

New supersymmetry index for the D1-D5 conformal field theories

Marcel R. R. Hughes¹ and Masaki Shigemori^{1,2}

¹*Department of Physics, Nagoya University, Furo-cho, Chikusa-ku, Nagoya 464-8602, Japan*

²*Center for Gravitational Physics, Yukawa Institute for Theoretical Physics, Kyoto University, Kitashirakawa Oiwakecho, Sakyo-ku, Kyoto 606-8502, Japan*



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We propose a new supersymmetry index for the D1-D5 conformal field theory (CFT) for T^4 , relevant to the $\text{AdS}_3/\text{CFT}_2$ correspondence. In a novel formulation of symmetric orbifold CFTs based on the Schur-Weyl duality, we show how this index can be naturally described and its protection is argued based on the detailed nature of exactly marginal operators in these theories. This index is a one-parameter generalization of the standard index and gives more fine-grained information about the structure of microstates than previously available. We demonstrate precise matching of the new index between supergravity and CFT below the black-hole threshold, where the standard index—the modified elliptic genus—is trivial. Above the threshold, we uncover a decomposition of black-hole microstates into distinct sectors, invisible to the modified elliptic genus.

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I. INTRODUCTION

Symmetric orbifolds of two-dimensional conformal field theories (CFTs) [1,2] provide an invaluable landscape of theories useful for understanding holography; these universally exhibit many of the properties expected of holographic theories at large N , including the factorization of correlators [3–5], a Hawking-Page phase transition [6,7] and the form of thermal two-point functions [8]. While sufficient conditions for a CFT to be holographic remain elusive, large classes of symmetric orbifold CFTs were shown to be consistent with a holographic description somewhere in their moduli spaces [9–11].

A paradigmatic example of symmetric orbifold CFTs in holography is the D1-D5 CFT arising in the $\text{AdS}_3/\text{CFT}_2$ correspondence [12], which has been central to the development of the holographic dictionary. Below the black-hole threshold energy, agreement between the CFT and the bulk excitation spectra was established [13–16], and above the threshold, the CFT correctly reproduces the Bekenstein-Hawking entropy of the AdS black hole [17]. Supersymmetry indices [18–20] are key to such matching, being independent of the coupling relating the symmetric orbifold and gravity regimes.

Recent advances are beginning to reveal the finer structure of the matched spectra: fortuity [21,22] refines

the notion of typicality for black-hole microstates, while CFT technologies [23–30] allow one to determine whether given microstates remain supersymmetric when interactions are turned on. Existing supersymmetry indices are of limited use here: defined purely from the supersymmetry algebra, they are largely insensitive to the details of interactions.

In this paper, we introduce a new Schur-Weyl-based formulation for symmetric orbifold CFTs which decomposes the Bogomol’nyi-Prasad-Sommerfield (BPS) Hilbert space into sectors of definite symmetry with respect to the action of the symmetric group. For the D1-D5 CFT on T^4 , we demonstrate that this formulation leads to a new formula for the standard supersymmetry index of this theory—the modified elliptic genus (MEG)—which is more transparent with regards to the contributing states. In this framework, we are then motivated to propose a new supersymmetry index—the *resolved elliptic genus* (REG)—that incorporates selection rules based on the nature of the interaction in symmetric orbifold CFTs. For the D1-D5 CFT on T^4 , we demonstrate that the REG provides much more information about the structure of microstates than the MEG

II. SYMMETRIC ORBIFOLD CFTS

An M^N/S_N symmetric orbifold CFT with central charge $c = Nc_0$ can be obtained by starting with the “seed” CFT M with central charge c_0 , taking its N -fold tensor product, and viewing each copy (or “strand”) as a separate world sheet carrying the fields of M . Orbifolding by S_N then allows permutations of these copies as we go around

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the strand, effectively gluing some of the strands together into a longer strand. Requirement of S_N invariance leads [1] to the Hilbert space decomposing into “twist sectors” $\mathcal{H}(M^N/S_N) \cong \bigoplus_{[g]} \mathcal{H}_{[g]}$ labeled by conjugacy classes $[g]$. For S_N , these conjugacy classes are labeled by partitions $\{n_k\}$ of N , satisfying $\sum_{k=1}^N kn_k = N$, where n_k is the number of strands of length k . $\mathcal{H}_{[g]}$ has the interpretation [31] as a multistring Hilbert space, where a strand of length k represents a string wrapping k times around the S^1 on which the CFT lives. It is then natural to introduce a “covering” Hilbert space, or Fock space, $\mathcal{H}$, in which we include all values of N by summing over $n_k \in \mathbb{Z}_{\geq 0}$. The original Hilbert space at fixed N is obtained by projection onto the subspace with total strand number N .

Besides operators coming from the seed theory, symmetric orbifold CFTs contain twist operators σ_k , labeled by cyclic permutations of order k , which generate twist-sector ground states. In particular, σ_2 , which splits a strand into two shorter strands or glues together two strands into a longer one, is used to construct a marginal deformation operator and plays an essential role in linking symmetric orbifold CFTs to a holographic setup.

In the context of holography, the most studied example of symmetric orbifold CFTs is the D1-D5 CFT [32–34] where the seed theory is an $\mathcal{N} = (4, 4)$ supersymmetric σ model on $M = T^4$ or $K3$. Here we will focus on the T^4 theory, for which the bosonic symmetries include the left- and right-moving R-symmetries $SU(2)_L \times SU(2)_R$ and an $SU(2)_1 \times SU(2)_2$ symmetry broken [35] by the compactness of T^4 . Per strand, the field content consists of four free bosons and four free left- and right-moving fermions. The Ramond-Ramond (RR) ground states on a strand of length k are tensor products of the left-moving ground states $|\alpha\rangle_k, |\dot{\alpha}\rangle_k$ and the right-moving ground states $|\check{\alpha}\rangle_k, |\dot{\check{\alpha}}\rangle_k$, where $\alpha, \dot{\alpha}, \check{\alpha}$ are doublet indices for $SU(2)_L, SU(2)_R, SU(2)_2$, respectively. The states $|\alpha\rangle_k, |\dot{\alpha}\rangle_k$ are bosonic while the $|\check{\alpha}\rangle_k$ are fermionic in our conventions. Each of these ground states can be excited by oscillator modes of the free bosons and fermions. A state in the full theory is then obtained by tensoring states for all strands together and symmetrizing with respect to the strand label. In this paper, we focus on 1/4-BPS states whose right-moving sector is in an R ground state and the left-moving sector can be in an arbitrary state.

While the above is the standard description of the Ramond ground states of this theory and the representations in which they transform, for our purposes it will prove important to distinguish the action of $SU(2)_2$ on the left-moving and right-moving parts of states separately. We define the group $\widetilde{SU}(2)_2$ as the right-moving action of $SU(2)_2$, under which the right-moving $|*\rangle\dot{\check{\alpha}}$ Ramond ground states transform as a doublet and the analogous left-moving ground states as singlets.

III. A SCHUR-WEYL FORMULATION

An alternative construction [36] of the covering Hilbert space $\mathcal{H}$, and one that will prove useful to construction of the new, more refined, index with which this paper is concerned, first factors the theory into left- and right-moving sectors [41]. Let V_k be the left-moving Hilbert space on a length- k strand, for which the covering space is $V = \bigoplus_{k=1}^{\infty} V_k$. Containing infinitely many bosonic and fermionic states, V is naturally a fundamental representation space of $GL(\infty|\infty)$. For the right-moving sectors, we take a k -independent single-strand Hilbert space of R ground states, $\tilde{V}$: right-moving BPS Hilbert spaces for different strand lengths are assumed to be all isomorphic. If $\tilde{V}$ contains b bosonic and f fermionic states, it is naturally a fundamental representation space of $GL(b|f)$.

The n -strand Hilbert spaces for the left- and right-movers, by the Schur-Weyl duality, decompose as

$$V^{\otimes n} \cong \bigoplus_{\lambda \vdash n} V_{\lambda} \otimes M_{\lambda}, \quad \tilde{V}^{\otimes n} \cong \bigoplus_{\tilde{\lambda} \vdash n} \tilde{V}_{\tilde{\lambda}} \otimes M_{\tilde{\lambda}}, \quad (1)$$

where the sums are over Young diagrams $\lambda, \tilde{\lambda}$, each with n boxes (as denoted by $\lambda, \tilde{\lambda} \vdash n$). In Eq. (1), $V_{\lambda} \subset V^{\otimes n}$ and $\tilde{V}_{\tilde{\lambda}} \subset \tilde{V}^{\otimes n}$ are irreducible $GL(\infty|\infty)$ - and $GL(b|f)$ -modules with representations labeled by λ and $\tilde{\lambda}$, obtained by the action of the so-called Schur functor. More physically, if V is the space of vectors T^I where $I = 1, \dots, \dim V$, then V_{λ} is the subspace of tensors $T^{I_1 \dots I_n}$ with symmetry dictated by λ [43] (and similarly for $\tilde{V}_{\tilde{\lambda}}$). M_{λ} and $M_{\tilde{\lambda}}$ are irreducible S_n -representations called Specht modules, labeled by λ and $\tilde{\lambda}$, respectively.

Let us assume that the physical n -strand Hilbert space $\mathcal{H}_n$ is the S_n -invariant subspace of $V^{\otimes n} \otimes \tilde{V}^{\otimes n}$ (we will comment on this assumption later). Since the S_n product representation $M_{\lambda} \otimes M_{\tilde{\lambda}}$ contains the trivial representation if and only if $\lambda = \tilde{\lambda}$, we find [44]

$$\mathcal{H}_n = \bigoplus_{\lambda \vdash n} V_{\lambda} \otimes \tilde{V}_{\lambda}, \quad (2)$$

where we dropped trivial Specht modules. Then the covering Hilbert space, summed over all strand numbers, is

$$\mathcal{H} = \bigoplus_{\lambda} V_{\lambda} \otimes \tilde{V}_{\lambda}, \quad (3)$$

where the sum is over λ with arbitrary numbers of boxes. The Hilbert space for fixed N is obtained by projection onto the subspace with total strand number N .

Let the (super)character of V —or physically the left-moving single-strand (signed) partition function—be

$$z(x|x') = \sum_i x_i - \sum_{i'} x'_{i'}, \quad (4)$$

where i and i' run over bosonic and fermionic states respectively in V , and x_i and $x'_{i'}$ are the eigenvalues of an arbitrary operator $g \in GL(\infty|\infty)$ in principle, but for applications to indices, $g = p^{\hat{k}} q^{L_0 - c/24} y^{2\hat{J}_0^3}$ where $\hat{k}$ is the strand length operator, L_0 is a Virasoro generator, and $\hat{J}_0^3$ is the Cartan generator of $SU(2)_L$. The minus sign for the second term corresponds to having $(-1)^F$, with F being the fermion number operator, in the character's trace. Analogously, the character of $\tilde{V}$ (or the right-moving single-strand partition function), assumed to be k independent, is

$$\tilde{z}(\tilde{x}|\tilde{x}') = \sum_{\tilde{i}} \tilde{x}_{\tilde{i}} - \sum_{\tilde{i}'} \tilde{x}'_{\tilde{i}'}, \quad (5)$$

where the chosen operator is typically $\tilde{g} = \tilde{y}^{2\tilde{J}_0^3} \in GL(b|f)$ with $\tilde{J}_0^3$ being the Cartan generator of $SU(2)_R$. Then the character—or physically the (signed) partition function—of the covering Hilbert space (3) is

$$\mathcal{Z} = \sum_{\lambda} S_{\lambda}(x|x') \tilde{S}_{\lambda}(\tilde{x}|\tilde{x}'), \quad (6)$$

where $S_{\lambda}(x|x')$, $\tilde{S}_{\lambda}(\tilde{x}|\tilde{x}')$ are the characters of V_{λ} , $\tilde{V}_{\lambda}$, and are given by super Schur functions [45]; see Supplemental Material [46].

IV. THE NEW INDEX

Before applying the formalism of the previous section, we first review the standard index for the D1-D5 CFT. A supersymmetry index is invariant under continuous changes of the coupling; it is defined so that states that can recombine—and “lift”—contribute zero [18]. For the D1-D5 CFT on T^4 , the appropriate index is the MEG [20], defined by the RR sector trace over the free-theory BPS Hilbert space, as follows:

$$\mathcal{E}_N(q, y) = \mathcal{D} \text{tr} [(-1)^F q^{L_0 - c/24} y^{2\hat{J}_0^3} \tilde{y}^{2\tilde{J}_0^3}], \quad (7)$$

where $(-1)^F = (-1)^{2(\hat{J}_0^3 - \tilde{\hat{J}}_0^3)}$ provides a $\mathbb{Z}_2$ grading by spin statistics and the differential operator acts as $\mathcal{D} := \frac{1}{2}(\tilde{y}\partial_{\tilde{y}})^2$ followed by setting all right-moving fugacities to one: $\tilde{y} = \tilde{\eta} = 1$. Although the 1/4-BPS spectrum of the D1-D5 CFT is not invariant under turning on couplings of the 20 exactly marginal operators of this theory, the MEG is protected: quartets of free-theory short multiplets that combine into a long multiplet in the interacting theory contribute zero.

To compute this MEG, one first writes the single-strand left-moving character (4) as

$$z(p, q, y) = \sum_{k, m, l} c(k, m, l) p^k q^m y^l, \quad (8)$$

where $k \geq 1$, $m \geq 0$, $l \in \mathbb{Z}$. This is an expanded form of the condensed expression (4), where the eigenvalues $x_i, x'_{i'}$ correspond to $p^k q^m y^l$. The (signed) degeneracies satisfy $c(k, m, l) = c(km, l)$ [31], with $c(m, l)$ defined by the seed theory left-moving partition function

$$\sum_{m, l} c(m, l) q^m y^l = - \left(\frac{\partial_1(\nu, \tau)}{\eta(\tau)^3} \right)^2, \quad (9)$$

where $y = e^{2\pi i \nu}$, $q = e^{2\pi i \tau}$ [the definitions for the Jacobi θ and Dedekind η functions are given in Eq. (A24) of the Supplemental Material [46]]. Then the generating function for $\mathcal{E}_N$ is given by [20]

$$\mathcal{E}(p, q, y) := \sum_{N=0}^{\infty} p^N \mathcal{E}_N(q, y) = \sum_{k, m, l} \frac{c(k, m, l) p^k q^m y^l}{(1 - p^k q^m y^l)^2}. \quad (10)$$

While this is the typical presentation of the MEG, we can now instead work with the Schur-Weyl formalism of the previous section. In the D1-D5 CFT for T^4 , the left-moving covering Hilbert space is given by $V = \text{span}\{\mathcal{O}|\alpha\rangle_k, \mathcal{O}|\dot{\alpha}\rangle_k\}_{k, \mathcal{O}}$, where $\mathcal{O}$ represents arbitrary left-moving bosonic and fermionic excitations. The right-moving Hilbert space is taken to be the k -independent space $\tilde{V} = \text{span}\{|\dot{\alpha}\rangle, |\dot{\alpha}\rangle\}$. Having two bosonic and two fermionic states, $\tilde{V}$ is naturally a $GL(2|2)$ representation space and $\tilde{V}_{\lambda}$ vanishes unless $\lambda_3 \leq 2$, with λ_i being the i th row length of the Young diagram λ . In other words, λ is constrained to be of the $(2|2)$ hook type, which we denote by $\lambda \in H(2|2)$ [47]. Noting that the trace in Eq. (7) is nothing but the (signed) partition function for the theory at fixed N , the generating function $\mathcal{E}(p, q, y)$ can be obtained by the action of $\mathcal{D}$ on Eq. (6), the partition function of the covering Hilbert space. We then find a Schur-Weyl expression for the MEG generating function, as follows:

$$\mathcal{E}(p, q, y) = \sum_{\lambda \in H(2|2)} S_{\lambda}(p, q, y) \mathcal{D} \tilde{S}_{\lambda}, \quad (11)$$

where here $S_{\lambda}(p, q, y)$ is a Schur function for the single-strand character (8) and $\tilde{S}_{\lambda}(\tilde{y}, \tilde{\eta})$ is a Schur function for the $GL(2|2)$ character

$$\tilde{z}(\tilde{y}, \tilde{\eta}) = \tilde{y} + \tilde{y}^{-1} - \tilde{\eta} - \tilde{\eta}^{-1}. \quad (12)$$

In writing this, we have kept only the fugacities $\tilde{y}$ and $\tilde{\eta}$ respectively for the $SU(2)_R$ and $\widetilde{SU}(2)_2$ subgroups of this $GL(2|2)$, which in the language of Eq. (5) is a supertrace of the operator $\tilde{g} = \tilde{y}^{2\tilde{J}_0^3} \tilde{\eta}^{2\tilde{J}_0^3}$, where $\tilde{J}_0^3$ is a generator of $\widetilde{SU}(2)_2$. While the fugacity $\tilde{\eta}$ is not necessary for defining the MEG (and we set $\tilde{\eta} = 1$ when working with the MEG, as otherwise it would not be protected), we include it in the right-moving character for later convenience.

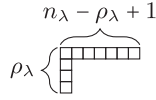


FIG. 1. Single-hook Young diagram $\lambda \in H(1|1)$ with n_λ boxes and ρ_λ rows.

While $\tilde{S}_\lambda(\tilde{y}, \tilde{\eta})$ is nonvanishing only if $\lambda_3 \leq 2$, $\mathcal{D}\tilde{S}_\lambda$ is nonvanishing only if $\lambda_2 \leq 1$, namely if $\lambda \in H(1|1)$ is a single-hook Young diagram (Fig. 1), and its value in that case is

$$\mathcal{D}\tilde{S}_\lambda = (-1)^{\rho_\lambda - 1} n_\lambda, \quad (13)$$

where ρ_λ is the number of rows and n_λ is the number of boxes in λ . Therefore, we arrive at an alternative expression for the MEG:

$$\mathcal{E}(p, q, y) = \sum_{\lambda \in H(1|1)} S_\lambda(p, q, y) (-1)^{\rho_\lambda - 1} n_\lambda. \quad (14)$$

While this new form of the generating function of the MEG is interesting in its own right, it also invites us to consider how the index can be refined by considering superselection sectors preserved by the deformed right-moving supercharge. The operator responsible for generating a long supersymmetry multiplet of the first-order deformed theory out of four short multiplets of the undeformed theory is referred to as the Gava-Narain operator $\mathcal{G}_0^{\pm A}$ (and its Hermitian conjugate $\sim \mathcal{G}_0^{\mp A}$) [23,27]. As denoted by its various indices, the Gava-Narain operator carries right-moving R-charge of $+\frac{1}{2}$ and transforms as a doublet of the $SU(2)_1$. Related to the first-order deformed right-moving supercharge, the Gava-Narain operator involves the twist operator σ_2 and therefore does not preserve the usual twist-sector decomposition of the free orbifold theory. It does, however, preserve the left-moving contracted large $\mathcal{N} = 4$ symmetry algebra, as well as the Clifford algebra cl_4 generated by the zero modes (in the Ramond sector) of the right-moving free fermions acting diagonally on the strand structure [25,27]. This Clifford algebra is a subalgebra of $gl(2|2)$, the algebra of the group $GL(2|2)$ acting on the space of right-moving Ramond ground states. The right-moving action of the Gava-Narain operator therefore maps between entire representations of cl_4 . Noting also that the Gava-Narain operator does not carry any charge under the $\widetilde{SU}(2)_2$; one is motivated to organize states into representations of the subalgebra $\mathcal{A} \equiv (su(2)_R \oplus \widetilde{su}(2)_2) \ltimes cl_4$ rather than of the full $gl(2|2)$ (here $h \ltimes g$ means a semidirect sum with g being the ideal). Irreducible representations of the algebra $\mathcal{A}$ are labeled using the spin $\tilde{j} \in \mathbb{Z}_{\geq 0}/2$ under $su(2)_R$ and $\tilde{j}_2 \in \mathbb{Z}_{\geq 0}/2$ under $\widetilde{su}(2)_2$, with the character of one such

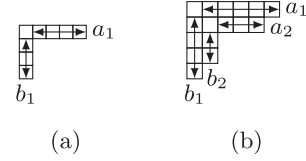


FIG. 2. Frobenius coordinates for Young diagrams in the case of (a) diagrams with $d_\lambda = 1$, $\lambda = (a_1|a_2) \in H(1|1)$ and (b) diagrams with $d_\lambda = 2$, $\lambda = (a_1, a_2|b_1, b_2) \in H(2|2)$ with $a_1 > a_2$ and $b_1 > b_2$.

representation being

$$\tilde{\chi}_{\tilde{j}, \tilde{j}_2}^A(\tilde{y}, \tilde{\eta}) = (-1)^{2\tilde{j}_2} [\tilde{\chi}_{\tilde{j}}(\tilde{y}) - \tilde{\chi}_{\tilde{j}}(\tilde{\eta})] \tilde{\chi}_{\tilde{j}}(\tilde{y}) \tilde{\chi}_{\tilde{j}_2}(\tilde{\eta}), \quad (15)$$

where $\tilde{\chi}_j(X) = X^{2j} + \dots + X^{-2j}$ is a spin j $su(2)$ character. We define $\tilde{\chi}_{\tilde{j}, \tilde{j}_2}^A$ to be vanishing if $\tilde{j} < 0$ or $\tilde{j}_2 < 0$.

We are then motivated to decompose the partition function (6) into the characters (15) and define sectors based upon the $\tilde{j}_2$ charge. Note that this $\widetilde{su}(2)_2$ representation is defined after factoring out the Clifford quartet structure, within which states have different $\widetilde{su}(2)_2$ (and $su(2)_R$) charges [48]. To do this, it is convenient to parametrize the Young diagram $\lambda \in H(2|2)$ labeling the representation of $gl(2|2)$ using Frobenius coordinates: *i.e.*, $\lambda = (a_1|b_1)$ for $d_\lambda = 1$ and $\lambda = (a_1, a_2|b_1, b_2)$ for $d_\lambda = 2$, where d_λ is the number of hooks in the diagram λ , or in other words, the largest integer s for which $\lambda_s \geq s$ (see Fig. 2). The decomposition of a $gl(2|2)$ -character S_λ into $\mathcal{A}$ characters can be shown to be given by

$$\tilde{S}_\lambda = \begin{cases} \tilde{\chi}_{\frac{a_1}{2}, \frac{b_1}{2}}^A + \tilde{\chi}_{\frac{a_1-1}{2}, \frac{b_1-1}{2}}^A & \text{if } d_\lambda = 1 \\ \tilde{\chi}_{\frac{a_{12}-1}{2}, \frac{b_{12}-2}{2}}^A + \tilde{\chi}_{\frac{a_{12}}{2}, \frac{b_{12}-1}{2}}^A \\ + \tilde{\chi}_{\frac{a_{12}}{2}, \frac{b_{12}-1}{2}}^A + \tilde{\chi}_{\frac{a_{12}-1}{2}, \frac{b_{12}}{2}}^A & \text{if } d_\lambda = 2 \end{cases}, \quad (16)$$

where $a_{12} = a_1 - a_2$ and $b_{12} = b_1 - b_2$. Therefore, the λ -sector decomposition of the MEG (11) further breaks down into $\mathcal{A}$ sectors as

$$\mathcal{E}(p, q, y) = \sum_{\lambda \in H(2|2)} \sum_{(\tilde{j}, \tilde{j}_2) \in \mathcal{A}} S_\lambda(p, q, y) \mathcal{D}\tilde{\chi}_{\tilde{j}, \tilde{j}_2}^A, \quad (17)$$

where $\sum_{(\tilde{j}, \tilde{j}_2) \in \mathcal{A}}$ means summation over $\mathcal{A}$ representations contained in the $gl(2|2)$ representation labeled by λ , according to Eq. (16). Here,

$$\mathcal{D}\tilde{\chi}_{\tilde{j}, \tilde{j}_2}^A = (-1)^{2\tilde{j}_2} (2\tilde{j} + 1)(2\tilde{j}_2 + 1). \quad (18)$$

As stated above, the Gava-Narain operator generates long multiplets of the deformed theory out of free-theory short multiplets in isomorphic $\widetilde{su}(2)_2$ representations. Therefore,

we can define a “resolved” version of the MEG by summing only over terms in Eq. (17) that have a particular value of $\tilde{j}_2$, as follows:

$$\begin{aligned}\mathcal{E}_{\tilde{j}_2}(p, q, y) &:= \sum_{\substack{\lambda \in H(2|2) \\ \tilde{j}_2 \subset \lambda}} \sum_{\tilde{j}_\lambda} S_\lambda(p, q, y) \mathcal{D}_{\tilde{\chi}_{\tilde{j}_\lambda, \tilde{j}_2}^A}(\tilde{y}, \tilde{\eta}) \\ &= \sum_{\substack{\lambda \in H(2|2) \\ \tilde{j}_2 \subset \lambda}} S_\lambda(p, q, y) (-1)^{\tilde{j}_2} (2\tilde{j}_2 + 1) \\ &\quad \times \sum_{\tilde{j}_\lambda} (2\tilde{j}_\lambda + 1),\end{aligned}\quad (19)$$

where the condition $\tilde{j}_2 \subset \lambda$ means that the $gl(2|2)$ representation labeled by λ must contain $\mathcal{A}$ representations with $\tilde{su}(2)_2$ charge $\tilde{j}_2$, and $\tilde{j}_\lambda$ runs over values of the $su(2)_R$ label $\tilde{j}$ that appear in the $gl(2|2)$ representation λ . We refer to this as the REG. Explicitly, first several terms

in Eq. (19) are

$$\begin{aligned}\mathcal{E}_{\tilde{j}_2=0} &= S_\square + 2S_{\square\square} + 3S_{\square\square\square} + S_{\square\square\square\square} + 4S_{\square\square\square\square\square} + 2S_{\square\square\square\square\square\square} + \dots, \\ \mathcal{E}_{\tilde{j}_2=\frac{1}{2}} &= -2S_{\square\square} - 4S_{\square\square\square} - 6S_{\square\square\square\square} - 2S_{\square\square\square\square\square} - 2S_{\square\square\square\square\square\square} + \dots, \\ \mathcal{E}_{\tilde{j}_2=1} &= 3S_{\square\square\square} + 6S_{\square\square\square\square} + \dots, \quad \mathcal{E}_{\tilde{j}_2=\frac{3}{2}} = -4S_{\square\square\square\square} + \dots, \quad \dots\end{aligned}\quad (20)$$

where we showed terms with only up to four boxes. Since $\mathcal{E}_{\tilde{j}_2}(p, q, y)$ is a generating function, the REG for fixed N , which we denote by $\mathcal{E}_{N, \tilde{j}_2}(q, y)$ (where $\tilde{j}_2 = 0, \frac{1}{2}, \dots, \frac{N-1}{2}$), can be found by expanding $\mathcal{E}_{\tilde{j}_2}(p, q, y)$ in p .

Another form for this generating function of REG—in the style of Eq. (10)—can be found by a Cauchy identity for the super Schur functions (see Supplemental Material [46]). One can derive the closed-form expression

$$\begin{aligned}\mathcal{E}(p, q, y, x) &:= \sum_{\tilde{j}_2} \mathcal{E}_{\tilde{j}_2}(p, q, y) \frac{x^{2\tilde{j}_2+1} + x^{-2\tilde{j}_2-1}}{2} \\ &= (x^{\frac{1}{2}} - x^{-\frac{1}{2}})^{-2} \prod_{k, m, l} \left(\frac{(1 - p^k q^m y^l x)(1 - p^k q^m y^l x^{-1})}{(1 - p^k q^m y^l)^2} \right)^{c(k, m, l)} \\ &\quad \times \left[1 + \frac{(x - x^{-1})^2}{2} \sum_{k, m, l} \frac{c(k, m, l) p^k q^m y^l}{(1 - p^k q^m y^l x)(1 - p^k q^m y^l x^{-1})} \right],\end{aligned}\quad (21)$$

where the $c(k, m, l)$ are defined in Eq. 8. Upon setting $x = 1$, this reduces to the MEG (10) (up to some irrelevant divergent terms at order p^0).

Since the arguments made in this section about lifting are made with reference to the first-order deformation of the symmetric orbifold theory by one of the twisted-sector moduli, the first-order BPS spectrum needs to be exact for the REG to be a fully protected index. While this has long been believed to be true in a wide number of theories, recent work [49] casts some doubt on this for $\mathcal{N} = 4$ super Yang-Mills. We also note that our use of representations of the $\tilde{su}(2)_2 \subset \mathcal{A}$ algebra to define superselection sectors for the first-order deformed supercharges and to label REGs with them is natural only from the free orbifold point. Away from this point the $\tilde{su}(2)_2$ charge is not a good quantum number, and so it is likely that the REG does not provide a meaningful separation of states at a generic point in moduli space. Nonetheless, the REG does provide a meaningful division of the free orbifold spectrum in a manner consistent with first-order lifting and which allows for detailed matching with the Kaluza-Klein spectrum in the bulk, as we now demonstrate.

V. APPLICATION TO $\text{AdS}_3/\text{CFT}_2$

Given the new REG defined above for the symmetric orbifold CFT of T^4 , one can also apply it to the space of

states dual to supergravitons in $\text{AdS}_3 \times S^3$ (or for large enough conformal dimensions, to superstrata backgrounds [50]). Supergraviton states are naturally defined in the Neveu-Schwarz (NS) sector of the CFT and the left-moving single-strand supergraviton Hilbert space is a restriction of the full NS Hilbert space, $V_{\text{SG}} = \text{span}\{\mathcal{O}_g|\alpha\rangle_k, \mathcal{O}_g|\dot{A}\rangle_k\}_{k, \mathcal{O}_g}$, where the $\mathcal{O}_g$ are excitations using modes only of the $SU(1, 1|2)$ global (anomaly-free) subalgebra of the full $\mathcal{N} = 4$ symmetry algebra [20]. The right-moving single-strand Hilbert space $\tilde{V}$ is the same as the CFT’s Ramond ground-state space [51]. The REG generating function for supergraviton states is again given by Eq. (19) [or by Eq. (21)], but now $S_\lambda(p, q, y)$ is the Schur function for the single-strand supergraviton character

$$\begin{aligned}z_{\text{SG}}(p, q, y) &= \sum_k p^k \left[\phi_{\frac{k-1}{2}}^{(s)}(q, y) - 2\phi_{\frac{k}{2}}^{(s)}(q, y) + \phi_{\frac{k+1}{2}}^{(s)}(q, y) \right] \\ &=: \sum_{k, m, l} c_{\text{SG}}(k, m, l) p^k q^m y^l,\end{aligned}\quad (22)$$

where $\phi_j^{(s)}$ are characters of short representations of $SU(1, 1|2)$ (see Supplemental Material [46]).

In Ref. [20], the supergraviton spectrum was compared with the full CFT spectrum using the MEG (7), and

agreement was found below the black-hole threshold, $h < h_{\text{BH}} = \frac{N}{4}$. However, this agreement is rather empty, as both the CFT and supergraviton MEGs in fact vanish below this threshold (except for the contribution of the global NS vacuum at order $q^0 y^0$). We will now see that the REG (19) turns this “0 = 0” statement into a meaningful comparison, by resolving each side of this equation into a sum of nonvanishing $\tilde{j}_2$ -sector contributions [52], $\mathcal{E}_{\tilde{j}_2}$, with $\tilde{j}_2 = 0, \frac{1}{2}, \dots, \frac{N-1}{2}$. In order to make this comparison, the R-sector left-moving character of the CFT (8) should be flowed to the NS sector via $z_{\text{NS}}(p, q, y) = z_{\text{R}}(pq^{\frac{1}{2}}y, q, yq^{\frac{1}{2}})$ [53].

At $N = 3$, for example, the CFT and supergraviton REGs have the q expansions [54]

$$\begin{aligned}\mathcal{E}_{3,0}^{\text{CFT}} &= 3 + q^{\frac{1}{2}}(-6y - 6y^{-1}) + q(+4y^2 + 28 + 4y^{-2}) + \dots \\ \mathcal{E}_{3,\frac{1}{2}}^{\text{CFT}} &= q^{\frac{1}{2}}(+6y + 6y^{-1}) + q(-12y^2 - 32 - 12y^{-2}) + \dots \\ \mathcal{E}_{3,1}^{\text{CFT}} &= q(+9y^2 + 12 + 9y^{-2}) + \dots \\ \mathcal{E}_{3,0}^{\text{SG}} &= 3 + q^{\frac{1}{2}}(-6y - 6y^{-1}) + q(+4y^{-2} + 25 + 4y^{-2}) + \dots \\ \mathcal{E}_{3,\frac{1}{2}}^{\text{SG}} &= q^{\frac{1}{2}}(+6y + 6y^{-1}) + q(-12y^2 - 36 - 12y^{-2}) + \dots \\ \mathcal{E}_{3,1}^{\text{SG}} &= q(+9y^2 + 12 + 9y^{-2}) + \dots,\end{aligned}$$

where we see not only detailed nontrivial matching up to the expected order $O(q^{\frac{1}{2}})$, but also an enhanced matching for the $\tilde{j}_2 = 1$ sector, in which $\mathcal{E}_{3,\frac{1}{2}}^{\text{CFT}} - \mathcal{E}_{3,\frac{1}{2}}^{\text{SG}} = O(q^{\frac{3}{2}})$. We have explicitly checked that these features (matching below the threshold and enhanced matching in certain $\tilde{j}_2$ sectors) of the REG hold up to $N = 12$. This matching with the supergraviton spectrum is further evidence that lifted states cancel from the REG separately for each $\tilde{j}_2$ sector and therefore that the REG is protected. In order to study states well above the black-hole threshold, we consider the logarithmic degeneracies of the CFT REG $d_{N,\tilde{j}_2}^{\text{CFT}}(h) := \log |\mathcal{E}_{N,\tilde{j}_2}^{\text{CFT}}(q, 1)|_{q^h}$ as well as of the difference between the CFT and supergraviton REGs $d_{N,\tilde{j}_2}^{\text{BH}}(h) := \log |\mathcal{E}_{N,\tilde{j}_2}^{\text{CFT}}(q, 1)|_{q^h} - \mathcal{E}_{N,\tilde{j}_2}^{\text{SG}}(q, 1)|_{q^h}|$, where setting $y = 1$ has the effect of summing over values of J_0^3 . From the logarithmic degeneracies (shown for $N = 5$ in Fig. 3) we see that each $\tilde{j}$ sector of the REG appears to have the same leading-order growth of states as the full MEG in the regime $h \gg N$, namely Cardy growth [56]. Since $\tilde{j}$ sectors contain a mixture of twist sectors, we conclude that black-hole states are distributed among twist sectors. The sub-leading growths of the REG should, however, depend on $\tilde{j}$. It would be interesting to see what the more refined structure of the REG implies for the regime $h \sim N \gg 1$, in which the “long-string sector” (the minimal n_λ sector) is believed to be dominant [57,58]. This long-string sector is contained within the $\tilde{j} = 0$ sector of the REG. We also observe that for quantum numbers well into the black-hole

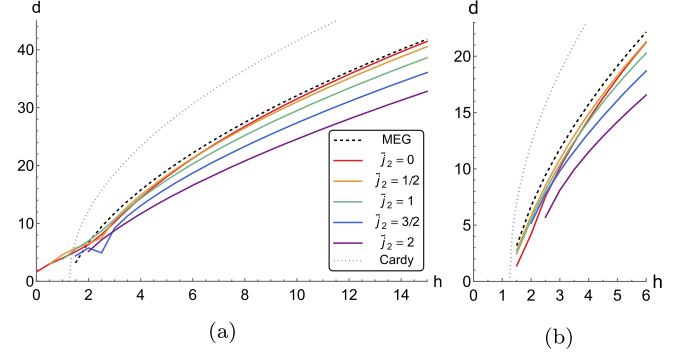


FIG. 3. Plots of the REG logarithmic degeneracies $d_{N=5,\tilde{j}_2}^{\text{CFT}}$ in (a) and $d_{N=5,\tilde{j}_2}^{\text{BH}}$ in (b). For comparison, we show the analogous quantities obtained using the MEG, along with the universal Cardy growth. The first states contributing to the REG are at larger h for sectors with larger $\tilde{j}$.

regime—that is, well within the parabola $h = \frac{\tilde{j}^2}{N} + \frac{N}{4}$ [59]—the coefficients of $q^h y^{2j}$ for fixed h and j appearing in the different $\tilde{j}$ sectors of the REG are all of the same sign.

VI. OUTLOOK

We constructed a new supersymmetry index for the D1-D5 CFT on T^4 —the resolved elliptic genus—and argued for its protection using the precise form of the twisted-sector exactly marginal operators. We demonstrated detailed matching between the CFT and supergraviton spectra in each $\tilde{j}$ sector below the black-hole threshold—the first such matching since the MEG is trivial in this region [20]. This is further evidence that the REG is protected. In fact, we found that many $\tilde{j}$ sectors even enjoy an enhanced CFT-supergraviton matching. Above the threshold, we showed that the REG resolves the black-hole microstates visible in the MEG into multiple distinct $\tilde{j}$ sectors.

The REG finds a natural description in the Schur-Weyl formalism developed in this paper which, with minimal modifications, should be applicable to general symmetric orbifold theories—for studies spaces of such theories relevant to holography; see, i.e., Ref. [9]. In particular, it should be applicable to the D1-D5 CFT on $K3$; while the theory’s Hilbert space (as defined from its torus orbifold description) does not totally factorize into left- and right-movers [contrary to the assumption of Eq. (2)], a slightly modified formalism should still apply. We hope to report progress on this in the near future.

An interesting question is whether the REG has a gravity dual [60]. The fugacity x in Eq. (21) counts representations of $\widehat{\mathfrak{su}}(2)_2$ associated with the bulk T^4 , but its bulk meaning is unclear and deserves further study.

Because of the D1-D5 CFT’s central role in the study of black hole microstate physics, the REG opens a broad avenue of applications and investigations, potentially

providing a new handle on longstanding and actively pursued problems—including the lifting problem [26–30], fortuity [22], BPS chaos [62], the holographic dictionary for multicenter configurations [63–65], $\text{AdS}_3 \times S^3 \times S^3 \times S^1$ holography [66–68], and the relation to the generalized supergravity index [69]. We hope to report progress on these subjects in the future.

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DATA AVAILABILITY

No data were created or analyzed in this study.

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