildener-Weinberg approximation, and demonstrate that, in addition to the SM-like Higgs, H_1 , and the dilaton, H_4 , the model contains two CP -even scalars, $H_{2,3}$, and one complex scalar, S_d .

We found that the perturbative unitarity constraints, discussed in Sec. IV, impose critical bounds on the quartic couplings and other parameters of the model, ensuring the theoretical consistency of the framework. By analyzing two-to-two scattering amplitudes at high energy, we derived explicit conditions that all quartic couplings and eigenvalues of the scalar coupling matrix must satisfy. These constraints play a vital role in limiting the parameter space, directly influencing the scalar masses, mixing angles, and VEVs within the model. In particular, these conditions require the mixing angles α_{12} , α_{13} , and α_{23} to be small [see Eq. (86)].

Furthermore, in analogy to Ref. [18], the introduction of vectorlike quarks and leptons, together with a natural choice of the different VEVs, partially addresses the hierarchy among the charged lepton and quark masses through a universal seesaw mechanism, facilitating the generation of masses for all SM particles as well as new exotic states at energy scales compatible with scale invariance. This framework surpasses the model proposed in Ref. [18], as it also provides an explanation for the hierarchy among the charged lepton masses, as shown in Sec. VI. Charged lepton flavor-violating effects are present in the model due to mixing between SM charged leptons and the exotic leptons. For scalar mediated charged lepton flavor-violating processes the constraints are avoided due to heavy mass of exotic particles, as well as relations between the couplings due to seesaw mechanism. The gauge mediated charged lepton flavor-violating processes, on the other hand, are avoided due to small couplings.

The gauge sector in the scale-invariant 3-3-1-1 model, treated in Sec. V, features a rich structure derived from the kinetic terms of the scalar bosons and their interactions with the gauge bosons. The charged (non-Hermitian) bosons arise from the nondiagonal generators of $\text{SU}(3)_L$ and acquire masses proportional to v_η (which in this model

is equal to v_{SM}) in the case of $W^\pm$, and proportional to the 3-3-1 energy scale, v_χ . In the neutral boson sector, mixing among gauge fields produces four physical states: the massless photon and the Z , Z' , and Z'' bosons, whose masses are linked to the VEVs of the η , χ , and σ fields. Among these, the Z'' boson is particularly important as it stabilizes the scalar potential, enabling dynamical symmetry breaking and ensuring the model's theoretical consistency at high energy scales. Furthermore, experimental bounds on the ρ_0 parameter, which measures the ratio of $W^\pm$ and Z boson masses, impose a lower limit on the $\text{SU}(3)_L$ symmetry-breaking scale, $v_\chi \gtrsim 3.6$ TeV. This result underscores the model's compatibility with current data and its potential for future experimental validation. However, in our analysis, we adopted a more conservative value of 10 TeV for v_χ , motivated by the experimental limits on the Z' boson mass.

A key strength of the proposed model lies in its ability to naturally explain dark matter phenomenology. The model incorporates a dark matter candidate stabilized due to the P_M symmetry, which emerges after symmetry breaking. Among the possible P_M -odd neutral particles, the fermion f_d is identified as a viable dark matter candidate. Our analysis shows that f_d satisfies the relic density constraint within the mass range $220 \text{ GeV} \lesssim m_{f_d} \lesssim 555 \text{ GeV}$, as shown in Fig. 4. This result primarily depends on the scalar mixing parameters α_{12} and the v_χ . Spin-independent scattering cross sections for f_d are consistent with experimental limits from LZ and PandaX-4T in certain regions of the parameter space (see Fig. 5). Moreover, some parts of the parameter space lie below the neutrino floor, making them inaccessible to current detection technologies.

Other dark matter candidates, such as the vector boson V^0 and the scalar S_d , are excluded due to their instability or incompatibility with direct detection constraints. Furthermore, a substantial portion of the parameter space can be probed in upcoming experiments, such as XLZD and PandaX-xT, offering opportunities for further validation of the model's predictions.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

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