

GTMDs, orbital angular momentum, and pretzelosity

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The leading generalized transverse momentum dependent parton distributions (GTMDs) are studied in the bag model. The model description is shown to be theoretically consistent. The orbital angular momentum is studied in terms of the GTMD $F_{1,4}^q$ and Ji sum rule. Analytical proofs of the associated sum rules are given. A deeper relationship between orbital angular momentum and the pretzelosity TMD is established in this model. Numerical results for quark GTMDs are given in dependence of all five variables for all values of x and ξ .

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I. INTRODUCTION

Generalized transverse momentum dependent parton distributions (GTMDs) emerged as an overarching concept [1,2] unifying the description of hadrons in terms of parton distribution functions (PDFs), transverse momentum dependent PDFs (TMDs) [3], and generalized parton distribution functions (GPDs) [4]. GTMDs contain Wigner functions [5,6] as a limiting case, providing a key to learn about the motion and spatial distribution of partons in the transverse plane. Certain GTMDs describe completely novel properties not associated with TMDs or GPDs. One such interesting property is parton orbital angular momentum [7,8]. A pedagogical exposition of the interrelations between PDFs, TMDs, GPDs, and GTMDs can be found in [9].

The theory of the description of high energy processes in QCD in terms of PDFs, TMDs and GPDs has reached an advanced stage [10–19], though their phenomenological understanding differs from case to case, see, e.g., [20–29]. In contrast to this, the situation for GTMDs is still at a pioneering stage with the first processes sensitive to gluon or quark GTMDs proposed in Refs. [30–35]. Our current knowledge of their nonperturbative properties is very limited. This makes GTMDs compelling objects for model studies.

Models can provide valuable lessons about hadron structure, in some instances crucially deepening our knowledge, e.g., see the review on the role of models for the understanding of TMDs in QCD in Sec. 7 of Ref. [3]. In other instances, one finds model-dependent but nevertheless interesting insights. One example for this is the

integral relation of orbital angular momentum to the pretzelosity TMD [36–38] supported in a class of quark models [38–42]. Although some models [43,44] and QCD do not support it, this relation revealed a first connection of orbital angular momentum to TMDs and is one of the motivations for the present work.

In this work, we will study the leading order GTMDs of the nucleon known as $F_{1,1}^q$, $F_{1,2}^q$, $F_{1,3}^q$, $F_{1,4}^q$ in the bag model [45–47]. This is a quark model, i.e., a model with no explicit gauge degrees of freedom, a simplification allowing one to omit Wilson lines. In such models, time-reversal odd TMDs vanish [48] and GTMDs are real valued. The bag model has a long track record of often serving for the first exploratory investigations of hadronic properties including structure functions [49,50], transversity PDFs [51], GPDs [52–54], TMDs [36,39,55], double parton distributions [56,57], and orbital angular momentum [39,58,59].

After introducing GTMDs and their properties in Sec. II and the bag model in Sec. III, we will derive the expressions for $F_{1,1}^q$, $F_{1,2}^q$, $F_{1,3}^q$, $F_{1,4}^q$ in the bag model, demonstrate their theoretical consistency in Sec. IV, and present numerical results in Sec. V. Hereby, we will work in the large- N_c limit which is necessary to consistently satisfy the Ji sum rule [52,60]. We will discuss the orbital angular momentum related to $F_{1,4}^q$ in Sec. VI and the one obtained from the Ji sum rule in Sec. VII. We will prove the Ji sum rule (along with quark number and momentum sum rules) analytically in Sec. VIII. Finally, in Sec. IX we will establish a deeper relationship between orbital angular momentum and pretzelosity through the framework of GTMDs, before presenting in Sec. X the concluding remarks.

We remark that $F_{1,4}^q$ was studied previously in the bag model using the Peierls-Yoccoz projection in [59]. Studies of nucleon GTMDs in other models can be found in Refs. [61–80] where the focus was on orbital angular

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momentum, Wigner distributions, GTMDs restricted to $x > \xi$, or gluons. To the best of our knowledge, we present the first study of the quark GTMDs as functions of all five variables for any x and ξ . For an exploration of a double distribution ansatz for the modeling of GTMDs, we refer to [81].

II. DEFINITION AND PROPERTIES OF LEADING GTMDS

In quark models with no gluon degrees of freedom, the fully unintegrated quark-quark correlator is defined as [2]

$$W_{ss'}^{[\Gamma]q}(P, k, \Delta) = \frac{1}{2} \int \frac{d^4 z}{(2\pi)^4} e^{ik \cdot z} \langle p', s' | \bar{\psi}^q \left(-\frac{z}{2} \right) \Gamma \psi^q \left(\frac{z}{2} \right) | p, s \rangle \quad (1)$$

where $P = \frac{1}{2}(p' + p)$, $\Delta = p' - p$, and nucleon states are normalized covariantly as $\langle p', s' | p, s \rangle = 2p^0 \delta^{(3)}(\vec{p}' - \vec{p}) \delta_{s's}$. We define the light-cone components of a generic four-vector a^μ as $a^\mu = (a^+, \vec{a}_T, a^-)$ with $a^\pm = \frac{1}{\sqrt{2}}(a^0 \pm a^3)$. In this notation, the GTMDs related to $\Gamma = \gamma^+$ are defined as [2]

$$\begin{aligned} \mathcal{M}_{ss'}^{[\gamma^+]q} &= \iint dk^+ dk^- \delta(k^+ - xP^+) W_{ss'}^{[\gamma^+]q}(P, k, \Delta) \\ &= \frac{1}{2M} \bar{u}(p', s') \left[F_{1,1}^q + \frac{i\sigma^{i+} k_T^i}{P^+} F_{1,2}^q + \frac{i\sigma^{i+} \Delta_T^i}{P^+} F_{1,3}^q \right. \\ &\quad \left. + \frac{i\sigma^{ij} k_T^i \Delta_T^j}{M^2} F_{1,4}^q \right] u(p, s), \end{aligned} \quad (2)$$

where the nucleon spinors are normalized as $\bar{u}(p, s') u(p, s) = 2M \delta_{s's}$ and M denotes the nucleon mass. The GTMDs are in general complex valued and depend on the five variables $x, \xi, \vec{k}_T^2, \vec{k}_T \cdot \vec{\Delta}_T, \vec{\Delta}_T^2$ with ξ and t defined as [2]

$$\xi = -\frac{1}{2} \frac{\Delta^+}{P^+} = \frac{p^+ - p'^+}{p^+ + p'^+}, \quad t = \Delta^2 = -\frac{\xi^2(4M^2 + \vec{\Delta}_T^2)}{1 - \xi^2} - \vec{\Delta}_T^2. \quad (3)$$

The variable t will be needed below for GPDs.

GTMDs possess multiple properties of particular interest. Hermiticity of the correlator implies the property [2]

$$F_{1,j}^{q*}(x, \xi, \vec{k}_T^2, \vec{k}_T \cdot \vec{\Delta}_T, \vec{\Delta}_T^2) = \pm F_{1,j}^q(x, -\xi, \vec{k}_T^2, -\vec{k}_T \cdot \vec{\Delta}_T, \vec{\Delta}_T^2) \quad (4)$$

with $\begin{cases} + & \text{for } j = 1, 3, 4, \\ - & \text{for } j = 2. \end{cases}$

In the forward limit $\Delta^\mu \rightarrow 0$, the variables $\xi, \vec{k}_T \cdot \vec{\Delta}_T, \vec{\Delta}_T^2$ become zero. In this limit, the GTMD $F_{1,1}^q$ is related to the unpolarized TMD [82] as follows [2]

$$F_{1,1}^q(x, 0, \vec{k}_T^2, 0, 0) = f_1^q(x, \vec{k}_T^2). \quad (5)$$

Integrating out transverse quark momenta, the GTMDs $F_{1,1}^q, F_{1,2}^q, F_{1,3}^q$ give rise to the unpolarized GPDs as follows

$$H^q(x, \xi, t) = \int d^2 \vec{k}_T \left[F_{1,1}^q + 2\xi^2 \left(\frac{\vec{k}_T \cdot \vec{\Delta}_T}{\vec{\Delta}_T^2} F_{1,2}^q + F_{1,3}^q \right) \right] \quad (6a)$$

$$E^q(x, \xi, t) = \int d^2 \vec{k}_T \left[-F_{1,1}^q + 2(1 - \xi^2) \left(\frac{\vec{k}_T \cdot \vec{\Delta}_T}{\vec{\Delta}_T^2} F_{1,2}^q + F_{1,3}^q \right) \right]. \quad (6b)$$

The GTMD $F_{1,4}^q$ is of particular interest because, upon integration over x and transverse quark momenta with the weight $\vec{k}_T^2/M^2$, it gives access to the contribution of quark orbital momentum to the nucleon spin according to [7,8]

$$\mathcal{L}_z^q = - \int_{-1}^1 dx \int d^2 k_T \frac{\vec{k}_T^2}{M^2} F_{1,4}^q(x, 0, \vec{k}_T^2, 0, 0), \quad (7)$$

which is often referred to as the canonical or Jaffe-Manohar orbital angular momentum because, through a gauge-invariant extension, it is related to the Jaffe-Manohar nucleon spin decomposition [83].

Choosing the Dirac matrices $\Gamma = \gamma^+ \gamma_5$ or $i\sigma^{j+} \gamma_5$ yields respectively the leading GTMDs $G_{1,j}$ with $1 \leq j \leq 4$ or $H_{1,j}$ with $1 \leq j \leq 8$. Choosing Dirac matrices other than that yields subleading GTMDs [2]. In this work, we will focus on the GTMDs related to $\Gamma = \gamma^+$ and feature one of the GTMDs related to $\Gamma = i\sigma^{j+} \gamma_5$. A study of all GTMDs in the bag model will be presented elsewhere [84].

III. THE BAG MODEL

In the bag model, $N_c = 3$ noninteracting quarks are confined in a color singlet state within a spherical cavity (“bag”) of radius R by means of appropriate boundary conditions [45–47]. The single-quark ground state wave function for a massless quark in momentum space is given by

$$\Psi_s(t, \vec{k}) = e^{-i\varepsilon_0 t} A \begin{pmatrix} t_0(k) \chi_s \\ t_1(k) \vec{k} \cdot \vec{\sigma} \chi_s \end{pmatrix}, \quad (8)$$

where $\varepsilon_0 = \omega_0/R$ is the ground-state energy, $\omega_0 \approx 2.04$ is the lowest positive solution of the transcendental equation $\omega = (1 - \omega) \tan \omega$, and σ^i are the Pauli matrices with χ_s denoting the two-component Pauli spinors. The functions t_ℓ for $\ell = 0, 1$ are defined in terms of spherical Bessel functions j_ℓ as follows

$$t_\ell(k) = \frac{\int_0^1 du u^2 j_\ell(ukR) j_\ell(u\omega_0)}{k^\ell}. \quad (9)$$

Notice that we define $t_1(k)$ with a factor of $1/k$ which differs from what is often used in literature such that in our work $t_0(k)$ is dimensionless while $t_1(k)$ has dimension $1/\text{mass}$. This definition will simplify the notation later on. The wave function is normalized as

$$\int d^3k \Psi_{s'}^\dagger(\vec{k}) \Psi_s(\vec{k}) = A^2 \int d^3k (t_0^2 + \vec{k}^2 t_1^2) \delta_{ss'} = \delta_{ss'} \quad (10)$$

with the normalization constant $A = \frac{1}{2\pi} [(\omega_0 R^3)/((\omega_0 - 1)j_0^2(\omega_0))]^{1/2}$.

The bag radius is fixed in terms of the nucleon mass M via the relation $M = \frac{4}{3}N_c \epsilon_0$. The calculations are conveniently carried out for “flavorless” quarks with the flavor dependence assigned based on the SU(4) spin-flavor symmetry by means of the spin-flavor factors [85], with N_q for spin-independent and P_q for spin-dependent nucleon matrix elements, given for the proton by (with u and d being interchanged for the case of the neutron)

$$\begin{aligned} N_u &= \frac{N_c + 1}{2}, & N_d &= \frac{N_c - 1}{2}, & P_u &= \frac{N_c + 5}{6}, \\ P_d &= \frac{-N_c + 1}{6}. \end{aligned} \quad (11)$$

For more detailed introductions to the model, we refer to Refs. [45–47].

IV. CALCULATION OF GTMDs IN THE MODEL

In the bag model, it is convenient to use the Breit frame where $P^\mu = (P^0, 0, 0, 0)$ and $\Delta^\mu = (0, \vec{\Delta})$ and $t = -\vec{\Delta}^2$. In order to ensure the validity of the Ji sum rule in the bag model, we will work in the large- N_c limit [52,60] where $M = \mathcal{O}(N_c)$ and $P^0 = M + \mathcal{O}(N_c^{-1})$ and we will neglect $\mathcal{O}(N_c^{-1})$ corrections. The behavior of the variables is

$$\begin{aligned} k_T^i &= \mathcal{O}(N_c^0), & \Delta_T^i &= \mathcal{O}(N_c^0), & x &= \mathcal{O}(N_c^{-1}), \\ \xi &= -\Delta^3/(2M) = \mathcal{O}(N_c^{-1}), & t &= \mathcal{O}(N_c^0). \end{aligned} \quad (12)$$

The expressions in Eq. (2) can be understood as 2×2 matrices in nucleon helicity indices. In order to evaluate these expressions, it is convenient to take traces with the unit and Pauli matrices. Evaluating in this way, in the large N_c limit, the nucleon spinor expressions on the right-hand side of Eq. (2) for “flavorless quarks” yields the results

$$\frac{1}{2} \text{tr}_{\text{spin}}[\mathcal{M}^{[\gamma^+]}] = F_{1,1}, \quad (13a)$$

$$\frac{1}{2} \text{tr}_{\text{spin}}[\mathcal{M}^{[\gamma^+]} \sigma_x] = \frac{ik_y}{M} F_{1,2} + \frac{i\Delta_y}{M} F_{1,3}, \quad (13b)$$

$$\frac{1}{2} \text{tr}_{\text{spin}}[\mathcal{M}^{[\gamma^+]} \sigma_y] = -\frac{ik_x}{M} F_{1,2} - \frac{i\Delta_x}{M} F_{1,3}, \quad (13c)$$

$$\frac{1}{2} \text{tr}_{\text{spin}}[\mathcal{M}^{[\gamma^+]} \sigma_z] = \frac{i(k_x \Delta_y - k_y \Delta_x)}{M^2} F_{1,4}, \quad (13d)$$

where we write for more clarity $\vec{k} = (k_x, k_y, k_z)$ and similarly for $\vec{\Delta}$ and neglect terms suppressed in large- N_c limit. Taking into account the quark flavor factors (11), the large- N_c scaling of GTMDs obtained from this expansion reads

$$\begin{aligned} F_{1,1}^q &= N^q \mathcal{O}(N_c), & F_{1,2}^q &= P^q \mathcal{O}(N_c^2), & F_{1,3}^q &= P^q \mathcal{O}(N_c^2), \\ F_{1,4}^q &= P^q \mathcal{O}(N_c^2). \end{aligned} \quad (14)$$

Note that different flavor combinations have different N_c -orders. In unpolarized case, the isoscalar flavor combination $N^u + N^d = N_c$ is leading while the isovector flavor combination $N^u - N^d = 1$ is subleading. In polarized case it is vice versa with $P^u + P^d = 1$ and $P^u - P^d = \frac{N_c + 2}{3}$. Our choice to apply the large- N_c expansion in Eq. (13) before projecting on definite flavors simplifies the model evaluation and corresponds to the scheme used in [60].¹

The correlator on the left-hand side of Eq. (2) is given in the bag model (for flavorless quarks) in the large- N_c limit by the expression

$$\mathcal{M}_{ss'}^{[\gamma^+]} = M \bar{\phi}_{s'} \left(\vec{k} + \frac{1}{2} \vec{\Delta} \right) (\gamma^0 + \gamma^3) \phi_s \left(\vec{k} - \frac{1}{2} \vec{\Delta} \right) \Big|_{k_z = xM - \epsilon_0}, \quad (15)$$

where we indicate that $k_z = xM - \epsilon_0$ is fixed. Evaluating this expression in the model yields the results

$$\begin{aligned} &\frac{1}{2} \text{tr}_{\text{spin}}[\mathcal{M}^{[\gamma^+]}] \\ &= MA^2 \left[t_0^- t_0^+ + \left(k_z - \frac{1}{2} \Delta_z \right) t_1^- t_0^+ + \left(k_z + \frac{1}{2} \Delta_z \right) t_0^- t_1^+ \right. \\ &\quad \left. + \left(\vec{k}^2 - \frac{1}{4} \vec{\Delta}^2 \right) t_1^- t_1^+ \right]_{k_z = xM - \epsilon_0}, \end{aligned} \quad (16a)$$

$$\begin{aligned} &\frac{1}{2} \text{tr}_{\text{spin}}[\mathcal{M}^{[\gamma^+]} \sigma_x] \\ &= MA^2 \left[ik_y (t_0^- t_1^+ - t_1^- t_0^+ - \Delta_z t_1^- t_1^+) \right. \\ &\quad \left. + i\Delta_y \left(\frac{1}{2} t_0^- t_1^+ + \frac{1}{2} t_1^- t_0^+ + k_z t_1^- t_1^+ \right) \right]_{k_z = xM - \epsilon_0}, \end{aligned} \quad (16b)$$

¹In an alternative large- N_c scheme, one could include the SU(4) spin-flavor in Eq. (13) before expanding the spinor expressions. Then one would proceed with separate model calculations for the $(u \pm d)$ cases. In the strict large- N_c limit, both schemes yield the same results.

$$\begin{aligned} & \frac{1}{2} \text{tr}_{\text{spin}}[\mathcal{M}^{[\gamma^+]} \sigma_y] \\ &= MA^2 \left[ik_x(t_1^- t_0^+ - t_0^- t_1^+ + \Delta_z t_1^- t_1^+) \right. \\ & \quad \left. - i\Delta_x \left(\frac{1}{2} t_0^- t_1^+ + \frac{1}{2} t_1^- t_0^+ + k_z t_1^- t_1^+ \right) \right]_{k_z=xM-\epsilon_0}, \quad (16c) \end{aligned}$$

$$\frac{1}{2} \text{tr}_{\text{spin}}[\mathcal{M}^{[\gamma^+]} \sigma_z] = MA^2 [-i(k_x \Delta_y - k_y \Delta_x) t_1^+ t_1^-]_{k_z=xM-\epsilon_0} \quad (16d)$$

where we introduced the notation

$$t_\ell^\pm = t_\ell \left(\left| \vec{k} \pm \frac{1}{2} \vec{\Delta} \right| \right). \quad (17)$$

Comparing the coefficients in Eqs. (13), (16), we obtain the following results for the GTMDs

$$\begin{aligned} F_{1,1}^q &= N^q A^2 M \left[t_0^- t_0^+ + k_z(t_0^- t_1^+ + t_1^- t_0^+) + \frac{1}{2} \Delta_z(t_0^- t_1^+ - t_1^- t_0^+) \right. \\ & \quad \left. + \left(\vec{k}^2 - \frac{1}{4} \vec{\Delta}^2 \right) t_1^- t_1^+ \right]_{k_z=xM-\epsilon_0}, \quad (18a) \end{aligned}$$

$$F_{1,2}^q = P^q A^2 M^2 [t_0^- t_1^+ - t_1^- t_0^+ - \Delta_z t_1^- t_1^+]_{k_z=xM-\epsilon_0}, \quad (18b)$$

$$F_{1,3}^q = P^q A^2 M^2 \left[\frac{1}{2} t_0^- t_1^+ + \frac{1}{2} t_1^- t_0^+ + k_z t_1^- t_1^+ \right]_{k_z=xM-\epsilon_0}, \quad (18c)$$

$$F_{1,4}^q = P^q A^2 M^3 [-t_1^+ t_1^-]_{k_z=xM-\epsilon_0}. \quad (18d)$$

In order to demonstrate the theoretical consistency of the model expressions, we first note that

$$\left| \vec{k} \pm \frac{1}{2} \vec{\Delta} \right| = \sqrt{\vec{k}_T^2 + \frac{1}{4} \vec{\Delta}_T^2 \pm \vec{\Delta}_T \cdot \vec{k}_T + \left(k_z \pm \frac{1}{2} \Delta_z \right)^2}, \quad (19)$$

which shows that the model GTMDs are explicit functions of the variables $F_{1,j} = F_{1,j}(x, \xi, \vec{k}_T^2, \vec{k}_T \cdot \vec{\Delta}_T, \vec{\Delta}_T^2)$. We recall that $t_0^\pm$ is dimensionless while $t_1^\pm$ has dimension 1/mass. Therefore, all GTMDs have the correct dimension 1/mass² and one obtains (in the specific cases) dimensionless GPDs upon integration over transverse momenta. Next we note that replacing $\xi \rightarrow (-\xi)$ and $\vec{\Delta}_T \rightarrow (-\vec{\Delta}_T)$ has the effect of $t_\ell^\pm \rightarrow t_\ell^\mp$. This observation immediately shows us that the model results in Eqs. (18a)–(18d) comply with the hermiticity property according to Eq. (4).

Setting $\xi \rightarrow 0$ and $\vec{\Delta}_T \rightarrow 0$ in the model expression for $F_{1,1}^q$ in Eq. (18a) implies $t_\ell^\pm \rightarrow t_\ell$ and we obtain

$$\begin{aligned} F_{1,1}^q(x, 0, \vec{k}_T^2, 0, 0) &= N^q A^2 M [t_0^2 + 2k_z t_0 t_1 \\ & \quad + \vec{k}^2 t_1^2]_{k_z=xM-\epsilon_0} = f_1^q(x, \vec{k}_T^2), \quad (20) \end{aligned}$$

where in the last step, we recovered the expression for the unpolarized TMD $f_1^q(x, \vec{k}_T^2)$ from Ref. [39]. In order to check the relation to GPDs, we first note that in the large- N_c limit the Eqs. (6a), (6b) simplify as

$$H^q(x, \xi, t) = \int d^2 \vec{k}_T F_{1,1}^q, \quad (21a)$$

$$E^q(x, \xi, t) = \int d^2 \vec{k}_T \left[-F_{1,1}^q + 2 \frac{\vec{k}_T \cdot \vec{\Delta}_T}{\vec{\Delta}_T^2} F_{1,2}^q + 2 F_{1,3}^q \right]. \quad (21b)$$

Inserting in Eq. (21) the expressions for the $F_{1,j}$ from Eq. (18), we obtain results for the GPDs which are in agreement with Ref. [52] if one considers the large- N_c limit, see also [86].

V. MODEL RESULTS FOR GTMDS

GTMDs are functions of five variables. In order to gain some intuition how these functions look like, we present plots in Fig. 1. The four columns show the four GTMDs $F_{1,1}^u, F_{1,2}^u, F_{1,3}^u, F_{1,4}^u$. The five rows show the dependencies on the five variables $x, \xi, k_T = |\vec{k}_T|, \vec{k}_T \cdot \vec{\Delta}_T, \Delta_T^2 = \vec{\Delta}_T^2$. Hereby we keep $x = 0.3, \xi = 0.2, k_T^2 = 0.1 \text{ GeV}^2, \vec{k}_T \cdot \vec{\Delta}_T = 0, \Delta_T^2 = 0.1 \text{ GeV}^2$ fixed, except for the one respectively varied variable. These values are chosen to be in the region of the applicability of the model, for instance valence- x or $\Delta_T^2 \ll M^2$ needed for the validity of the large- N_c approach. We will comment more on these choices in the following paragraphs. The d -quark GTMDs are not shown as the flavor dependence can be inferred from the SU(4) spin flavor factors in Eq. (11).

The bag model results refer to a low renormalization scale μ_0 [87]. In general, there is no unambiguous way to determine the model scale. In one method, one takes the fraction of nucleon momentum carried by valence quarks known at high scales from phenomenological PDF parametrizations and evolves it backward to a scale μ_0 where valence quarks carry 100% of the nucleon momentum assuming perturbative evolution is applicable down to μ_0 [88]. The result depends on the used parametrization, $\alpha_s(\mu)$ value adapted for the calculation, perturbative order, and scheme when working beyond leading order evolution. One obtains $300 \text{ MeV} \lesssim \mu_0 \lesssim 600 \text{ MeV}$ depending on technical details [87–90]. TMDs and GTMDs depend in QCD on an additional scale associated with rapidity divergences [3] which is also low and not unambiguously defined in models [91–93]. In short, the estimate of the model scale(s) is part of the modeling. While the exact value is

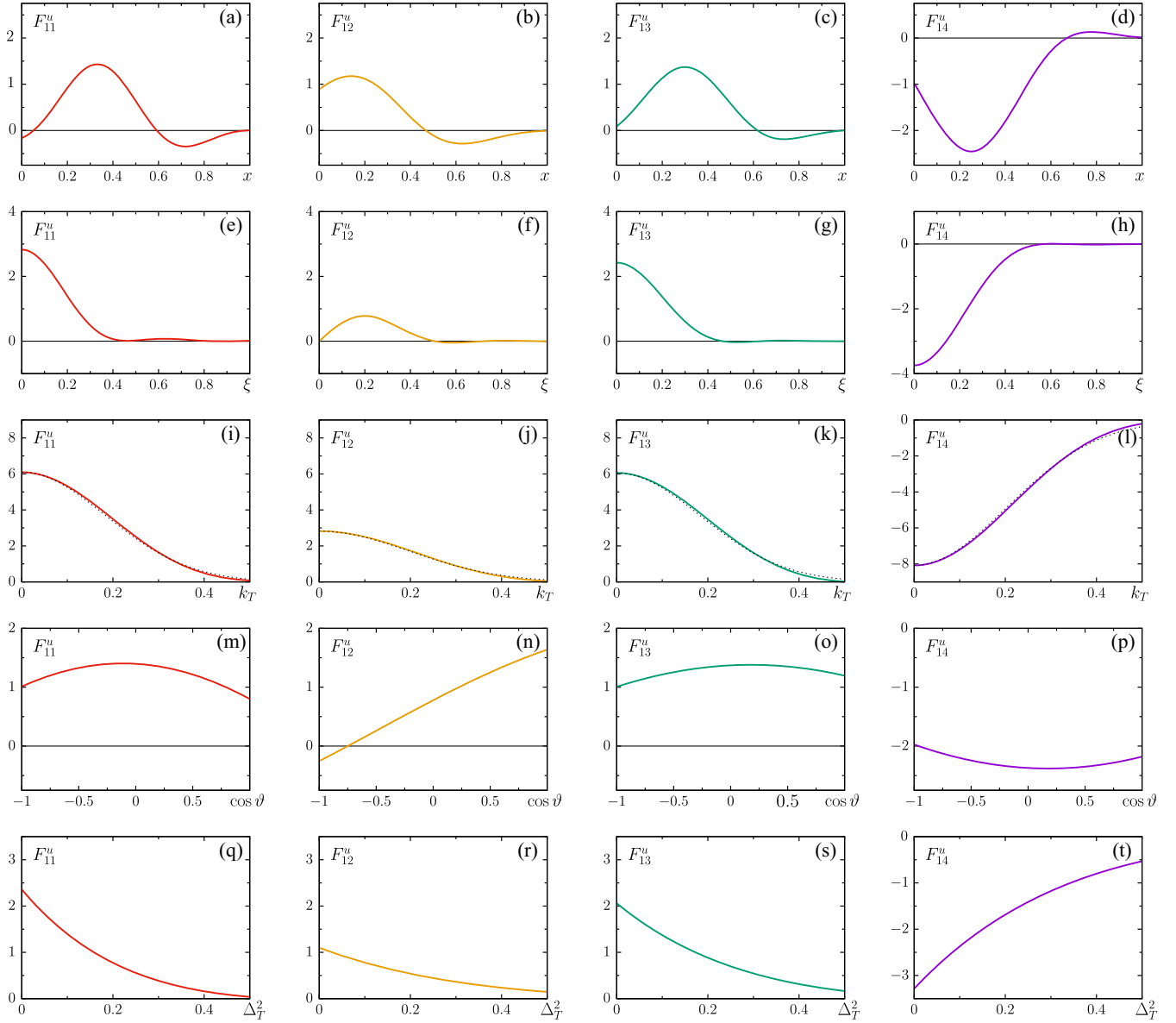


FIG. 1. u -quark GTMDs $F_{1,1}^u$, $F_{1,2}^u$, $F_{1,3}^u$, $F_{1,4}^u$ in the bag model at the low hadronic scale. One variable is, respectively, varied, while four out of the other five variables are fixed as $x = 0.3$, $\xi = 0.2$, $k_T^2 = 0.1 \text{ GeV}^2$, $\vec{k}_T \cdot \vec{\Delta}_T = 0$, $\Delta_T^2 = 0.1 \text{ GeV}^2$. (a)–(d) show the dependence on x ; (e)–(h) on ξ ; (i)–(l) on k_T ; (m)–(p) on $\cos \vartheta$ defined through $\vec{k}_T \cdot \vec{\Delta}_T = \cos \vartheta k_T \Delta_T$; and (q)–(t) the dependence on Δ_T^2 . k_T and Δ_T are given in units of GeV.

not essential, in the following it is important to keep in mind that our results refer to a low scale $\mu_0 \sim \mathcal{O}(500 \text{ MeV})$ well below 1 GeV.

In Figs. 1(a)–1(d), we show the x dependence. In the valence x region, $F_{1,1}^u$, $F_{1,2}^u$, $F_{1,3}^u$ are positive and exhibit global maxima between $0.1 < x < 0.4$, while $F_{1,4}^u$ is negative and exhibits a global minimum. The GTMDs change sign after the valence- x region between $0.4 < x < 0.7$ with the exact position of the node depending on the GTMD and the values of the other variables. For reference, we set $\xi = 0.2$ in Fig. 1. For smaller (larger) ξ -values, the nodes shift toward larger (smaller) x values. The

description of the small- x and large- x limits is beyond the range of applicability of the bag model. Although they become very small, the GTMDs do not go to zero beyond $x > 1$. This was the case also for PDFs, TMDs, and GPDs and happens because the “initial and final nucleons are not good momentum eigenstates” [52].

In Figs. 1(e)–1(h), the ξ dependence is shown. The GTMDs $F_{1,1}^u$, $F_{1,3}^u$, $F_{1,4}^u$ are largest for $\xi = 0$ while $F_{1,2}^u$ is exactly zero due to Eq. (4) (recall that we set $\vec{k}_T \cdot \vec{\Delta}_T = 0$ and the model GTMDs are real). It is interesting that the ξ -dependence of GTMDs is pronounced in contrast to the ξ -dependence of GPDs which was found to be very weak in

this model [52–54]. $F_{1,1}^u$ and $F_{1,3}^u$ are positive and monotonically decreasing in the region $0 < \xi \lesssim 0.4$, while $F_{1,4}^u$ is positive and monotonically increasing. $F_{1,2}^q$ increases from zero at $\xi = 0$ until reaching a global maximum around $\xi \approx 0.2$ before it also starts decreasing. It is not shown in the plots, but $F_{1,1}^u, F_{1,3}^u, F_{1,4}^u$ are even functions of ξ while $F_{1,2}^u$ is odd due to Eq. (4) and the choice $\vec{k}_T \cdot \vec{\Delta}_T = 0$. Beyond $\xi > 0.4$, all $F_{1,j}^u$ are very small.

In Figs. 1(i)–1(l), we display the dependence on k_T . In view of the low model scale $\mu_0 \sim \mathcal{O}(500 \text{ MeV})$, we focus on k_T -values below this scale. In this region, the k_T -behavior of the GTMDs is approximately Gaussian. To show this, we have plotted in Figs. 1(i)–1(l) in each case two curves: the exact model result and a Gaussian approximation proportional to $\exp(-k_T^2/\langle k_T^2 \rangle)$ with the Gaussian widths $\langle k_T^2 \rangle = (260 \text{ MeV})^2, (280 \text{ MeV})^2, (260 \text{ MeV})^2, (285 \text{ MeV})^2$, for, respectively, $F_{1,1}^u, F_{1,2}^u, F_{1,3}^u, F_{1,4}^u$. We stress that the k_T behavior is not exactly Gaussian, and it really is an approximation that breaks down for larger $k_T \gtrsim 400 \text{ MeV}$. However, until then the Gaussian approximation works reasonably well. Very similar observations were made for TMDs in the bag model [39]. In some other quark models, e.g., the light-front constituent quark model [91,92], the Gaussian approximation works similarly well. The Gaussian ansatz is approximately supported by TMD phenomenology [94,95].

The Figs. 1(m)–1(p) showcase the dependence on the variable $\vec{k}_T \cdot \vec{\Delta}_T$. Since $\vec{k}_T \cdot \vec{\Delta}_T = \cos \vartheta k_T \Delta_T$ and the values of k_T and Δ_T are fixed, this dependence can be conveniently expressed through the variable $\cos \vartheta$ where ϑ denotes the angle between the transverse vectors $\vec{k}_T$ and $\vec{\Delta}_T$. (Since $\cos \vartheta$ has support in the interval $[-1, 1]$ we have chosen the middle point of this integral, i.e., $\vec{k}_T \cdot \vec{\Delta}_T = 0$, for display of the dependencies in the other figures.) The dependence of $F_{1,1}^u, F_{1,3}^u, F_{1,4}^u$ on $\cos \vartheta$ is mild for the choice of the other variables in Fig. 1, but becomes much more pronounced when one chooses, e.g., a larger ξ value. Had we chosen $\xi = 0$, the $F_{1,1}^u, F_{1,3}^u, F_{1,4}^u$ would be even functions of the variable $\cos \vartheta$, while $F_{1,2}^u$ would be odd due to Eq. (4). Thus, $\xi \neq 0$ in Figs. 1(m)–1(p) distorts this symmetry.

The Figs. 1(q)–1(t) show the Δ_T^2 -dependence which resembles the $(-t)$ dependence of form factors in the bag model. In fact it is $(-t) = (2\xi M)^2 + \Delta_T^2$ in our large- N_c approach where it is important to keep the condition $\Delta_T^2 \ll M^2$ in mind. In this range, one in principle could approximate the Δ_T^2 -behavior of GTMDs in terms of multipole shapes proportional to $1/(1 + \Delta_T^2/\Lambda_n^2)^n$ with $n = 2$ or 3 as it is often done for form factors. But there is no factorization from the other variables and the multipole masses Λ_n would in general depend on the other variables.

Overall, we make the interesting observation that the GTMDs $F_{1,1}^u$ and $F_{1,3}^u$, while clearly different and linearly

independent, have similar shapes and magnitudes over a wide range of their variables. In contrast to this, $F_{1,4}^u$ exhibits an opposite sign, about twice the magnitude, and a more noticeably different dependence on the variables as compared to $F_{1,1}^u$ and $F_{1,3}^u$, albeit it exhibits similar generic features. These three GTMDs have in common that they are even under $\xi \rightarrow -\xi$ and $\vec{k}_T \cdot \vec{\Delta}_T \rightarrow -\vec{k}_T \cdot \vec{\Delta}_T$ according to Eq. (4). In contrast to this, $F_{1,2}^u$ is odd under this symmetry and exhibits in many respects distinctly different dependencies on the variables.

The d -quark distributions follow from the SU(4) spin-flavor factors in Eq. (11). In particular, $F_{1,1}^d$ has the same sign as $F_{1,1}^u$ and half its size. $F_{1,2}^q, F_{1,3}^q, F_{1,4}^q$ follow a different flavor pattern: the d -quark distributions have opposite signs to the u -quark distributions and are four times smaller. This needs to be kept in mind when comparing $F_{1,1}^q$ to other GTMDs. For instance, we roughly observe in the proton $F_{1,1}^u \approx F_{1,3}^u$ for u -flavor, but $F_{1,1}^d \approx -2F_{1,3}^d$ for d -flavor. More details on the GTMD properties will be discussed in [84].

VI. ORBITAL ANGULAR MOMENTUM FROM GTMD $F_{1,4}^q$

It is customary to define an x -dependent orbital angular momentum distribution function $\mathcal{L}_z^q(x)$ in terms of $F_{1,4}^q$ with $\xi \rightarrow 0$ and $\Delta_T^i \rightarrow 0$ as follows

$$\mathcal{L}_z^q(x) = - \int d^2 k_T \frac{\vec{k}_T^2}{M^2} F_{1,4}^q(x, 0, \vec{k}_T^2, 0, 0), \quad (22)$$

such that upon integration over x , one obtains $\mathcal{L}_z^q = \int dx \mathcal{L}_z^q(x)$ in Eq. (7). When evaluating x integrals in the bag model, it is convenient to proceed as follows [$R(\vec{k})$ below denotes a generic function of $\vec{k}$ and $\vec{\Delta}$]

$$\begin{aligned} M \int_{-1}^1 dx \int d^2 k_T [R(\vec{k})]_{k_z = xM - \epsilon_0} \\ = M \int_{-1}^1 dx \int d^3 k R(\vec{k}) \delta(k_z + \epsilon_0 - xM) \\ = \int d^3 k R(\vec{k}). \end{aligned} \quad (23)$$

The final result is obtained by substituting $x \rightarrow y = xM$ and using the large- N_c limit to replace the integration interval $-M \leq y \leq M$ by the whole y axis. In this way, using the bag model expression for $F_{1,4}^q(z, 0, \vec{k}_T^2, 0, 0)$ from Eq. (18d) and simplifying $\int d^3 k k_i^2 t_1^2 = \frac{1}{3} \int d^3 k \vec{k}^2 t_1^2$ for $i = \{x, y, z\}$, we obtain

$$\mathcal{L}_z^q = P^q A^2 \int d^3 k \left[\frac{2}{3} \vec{k}^2 t_1^2 \right]. \quad (24)$$

This is the contribution to the nucleon spin originating from the canonical quark orbital angular momentum.

To demonstrate the theoretical consistency of the model, we need to take into account the quark spin contribution S^q to the nucleon spin which is defined in terms of the helicity PDF as

$$S^q = \frac{1}{2} \int dx g_1^q(x), \quad g_1^q(x) = \int d^2 k_T g_1^q(x, k_T) \quad (25)$$

where we expressed the helicity PDF in terms of the helicity TMD $g_1^q(x, k_T)$, which is possible in simple quark models though not in QCD [3]. The latter is given in the bag model by [36,39]

$$g_1^q(x, k_T) = P^q A^2 M [t_0^2 + 2k_z t_0 t_1 - (\vec{k}_T^2 - k_z^2) t_1^2]_{k_z = xM - \epsilon_0}. \quad (26)$$

Evaluating S_z^q in the bag model according to Eq. (23) where the term proportional to k_z in Eq. (26) drops out due to symmetry reasons, we obtain

$$S_z^q = \frac{1}{2} \int dx \int d^2 k_T g_1^q(x, k_T) = \frac{1}{2} P^q A^2 \int d^3 k \left[t_0^2 - \frac{1}{3} \vec{k}^2 t_1^2 \right]. \quad (27)$$

Adding up the results in Eqs. (24), (27) and making use of the normalization in Eq. (10) yields for the total angular momentum contribution J_z^q to the nucleon spin the result

$$S_z^q + \mathcal{L}_z^q = \frac{1}{2} P^q A^2 \int d^3 k [t_0^2 + \vec{k}^2 t_1^2] = \frac{1}{2} P^q. \quad (28)$$

If we sum over u - and d quarks with the SU(4) spin-flavor factors in Eq. (11), we obtain $\sum_q (S_z^q + \mathcal{L}_z^q) = \frac{1}{2}$ as expected, because in the bag model the nucleon spin is entirely due to the spin and orbital angular momentum of quarks.

TABLE I. Contributions of quark orbital angular momentum and quark spin (in %) to the nucleon spin in quark models at the low scales: bag model [39]; light-front constituent quark model (LFCQM) [7,62]; unquenched constituent quark model (UCQM) [96]; Nambu–Jona-Lasinio model (NJL) [97]; chiral quark-soliton model (χ QSM) [98].

Model	Bag model	LFCQM	UCQM	NJL	χ QSM
Scale μ_0	500 MeV	510 MeV	400 MeV	400 MeV	600 MeV
$\sum_q 2\mathcal{L}_z^q$	35.0%	25.2%	32.4%	28.4%	65.0%
$\sum_q 2S_z^q$	65.0%	74.8%	67.6%	71.6%	35.0%

Our result for $2\mathcal{L}_z^u(x)$ is shown in Fig. 2(a) and that for $g_1^u(x)$ in Fig. 2(b). These results refer to a low hadronic scale estimated to be below 1 GeV, see Sec. V. The results for d flavor have opposite signs and are four times smaller according to the spin-flavor factors in Eq. (11). $F_{1,4}^q$ was studied previously in bag model without using large- N_c limit in [59]. Our results agree well with [59] indicating that $1/N_c$ -corrections are moderate in the case of $F_{1,4}^q$. In the bag model, the quark spin contributes $\sum_q 2S^q = 0.65$, i.e., 65% to the nucleon spin, while the angular momentum contribution is $\sum_q 2\mathcal{L}_z^q = 0.35$, i.e., 35% [39]. These are typical results in quark models at low scales, see Table I.

In Table I, we quote results from several quark models at low initial scales between 400–600 MeV. The results from the bag model [39] (see also [52,55] and this work), light-front constituent quark model [7,62], unquenched constituent quark model [96], and Nambu–Jona-Lasinio model [97] are in agreement with each other in that around 25%–35% of the nucleon spin is due to quark orbital angular momentum. A common feature of these models is that they are based on valence quarks degrees of freedom. Compared to these models, the chiral quark-soliton model [98] in Table I is an outlier with 65% of the nucleon spin being due to quark orbital angular momentum. The chiral quark soliton model is also the only model in Table I which incorporates not solely valence, but also sea quarks which may explain the different trend. For discussions on how to

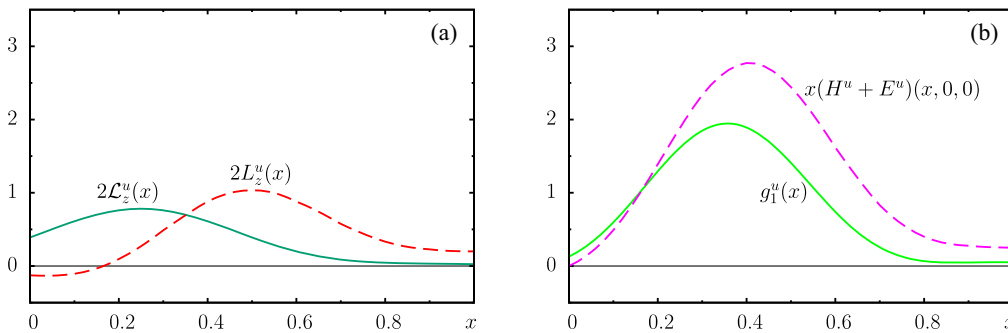


FIG. 2. (a) Orbital angular momentum distributions in the bag model: $2\mathcal{L}_z^u(x)$ obtained from GTMD $F_{1,4}^q$ via Eq. (22) and $2L_z^u(x)$ obtained from Ji decomposition via Eq. (30). (b) For comparison: helicity PDF $g_1^u(x)$ and twice the total angular momentum distribution $(H^u + E^u)(x, 0, 0)$. The results refer to a low scale below 1 GeV².

reconcile the valence-quark picture of the nucleon spin decomposition with phenomenological observations, we refer to [99–101]. For completeness, we note that μ_0 of the unquenched constituent quark model was not specified in Ref. [96], and the value in Table I is estimated based on comparison to similar quark models [102].

Before ending this section, we note that in QCD the x integrals extend over $-1 \leq x \leq 1$, with the understanding that negative x describe antiquarks, e.g., $g_1^q(-x) = g_1^{\bar{q}}(x)$ in Eq. (25). The antiquark distributions in the bag model are unphysical but must be included when evaluating sum rules [52]. Note also that the bag model gives rise to small but nonzero contributions outside the region $|x| < 1$, which must also be included when evaluating sum rules [52].

VII. ORBITAL ANGULAR MOMENTUM FROM JI SUM RULE

A different (historically earlier) way to define quark orbital angular momentum is based on the Ji sum rule which states that the total angular momentum contributions J_z^q of quarks can be accessed via GPDs as [12]

$$J^q = \frac{1}{2} \lim_{t \rightarrow 0} \int dx x (H^q(x, \xi, t) + E^q(x, \xi, t)). \quad (29)$$

where $f_1^q(x) = \int d^2 k_T f_1^q(x, k_T)$ with $f_1^q(x, k_T)$ given by Eq. (20). The derivatives $t'_\ell = dt_\ell(k)/dk$ in Eq. (32b) come from applying L'Hôpital's rule to the term $\vec{k} \cdot \vec{\Delta}_T F_{1,2}/\vec{\Delta}_T^2$ in Eq. (21b) in the limit $\vec{\Delta}_T \rightarrow 0$ [notice that in the model, $F_{1,2}^q$ is real and therefore an odd function of Δ_T^i due to the property (4), which guarantees that the limit exists].

From the expressions in Eqs. (26), (30), (32), we obtain the result for $2L_z^u(x)$ shown in Fig. 2(a). For comparison, we display the forward limit $x(H^u + E^u)(x, 0, 0)$ in Fig. 2(b). The results for d quarks are again of opposite signs and four times smaller according to Eq. (11). In Fig. 2(a) we see that $L_z^q(x) \neq \mathcal{L}_z^q(x)$ in the bag model as in other models.

The Ji sum rule is satisfied in the bag model, which was checked numerically in Refs. [52,59,60]. Given the importance of this sum rule, we present an analytical proof in the next section which will show an important difference to the spin sum rule in Eq. (28).

VIII. PROOF OF JI SUM RULE

There is an important difference between the spin sum rule in Eq. (28) and the Ji sum rule. In order to illustrate this, we present here explicit analytical proofs of the quark

By generalizing the integrand in Eq. (29) to an x -dependent distribution of total angular momentum and subtracting the spin contribution described by one half of the helicity PDF $g_1^q(x)$, it was proposed to define the orbital angular momentum distribution $L_z^q(x)$ in the Ji decomposition as [103], see also the reviews [104,105],

$$L_z^q(x) = \frac{1}{2} x (H^q + E^q)(x, 0, 0) - \frac{1}{2} g_1^q(x). \quad (30)$$

In quark models (but not in QCD), both orbital angular momentum distributions have the same normalization [106],

$$\int dx L_z^q(x) = \int dx \mathcal{L}_z^q(x) = \mathcal{L}_z^q, \quad (31)$$

but their x dependence is in general different, $L_z^q(x) \neq \mathcal{L}_z^q(x)$.

To investigate this matter, we need the model expression for $H^q(x, \xi, t) + E^q(x, \xi, t)$ in the forward limit $\xi \rightarrow 0$ and $t \rightarrow 0$. Taking this limit in Eq. (21) yields

$$H^q(x, 0, 0) = f_1^q(x), \quad (32a)$$

$$E^q(x, 0, 0) = -f_1^q(x) + P^q A^2 M^2 \int d^2 k_T \left[\frac{\vec{k}_T^2}{k} (t_0 t'_1 - t'_0 t_1) + 2(t_0 t_1 + k_z t_1^2) \right]_{k_z = xM - \epsilon_0}, \quad (32b)$$

number, momentum, and Ji sum rules which are given by

$$\int dx f_1^q(x) = N^q, \quad (33)$$

$$\int dx \sum_q x f_1^q(x) = 1, \quad (34)$$

$$\int dx x (H^q + E^q)(x, 0, 0) = 2J^q. \quad (35)$$

To evaluate the quark number sum rule (33), we proceed as in Eq. (23) whereby an odd term (under $k_z \rightarrow -k_z$) drops out, and we obtain $\int dx x f_1^q(x) = N^q A^2 \int d^3 k (t_0^2 + \vec{k}^2 t_1^2)$, which proves the sum rule (33) owing to the wave function normalization in Eq. (10). The proof of the spin sum rule in Eq. (28) is conceptually similar in that it can be traced back to the normalization of the quark wave function. In other words, in the bag model the proofs of these two sum rules rely on the normalization of the quark wave function in Eq. (10) and SU(4) spin-flavor factors. This alone is not sufficient to prove the sum rules (34) and (35).

In order to evaluate the momentum sum rule, we use the expression for the unpolarized TMD in Eq. (20) and carry out steps which are analogous to those in Sec. VI, which yields

$$\int dx x f_1^q(x) = \frac{N^q A^2}{M} \int d^3 k \left[\epsilon_0 (t_0^2 + \vec{k}^2 t_1^2) + \frac{2}{3} k^2 t_0 t_1 \right]. \quad (36)$$

In order to proceed, we explore the equation of motion which is given in coordinate space by $(i\cancel{D} - m_q)\Psi_s(t, \vec{r}) = 0$ for $|\vec{r}| < R$ where $\Psi_s(t, \vec{r}) = \int d^3 k e^{-i(\epsilon_0 t - \vec{k} \cdot \vec{r})} \tilde{\phi}(\vec{k})$ is the Fourier transform of the flavorless wave function in Eq. (8) (we use the same notation for the function and its Fourier transform) and m_q is the quark mass which we will eventually set to zero. With the understanding that $\Psi_s(t, \vec{r}) = 0$ for $|\vec{r}| > R$, the following trivial identity

$$\int d^3 r F(t, \vec{r}) \Gamma (i\cancel{D} - m_q) \Psi_s(t, \vec{r}) = 0 \quad (37)$$

holds for any function $F(t, \vec{r})$ and any Dirac matrix Γ . Choosing $F(t, \vec{r}) = \tilde{\Psi}_{s'}(t, \vec{r}) = \int d^3 k' e^{i(\epsilon_0 t - \vec{k}' \cdot \vec{r})} \tilde{\phi}(\vec{k}')$ and $\Gamma = \mathbb{1}$ yields (so far we have kept $m_q \neq 0$ for didactic purposes to indicate the Dirac equation, but now we set it to zero)

$$\begin{aligned} & \int \frac{d^3 r}{(2\pi)^3} \tilde{\Psi}_{s'}(t, \vec{r}) (i\cancel{D} - m_q) \Psi_s(t, \vec{r}) \\ &= \int d^3 k \tilde{\phi}_{s'}(\vec{k}) (\epsilon_0 \gamma^0 - \vec{k} \cdot \vec{\gamma}) \phi_s(\vec{k}) \\ &= A^2 \int d^3 k [\epsilon_0 (t_0^2 + \vec{k}^2 t_1^2) - 2k^2 t_0 t_1] \delta_{ss'} = 0, \end{aligned}$$

which implies the identity $\int d^3 k [2k^2 t_0 t_1] = \epsilon_0 \int d^3 k (t_0^2 + \vec{k}^2 t_1^2)$. Making use of this identity in Eq. (36) yields

$$\int dx x f_1^q(x) = N^q \frac{4\epsilon_0}{3M} A^2 \int d^3 k [t_0^2 + \vec{k}^2 t_1^2] = \frac{N^q}{N_c} \quad (38)$$

where in the last step we used Eq. (10) and $M = \frac{4}{3} N_c \epsilon_0$. Summing over flavors yields the momentum sum rule (34).

Turning our attention to the Ji sum rule, we first note that $H^q(x, 0, 0)$ and $E^q(x, 0, 0)$ both contain a contribution from $f_1^q(x)$ but with opposite signs, see Eqs. (32a), (32b). Thus, the only piece in these GPDs that could have been literally related to the momentum sum rule cancels out exactly in the Ji sum rule which appears to be a manifestly genuine off-forward effect in the sense that nothing known from the forward case—not even in a simple quark model—can help one to prove this sum rule. However, we shall see that upon invoking an additional ingredient, namely the explicit consideration of off-forward kinematics, a proof of the Ji

sum rule can be formulated analogously to the momentum sum rule. Note that both sum rules are ultimately rooted in the conservation of the energy momentum tensor, but in complimentary ways.

Starting with the model expressions for $H^q(x, 0, 0)$ and $E^q(x, 0, 0)$ in Eq. (32) and using symmetry arguments yields

$$\begin{aligned} & \int dx x (H^q(x, 0, 0) + E^q(x, 0, 0)) \\ &= P^q A^2 \int d^3 k \left[\frac{2}{3} \epsilon_0 k (t_0 t_1' - t_0' t_1) \right. \\ & \quad \left. + 2\epsilon_0 t_0 t_1 + \frac{2}{3} \vec{k}^2 t_1^2 \right]. \end{aligned} \quad (39)$$

Next we use Eq. (37) with $F(t, \vec{r}) = \tilde{\Psi}_{s'}(t, \vec{r}) e^{-i\vec{\Delta} \cdot \vec{r}}$ and $\Gamma = \gamma^3$. After evaluating the model expression and tracing with the unit matrix in nucleon spin indices (and setting $m_q = 0$) we obtain the identity

$$\begin{aligned} K(\vec{\Delta}) &= A^2 \int d^3 k [2k_z (-t_0^- t_0^+ + \epsilon_0 t_0^+ t_1^- - \epsilon_0 t_0^- t_1^+ + t_1^- t_1^+) \\ & \quad + \Delta_z (t_0^- t_0^+ - \epsilon_0 t_0^+ t_1^- - \epsilon_0 t_0^- t_1^+ + t_1^- t_1^+)] = 0. \end{aligned} \quad (40)$$

Taking the derivative of the above expression with respect to Δ_z and setting $\vec{\Delta}$ to zero yields the identity

$$\begin{aligned} \left. \frac{\partial K(\vec{\Delta})}{\partial \Delta_z} \right|_{\vec{\Delta}=0} &= A^2 \int d^3 k \left[t_0^2 - 2\epsilon_0 t_0 t_1 + \frac{1}{3} k^2 t_1^2 \right. \\ & \quad \left. + \frac{2}{3} \epsilon_0 k t_0' t_1 - \frac{2}{3} \epsilon_0 k t_0 t_1' \right] = 0. \end{aligned} \quad (41)$$

Multiplying Eq. (41) by P^q and adding this “zero” to Eq. (39), we obtain

$$\begin{aligned} & \int dx x (H^q(x, 0, 0) + E^q(x, 0, 0)) \\ &= P^q A^2 \int d^3 k [t_0^2 + \vec{k}^2 t_1^2] = P^q, \end{aligned} \quad (42)$$

where we made use of Eq. (10) in the final step. This proves the Ji sum rule in the bag model.

Note that in the proof of the momentum sum rule (34), we had to make use of the bag model expression for the nucleon mass $M = \frac{4}{3} N_c \epsilon_0$. It is instructive to recall how this result is obtained. Namely, it follows from minimizing the nucleon mass as a function of R , $M(R) = N_c \epsilon_0 + \frac{4}{3} \pi R^3 B$ where $\epsilon_0 = \omega_0/R$ and B is the bag constant. The result $M = \frac{4}{3} N_c \epsilon_0$ follows from setting $M'(R) = 0$ and eliminating B . The condition $M'(R) = 0$ is the virial theorem in the bag model and is equivalent to the von Laue condition, which is ultimately connected to the conservation of the

energy-momentum tensor in the bag model [60] and in general theory [107].

This illustrates the important difference between the spin (28) and quark number (33) sum rules on the one hand and the momentum (34) and Ji (35) sum rules on the other hand. The former merely require the normalization condition for the quark wave functions (10), i.e., are based on the mere construction of quark flavor and spin quantum numbers within the SU(4) spin-flavor symmetry underlying this model. The latter require, in addition to that, the use of the bag model equation of motion for the single quarks, $(i\cancel{D} - m_q)\Psi_s(t, \vec{r}) = 0$, i.e., explicitly invokes the dynamics of the model. For the momentum sum rule, one further ingredient is needed, namely the determination of the nucleon mass, $M'(R) = 0$, where all N_c quarks collectively contribute along with the bag constant.

Finally, we discuss the sum rule for the anomalous gravitomagnetic moment related to the energy-momentum tensor form factor $B(t)$ [12]. Based on the results for the momentum sum rule (38) and Ji sum rule (42), we obtain

$$B^q(0) = \int dx x E^q(x, 0, 0) = P^q - \frac{N^q}{N_c}. \quad (43)$$

Numerically, we obtain the results $B^u(0) = \frac{2}{3}$ and $B^d(0) = -\frac{2}{3}$ for the proton and three colors. Summing over all flavors in the bag model, we find that the total anomalous gravitomagnetic moment of the nucleon vanishes,

$$B(0) = \sum_q B^q(0) = 0, \quad (44)$$

which is expected and equivalent to the Ji sum rule [12].

At this point, it is instructive to comment on the large- N_c limit, which we used when deriving the results in (13). Thereafter, we included in Eq. (18) the spin-flavor factors (11), left N_c general, and set it equal to its physical value when presenting results in Fig. 2, which constitutes our “large- N_c expansion scheme” as described below Eq. (14). One could be concerned if this is a consistent scheme. In fact, other schemes could be applied as well, see footnote 1. A good test of the theoretical consistency of a model calculation is compliance with sum rules, and we have explicitly demonstrated that our results correctly satisfy them. Moreover, inserting the spin-flavor factors (11) in Eq. (43) yields the $1/N_c$ expansion of the quark contributions to the anomalous gravitomagnetic moment of the proton

$$B^u(0) = -B^d(0) = \frac{1}{6}N_c + \frac{1}{3}N_c^0 - \frac{1}{2}N_c^{-1}. \quad (45)$$

This shows that one may use our scheme or truncate the expansion after leading (or subleading) order. This will affect the numerical values for $B^q(0)$, but the total

anomalous gravitomagnetic moment of the proton is always zero, and the Ji sum rule is always correctly satisfied.

For completeness, we remark that orbital angular momentum can also be accessed through the twist-3 GPD $G_2^q(x, \xi, t)$ [108,109]. The connection of this GPD to orbital angular momentum in the bag model was studied in [59]. Another expression for the orbital angular momentum involving two twist-3 GPDs, $G_2^q(x, \xi, t)$ and $G_4^q(x, \xi, t)$, was proposed in Ref. [110].

IX. ORBITAL ANGULAR MOMENTUM AND THE PRETZELOCITY GTMD

The pretzelocity TMD $h_{1T}^{\perp q}(x, k_T^2)$ is related in certain quark models to orbital angular momentum. It is instructive to briefly review the model studies which lead to this insight. In the light-cone representation and taking into account effects of Melosh-Wigner rotations, it was observed in Ref. [37] that the difference between the transversity PDF $h_1^q(x)$ and helicity PDF $g_1^q(x)$ is related to quark orbital angular momentum. In Ref. [36] it was shown that, in the bag model, the difference of helicity and transversity TMDs is given in terms of the transverse moment of pretzelocity TMD, $g_1^q(x, k_T^2) - h_1^q(x, k_T^2) = h_{1T}^{\perp(1)q}(x, k_T^2)$ where $h_{1T}^{\perp(1)q}(x, k_T^2) = \frac{k_T^2}{2M^2} h_{1T}^{\perp q}(x, k_T^2)$. This in turn paved the way to the insight that pretzelocity is related to quark orbital angular momentum via the integral relation [38]

$$\mathcal{L}_z^q = - \int dx \int d^2\vec{k}_T \frac{\vec{k}_T^2}{2M^2} h_{1T}^{\perp q}(x, \vec{k}_T^2). \quad (46)$$

This result is supported in the light-cone SU(6) quark-spectator model [38], bag model [39], covariant parton model [40,41], light-front constituent quark model [7], and chiral quark-soliton model restricted to the three-quark sector [7]. Models like the spectator model of Ref. [43] or the light-front models of Refs. [44] do not support Eq. (46). Spherical symmetry of quark wave functions was shown to be an important prerequisite for Eq. (46) to hold in a quark model [111]. In models where this symmetry is not present and in QCD, Eq. (46) is not supported [3].

Although not general, it is nevertheless interesting to ask the question why the Eq. (46) holds in certain models. The observation that some model symmetry must be present [111] sheds light on a necessary technical requirement. But it would be interesting to gain further insights. Moreover, it is fair to ask whether the connection of $h_{1T}^{\perp q}$ to $\mathcal{L}_z^q$ could be accidental: in simple models, in one way or another, all quantities are expressed in terms of a limited number of structures, $t_0(k)$ and $t_1(k)$ in Eqs. (8), (9) in our case, and integrals of different functions could coincide by accident. To investigate this point, we inspect the x dependence.

In the bag model, the expression for the pretzelosity TMD (rewritten in the notation of this work) is given by [39]

$$h_{1T}^{\perp q}(x, \vec{k}_T^2) = P^q A^2 M^3 [-2t_1^+]. \quad (47)$$

Comparing to the model expression of the GTMD $F_{1,4}^q(x, \xi, \vec{k}_T^2, \vec{k}_T \cdot \vec{\Delta}_T, \vec{\Delta}_T^2)$ in Eq. (18d) in the limit $\xi \rightarrow 0$ and $\vec{\Delta}_T \rightarrow 0$, we find in the bag model the following relation

$$2F_{1,4}^q(x, 0, \vec{k}_T^2, 0, 0) = h_{1T}^{\perp q}(x, \vec{k}_T^2). \quad (48)$$

This is an interesting observation. It shows that the relation is deeper than what one might conclude from the mere observation that the same model expression $\mathcal{L}_z^q$ results from integrating $2F_{1,4}^q(x, 0, \vec{k}_T^2, 0, 0)$ and $h_{1T}^{\perp q}(x, \vec{k}_T^2)$ over dx and d^2k_T in Eqs. (7), (46). In fact, the relation holds at the level of the integrands in Eqs. (7), (46) before integrations.

At this point, one may wonder whether the relation of pretzelosity and orbital angular momentum in the bag model could be still deeper. In fact, the pretzelosity TMD can be obtained as the forward limit of one of the GTMDs related to the Dirac structure $\Gamma = i\sigma^{j+}\gamma_5$. This Dirac structure defines 8 leading GTMDs $H_{1,l}^q$ with $l = 1, \dots, 8$. The GTMD containing the pretzelosity TMD as a forward limit is $H_{1,4}^q$, and the relation is given by

$$h_{1T}^{\perp q}(x, \vec{k}_T^2) = H_{1,4}^q(x, 0, \vec{k}_T^2, 0, 0). \quad (49)$$

In order to investigate if there is deeper relation, it is instructive to derive the bag model expression for $H_{1,4}^q$ by repeating the steps in Eqs. (13)–(18) for $\Gamma = i\sigma^{j+}\gamma_5$. Details of this calculation (and results for the other $H_{1,l}^q$, $l \neq 4$) will be reported elsewhere [84]. Here we shall content ourselves with quoting the result for $H_{1,4}^q$ which is given by

$$H_{1,4}^q = P^q A^2 M^3 [-2t_1^+ t_1^-]. \quad (50)$$

In the forward limit, we reproduce Eq. (49) from (50). A comparison of Eqs. (18d) and (50) yields the relation

$$H_{1,4}^q(x, \xi, \vec{k}_T^2, \vec{k}_T \cdot \vec{\Delta}_T, \vec{\Delta}_T^2) = 2F_{1,4}^q(x, \xi, \vec{k}_T^2, \vec{k}_T \cdot \vec{\Delta}_T, \vec{\Delta}_T^2). \quad (51)$$

This is the desired result revealing a deeper connection: in the bag model, pretzelosity appears to be connected to orbital angular momentum because the GTMDs $F_{1,4}^q$ and $H_{1,4}^q$, as functions of all five variables, are related to each other according to Eq. (51). To the best of our knowledge, this relation has not been observed in a model before.

It will be interesting to explore along the lines of Ref. [111] whether some symmetry of quark wave functions might be a necessary (sufficient) condition for the existence of the relation (51) in a model, and whether there are potentially other models which support such a relation.

X. CONCLUSIONS

We presented a study of the leading GTMDs $F_{1,1}^q, F_{1,2}^q, F_{1,3}^q, F_{1,4}^q$ associated with $\Gamma = \gamma^+$ in the bag model in the large- N_c limit. The results satisfy Hermiticity properties and correctly reproduce known relations to TMDs and GPDs with $F_{1,1}^q$ relating to the TMD $f_1^q(x, \vec{k}_T^2)$ and $F_{1,1}^q, F_{1,2}^q, F_{1,3}^q$ relating to the GPDs $H^q(x, \xi, t)$ and $E^q(x, \xi, t)$. This demonstrates that the bag model can be successfully extended to the description of GTMDs and, within its limitations, describe them in a theoretically consistent way. This includes the polynomiality property, which was not explicitly discussed in this work, but verified in [52,54].

Only $F_{1,4}^q$ was studied previously in the bag model [59]. Our results for $F_{1,4}^q$ are in agreement with those of Ref. [59]. We note that in Ref. [59], Peierls-Yoccoz projection techniques were used while our approach is based on the large- N_c limit, and it is reassuring to observe that the bag model makes robust predictions for $F_{1,4}^q$ independent of such technical details. $F_{1,1}^q, F_{1,2}^q, F_{1,3}^q$ have not been studied in the bag model before and constitute novel results. Moreover, to the best of our knowledge, we presented the first study of the quark GTMDs as functions of all five variables providing insights on the shape of GTMDs for any x and ξ . The model results refer to a low scale around $\mathcal{O}(500 \text{ MeV})$.

We applied the obtained results to study the orbital angular momentum distributions as defined via $F_{1,4}^q$ and the Ji sum rule. As observed in prior model studies, the x -dependencies of these two partonic distributions differ, but the integrated contributions $\mathcal{L}_z^q$ coincide, which is expected in quark models. The numerical results for the u - and d -quark contribution to $\mathcal{L}_z^q$ agree with what is typically found in models based on valence quark degrees of freedom at low hadronic scales. We furthermore obtained several interesting new insights.

One interesting new insight is based on our analytical proofs for the total quark angular momentum sum rule for the J^q contribution to the nucleon spin (i) as the sum of quark orbital angular momentum (obtained via $F_{1,4}^q$) and quark spin (obtained via the helicity PDF g_1^q) which is often referred to as the Jaffe-Manohar sum rule, and (ii) using the Ji sum rule. Both spin sum rules were previously tested numerically in the bag model within an accuracy of better than 1%, the former in Ref. [59] and the latter in Refs. [52,59,60].

Analytical proofs show something numerical tests cannot reveal. The Jaffe-Manohar sum rule is satisfied in the

bag model based on a correct normalization of the quark wave functions analogously to the quark number sum rule. The Ji sum rule is satisfied if and only if, in addition to that, the dynamics of the model is correctly solved. From this point of view, the proof of the Ji sum rule resembles that of the momentum sum rule. It would be interesting to see whether similar distinctions between Jaffe-Manohar and Ji sum rules can also be made in other models.²

Another interesting insight concerns the connection between orbital angular momentum and the pretzelosity TMD. It has been observed in several quark models that the x -integral of the (1)-transverse moment of the pretzelosity TMD is related to $\mathcal{L}_z^q$. In this work, we have shown that the connection goes beyond this integral relation and holds on the level of the x - and k_T -integrands. In fact, in the bag model, the pretzelosity TMD $h_{1T}^{+q}(x, \vec{k}_T^2)$ coincides with $2F_{1,4}^q(x, 0, \vec{k}_T^2, 0, 0)$ in the forward limit. Finally, we went one step further and demonstrated that, in the bag model, the relation $H_{1,4}^q = 2F_{1,4}^q$ holds even when all the GTMD variables x , ξ , $\vec{k}_T^2$, $\vec{k}_T \cdot \vec{k}_T^2$ and $\vec{\Delta}_T^2$ are, respectively, kept nonzero and are not integrated over. Here, $H_{1,4}^q$ is the GTMD which contains the pretzelosity TMD as a limiting case in the forward limit. To the best of our knowledge, this extension of the relation between pretzelosity and orbital angular momentum in terms of GTMDs has not been encountered in model studies before.

It is important to keep in mind that models have limitations. Such relations can be found unsupported in quark models exhibiting a lesser degree of symmetry or can be spoiled by relaxing certain model assumptions. Models

with gauge field degrees of freedom cannot be expected to support such relations, and in QCD all GTMDs are independent functions and describe complementary aspects of nucleon structure. However, quark models have been found to catch some gross features of TMDs and GPDs observed in phenomenology or lattice QCD studies. It would therefore be interesting to see whether this might be true also in the case of the GTMDs $F_{1,4}^q$ and $H_{1,4}^q$ —at least in some region of the variables, for instance, valencelike x , small $\vec{k}_T^2 < M^2$, or moderate $\vec{\Delta}_T^2 < M^2$.

It will therefore be interesting to see if our findings can be confirmed in other quark model studies and perhaps even in lattice QCD—exactly or as approximate relations. The ultimate testing ground will be provided by phenomenological studies. The path to rigorous factorization proofs and phenomenological tests of GTMDs is long. Until then, such model relations can be helpful guidelines to prepare the first estimates for GTMD-related observables. We can look forward to learning more about nonperturbative properties of GTMDs from models.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

²Recently, we learned that the situation is similar (but not quite the same) in the scalar diquark model of [112].

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