

Forward cascade of large-scale primordial magnetic fields during structure formation

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The origin of large scale magnetic fields in the Universe is widely thought to be from early Universe processes, like inflation or phase transitions. These magnetic fields evolve via magnetohydrodynamic processes until the epoch of recombination. When structures begin to form in the later Universe, the conservation of magnetic flux amplifies the magnetic fields via the adiabatic collapse of gravitationally bound gas clouds hosting the magnetic fields and moves them to smaller scales. In this work, we have semianalytically studied this forward cascade effect, considering simple models of gravitational collapse of structures. We find that this simple model is able to reproduce the general qualitative features of the evolution of the magnetic field spectrum as seen from magnetized cosmological simulations.

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I. INTRODUCTION

Magnetic fields are observed throughout the Universe on scales ranging from planets and stars to galaxies and galaxy clusters [1]. The origin of these magnetic fields is an open question in modern astrophysics and cosmology. It is commonly assumed that the observed cosmic magnetic fields originated from small “seed” fields that were amplified during structure formation [2]. Such theories of field generation, or magnetogenesis, can be divided into two major classes: astrophysical (where weak initial seed fields from local sources in galaxies are amplified and transferred to larger scales, or are produced on large scales through charge separation processes; see, e.g., Refs. [2–4]) and cosmological or primordial (where a seed is generated prior

to galaxy formation in the prerecombination Universe on scales that are now large) [5].

The primordial magnetogenesis scenario is motivated by observations of TeV blazar emission spectra by NASA’s Fermi Gamma-ray Space Telescope (see Refs. [6–14] for pioneering works and Refs. [15–31] for later works, and references therein). That is, the existence of magnetic fields (no weaker than $\sim 10^{-16}$ G correlated on Mpc scales), could deflect the electron-positron cascade pairs triggered when multi-TeV photons from blazars interact with extragalactic background light. This would modify the rechanneling of the TeV emission into the multi-GeV energy range, thereby explaining the lack of a multi-GeV bump in some blazar emission spectra. For this interpretation of the observed blazar spectra features, at least 60% of the cosmological volume must be permeated by such fields, a degree of magnetization that is unattainable by magnetized outflows from normal or active galaxies [12,32]. Alternative explanations of the Fermi observations have been proposed including uncertainties in the measurements of blazar spectra [33], plasma instabilities [34–42], magnetized outflows by normal or active galaxies [43], return currents induced by cosmic rays escaping into the intergalactic medium from the first galaxies [44,45], and collective action of galactic dipole fields [46]. Some other alternatives have also been refuted; for example, see [46,47]. Interestingly, primordial magnetogenesis is supported through the direct detection of a high synchrotron

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emission flux coming from filaments [48] that requires a field strength of the order of 30–60 nG. Explanation of these limits [49], as well as the evolution of magnetic fields in cosmic filaments [50], remains challenging solely through dynamo amplification and astrophysical mechanisms [51,52], while cosmological magnetized simulations reveal compatibility with empirical data [53–57], favoring the scenario of tangled [51,58,59] or large-scale-correlated primordial magnetic fields [60].

In this paper, we focus on the evolution of primordial magnetic fields (PMFs) during the late stages of the expansion of the Universe, and present a semianalytical description of the interaction between the magnetic fields and large-scale structures during gravitational collapse. Previously, some of us numerically studied the dynamics of PMFs in the early Universe and the generation of prerecombination magnetohydrodynamic (MHD) turbulence using the publicly available PENCIL CODE [61]. However, this modeling did not include the effect of gravitational (density) perturbations—such a simplification is well justified for the linear stage, but the non-linearity of density perturbations might substantially change the picture [62].

It is natural to assume that PMFs that were frozen during the dark ages are affected by the physical processes during structure formation. Reference [63] explores some milestones in the evolution of PMFs during structure formation using the cosmological MHD code ENZO [64], accounting for different PMF spectra corresponding to different magnetogenesis scenarios and studying the dependence of the field growth on its coherence scale. It was shown that collapsing structures amplify magnetic fields and move their energy from large to small scales.

In this paper, we theoretically derive the evolution of the magnetic field energy spectrum and show that if the field dynamics is governed by gravitational collapse, then its spectrum should indeed show forward cascading toward smaller scales. Our goal is to trace the evolution of the PMF during the formation of large-scale structures in the late Universe. Specifically, our aim is to determine how the energy spectra of PMFs, characterized by different peak positions and spectral slopes, evolve and shift under the influence of gravitational collapse.

This paper is arranged as follows. In Sec. II, we briefly review the statistical modeling of PMFs and how their evolution is studied in cosmological simulations. We introduce the formalism that describes the forward cascade of magnetic energy via collapse of structures in Sec. III and present our numerical results in Sec. IV; finally we present our concluding discussions in Sec. V. Throughout this article, we work in natural units where $\hbar = c = k_B = 1$. In addition, the permeability of the free space is set to unity, $\mu_0 = 1$, expressing all electromagnetic quantities in Lorentz-Heaviside units.

II. MODELING COSMOLOGICAL MAGNETIC FIELDS

Primordial magnetogenesis scenarios are typically divided into two categories, depending on their coherence lengths at the moment of PMFs generation: (a) unlimited by the Hubble horizon length scale and (b) limited by the Hubble horizon: for example, inflationary magnetic fields that are generated through quantum-mechanical fluctuations are characterized by arbitrary correlation lengths, since their correlation lengths are increased by the inflationary expansion. In the opposite case, when the correlation lengths are bounded through the Hubble length (sometimes referred to as *causal*), PMFs might arise from primordial turbulence or turbulence associated with violent phase transitions [65–67].

There are two cosmological phase transitions of interest in the early Universe: the electroweak (EW) and the quantum chromodynamics (QCD) phase transitions (PTs), referred to as EWPT and QCDPT, respectively. In the standard model of particle physics and cosmology, these transitions are smooth crossovers rather than violent first-order PTs; however, much research has been devoted to extend the standard model to make these transitions first order (see [68,69] for discussion). Alternatively, magnetic field seeds, generated during the inflationary epoch and afterward that survive the reheating, unavoidably interact with the plasma, leading to the development of turbulence [70].

A. Statistical properties of magnetic fields

Magnetic fields generated in the early Universe are modeled to have a stochastic¹ isotropic and statistically homogeneous Gaussian distribution. Notably, the inflationary PMFs with a scale-invariant spectrum induce effects of the homogeneous, since most of the power is concentrated in superhorizon scales, allowing us to treat such a field as semihomogeneous [72]. The two-point correlation function of these magnetic fields in real space is defined as

$$\mathcal{B}_{ij}(\mathbf{x}, \mathbf{x} + \mathbf{r}) \equiv \langle B_i(\mathbf{x}) B_j(\mathbf{x} + \mathbf{r}) \rangle = \mathcal{B}_{ij}(\mathbf{r}), \quad (1)$$

where B_i denotes the i th component of the comoving magnetic field $\mathbf{B}$, $\mathbf{x}$ and $\mathbf{r}$ denote comoving coordinates, and the angle brackets denote ensemble averaging (which we achieve in practice by a spatial average over a comoving volume V). The statistical isotropy (rotational invariance) of the distribution of magnetic fields ensures that the

¹We do not consider constant spatially homogeneous magnetic fields due to their “pathological nature” as described in Ref. [71]. In particular, truly uniform magnetic fields do not demonstrate the familiar MHD decay properties that, for example, a statistically homogeneous distribution of fields does; see Ref. [71] for more details.

correlation function depends only on $r = |\mathbf{r}|$, i.e., we can write $\mathcal{B}_{ij}(\mathbf{r}) = \mathcal{B}_{ij}(r)$.

The three-dimensional power spectrum of magnetic fields $\mathcal{F}_{ij}(\mathbf{k})$, which is the Fourier transform of the real-space correlation function,

$$\mathcal{B}_{ij}(\mathbf{r}) = \frac{1}{(2\pi)^3} \int d^3\mathbf{k} e^{-i\mathbf{k}\cdot\mathbf{r}} \mathcal{F}_{ij}(\mathbf{k}), \quad (2)$$

can be decomposed into symmetric and helical parts,

$$\frac{\mathcal{F}_{ij}(\mathbf{k})}{(2\pi)^3} = P_{ij}(\hat{\mathbf{k}}) \frac{E_M(k)}{4\pi k^2} + i\epsilon_{ijl} k_l \frac{H_M(k)}{8\pi k^2}, \quad (3)$$

where $\hat{\mathbf{k}} = \mathbf{k}/k$, $P_{ij}(\hat{\mathbf{k}}) = \delta_{ij} - \hat{k}_i \hat{k}_j$ is the projection operator in momentum space, and $E_M(k)$ and $H_M(k)$ are, respectively, the spectral magnetic energy and helicity densities. In terms of spectral magnetic energy density, the mean magnetic energy density $\mathcal{E}_M = \langle \mathbf{B}^2(\mathbf{x}) \rangle / 2 = \mathcal{B}_{ii}(0)/2$ can be written as

$$\mathcal{E}_M = \int_0^\infty dk E_M(k), \quad (4)$$

and we can define an integral length scale ξ_M as

$$\xi_M = \frac{1}{\mathcal{E}_M} \int_0^\infty dk k^{-1} E_M(k), \quad (5)$$

which is the typical scale on which magnetic fields are correlated in the plasma.

Depending on the exact nature of the generation mechanism, magnetic fields can be helical or nonhelical. It has been shown that in the decaying MHD turbulence regime, applicable to the early Universe epoch, the correlation length of both helical and nonhelical fields increases [61]; the field strength and correlation length at the recombination epoch then depend on the magnetic field decay timescale and B – ξ_M evolutionary trend [73–76]. The authors of Ref. [75] argue that the decay timescale is governed by reconnection physics which leads to prolonged decay timescales and therefore to small-scale-correlated (of the order of tens of pc) fields; on the other hand, Ref. [76] finds magnetic field coherence scales reaching hundreds of kpc at the end of the recombination epoch.²

B. Magnetized cosmological simulations

Reconstructing the full picture of the evolution of PMFs, from their generation to the current epoch, remains challenging. MHD codes have been used extensively to study

their evolution in the prerecombination epoch (see [77,78] for reviews and references therein). On the other hand, the postrecombination evolution of PMFs has been explored using cosmological MHD codes; see, e.g., Refs. [60,79–84]. However, these simulations cannot resolve all scales relevant to the PMF evolution. For instance, resolving coherence scales down to kpc or pc requires extremely high-resolution simulations or enlarged simulations focused on specific objects, such as galaxies, where large-scale dynamics typically remain under-resolved.

Reference [63] instead studied the evolution of different PMFs in a cosmological setup without resolving the turbulent dynamo amplification within galaxy clusters (see, e.g., Ref. [83] where a subgrid dynamo model has been used to account for the turbulent amplification of PMFs in galaxy clusters). Nevertheless, such simulations can capture well enough global properties of PMFs and their evolution on large scales during the formation of large-scale structure.

In Ref. [63], some of us used the cosmological code ENZO [64] to study the evolution of PMFs in a $(67.7h^{-1} \text{ cMpc})^3$ comoving volume (“c” referring to comoving units) employing 512^3 grid cells and 512^3 dark matter particles, from redshift $z = 50$ to $z = 0$; this corresponds to a $132h^{-1} \text{ ckpc}$ spatial resolution and a dark matter particle mass of $m_{\text{DM}} = 2.53 \times 10^8 M_\odot$. We assumed Λ CDM cosmology with parameters $h = 0.674$, $\Omega_m = 0.315$, $\Omega_b = 0.0493$, $\Omega_\Lambda = 0.685$, and $\sigma_8 = 0.807$ [85] and focused on adiabatic dynamics, neglecting gas cooling, chemical evolution, star formation, feedback from active galactic nuclei, and modification of the matter power spectrum due to PMF-induced density perturbations (see e.g., recent work [86–88]). For a more detailed description of the simulation setup, numerical methods, and resolution studies, we refer the reader to Ref. [63], while in this Section we briefly summarize the main findings of that study.

In Fig. 1, we show the evolution of the PMF energy spectrum $E_M(k)$ from Ref. [63] for four different PMF models: (a) a spatially homogeneous model, i.e., a constant magnetic field, (b) a stochastic magnetic field initially (at $z = 50$) characterized by a $k^{-5/3}$ spectra, (c) a stochastic helical field with a $\sim 2h^{-1} \text{ cMpc}$ correlation length, and (d) a stochastic field with a coherence scale of $\sim 1h^{-1} \text{ cMpc}$. We refer to the latter two models, respectively, as helical and nonhelical, to be consistent with the labeling in Ref. [63]; we note, however, that in this work we do not study the effects of helicity.

From Fig. 1, we see that the spectra of different PMF models evolve in distinguishable ways. The uniform model exhibits homogeneous growth at small wave numbers from the first redshifts; the model with initial $k^{-5/3}$ spectra shows pronounced growth only from $z = 10$ (the onset of structure formation). In these two cases, the magnetic field growth is larger on large scales [compared to the growth observed for

²In this paper we are focused on nonhelical PMFs dynamics during the gravitational collapse, but to show a full picture of PMFs statistical properties we address helical magnetic fields in the Appendix A.

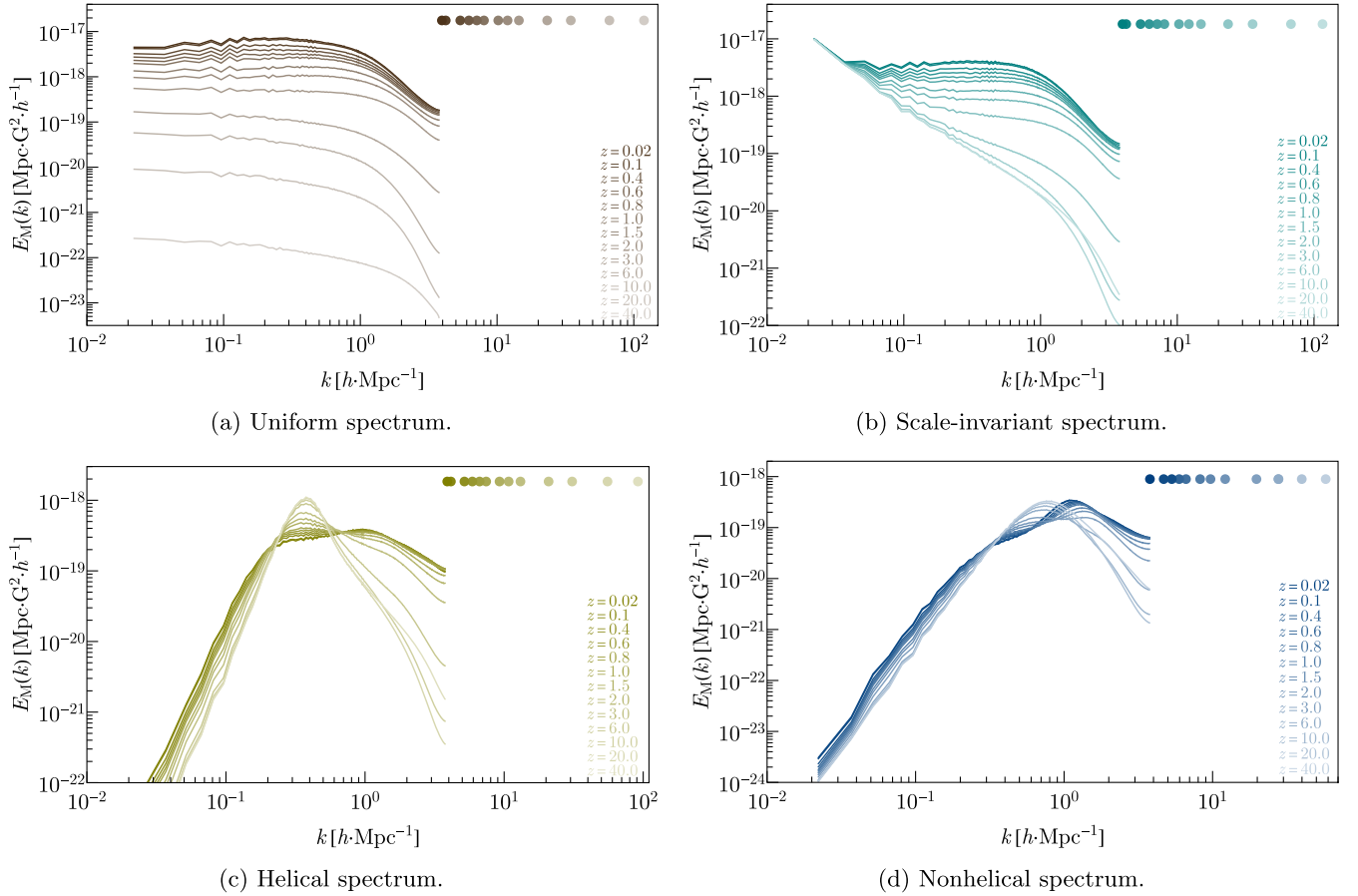


FIG. 1. Evolution of the magnetic energy spectra $E_M(k)$ with redshift z as affected by structure formation, calculated by numerical simulations in Ref. [63], for different initial spectra for seed PMFs—uniform, scale-invariant, helical, and nonhelical. The filled circles indicate the Jeans wave number at the different redshifts given by $k_J(z) = a(z)\sqrt{4\pi G\bar{\rho}(z)}/c_s$.

cases (c) and (d)] since these models have the most of the energy contained on large scales. Finally, for models (c) and (d), which have a characteristic peak within the simulation box, we see (i) moderate growth on large scales, (ii) a decay of the magnetic energy at the initial peak scale, and then (iii) a shift of magnetic energy toward smaller scales (at ~ 1 Mpc scales where galaxy clusters form).

In Appendix C, however, we also show that this shift of the characteristic peak in our simulations is resolution-dependent; this is the case, in general, for the amplification of magnetic fields (see also Refs. [89,90]) since as resolution increases, turbulent amplification also contributes to the growth of magnetic fields on smaller scales and, therefore, interplay between the adiabatic and turbulent growth of magnetic fields becomes more complex [91].

III. MAGNETIC FORWARD CASCADE FROM ADIABATIC COLLAPSE

A. Adiabatic collapse and flux conservation

After recombination, the physical magnetic fields dilute with redshift z as $B \propto (1+z)^2$, keeping the comoving

magnetic field strength constant. This is a consequence of the conservation of magnetic flux [92]. Once (nonlinear) cosmic structure formation sets in, the magnetic energy spectrum is affected by (i) flux freezing during gravitational collapse and (ii) nonlinear physics governing the injection and dissipation of energy [1,77,84,93]. If we consider comoving magnetic fields to be affected only by the adiabatic gravitational collapse of structures, we see that the magnetic fields are amplified as the structures shrink in size [94,95]. In particular, if a structure of typical size L_1 hosting a magnetic field of typical strength B_1 at redshift z_1 collapses to size L_F at a later redshift z_F , the strength of the magnetic fields increases to $B_F = B_1(L_1/L_F)^2$ [96]. This should lead to a forward cascade of magnetic energy to smaller length scales, i.e., toward larger wave numbers, which potentially counters the inverse cascade of the energy of helical magnetic fields during their MHD evolution, which occurred before recombination.

We can get an estimate of this forward cascade as follows. Let us choose the function $Q(k)$ to denote how structures collapse adiabatically, i.e., structures of typical wave number k collapse to wave numbers $Q(k)$. We assume

that $Q(k) \geq k \forall k$ to ensure that there is no dilution at any scale and that it is monotonically increasing with k to ensure that $Q^{-1}(k)$ exists uniquely.³

Let $d\mathcal{E}_M(k) = E_M(k)dk$ be the magnetic energy density contained between the scales k and $k + dk$. After collapse, this energy density moves to a scale $Q(k)$, and is replaced by the energy density $d\mathcal{E}_M(Q^{-1}(k)) = E_M(Q^{-1}(k))dk$ previously between the scales $Q^{-1}(k)$ and $Q^{-1}(k) + dk$. Due to amplification of the magnetic field, this energy density is amplified by a factor of $(k/Q^{-1}(k))^4$. The new energy density $d\mathcal{E}_M^{\text{New}}(k)$ on the scale k is then given by

$$\begin{aligned} d\mathcal{E}_M^{\text{New}}(k) &= \left(\frac{k}{Q^{-1}(k)}\right)^4 d\mathcal{E}_M(Q^{-1}(k)) \\ &= \left(\frac{k}{Q^{-1}(k)}\right)^4 E_M(Q^{-1}(k))dk. \end{aligned} \quad (6)$$

In terms of the spectral energy density $E_M^{\text{New}}(k)$ after the collapse, we can write the lhs of Eq. (6) as

$$d\mathcal{E}_M^{\text{New}}(k) = E_M^{\text{New}}(k)dk. \quad (7)$$

Comparing Eqs. (6) and (7), we can write

$$E_M^{\text{New}}(k) = \left(\frac{k}{Q^{-1}(k)}\right)^4 E_M(Q^{-1}(k)). \quad (8)$$

In the following Subsection, we derive the evolution equation for the forward cascade of the magnetic field energy spectrum $E_M(k, t)$.

B. The forward cascade equation

We start with the simplified assumption that all structures at scale k at time t collapse at a rate $\zeta(k, t)$. This means that a structure on scale k at time t becomes a structure on scale $k + \zeta(k, t)dt$ at time $t + dt$. In reality, different structures on the same scale may collapse at different rates depending on how they differ by their mass, density profile, thermodynamic properties, etc.; in the Appendix D, we generalize this discussion slightly by considering a mass dependence of the collapse rate.

Structures on scale k at time $t + dt$ were on scale k' at time t , which is related to k as $k' + \zeta(k', t)dt = k$. This implies

$$\begin{aligned} k' &= k - \zeta(k', t)dt \\ &\approx k - \zeta(k, t)dt, \end{aligned} \quad (9)$$

where the approximation in the second step induces an error of $\mathcal{O}(dt^2)$ that we ignore. Using the result in Eq. (8)

³Note here that $Q(k)$ represents a wave number, and so does $Q^{-1}(k)$; the latter *does not* indicate $1/Q(k)$.

we write the magnetic energy density spectrum at time $t + dt$ as

$$\begin{aligned} E_M(k, t + dt) &= \left[\frac{k}{k - \zeta(k, t)dt}\right]^4 E_M(k - \zeta(k, t)dt, t) \\ &= \left[1 - \frac{\zeta(k, t)dt}{k}\right]^{-4} E_M(k - \zeta(k, t)dt, t). \end{aligned} \quad (10)$$

With appropriate binomial and Taylor expansions, and keeping terms up to $\mathcal{O}(dt)$, we can write

$$\begin{aligned} E_M(k, t) + \frac{\partial E_M(k, t)}{\partial t} dt &= \left[1 + \frac{4\zeta(k, t)dt}{k}\right] \\ &\times \left[E_M(k, t) - \frac{\partial E_M(k, t)}{\partial k} \zeta(k, t)dt\right] \\ &= E_M(k, t) + \zeta(k, t) \left[\frac{4}{k} - \frac{\partial}{\partial k}\right] E_M(k, t) dt + \mathcal{O}(dt^2). \end{aligned} \quad (11)$$

Simplifying the above equation, we can express the evolution of the spectral magnetic energy density as the *forward cascade equation*

$$\frac{\partial E_M(k, t)}{\partial t} + \zeta(k, t) \left[\frac{\partial}{\partial k} - \frac{4}{k}\right] E_M(k, t) = 0. \quad (12)$$

IV. NUMERICAL SOLUTIONS

It is convenient to normalize the various equations and quantities using the free-fall time t_{ff} , the Jeans wave number k_J , and the thermal energy density u_{th} , where the density and the speed of sound are obtained from Ref. [63],

$$t_{\text{ff}} = \sqrt{\frac{3\pi}{32G\rho}}, \quad k_J = \sqrt{\frac{4\pi G\rho}{c_s^2}}, \quad u_{\text{th}} \equiv \rho c_s^2. \quad (13)$$

Some quantities are normalized in a straightforward way,

$$\tilde{t} \equiv t/t_{\text{ff}}, \quad \tilde{k} \equiv k/k_0, \quad (14)$$

where k_0 is the location of the peak of the spectrum at the initial time. Since the magnetic energy spectrum E_M has units of energy density per wave number, it can be normalized as

$$\tilde{E}_M(\tilde{k}, \tilde{t}) \equiv \frac{k_0}{u_{\text{th}}} E_M(k, t). \quad (15)$$

With this normalization,

$$\mathcal{E}_M(t) = u_{\text{th}} \int_0^\infty \tilde{E}_M(\tilde{k}, \tilde{t}) d\tilde{k}. \quad (16)$$

The collapse rate $\zeta(k, t)$ has units of k/t , and can thus be normalized as

$$\tilde{\zeta}(\tilde{k}, \tilde{t}) \equiv \frac{t_{\text{ff}}}{k_0} \zeta(k, t), \quad (17)$$

and the dimensionless form of the forward cascade equation Eq. (12) is

$$\frac{\partial \tilde{E}_M(\tilde{k}, \tilde{t})}{\partial \tilde{t}} + \tilde{\zeta}(\tilde{k}, \tilde{t}) \left[\frac{\partial}{\partial \tilde{k}} - \frac{4}{\tilde{k}} \right] \tilde{E}_M(\tilde{k}, \tilde{t}) = 0. \quad (18)$$

In order to numerically solve Eq. (18), we use the *finite difference method* which parametrizes partial differentials as finite differentials utilizing a grid representing the solution at some time $\tilde{t}$ and some wave number $\tilde{k}$, and solving for times $\tilde{t} + 1$ and wave numbers $\tilde{k} + 1$ based on previous temporal and spatial solutions. We construct a grid of dimensions (T, K) , where $\partial \tilde{E}_M / \partial \tilde{k} \rightarrow (\tilde{E}_{M, \tilde{i}}^{\tilde{k}+1} - \tilde{E}_{M, \tilde{i}}^{\tilde{k}}) / d\tilde{k}$, and $\partial \tilde{E}_M / \partial \tilde{t} \rightarrow (\tilde{E}_{M, \tilde{i}+1}^{\tilde{k}} - \tilde{E}_{M, \tilde{i}}^{\tilde{k}}) / d\tilde{t}$; here, $d\tilde{k}$ and $d\tilde{t}$ are the respective step sizes of the grid. For the left and right boundaries of the grid, the respective update conditions are

$$\tilde{E}_{M, \tilde{i}+1}^{\tilde{k}} = \tilde{E}_{M, \tilde{i}}^{\tilde{k}} - \left(\frac{\tilde{E}_{M, \tilde{i}}^{\tilde{k}+1} - \tilde{E}_{M, \tilde{i}}^{\tilde{k}}}{d\tilde{k}} - \frac{4}{\tilde{k}} \tilde{E}_{M, \tilde{i}}^{\tilde{k}} \right) \tilde{\zeta}_0 d\tilde{t} \quad (19)$$

and

$$\tilde{E}_{M, \tilde{i}+1}^{\tilde{k}} = \tilde{E}_{M, \tilde{i}}^{\tilde{k}} - \left(\frac{\tilde{E}_{M, \tilde{i}}^{\tilde{k}} - \tilde{E}_{M, \tilde{i}}^{\tilde{k}-1}}{d\tilde{k}} - \frac{4}{\tilde{k}} \tilde{E}_{M, \tilde{i}}^{\tilde{k}} \right) \tilde{\zeta}_0 d\tilde{t}. \quad (20)$$

Otherwise, the general update condition is

$$\tilde{E}_{M, \tilde{i}+1}^{\tilde{k}} = \tilde{E}_{M, \tilde{i}}^{\tilde{k}} - \left(\frac{\tilde{E}_{M, \tilde{i}}^{\tilde{k}+1} - \tilde{E}_{M, \tilde{i}}^{\tilde{k}-1}}{2d\tilde{k}} - \frac{4}{\tilde{k}} \tilde{E}_{M, \tilde{i}}^{\tilde{k}} \right) \tilde{\zeta}_0 d\tilde{t}, \quad (21)$$

where the factor of 2 in the denominator of the partial derivative of $\tilde{E}_M$ with respect to $\tilde{k}$ accounts for the larger step size taken in this regime compared to that taken in the boundaries. For the results demonstrated in the subsequent figures in this Section, we have chosen $d\tilde{k} = 0.002$ and $d\tilde{t} = 0.002$. The initial energy spectrum follows a $\propto \tilde{k}^4$ power law from $\tilde{k} = 0$ to $\tilde{k} = 1$. From $\tilde{k} = 1$ to $\tilde{k} = 10$, the energy spectrum follows a $\propto \tilde{k}^{-1}$ power law; and for $\tilde{k} > 10$, the energy spectrum follows a $\propto \tilde{k}^{-5/3}$ power law. In Fig. 2, we show the evolution of the spectrum $\tilde{E}_M(\tilde{k})$, according to Eq. (18) for a constant collapse rate $\zeta = \zeta_0$ and the initial conditions above. We see a clear indication of the forward collapse of the magnetic energy with the collapse of

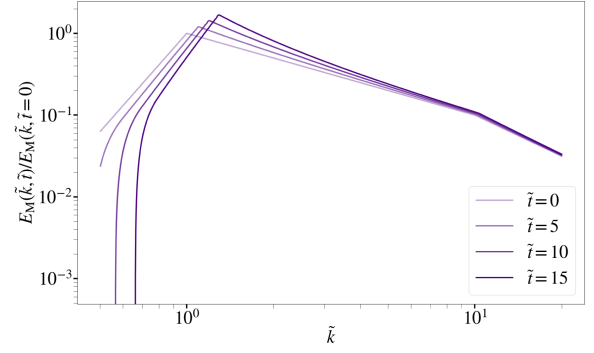


FIG. 2. Evolution of the magnetic energy density $E_M(\tilde{k}, \tilde{t}) / E_M(\tilde{k}, \tilde{t} = 0)$ according to Eq. (12) for a fixed normalized collapse rate, $\tilde{\zeta}_0 = 0.01$.

structures. However, this constant collapse rate implies that all structures collapse at the same rate at any given time, which is unphysical and does not realistically model the formation of structures. To have a collapse at a given Jeans wave number k_J , the collapse rate is defined from $k = k_J$ to $k \rightarrow \infty$ to be

$$\zeta(k) = \zeta_0 [1 - \tanh \{a(k - k_J)\}], \quad (22)$$

where the parameter a controls the smoothness of transition between the two regimes. In a more realistic model of collapse, this parameter will likely be superseded by new ones, since a new functional form of $\zeta(k)$ will be needed to describe the collapse. This is a simple toy model of collapse, intended to check that the formalism is self-consistent, and the predictions agree qualitatively with our expectations. Numerically for small k , to ensure that the $k = 0$ to $k = 1$ regime has roughly a power spectrum of k^4 , changing the evolution of ζ_0 to become a function of k can resolve this issue. We define $\zeta(k)$ for the range $k \in [0, k_J]$ to be

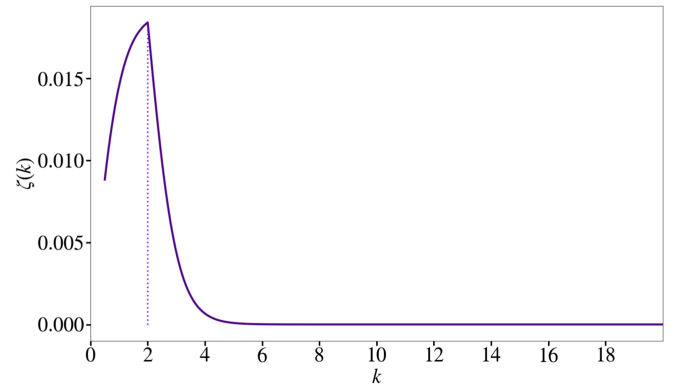


FIG. 3. Collapse rate $\zeta(k)$, as defined by Eqs. (22) and (23), plotted as a function of k , with $k_J = 2$, as indicated by the vertical line.

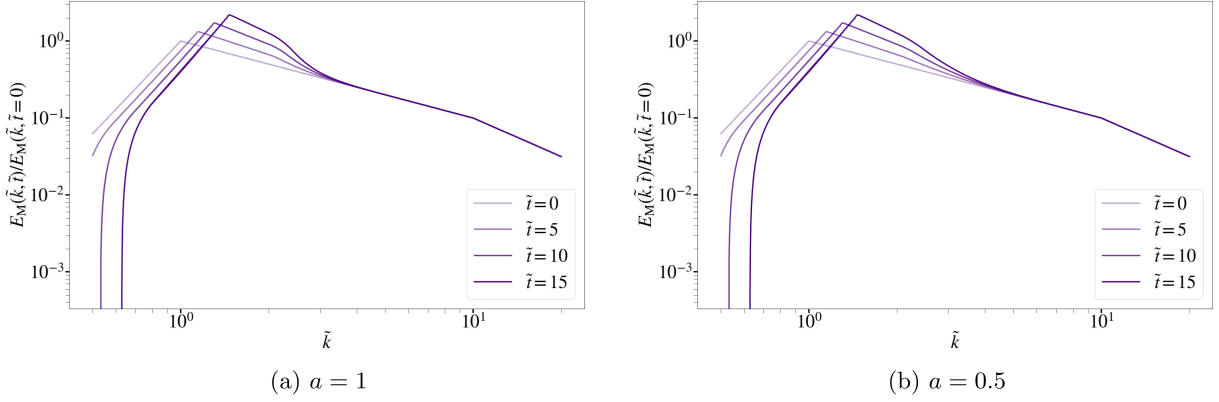


FIG. 4. Evolution of the normalized magnetic energy density $\tilde{E}_M(\tilde{k}, \tilde{t})$ according to Eq. (18), and a realistic collapse rate given by Eqs. (23) and (22). We show the results for two different values of a , which characterizes the transition of the collapse rate across the Jeans wave number.

$$\zeta(k) = \frac{\pi\zeta_0}{2} \arctan(k). \quad (23)$$

The resulting collapse rate is shown in Fig. 3.

Then we model the evolution of E_M , using Eq. (12). In Fig. 4, we show the evolution of $E_M(k)$ for this realistic collapse rate, using $k_J = 2$ and $\zeta_0 = 0.03$, for two different values of a . We note here that we choose these values for visual purposes; for other choices of k_J , ζ_0 , and a , the solution behaves in an analogous way. The plots use a time defined as the location along the length grid $N_t = 10000$. Indeed, we see a downward collapse for both values of a , which is consistent with expectations. In contrast to the case of the constant collapse rate ζ , as shown in Fig. 2, the growth of magnetic energy in the present case is suppressed at small scales (i.e., large k) due to the existence of the Jeans scale in the realistic collapse rate $\zeta(k)$, as structures smaller than the Jeans scale do not collapse. In both cases, the amplification of magnetic fields is the result of magnetic flux compactification due to the gravitational collapse of structures.

We now apply our forward cascade model to the magnetic energy spectra obtained from cosmological simulations in Figs. 5 and 6; i.e., we take the spectrum from simulations at $z = 10$ and evolve it using Eqs. (18), (22), and (23) for two different values of ζ_0 . These figures show a comparison of these two spectra for different PMF models at the same t/t_{ff} , with t_{ff} being the dynamical time measured in cosmological simulations and in our calculations using Eq. (F1). We note that the presented comparison is only semiquantitative as several aspects of our numerical calculations and those conducted using cosmological simulations differ (e.g., the dynamical time in cosmological simulations varies at each timestep).

In Figs. 5 and 6, we notice some similarities between the results of cosmological simulations and the semianalytic solutions. Specifically, growth of the energy power

spectrum on large scales (small wave numbers) for the uniform model occurs in a similar fashion as is observed in cosmological simulations (although the growth is larger in the latter case). We also see a forward cascade only for the smaller-scale models (helical and nonhelical), although the shift of the energy peak from large to small scales is faster in simulations. The energy growth in the semianalytic case is most efficient near the initial peak scale of the power spectrum, likely caused by the constant free-fall time used in this case. Accounting for a more realistic collapse rate is expected to yield better agreement with simulation results, which will be explored in a future work. It should also be noticed that at small wave numbers, i.e., those below the peak of the magnetic energy spectra, and later times, the magnetic energy slightly increases in the simulations [Fig. 1, panels (c) and (d)], an effect not seen in our semianalytic model. This is because in the simulations, the growth of the field in all stochastic cases is dominated by nonlinear effects, which we think leads to the transfer of energy to both large and small scales, and also the fact that dissipative effects are ignored in the semianalytic model in order to focus on the amplification effects.

The absence of forward cascade for large-scale models is caused, in our understanding, by a small coupling between the small- and large-scale modes, since for these cases most of the magnetic energy is concentrated on large scales. We also observe from Figs. 5 and 6 that the energy uniformly increases on large scales for uniform and scale-invariant models. This uniform growth is also seen for the evolution of the uniform model [Fig. 1, panel (a)] in simulations. In simulations, this happens because the amplification of the uniform model is governed by the (linear) growth of density perturbations. In the scale-invariant case [Fig. 1, panel (b)], on the other hand, from the simulations we see that growth is not uniform and is larger on smaller scales (intermediate and large wave numbers).

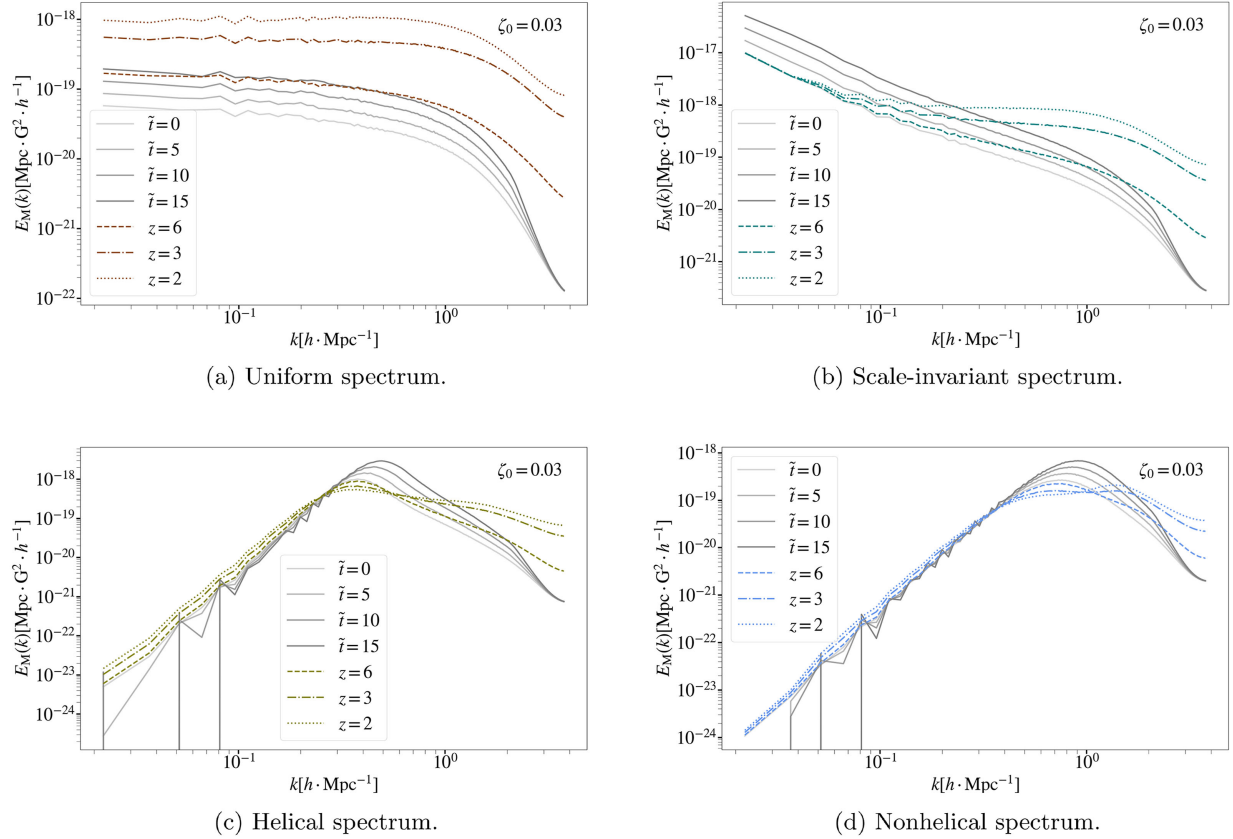


FIG. 5. Comparison between the time evolution of the magnetic energy spectra, $E_M(k)$, obtained from the $z = 10$ snapshot of the simulations in Ref. [63] and evolved using the analytical model [solid lines; Eq. (18)], and the evolution of $(E_M(k))$ at different redshifts in the numerical simulations (dashed, dashed-dotted, and dotted lines). The different panels correspond to different initial spectra for the seed PMFs: uniform (a), scale invariant (b), helical (c), and nonhelical (d). The spectra obtained from the analytical model show indications of numerical instability, likely caused by the relatively large dk used in these simulations. For the result shown here, the collapse-rate parameter in the analytical model was set to $\zeta_0 = 0.03$.

V. CONCLUSIONS

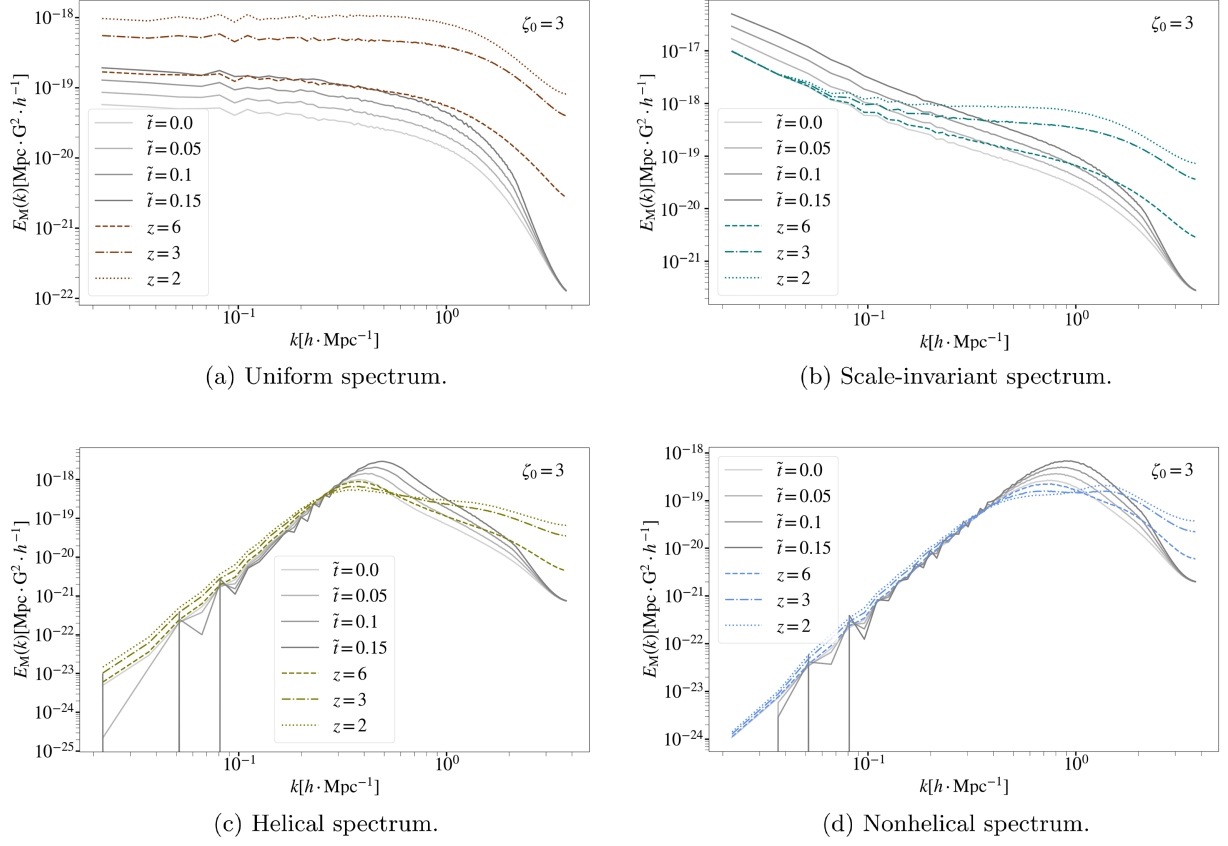
In this work, we semianalytically computed the forward cascade of the energy spectrum of PMFs in the late stages of the evolution of the Universe. This evolution is a result of the adiabatic collapse of structures, which, via the conservation of magnetic flux, strengthens the magnetic fields and pushes them to smaller scales. We expect that this forward cascade, which occurs during the era of structure formation, partially compensates for the inverse cascade sometimes undergone by the PMFs in the primordial plasma prior to recombination, i.e., at much earlier times.⁴

We showed that this forward cascade depends on the characteristics (i.e., the spectral shape, fractional helicity, etc.) of the magnetic field itself. For fields that have small coherence scales ($\lesssim 2$ Mpc) we observe the forward cascade to be pronounced on scales below the Jeans scale for large-scale correlated fields we do not observe the forward

cascade. Similarly to the results from cosmological simulations [63], we see that the power spectrum for the uniform model increases homogeneously on large and intermediate scales. Our analysis, however, does not seem to be applicable on smaller scales where nonlinear growth dominates over linear growth; this is true for all different magnetic field models studied in this work. We must also note that uncertainties in the evolution of magnetic fields at high wave numbers also exist in simulations, e.g., due to limited resolution [97].

An accurate understanding of this forward cascade has important implications for understanding and constraining models and mechanisms of the generation and evolution of PMFs. From an observational standpoint, the forward cascade of magnetic energy implies that the effects of PMF on large scale structures would be on smaller scales than in the absence of a forward cascade, and an accurate modeling of this cascade will tell us the relevant scales on which the effects of PMF on large scale structures are to be studied. Importantly, since the forward cascade would occur at a much later era compared to the epoch of

⁴This inverse cascade depends on many factors, and importantly on the presence of magnetic helicity; see Appendix A for more details.

FIG. 6. Same as Fig. 5 but for $\zeta_0 = 3$.

recombination, these scales would be smaller than the analogous scales on the CMB sky on which PMFs potentially leave their imprint.

In this work, we have considered a simple model of structure collapse, namely a uniform collapse rate of all structures larger than the Jeans length scale. This approximation already highlights the general qualitative features seen in numerical simulations. There, a forward cascade emerges at the onset of structure formation around $z \sim 10$. In the cosmological simulations where the spectral peak lies within the simulation domain, we observe moderate large-scale growth, a decline in magnetic energy near the initial peak, and a gradual transfer of energy to smaller spatial scales. This latter behavior is well captured by our analytical toy model.

In order to model the interplay between structures and PMFs more realistically, we have to relax our simple assumption of the form of the collapse rate $\zeta(k)$ we considered in our analysis above, and allow ζ to depend on other parameters; one such simple extension, where structures on the same scale may collapse at different mass-dependent rates, is discussed in the Appendix D. Such an analysis, which we defer to future work, is expected to capture the effect of variable collapse rate on the forward cascade of magnetic energy and would be a more realistic comparison to the simulations. In the present

work, we have restricted our scope to a proof-of-concept demonstration of our hypothesis that the adiabatic collapse of large-scale structures leads to a forward cascade of the PMF power spectrum, and that the solutions obtained from Eq. (12) using the simplest models of collapse are consistent with the principle of flux conservation and our expectations of PMF amplifications based on that and is able to qualitatively reproduce some of the evolutionary behavior of the PMF spectra seen in numerical simulations, as discussed above. The evolution of the spectrum is also affected by the various conservation conditions obeyed by the fluid quantities, like helicity, the Loitsianky and Saffman integrals, etc. depending on the turbulence conditions. It would therefore be instructive to study the solution of the forward cascade equations developed here subject to such conservation conditions for a more accurate quantitative comparison with simulations. In our recent work [91] we have conducted numerical simulations to understand the interplay between forward cascade and dynamo action; nonetheless, in the latter work, we used isolated spherical collapse setup, while understanding such interplay in a cosmological context, is still missing. For instance, although in [91] we have shown that even with high-resolution, self-gravitating setup we do obtain a forward cascade behavior in the evolution of magnetic fields (later on accompanied by dynamo action) it remains

an open question which magnetic coherence scales leads to this behaviour (see e.g., for comparison Fig. 8 and Fig. C.1 in [91]), which might be important in constraining the coherence scales of PMFs.

The results of our work even being phenomenological, open a new direction in the study of the amplification of large-scale magnetic fields due to budgetary use of computational resources. To better understand how galaxy-cluster-scale fields have been amplified, we need to account for the forward cascade of small-scale correlated fields (possibly originating from phase transitions) during structure formation. The forward cascade of magnetic fields not only amplifies such fields on scales of collapsing regions, but also drives turbulence. Therefore, an interplay between field amplification due to forward cascade and turbulence should be carefully studied. This, in turn, may help us understand how efficiently magnetic fields grow during structure formation and how strong (or weak) initial seed magnetic fields must have been.

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DATA AVAILABILITY

The source codes used for the numerical solutions in this study, and data for the different plots, are freely available at this repository <https://github.com/mpa710/pmf>.

APPENDIX A: MAGNETIC HELICITY AND REALIZABILITY CONDITION

Magnetic fields generated during inflation or PTs may be helical if the underlying generation processes involve parity violating processes (see Refs. [98–102] for a few examples and Ref. [5] for a detailed census of models). Magnetic

helicity is an important quantity that impacts the evolution of magnetic fields in a charged plasma. We define the mean magnetic helicity density over a comoving volume V , with fully contained field lines, as

$$\mathcal{H}_M = \frac{1}{V} \int_V d^3\mathbf{x} \mathbf{A} \cdot \mathbf{B}, \quad (\text{A1})$$

where the comoving magnetic vector potential $\mathbf{A}$ is defined such that⁵ $\mathbf{B} = \nabla \times \mathbf{A}$. Magnetic helicity is a nonlocal quantity and can be understood from a topological point of view in terms of the linkage of nonoverlapping magnetic flux tubes [103]; in the mathematical literature, this quantity is typically called a *generalized asymptotic form of the Hopf invariant* [104]. Analogously to Eq. (4), we can express the mean magnetic helicity density in terms of the spectral magnetic helicity density as

$$\mathcal{H}_M = \int_0^\infty dk H_M(k). \quad (\text{A2})$$

Application of the *Cauchy-Buyanovsky-Schwarz inequality* to the distribution of helical magnetic fields implies the existence of an upper bound on the magnetic helicity given by the realizability condition⁶

$$|\mathcal{H}_M| \leq 2\xi_M \mathcal{E}_M; \quad (\text{A4})$$

we can thus define a quantity called fractional magnetic helicity,

$$\epsilon_M \equiv \frac{\mathcal{H}_M(t)}{2\xi_M(t)\mathcal{E}_M(t)} \in [-1, 1]. \quad (\text{A5})$$

During the magnetic field decay the field lines are redistributed in such a way that the fractional helicity reaches its maximal value, i.e. $\epsilon_M = 1$. The decay time necessary to obtain the maximal helical configuration is determined by the initial values of ϵ_M (inversely proportional ϵ_M^2 [105]).

The presence of magnetic helicity strongly affects the evolution of turbulent magnetic fields: the conservation of magnetic helicity in the highly conductive plasma of the early universe leads to the inverse cascade of energy in the helical magnetic fields, where magnetic energy is

⁵The *physical* field $\mathbf{B}_{\text{phys}}$ and the vector potential $\mathbf{A}_{\text{phys}}$ are given by $\mathbf{B} = a^2 \mathbf{B}_{\text{phys}}$, and $\mathbf{A} = a \mathbf{A}_{\text{phys}}$, and are related as $\mathbf{B}_{\text{phys}} = \nabla_{\mathbf{y}} \times \mathbf{A}_{\text{phys}}$, where $\mathbf{y} = a\mathbf{x}$ denotes the physical coordinates.

⁶Alternatively, we can state that there exists a lower bound on ξ_M for a given helicity,

$$\xi_M^{\text{min}}(t) \equiv \frac{|\mathcal{H}_M(t)|}{2\mathcal{E}_M(t)}, \quad (\text{A3})$$

which implies $\xi_M(t) \geq \xi_M^{\text{min}}(t)$.

transferred from small to larger length scales; helical magnetic fields also decay more slowly compared to non-helical fields [74,106–110]. The time evolution of the energy density and the correlation length on dimensional grounds have been studied by several authors. In particular, if helicity is conserved, the characteristic scale of magnetic fields increases with time, i.e. an inverse cascade of magnetic energy to larger length scales occurs during the MHD evolution [74,106,108].

APPENDIX B: JEANS WAVE NUMBER IN SIMULATIONS

In Fig. 7, we show the comoving Jeans wave number $k_J(z) = a(z)\sqrt{4\pi G\bar{\rho}(z)}/c_s$ as a function redshift z for all four cases in Fig. 1; here, $a(z)$ is the scale factor, c_s is the speed of sound, and $\bar{\rho}$ is the average *physical* density of matter (i.e., of gas and dark matter combined).

APPENDIX C: RESOLUTION EFFECT FOR SIMULATIONS

In Fig. 8 we see that for our higher-resolution run (with 1024^3 grid points) the final peak of the magnetic energy is not as pronounced as for the lower-resolution case. However, we still see a less pronounced bump at $k \sim 4h \text{ cMpc}^{-1}$. This bump is at higher k than the peak for the 512^3 run. This shift to further higher wave numbers might be governed by the shift of the peak of the density field in the higher-resolution case.

APPENDIX D: GENERALIZATION CONSIDERING MASS-DEPENDENT COLLAPSE RATE

We generalize $\zeta(k, t)$ to define a mass-dependent rate of collapse of the structure $\zeta(m, k, t)$ that is the rate at which a mass m on the scale k shrinks over time t . We also define $\nu(m, k, t)$ to be the number density of the structures of mass m on scale k on time t , i.e., $\nu(m, k, t)dm$ is the number of structures on scale k with masses between m and $m + dm$.

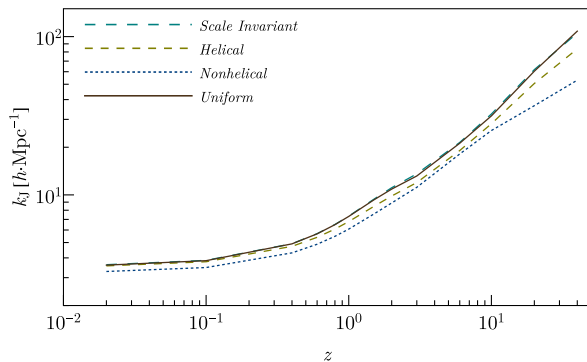


FIG. 7. Evolution of the Jeans wave number $k_J(z) = a(z)\sqrt{4\pi G\bar{\rho}(z)}/c_s$ with redshift z for all four cases in Fig. 1.

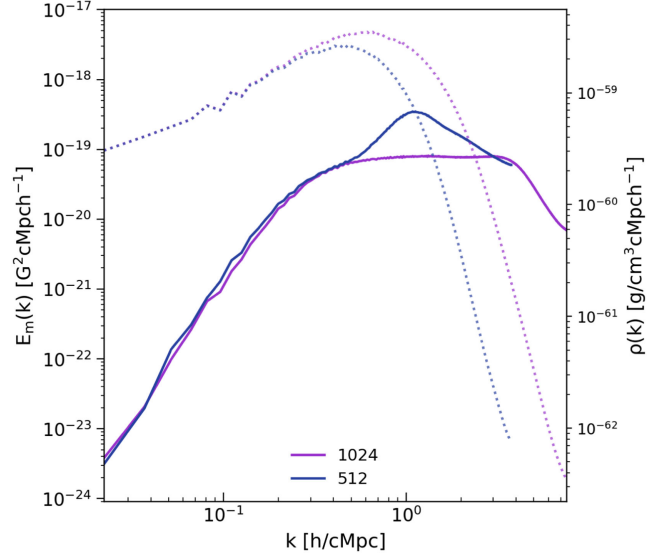


FIG. 8. Density (lower-opacity dotted lines) and magnetic energy power spectrum for two different resolutions, for 512^3 and 1024^3 grid points, from our cosmological simulations.

Thus, the number of structures at scale k at time t is given by

$$N(k, t) = \int dm \nu(m, k, t). \quad (\text{D1})$$

We can work out a time evolution of the number density by noting from a generalized form of Eq. (9) that a structure of the mass m at time $t + dt$ at scale k was at a scale $k - \zeta(m, k, t)dt$ at time t ; this collapse preserves the number of structures with masses between m and $m + dm$, i.e.,

$$\nu(m, k, t + dt) = \nu(m, k - \zeta(m, k, t)dt, t), \quad (\text{D2})$$

leading to the evolution equation [suppressing the (m, k, t) dependence of ζ and ν]

$$\frac{\partial \nu}{\partial t} + \zeta \frac{\partial \nu}{\partial k} = 0. \quad (\text{D3})$$

It is convenient to define a fractional density $\pi(m, k, t)$ of structures of mass m at scale k at time t as

$$\pi(m, k, t) \equiv \frac{\nu(m, k, t)}{N(k, t)}, \quad (\text{D4})$$

so that from Eq. (D1) we get

$$\int dm \pi(m, k, t) = 1. \quad (\text{D5})$$

Considering that the rate of structure collapse is mass dependent, we can generalize Eq. (10) as

$$\begin{aligned}
E_M(k, t + dt) &= \int dm \pi(m, k - \zeta(m, k, t)dt, t) \\
&\times \left[1 - \frac{\zeta(m, k, t)dt}{k} \right]^{-4} \\
&\times E_M(k - \zeta(m, k, t)dt, t). \quad (D6)
\end{aligned}$$

Again, with appropriate binomial and Taylor expansions up to $\mathcal{O}(dt)$, we can write the above as

$$\begin{aligned}
&E_M(k, t) + \frac{\partial E_M(k, t)}{\partial t} dt \\
&= \int dm \left\{ \left[\pi(m, k, t) - \frac{\partial \pi(m, k, t)}{\partial k} \zeta(m, k, t) \right] \right. \\
&\quad \times \left[1 + \frac{4\zeta(m, k, t)dt}{k} \right] \left[E_M(k, t) - \frac{\partial E_M(k, t)}{\partial k} \zeta(m, k, t)dt \right] \Big\} \\
&= E_M(k, t) \\
&\quad + \int dm \zeta(m, k, t) \left\{ \pi(m, k, t) \left[\frac{4}{k} - \frac{\partial}{\partial k} \right] - \frac{\partial \pi(m, k, t)}{\partial k} \right\} \\
&\quad \times E_M(k, t) dt + \mathcal{O}(dt^2), \quad (D7)
\end{aligned}$$

where in the second step, we have used Eq. (D5). We can simplify Eq. (D7) to get the mass-dependent forward cascade equation

$$\begin{aligned}
&\frac{\partial E_M(k, t)}{\partial t} + \int dm \zeta(m, k, t) \\
&\times \left\{ \frac{\partial \pi(m, k, t)}{\partial k} - \pi(m, k, t) \left[\frac{4}{k} - \frac{\partial}{\partial k} \right] \right\} E_M(k, t) = 0, \quad (D8)
\end{aligned}$$

which is a generalized version of the cascade equation Eq. (12).

APPENDIX E: IMPLICIT NUMERICAL METHODS AND METHOD OF CHARACTERISTICS

To check the numerical solutions as solved with the explicit finite difference method, we additionally use the implicit finite difference method. This method uses the “inverse” stencil for the explicit method, generating an equation of the form

$$\frac{E_{M,t+1}^k - E_{M,t}^k}{\Delta t} + \zeta_0 \frac{E_{M,t+1}^k - E_{M,t+1}^{k-1}}{\Delta k} - \frac{4\zeta_0}{k} E_{M,t+1}^k = 0, \quad (E1)$$

which yields a system of linear equations for $E_{M,t+1}^k$,

$$E_{M,t+1}^k \left(1 + \lambda - \frac{4\lambda\Delta k}{k} \right) - \lambda E_{M,t+1}^{k-1} = E_{M,t}^k. \quad (E2)$$

Here $\lambda \equiv \zeta_0 \Delta t / \Delta k$ and in matrix form,

$$A = \begin{bmatrix} 1 + \lambda - \frac{4\lambda\Delta k}{k} & 0 & \dots \\ -\lambda & 1 + \lambda - \frac{4\lambda\Delta k}{k} & \dots \\ \dots & -\lambda & 1 + \lambda - \frac{4\lambda\Delta k}{k} \end{bmatrix}; \quad (E3)$$

this yields

$$A E_{M,t+1}^k = E_{M,t}^k, \quad (E4)$$

and the solution of this system of linear equations is shown in Fig. 9.

The method of characteristics solves for the evolution of the magnetic energy density with a fixed rate of collapse ζ_0 . We start with the Lagrange-Charpit equations,

$$\frac{dt}{1} = \frac{dk}{\zeta_0} = \frac{k dE_M}{4\zeta_0 E_M}. \quad (E5)$$

Substituting

$$k(t) = \zeta_0 t + k_0 \quad (E6)$$

into Eq. (E5) and solving, we get

$$\ln(E_M) = 4 \ln(k(t)) + C' \Rightarrow E_M = C k^4. \quad (E7)$$

The initial conditions imply that

$$C = \frac{E_0(k_0)}{k_0^4}, \quad (E8)$$

where $E_0(k)$ is the initial condition function, solved at $t = 0$. We therefore arrive at the final equation,

$$E_M(k, t) = E_0(k - \zeta t) \left[\frac{k}{k - \zeta_0 t} \right]^4, \quad (E9)$$

which is plotted in Fig. 10.

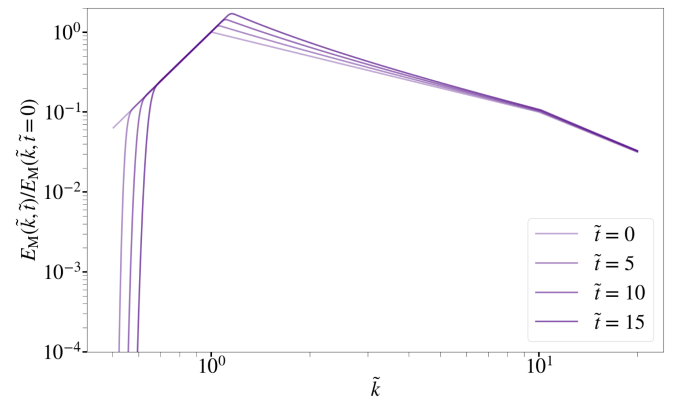


FIG. 9. Implicit numerical solution for Eq. (12).

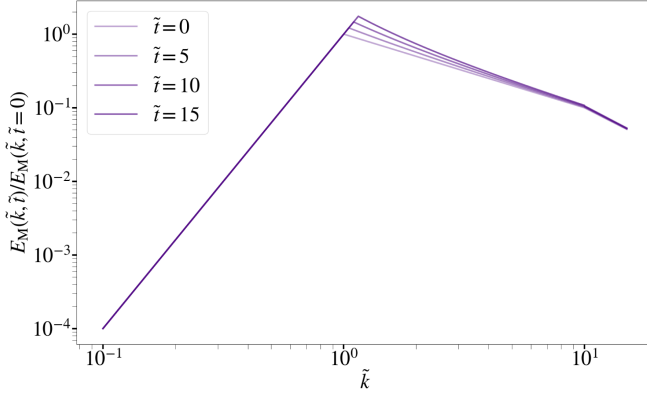


FIG. 10. Method of Characteristics solution for Eq. (12).

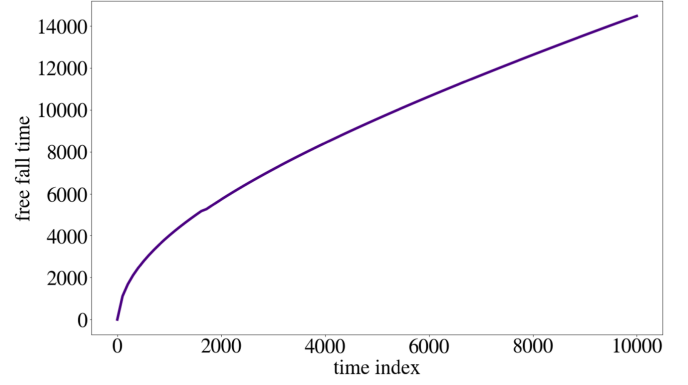


FIG. 11. Free fall time versus time index for Fig. 2, using Eq. (F5).

APPENDIX F: COMPARISON OF FREE FALL TIME IN NUMERICAL SIMULATIONS TO CLASSICAL FREE FALL TIME

To make a crude estimate of the collapse of this material, we start with the known free fall time for a pressureless, uniform sphere,

$$t_{\text{ff}} = \sqrt{\frac{3\pi}{32G\rho_{\text{max}}}}, \quad (\text{F1})$$

and also write the Newtonian force equation for an arbitrary particle of mass m on the surface of the sphere with mass M , where $r(t)$ is taken as the radius of the sphere, as

$$\ddot{r}(t) = -\frac{GM}{r(t)^2}. \quad (\text{F2})$$

The mass M can be rewritten in terms of density and radius as $\frac{4}{3}\pi r^3(t)\rho_{\text{max}}$, and we assume that there is a uniform density $\rho=\rho_{\text{max}}$ throughout the space occupied by the material.

Substituting the above definition of M in Eq. (F2), we obtain

$$\ddot{r}(t) = -\frac{4\pi\rho_{\text{max}}Gr(t)}{3}. \quad (\text{F3})$$

Defining $\omega^2 = 4\pi\rho_{\text{max}}G/3$, we see that our solution is

$$r(t) = r_0 \cos(\omega t), \quad (\text{F4})$$

where r_0 is the radius at $t = 0$, and the natural sinusoidal term vanishes due to the initial collapse rate of the sphere being assumed to be zero. Under the assumption that, in our solutions, $\lambda \sim r \Rightarrow \frac{2\pi}{k} \sim r$, and taking $r_0 \sim k_0$, where k_0 is unity in our simulations, we rewrite this equation as follows.

$$\arccos\left(\frac{1}{k}\right) = \omega t = \frac{\pi t}{2\sqrt{2}t_{\text{ff}}}, \quad (\text{F5})$$

where k is the location of the peak in Fig. 2 for a given time t , where the time index versus the time of free fall is plotted in Fig. 11. We note an increase in free fall time with respect to time, an error likely attributable to the assumption of fixed density throughout space.

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- [1] K. Subramanian, *Rep. Prog. Phys.* **79**, 076901 (2016).
 - [2] R. M. Kulsrud and E. G. Zweibel, *Rep. Prog. Phys.* **71**, 0046091 (2008).
 - [3] S. Naoz and R. Narayan, *Phys. Rev. Lett.* **111**, 051303 (2013).
 - [4] M. Langer and J.-B. Durrieu, *Galaxies* **6**, 124 (2018).
 - [5] A. Kandus, K. E. Kunze, and C. G. Tsagas, *Phys. Rep.* **505**, 1 (2011).
 - [6] A. Neronov and I. Vovk, *Science* **328**, 73 (2010).
 - [7] F. Tavecchio, G. Ghisellini, G. Bonnoli, and L. Foschini, *Mon. Not. R. Astron. Soc.* **414**, 3566 (2011).

- [8] W. Essey, S. Ando, and A. Kusenko, *Astropart. Phys.* **35**, 135 (2011).
- [9] A. M. Taylor, I. Vovk, and A. Neronov, *Astron. Astrophys.* **529**, A144 (2011).
- [10] H. Huan, T. Weisgarber, T. Arlen, and S. P. Wakely, *Astrophys. J. Lett.* **735**, L28 (2011).
- [11] I. Vovk, A. M. Taylor, D. Semikoz, and A. Neronov, *Astrophys. J. Lett.* **747**, L14 (2012).
- [12] K. Dolag, M. Kachelriess, S. Ostapchenko, and R. Tomas, *Astrophys. J. Lett.* **727**, L4 (2011).

- [13] K. Takahashi, M. Mori, K. Ichiki, and S. Inoue, *Astrophys. J. Lett.* **744**, L7 (2012).
- [14] C. D. Dermer, M. Cavadini, S. Razzaque, J. D. Finke, J. Chiang, and B. Lott, *Astrophys. J. Lett.* **733**, L21 (2011).
- [15] J. D. Finke, L. C. Reyes, M. Georganopoulos, K. Reynolds, M. Ajello, S. J. Fegan, and K. McCann, *Astrophys. J.* **814**, 20 (2015).
- [16] S. Archambault *et al.* (VERITAS Collaboration), *Astrophys. J.* **835**, 288 (2017).
- [17] M. Ackermann *et al.* (Fermi-LAT Collaboration), *Astrophys. J. Suppl. Ser.* **237**, 32 (2018).
- [18] A. Saveliev and R. Alves Batista, *Mon. Not. R. Astron. Soc.* **500**, 2188 (2020).
- [19] R. Alves Batista and A. Saveliev, *Astrophys. J. Lett.* **902**, L11 (2020).
- [20] A. Korochkin, A. Neronov, G. Lavaux, M. Ramsøy, and D. Semikoz, *J. Exp. Theor. Phys.* **134**, 498 (2022).
- [21] E. I. Podlesnyi, T. A. Dzhatdov, and V. I. Galkin, *Mon. Not. R. Astron. Soc.* **516**, 5379 (2022).
- [22] V. A. Acciari *et al.* (MAGIC Collaboration), *Astron. Astrophys.* **670**, A145 (2023).
- [23] P. Da Vela, G. Martí-Devesa, F. G. Saturni, P. Veres, A. Stamerra, and F. Longo, *Phys. Rev. D* **107**, 063030 (2023).
- [24] F. Aharonian *et al.* (Fermi-LAT, H. E. S. S. Collaborations), *Astrophys. J. Lett.* **950**, L16 (2023).
- [25] T. A. Dzhatdov, E. I. Podlesnyi, and G. I. Rubtsov, *Mon. Not. R. Astron. Soc.* **527**, L95 (2023).
- [26] Y.-Y. Huang, C.-Y. Dai, H.-M. Zhang, R.-Y. Liu, and X.-Y. Wang, *Astrophys. J. Lett.* **955**, L10 (2023).
- [27] I. Vovk, A. Korochkin, A. Neronov, and D. Semikoz, *Astron. Astrophys.* **683**, A25 (2024).
- [28] A. Pagliotta, C. J. Riseley, A. Bonafede, C. Stuardi, and F. Loi, *Astron. Astrophys.* **700**, A139 (2025).
- [29] T. Keita, R. Belmont, and T. Stolarczyk, *Proc. Sci. ICRC2025* (2025) 710, [arXiv:2509.16092], <https://pos.sissa.it/501/710>.
- [30] L. Burmeister, P. Da Vela, F. Longo, G. Martí-Devesa, M. Meyer, F. G. Saturni, A. Stamerra, and P. Veres, *Phys. Rev. D* **113**, 043041 (2026).
- [31] I. Vovk, *J. High Energy Astrophys.* **52**, 100577 (2026), <https://www.sciencedirect.com/science/article/pii/S2214404826000297>.
- [32] J. Tjemsland, M. Meyer, and F. Vazza, *Astrophys. J.* **963**, 135 (2024).
- [33] T. C. Arlen, V. V. Vassiliev, T. Weisgarber, S. P. Wakely, and S. Y. Shafi, *Astrophys. J.* **796**, 18 (2014).
- [34] A. E. Broderick, P. Tiede, P. Chang, A. Lamberts, C. Pfrommer, E. Puchwein, M. Shalaby, and M. Werhahn, *Astrophys. J.* **868**, 87 (2018).
- [35] D. Yan, J. Zhou, P. Zhang, Q. Zhu, and J. Wang, *Astrophys. J.* **870**, 17 (2019).
- [36] R. Alves Batista, A. Saveliev, and E. M. de Gouveia Dal Pino, *Mon. Not. R. Astron. Soc.* **489**, 3836 (2019).
- [37] R. Perry and Y. Lyubarsky, *Mon. Not. R. Astron. Soc.* **503**, 2215 (2021).
- [38] M. Alawashra and M. Pohl, *Astrophys. J.* **929**, 67 (2022).
- [39] C. Blanco, O. Ghosh, S. Jacobsen, and T. Linden, *arXiv*: 2303.01524.
- [40] L. E. Espinosa Castro, S. Rossoni, and G. Sigl, *arXiv*: 2405.15390.
- [41] M. Alawashra, I. Vovk, and M. Pohl, *Astrophys. J.* **978**, 95 (2025).
- [42] S. Dey and G. Sigl, *Proc. Sci. ICRC2025* (2025) 627, [arXiv:2509.17104], <https://pos.sissa.it/501/627>.
- [43] K. Bondarenko, A. Boyarsky, A. Korochkin, A. Neronov, D. Semikoz, and A. Sokolenko, *Astron. Astrophys.* **660**, A80 (2022).
- [44] F. Miniati and A. R. Bell, *Astrophys. J.* **729**, 73 (2011).
- [45] M. Fernández Alonso, A. D. Supanitsky, and A. C. Rovero, *Astrophys. J.* **869**, 43 (2018).
- [46] D. Garg, R. Durrer, and J. Schober, *arXiv*:2505.14774.
- [47] O. Ghosh, A. Brandenburg, C. Caprini, A. Neronov, and F. Vazza, *Phys. Rev. D* **113**, 023523 (2026).
- [48] T. Vernstrom, G. Heald, F. Vazza, T. J. Galvin, J. West, N. Locatelli, N. Fornengo, and E. Pinetti, *Mon. Not. R. Astron. Soc.* **505**, 4178 (2021).
- [49] T. Hodgson, F. Vazza, M. Johnston-Hollitt, and B. McKinley, *Pub. Astron. Soc. Aust.* **39**, e033 (2022).
- [50] E. Carretti, S. P. O'Sullivan, V. Vacca, F. Vazza, C. Gheller, T. Vernstrom, and A. Bonafede, *Mon. Not. R. Astron. Soc.* **518**, 2273 (2022).
- [51] E. Carretti, F. Vazza, S. P. O'Sullivan, V. Vacca, A. Bonafede, G. Heald, C. Horellou, S. Mtchedlidze, and T. Vernstrom, *Astron. Astrophys.* **693**, A208 (2025).
- [52] F. Vazza, C. Gheller, F. Zanetti, M. Tsizh, E. Carretti, S. Mtchedlidze, and M. Brüggén, *Astron. Astrophys.* **696**, A58 (2025).
- [53] S. P. O'Sullivan *et al.*, *Mon. Not. R. Astron. Soc.* **495**, 2607 (2020).
- [54] E. Carretti *et al.*, *Mon. Not. R. Astron. Soc.* **512**, 945 (2022).
- [55] V. Pomakov, S. O'Sullivan, M. Brüggén, F. Vazza, E. Carretti, G. Heald, C. Horellou, T. Shimwell, A. Shulevski, and T. Vernstrom, *Mon. Not. R. Astron. Soc.* **515**, 256 (2022).
- [56] A. Aramburo-García, K. Bondarenko, A. Boyarsky, A. Neronov, A. Scaife, and A. Sokolenko, *Mon. Not. R. Astron. Soc.* **515**, 5673 (2022).
- [57] S. Mtchedlidze, F. Vazza, X. Du, E. Carretti, C. Stuardi, and S. P. O'Sullivan, *Astron. Astrophys.* **708**, A336 (2026).
- [58] E. Carretti, S. P. O'Sullivan, V. Vacca, F. Vazza, C. Gheller, T. Vernstrom, and A. Bonafede, *Mon. Not. R. Astron. Soc.* **518**, 2273 (2023), .
- [59] F. Vazza *et al.*, *Galaxies* **9**, 109 (2021).
- [60] S. Mtchedlidze, P. Domínguez-Fernández, X. Du, E. Carretti, F. Vazza, S. P. O'Sullivan, A. Brandenburg, and T. Kahniashvili, *Astrophys. J.* **977**, 128 (2024).
- [61] A. Brandenburg and T. Kahniashvili, *Phys. Rev. Lett.* **118**, 055102 (2017).
- [62] S. Hutschenreuter, S. Dorn, J. Jasche, F. Vazza, D. Paoletti, G. Lavaux, and T. A. Enßlin, *Classical Quantum Gravity* **35**, 154001 (2018).
- [63] S. Mtchedlidze, P. Domínguez-Fernández, X. Du, A. Brandenburg, T. Kahniashvili, S. O'Sullivan, W. Schmidt, and M. Brüggén, *Astrophys. J.* **929**, 127 (2022).
- [64] G. L. Bryan *et al.* (ENZO Collaboration), *Astrophys. J. Suppl. Ser.* **211**, 19 (2014).
- [65] T. W. B. Kibble, *J. Phys. A* **9**, 1387 (1976).
- [66] C. J. Hogan, *Phys. Rev. Lett.* **51**, 1488 (1983).

- [67] E. Witten, *Phys. Rev. D* **30**, 272 (1984).
- [68] D. J. Schwarz and M. Stuke, *J. Cosmol. Astropart. Phys.* **11** (2009) 025; **10** (2010) E01.
- [69] V. A. Rubakov and D. S. Gorbunov, *Introduction to the Theory of the Early Universe: Hot big bang theory* (World Scientific, Singapore, 2017).
- [70] T. Kahniashvili, A. Brandenburg, R. Durrer, A. G. Tevzadze, and W. Yin, *J. Cosmol. Astropart. Phys.* **12** (2016) 002.
- [71] A. Brandenburg, R. Durrer, Y. Huang, T. Kahniashvili, S. Mandal, and S. Mukohyama, *Phys. Rev. D* **102**, 023536 (2020).
- [72] T. Kahniashvili, A. G. Tevzadze, S. K. Sethi, K. Pandey, and B. Ratra, *Phys. Rev. D* **82**, 083005 (2010).
- [73] R. Banerjee and K. Jedamzik, *Phys. Rev. D* **70**, 123003 (2004).
- [74] A. Brandenburg, T. Kahniashvili, S. Mandal, A. Roper Pol, A. G. Tevzadze, and T. Vachaspati, *Phys. Rev. D* **96**, 123528 (2017).
- [75] D. N. Hosking and A. A. Schekochihin, *Nat. Commun.* **14**, 7523 (2023).
- [76] A. Brandenburg, A. Neronov, and F. Vazza, *Astron. Astrophys.* **687**, A186 (2024).
- [77] R. Durrer and A. Neronov, *Astron. Astrophys. Rev.* **21**, 62 (2013).
- [78] T. Kahniashvili, A. Brandenburg, and A. G. Tevzadze, *Phys. Scr.* **91**, 104008 (2016).
- [79] F. Marinacci, M. Vogelsberger, P. Mocz, and R. Pakmor, *Mon. Not. R. Astron. Soc.* **453**, 3999 (2015).
- [80] J. Donnert, F. Vazza, M. Brüggen, and J. ZuHone, *Space Sci. Rev.* **214**, 122 (2018).
- [81] S. Martin-Alvarez, H. Katz, D. Sijacki, J. Devriendt, and A. Slyz, *Mon. Not. R. Astron. Soc.* **504**, 2517 (2021).
- [82] E. Garaldi, R. Pakmor, and V. Springel, *Mon. Not. R. Astron. Soc.* **502**, 5726 (2021).
- [83] F. Vazza, D. Paoletti, S. Banfi, F. Finelli, C. Gheller, S. P. O'Sullivan, and M. Brüggen, *Mon. Not. R. Astron. Soc.* **500**, 5350 (2020).
- [84] M. Sanati, S. Martin-Alvarez, J. Schober, Y. Revaz, A. Slyz, and J. Devriendt, *Astron. Astrophys.* **690**, A59 (2024).
- [85] N. Aghanim *et al.* (Planck Collaboration), *Astron. Astrophys.* **641**, A6 (2020).
- [86] T. Adi, H. A. G. Cruz, and M. Kamionkowski, *Phys. Rev. D* **108**, 023521 (2023).
- [87] P. Ralegankar, M. Pavičević, and M. Viel, *J. Cosmol. Astropart. Phys.* **07** (2024) 027.
- [88] P. Ralegankar, E. Garaldi, and M. Viel, *J. Cosmol. Astropart. Phys.* **08** (2025) 011.
- [89] F. Vazza, M. Brüggen, C. Gheller, and P. Wang, *Mon. Not. R. Astron. Soc.* **445**, 3706 (2014).
- [90] U. P. Steinwandel, L. M. Boess, K. Dolag, and H. Lesch, *Astrophys. J.* **933**, 131 (2022).
- [91] J. Schober, M. Abramson, S. Mandal, S. Mchedlidze, and T. Kahniashvili, *Astron. Astrophys.* **711**, A147 (2026).
- [92] H. Alfvén, *Nature (London)* **150**, 405 (1942).
- [93] T. Kahniashvili, Y. Maravin, A. Natarajan, N. Battaglia, and A. G. Tevzadze, *Astrophys. J.* **770**, 47 (2013).
- [94] L. Mestel, *Q. J. R. Astron. Soc.* **6**, 161 (1965).
- [95] L. Mestel, *Q. J. R. Astron. Soc.* **6**, 265 (1965).
- [96] Y. B. Zel'dovich, A. A. Ruzmaikin, and D. D. Sokoloff, *Magnetic Fields in Astrophysics* (Gordon and Breach, New York, 1983).
- [97] F. Vazza, M. Brüggen, C. Gheller, and P. Wang, *Mon. Not. R. Astron. Soc.* **445**, 3706 (2014).
- [98] C. Caprini and L. Sorbo, *J. Cosmol. Astropart. Phys.* **10** (2014) 056.
- [99] J. M. Cornwall, *Phys. Rev. D* **56**, 6146 (1997).
- [100] G. B. Field and S. M. Carroll, *Phys. Rev. D* **62**, 103008 (2000).
- [101] T. Vachaspati, *Phys. Rev. Lett.* **87**, 251302 (2001).
- [102] G. Sigl, *Phys. Rev. D* **66**, 123002 (2002).
- [103] A. Brandenburg and K. Subramanian, *Phys. Rep.* **417**, 1 (2005).
- [104] V. I. Arnold, The asymptotic hopf invariant and its applications, in *Vladimir I. Arnold—Collected Works: Hydrodynamics, Bifurcation Theory, and Algebraic Geometry 1965–1972*, edited by A. B. Givental, B. A. Khesin, A. N. Varchenko, V. A. Vassiliev, and O. Y. Viro (Springer, Berlin, Heidelberg, 2014), pp. 357–375.
- [105] A. G. Tevzadze, L. Kisslinger, A. Brandenburg, and T. Kahniashvili, *Astrophys. J.* **759**, 54 (2012).
- [106] T. Kahniashvili, A. G. Tevzadze, A. Brandenburg, and A. Neronov, *Phys. Rev. D* **87**, 083007 (2013).
- [107] C. J. Copi, F. Ferrer, T. Vachaspati, and A. Achucarro, *Phys. Rev. Lett.* **101**, 171302 (2008).
- [108] A. Brandenburg, T. Kahniashvili, and A. G. Tevzadze, *Phys. Rev. Lett.* **114**, 075001 (2015).
- [109] A. Brandenburg, T. Kahniashvili, S. Mandal, A. R. Pol, A. G. Tevzadze, and T. Vachaspati, *Phys. Rev. Fluids* **4**, 024608 (2019).
- [110] A. Brandenburg, R. Durrer, T. Kahniashvili, S. Mandal, and W. W. Yin, *J. Cosmol. Astropart. Phys.* **08** (2018) 034.