

Longitudinal structure function in the limit $x = Q^2/s$

G. R. Boroun^{*}*Department of physics, Razi University, Kermanshah 67149, Iran*

(Received 7 July 2025; accepted 1 October 2025; published 17 October 2025)

The longitudinal structure function for nucleons and nuclei is considered at fixed $\sqrt{s}$ and Q^2 to the minimum value of x given by Q^2/s . This is done using the expansion method and color dipole model in the next-to-leading order approximation. The extracted longitudinal structure functions were consistent with HERA hepdata and the determination of F_L at $x = Q^2/s$ [Taylor, *Phys. Rev. D* **111**, 052001 (2025)] at moderate and large Q^2 values. The results, consistent with the dipole picture at low Q^2 values, show that the longitudinal structure function is small as expected due to the transverse polarization of the exchanged photon and the strong suppression of the dominant gluon component. Nonlinear corrections to the nuclear longitudinal structure function at low values of x and Q^2 are also considered. These results may enhance the deep inelastic scattering neutral current data in future colliders at low x and low Q^2 . The longitudinal structure functions for deuterium at low four-momentum transfer squared, $Q^2 < 1 \text{ GeV}^2$ at the leading order and next-to-leading order approximations are determined and compared with the JLab E00-002 data.

DOI: [10.1103/b1ml-2fht](https://doi.org/10.1103/b1ml-2fht)

I. INTRODUCTION

The longitudinal structure function, $F_L(x, Q^2)$, is a fundamental observable in deep inelastic scattering (DIS) experiments, as in the first approximation of the parton model, it is equal identically zero [1] since in the naive quark-parton model (QPM) the massless spin- $\frac{1}{2}$ partons cannot absorb the longitudinally polarized photon. The knowledge of the longitudinal structure function at small values of the Bjorken variable x is important for understanding the inside structure of hadrons (like protons) using high-energy lepton collisions in future such as the Large Hadron electron Collider (LHeC) [2] at CERN and the Electron-Ion Collider (EIC) [3] at Brookhaven National Laboratory (BNL) in e-A scattering. The longitudinal structure function helps test quantum chromodynamics (QCD) predictions beyond leading order (LO) and provides constraints on the gluon density crucial for LHC predictions. It is sensitive to higher-twist effects and nonlinear QCD dynamics at very small x , and it is important for precision Standard Model tests and searches for new physics. In the QCD improved parton model, it provides important information about the quarks and gluon distribution functions of the target. The measured longitudinal

structure function $F_L(x, Q^2)$ is related to the cross section σ_L for absorption of longitudinally polarized virtual photon by

$$F_L(x, Q^2) = \frac{Q^2}{4\pi^2\alpha_{em}}(1-x)\sigma_L(x, Q^2), \quad (1)$$

and is related to the transverse structure functions by

$$F_L(x, Q^2) = F_2(x, Q^2) - 2xF_1(x, Q^2). \quad (2)$$

The longitudinal structure function has been measured at HERA (electron-proton collider) and in fixed-target experiments. It is extracted from cross-section measurements at different beam energies and determined in the high inelasticity (y) at the HERA collider. HERA collected the electron-proton (ep) data at various x and Q^2 values for different $\sqrt{s}$ values, where s is the total energy squared of the electron-proton scattering. These analyses have been conducted by the H1 [4] and ZEUS [5] collaborations. H1 covered a kinematic range from $Q^2 = 1.5 \text{ GeV}^2$ and $x = 0.279 \times 10^{-4}$ up to $Q^2 = 800 \text{ GeV}^2$ and $x = 0.0322$. ZEUS data has been taken in much smaller region from $Q^2 = 9 \text{ GeV}^2$ up to $Q^2 = 110 \text{ GeV}^2$. These kinematics will be extended down to $x \simeq 10^{-7}$ at the Future Circular Collider electron-hadron (FCC-eh) [6] with a center-of-mass energy of $\sqrt{s} \simeq 3.5 \text{ TeV}$ at a similar luminosity as the LHeC with $\sqrt{s} \simeq 1.3 \text{ TeV}$. This is about four times the center-of-mass energy range of ep collisions at HERA [7,8]. On the other hand, the center-of-mass energy at the EIC will be approximately $\sqrt{s} \simeq 140 \text{ GeV}$ which is lower

^{*}Contact author: boroun@razi.ac.ir

Published by the American Physical Society under the terms of the [Creative Commons Attribution 4.0 International](https://creativecommons.org/licenses/by/4.0/) license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Funded by SCOAP³.

than the flagship HERA data. These new colliders will enable the investigation of lepton-hadron processes in ultrahigh energy (UHE) neutrino astroparticle physics. One of the main topics in hadron physics in the new accelerators at small x limit is the color glass condensate (CGC) effective field theory [9,10] in the dipole frame [11,12]. Saturation models [13,14] and geometrical scaling [15,16] can successfully describe HERA data in the small x and low Q^2 region. Future electron-nucleus colliders are the best candidates for discriminating between these models and the CGC physics. The CGC forms the initial state, which is important in itself as a new state of matter that depends on the unintegrated gluon distribution (UGD). The next-to-leading-order (NLO) corrections to the cross section for the inclusive production of a pair of hard jets using the color dipole picture for photon-nucleus interactions at small x are discussed in Ref. [17]. The importance of including a finite size for the target on observables sensitive to small- x evolution within the CGC is discussed in Ref. [18]. The prospects for extracting the longitudinal proton structure function at the EIC, which is a highly competitive direct probe of the proton gluon density, are explored in Ref. [19]. The reduced cross section for

inclusive e-p neutral current deep inelastic scattering (NC DIS) depends on the DIS structure functions, F_2 and F_L , as they include terms of photon, photon-Z boson interference, and direct Z boson interaction terms. In the kinematic range discussed above, the factors dependent on Z-boson exchange are small. Therefore the reduced cross section is as follows:

$$\sigma_r(x, Q^2, s) = F_2(x, Q^2) - \frac{y^2}{Y_+} F_L(x, Q^2), \quad (3)$$

where $Y_+ = 1 + (1 - y)^2$ and $y = Q^2/(xs)$. Recently, Frank E. Taylor [20] has discussed the longitudinal structure function based on an extrapolation of the reduced neutral current cross section with HERA data at a fixed $\sqrt{s}$ and Q^2 at $x_{\min} = Q^2/s$, where indicate that the longitudinal polarization of the virtual photon at $y = 1$ is zero.¹ Therefore the measurement of the reduced cross section in this limit is a determination of the transversely polarized photon-parton scattering. The reduced cross section, in this limit, is given by

$$\lim_{y \rightarrow 1} [\sigma_r(x, Q^2, s)] = F_2(Q^2/s, Q^2) - F_L(Q^2/s, Q^2) = 2 \frac{Q^2}{s} F_1(Q^2/s, Q^2). \quad (4)$$

This limit shows several properties consistent with the dipole picture such as the effect of parton saturation [22] at very low Q^2 . A connection between the longitudinal DIS structure function and the differential cross section for single-inclusive jet production with transverse momentum dependent (TMD) is considered in Ref. [23] at small Bjorken x . They show that the single-inclusive jet cross section in longitudinally polarized DIS at leading power (LP) in $P_\perp/Q$ and small x depends on gathering the quark and gluon jet contributions. Some analytical solutions of the longitudinal DIS structure function have been reported in recent years [24–38]. In DIS on nucleon targets, it is well known that an additional constraint at $x_{\min}$ on the gluon distribution determines the transversely polarized photon-parton scattering, which comes from measuring the longitudinal structure function F_L . On the other hand, due to the poor determination of the nuclear gluon distribution, measurements of F_L on nuclear targets would be of great importance in EIC colliders. This would help constrain the

gluon distribution and study the nuclear dynamics at $x_{\min}$. In fact, studies within perturbative QCD and model calculations show that the corresponding nuclear effects closely follow those on the gluon distribution at $x_{\min}$ [8]. In this paper, an analytical distribution based on the expansion method at the next-to-leading order (NLO) approximation is extended to explore the longitudinal structure function at the kinematic point $y = 1$ (corresponding to $x_{\min} = Q^2/s$) in nucleons and nuclei. The longitudinal structure function approaches zero at this point, and this behavior could be beneficial for nucleon and nuclei parametrization groups.

II. METHOD

The longitudinal structure function can be related to the parton distribution functions at small x by ignoring the nonsinglet distribution function. This relationship can be written as [39,40]

$$F_L(x, Q^2) = \langle e^2 \rangle [C_{L,q}(x, a_s) \otimes x\Sigma(x, Q^2) + C_{L,g}(x, a_s) \otimes xg(x, Q^2)], \quad (5)$$

¹The virtual photon can be polarized into transverse and longitudinal polarizations at low values of x [21]. The longitudinal polarization of the virtual photon is given by $\epsilon = \frac{2(1-y)}{Y_+}$.

where $a_s = \alpha_s/4\pi$ and $\langle e^2 \rangle = \sum_{i=1}^{n_f} e_i^2/n_f$ with the charge e_i for the active quark flavors (n_f). Here $F_{2s}(x, Q^2) = x\Sigma(x, Q^2) = x \sum_{i=u,d,s,c,\dots} (q_i(x, Q^2) + \bar{q}_i(x, Q^2))$ and $G(x, Q^2) = xg(x, Q^2)$ are the singlet and gluon distribution functions in the fractional hadron momentum. The perturbative expansion of the coefficient functions is as follows

$$C_{L,i}(x, a_s) = \sum_{n=1} \left(\frac{\alpha_s}{4\pi} \right)^n c_{L,i}^{(n)}(x). \quad (6)$$

where the coefficient functions in the LO approximation read [41]

$$C_{L,q}^{(1)}(x) = 4C_F x, \quad C_{L,g}^{(1)}(x) = 8n_f x(1-x), \quad (7)$$

where $C_F = 4/3$ in QCD. The coefficient functions in the NLO approximation at small fraction momentum read

$$C_{L,q}^{(2)}(x)_{x \ll 1} \simeq -2.370 \frac{n_f}{x}, \quad C_{L,g}^{(2)}(x)_{x \ll 1} \simeq -5.333 \frac{n_f}{x}. \quad (8)$$

By applying the convolution integral, the longitudinal structure function can be rewritten as

$$F_L(x, Q^2) = \sum_{n=1} \left(\frac{\alpha_s}{4\pi} \right)^n \int_0^{1-x \approx 1} dz \left[c_{L,q}^{(n)}(1-z) F_2\left(\frac{x}{1-z}, Q^2\right) + \langle e^2 \rangle c_{L,g}^{(n)}(1-z) G\left(\frac{x}{1-z}, Q^2\right) \right]. \quad (9)$$

Expanding the distribution functions around the point $z = a$ [42–47] (where $0 \leq z \lesssim 1$) can be done in the following form

$$f\left(\frac{x}{1-z}\right) \Big|_{z=a} = f\left(\frac{x}{1-a}\right) + \frac{x}{1-a} (z-a) \frac{\partial f\left(\frac{x}{1-a}\right)}{\partial x} + \mathcal{O}(z-a)^2. \quad (10)$$

By integrating and then collecting terms, we ensure that the following series converges for $|z-a| < 1$ as

$$\frac{x}{1-z} \Big|_{z=a} = \frac{x}{1-a} \sum_{k=1}^{\infty} \left[1 + \frac{(z-a)^k}{(1-a)^k} \right]. \quad (11)$$

Therefore, we find

$$F_L(x, Q^2) = \frac{2\alpha_s}{3\pi} F_2\left(\frac{x}{1-a} \left(\frac{4}{3} - a\right), Q^2\right) + \frac{10\alpha_s}{27\pi} G\left(\frac{x}{1-a} \left(\frac{3}{2} - a\right), Q^2\right) - 9.480 \left(\frac{\alpha_s}{4\pi}\right)^2 \frac{\frac{3}{2} - 2a}{(1-a)^2} F_2\left(\frac{\frac{3}{2} - 2a}{(1-a)^2} x, Q^2\right) - 5.926 \left(\frac{\alpha_s}{4\pi}\right)^2 \frac{\frac{3}{2} - 2a}{(1-a)^2} G\left(\frac{\frac{3}{2} - 2a}{(1-a)^2} x, Q^2\right). \quad (12)$$

At the point where $z = 0.5$ is expanding (for further expanding points refer to the Appendix), Eq. (12) can be rewritten in the following simplest form:

$$\begin{aligned} F_L(x, Q^2) &= \frac{2\alpha_s}{3\pi} F_2\left(\frac{5}{3}x, Q^2\right) + \frac{10\alpha_s}{27\pi} G(2x, Q^2) - \frac{18.960\alpha_s^2}{16\pi^2} F_2(2x, Q^2) - \frac{11.852\alpha_s^2}{16\pi^2} G(2x, Q^2) \\ &= \frac{10\alpha_s}{27\pi} G(2x, Q^2) \left[1 + \frac{9}{5} \frac{F_2\left(\frac{5}{3}x, Q^2\right)}{G(2x, Q^2)} \right] - \frac{11.852\alpha_s^2}{16\pi^2} G(2x, Q^2) \left[1 + \frac{18.960}{11.852} \frac{F_2(2x, Q^2)}{G(2x, Q^2)} \right] \\ &= \frac{10\alpha_s}{27\pi} G(2x, Q^2) [1 + \eta] - \frac{11.852\alpha_s^2}{16\pi^2} G(2x, Q^2) [1 + \xi]. \end{aligned} \quad (13)$$

At small x , the right-hand side is dominated by the gluon contribution. In fact, the ratio of F_2/G is less than 0.1 at small x . The singlet and gluon distributions are related to each other at small x by the function $C(\alpha_s \ln Q^2, \alpha_s, \alpha_s \ln 1/x)$ which describes the splitting of the virtual photon into the quark-antiquark pair by the

following form [48,49]:

$$F_{2s}(x, Q^2) = C(\alpha_s \ln Q^2, \alpha_s, \alpha_s \ln 1/x) xg(x, Q^2). \quad (14)$$

Using the leading order DGLAP evolution equation that gives for the proton structure function

$$F_2(x, Q^2) = \langle e^2 \rangle \frac{C_F \alpha_s}{2\pi} \int_0^v dv' \int_x^1 dz P_{qG}(z) \left(\frac{x}{z} g\left(\frac{x}{z}, v'\right) \right), \quad (15)$$

with $P_{qG} = (1 + (1 - z)^2)/z$ and $v = \ln Q^2$. Here $C_F = (N_c^2 - 1)/2N_c$ where N_c is the number of colors. At small values of x , the gluon distribution function takes the form

$$xg(x, v) = \int_{\epsilon - i\infty}^{\epsilon + i\infty} \frac{d\gamma}{2\pi i} \left(\frac{1}{x} \right)^{\alpha_s \frac{N_c}{\pi} \chi(\gamma)} e^{\gamma v} g(\gamma)_{\text{initial}}, \quad (16)$$

where the BFKL kernel $\chi(\gamma)$ has the following form

$$\chi(\gamma) = 2\psi(1) - \psi(\gamma) - \psi(1 - \gamma), \quad (17)$$

where for $\gamma \rightarrow \frac{1}{2}$

$$\chi(\gamma) \rightarrow \omega_0 + D \left(\gamma - \frac{1}{2} \right)^2, \quad (18)$$

with $\omega_0 = 4 \ln 2$ and $D = 14\zeta(3)$. After integrating and applying the diffusion approximation and the method of steepest descent [49], the simplest relation between the proton structure function and the gluon distribution function is obtained in the following form:

$$\frac{F_2(x, Q^2)}{G(x, Q^2)} = \langle e^2 \rangle \frac{2C_F}{\omega_0 N_c} \simeq 0.09, \quad (19)$$

with $n_f = 4$. Instead of starting with a theoretically motivated form of the color dipole model (CDM) in the dipole picture of deep inelastic scattering [50], we will begin with a parametrization of the deep inelastic structure function for electromagnetic scattering with protons due to the Bjorken rescaling of the functions in Eqs. (13), (A1), and (A2). We can use the parametrizations of F_2 and G by Donnachie-Landshoff (DL) [51] and Block-Durand-Ha (BDH) [52]. The DL parametrization [51] meets the requirement of being defined for all Q^2 , even if not valid for the full range, but it has a limited range of applicability. The BDH parametrization applies to large and small Q^2 and small x , where it has an expression for the asymptotic part of F_2 (no-valence) that accounts for the asymptotic behavior ($W^2 \rightarrow \infty$ with Q^2 fixed). For small x , where W is the invariant mass of the final state, $F_2(W^2, Q^2) \propto \ln^2(W^2/Q^2) \simeq \ln^2(1/x)$. Taking the BDH parametrization, we find the coefficients $\eta_a(Q_s^2, Q^2)$ and $\xi_a(Q_s^2, Q^2)$ in Fig. 1 over a wide range of Q^2 at $x = Q^2/s$ with $\sqrt{s} = 318$ GeV. In Fig. 1, we observe that the ratio

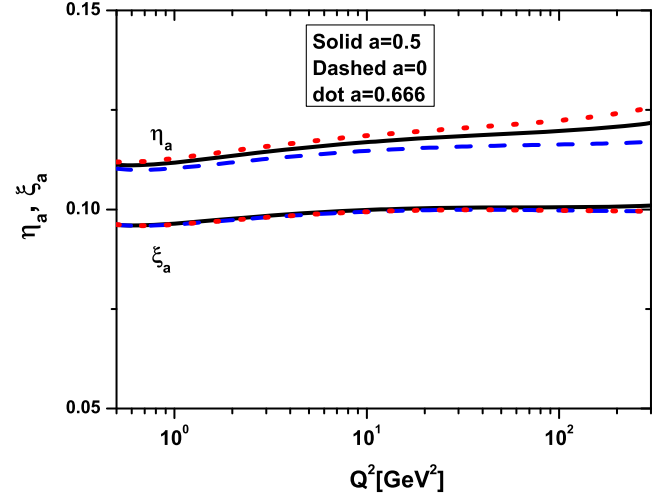


FIG. 1. Ratios of $\eta_a(Q_s^2, Q^2)$ and $\xi_a(Q_s^2, Q^2)$ at $\sqrt{s} = 318$ GeV as a function of Q^2 with $a = 0, 0.5$ and 0.666 according to Eqs. (A1), (13), and (A2), respectively.

$\xi_a \lesssim 0.1$ and $\eta_a \lesssim 0.12$ over a wide range of Q^2 values. However, we can ignore the effects of these coefficients at small values of x . Therefore, Eq. (13) reduced and rewritten by the following form

$$F_L(Q^2/s, Q^2) \simeq \left[\frac{10\alpha_s}{27\pi} - \frac{11.852\alpha_s^2}{16\pi^2} \right] G(2Q^2/s, Q^2), \quad (20)$$

where the gluon distribution is obtained using a dipole model fit to low- x data on F_2 in Ref. [53]. To predict the rates of the various processes, a set of gluon distribution functions corresponding to the active flavor number n_f is required. When Q^2 increases above m_c^2 and then above m_b^2 , the number of active flavors increases from $n_f = 3$ to $n_f = 4$ and then to $n_f = 5$, which corresponds to the variable-flavor-number scheme (VFNS) [54]. The charm and bottom quarks are considered as infinitely massive below $Q^2 = m_{c,b}^2$ and massless above this threshold. For realistic kinematics it has to be extended to the case of a general-mass VFNS (GM-VFNS) which is defined similarly to the zero-mass VFNS (ZM-VFNS) in the $Q^2/m_{c,b}^2 \rightarrow \infty$ limit. For scales $Q^2 < m_c^2$ the fixed-flavor-number scheme (FFNS) is valid and for $Q^2 > m_c^2$, the approach outlined above to define a VFNS is valid [55]. On the threshold of heavy quark production, the perturbative predictions for longitudinal structure function are in accordance with the heavy quark production threshold, which can be written as [34,56]

$$F_L(x, Q^2) = C_{L,q}(x, a_s) \otimes F_2(x, Q^2) + \frac{2}{9} C_{L,g}(x, a_s) \otimes G_{n_f=3}(x, Q^2) + F_L^c(x, Q^2), \quad \text{for } Q^2 \geq m_c^2 \quad (21)$$

and

$$F_L(x, Q^2) = C_{L,q}(x, a_s) \otimes F_2(x, Q^2) + \frac{2}{9} C_{L,g}(x, a_s) \otimes G_{n_f=3}(x, Q^2) + F_L^c(x, Q^2) + F_L^b(x, Q^2), \quad \text{for } Q^2 \geq m_b^2. \quad (22)$$

The HERA available data are insufficient to accurately extract the $F_L^{c,b}$ values. Therefore, we include both charm and bottom quark contributions to F_L , which are generated radiatively from gluons in the color dipole model. The relationship between the gluon distribution and the dipole cross section is defined by the dipole picture and the k_T -factorization theorem [57,58]. The basic theoretical formula for F_L in terms of the longitudinal photon polarization is given by

$$F_L(x, Q^2) = \frac{Q^2}{4\pi^2\alpha_{\text{em}}} \sigma_L^{*p}(x, Q^2). \quad (23)$$

When the virtual photon dissociates into a quark-antiquark pair (a $q\bar{q}$ dipole), it subsequently interacts with the proton as

$$\sigma_L(x, Q^2) = \sum_f \int_0^1 dz \int d^2r |\Psi_L(r, z, Q)|^2 \hat{\sigma}(x, r^2), \quad (24)$$

where the sum over quark flavors f is performed as it depends on the contributions from the light quark pairs ($u\bar{u}$, $d\bar{d}$ and $s\bar{s}$) as well as the contributions from the $c\bar{c}$ and $b\bar{b}$ pairs. The photon wave function depends on the mass of the quarks in the $q\bar{q}$ dipole. Therefore, we can consider contributions to the longitudinal structure function from the individual quark flavor pairs according to Eqs. (21) and (22) where the Bjorken variable x is modified to include the dipole cross section into the quark mass using the following form

$$x \rightarrow \tilde{x} = x \left(1 + \frac{4m_f^2}{Q^2} \right). \quad (25)$$

Indeed, the Bjorken variable x is modified at the limit $x = Q^2/s$ to the rescaling variable $\tilde{x}$, as $\tilde{x} = \frac{Q^2}{s} + \frac{4m_f^2}{s}$ which reduces to the Bjorken variable x at high Q^2 values. This modification is consistent with the GM-VFNS in parametrization models. The dipole cross section is

$$\hat{\sigma}(x, r^2) = \frac{4\pi^2}{3} \int \frac{dk^2}{k^4} \alpha_s f(x, k^2) (1 - J_0(kr)), \quad (26)$$

where $J_0(kr)$ is the Bessel function of the first kind and $f(x, k^2)$ is the unintegrated gluon distribution. The unintegrated gluon distribution is obtained in the following form [53]

$$f(x, k^2) = \frac{3\sigma_0}{4\pi^2\alpha_s} k^4 (x/x_0)^\lambda e^{-k^2(x/x_0)^\lambda}. \quad (27)$$

The three parameters (σ_0 , x_0 and λ) of the fits with the GBW model using the quark masses are listed in Table I. The integrated gluon distribution is obtained using the leading twist relation

$$xg(x, Q^2) = \int_0^{Q^2} \frac{dk^2}{k^2} f(x, k^2). \quad (28)$$

Therefore the integrated gluon distribution is obtained in the following form

$$G(x, Q^2) = \frac{3\sigma_0}{4\pi^2\alpha_s} [-Q^2 e^{-Q^2(x/x_0)^\lambda} + (x_0/x)^\lambda (1 - e^{-Q^2(x/x_0)^\lambda})]. \quad (29)$$

The effect of charm and bottom quark contributions to the gluon distribution are included by applying the rescaling variable as

TABLE I. The coefficient values are obtained in Ref. [59].

Fit	m_l [GeV]	m_c [GeV]	m_b [GeV]	σ_0 [mb]	λ	$x_0/10^{-4}$	χ^2/N_{dof}
0	0.14	...	...	23.58 ± 0.28	0.270 ± 0.003	2.24 ± 0.16	1.83
1	0.14	1.4	...	27.32 ± 0.35	0.248 ± 0.002	0.42 ± 0.04	1.60
2	0.14	1.4	4.6	27.43 ± 0.35	0.248 ± 0.002	0.40 ± 0.04	1.61

$$G(x, Q^2) \rightarrow G(\tilde{x}, Q^2) = \frac{3\sigma_0}{4\pi^2\alpha_s} [-Q^2 e^{-Q^2(\tilde{x}/x_0)^\lambda} + (x_0/\tilde{x})^\lambda (1 - e^{-Q^2(\tilde{x}/x_0)^\lambda})]. \quad (30)$$

Therefore, we find that the longitudinal structure function at the limit $x = Q^2/s$ depends on the mass of the quarks in the $q\bar{q}$ dipole in the following form

$$F_L(Q^2/s, Q^2) = \left[\frac{10\alpha_s}{27\pi} - \frac{11.852\alpha_s^2}{16\pi^2} \right] \frac{3\sigma_0}{4\pi^2\alpha_s} [-Q^2 e^{-Q^2(\frac{2}{x_0}(Q^2/s+4m_f^2/s))^\lambda} + (x_0/(2(Q^2/s+4m_f^2/s)))^\lambda (1 - e^{-Q^2(\frac{2}{x_0}(Q^2/s+4m_f^2/s))^\lambda})]. \quad (31)$$

Recently, the authors in Ref. [60] modified the dipole cross section with simulated nonlinear QCD evolution by introducing a rescaling in the saturation scale as $Q_s^2(x) \rightarrow k \cdot Q_s^2(x)$ where $Q_s^2(x) = Q_0^2(x/x_0)^{-\lambda}$ with $Q_0^2 = 1 \text{ GeV}^2$. The parameter k can be understood as a factor that controls the strength of the triple Pomeron vertex and, therefore, the significance of nonlinear dynamics. A value of $k = 0$ corresponds to the linear behavior of the dipole cross section, while $k = 1$ results in the dipole cross sections fitted to HERA data. Finally, for $k > 1$, there is an additional enhancement of nonlinear effects. The nuclear enhancement of the saturation scale, which corresponds to an increase in the density of gluons, is defined on a large nucleus rather than a single proton. In this case, $Q_s^2(x) \rightarrow Q_{s,A}^2(x)$, where $Q_{s,A}^2(x)$ represents the saturation scale for the nuclear target. If the color dipole scatters on a large nucleus instead of a single proton, then the saturation scales is modified by

$$Q_{s,A}^2(x) = kQ_s^2(x), \quad (32)$$

where the parameter k represents the enhancement of color sources in the large nucleus for a nucleus with mass number A , where $k = A^{1/3}$. The study of the authors in Ref. [60] is based on the Golec-Biernat Wusthoff (GBW) model [22]

and Bartels Golec-Biernat Kowalski (BGK) model [13]. This allows us to more directly access the relevance of nonlinear corrections for describing the energy dependence of the photoproduction cross section. These models are modified by the following forms:

$$\sigma_{q\bar{q}}^{\text{GBW}}(x, r, k) = \frac{\sigma_0^{\text{GBW}}}{k} \left[1 - \exp \left(-k \cdot \frac{r^2 Q_s^2(x)}{4} \right) \right], \quad (33)$$

and

$$\sigma_{q\bar{q}}^{\text{BGK}}(x, r, k) = \frac{\sigma_0^{\text{BGK}}}{k} \left[1 - \exp \left(-k \cdot \frac{r^2 \pi^2 \alpha_s(\mu_r^2) x g(x, \mu_r^2)}{3\sigma_0^{\text{BGK}}} \right) \right], \quad (34)$$

where the renormalization scale μ is usually identified with the factorization scale and taken to depend on the dipole size with $\mu^2 \sim 1/r^2$ for small dipole sizes. Now we use the DIS F_L on electron-proton collisions to make predictions for electron-ion collisions with the replacements [61] $S \rightarrow S_A = A^{2/3}S$, where S is the area of the target and the transverse size of the dipole cross section scaling is approximately proportional to $\sim A^{2/3}$, and $Q_s^2(x) \rightarrow Q_{s,A}^2(x) = A^{1/3}Q_s^2(x)$. The nuclear longitudinal structure function at the limit $x = Q^2/s$ is

$$F_L^A(Q^2/s, Q^2) = \left[\frac{10\alpha_s}{27\pi} - \frac{11.852\alpha_s^2}{16\pi^2} \right] \frac{3\sigma_0 A^{2/3}}{4\pi^2\alpha_s} [-Q^2 e^{-\frac{Q^2}{A^{1/3}x_0}(\frac{2}{x_0}(Q^2/s+4m_f^2/s))^\lambda} + A^{1/3}(x_0/(2(Q^2/s+4m_f^2/s)))^\lambda (1 - e^{-\frac{Q^2}{A^{1/3}x_0}(\frac{2}{x_0}(Q^2/s+4m_f^2/s))^\lambda})]. \quad (35)$$

The nuclear DIS structure functions in recent years by several methods have been proposed in Refs. [62–69]. In Ref. [62], the longitudinal structure function in nuclear DIS at small x is discussed within the framework of universal parton densities obtained in DGLAP analysis. The longitudinal structure function in pQCD in the $\overline{\text{MS}}$ scheme was calculated, and the ratio F_L^A/F_L^p was compared with PDF

sets such as EPS09 [70], nDS [71], HKN07 [72], and FGS10 [73] for lead ($A = 208$). In Ref. [23], the authors studied of TMD factorization in the target fragmentation region and found an interesting result concerning the leading power contribution to the unintegrated F_L distribution at NLO. The NLO computations of the unintegrated F_L at leading power are consistent with the Altarelli-Martinelli

relation at NLO which included a step in DGLAP for the gluon distribution at LO and the contribution from sea quarks coming from the splitting of gluon into a quark-antiquark pair. In Ref. [74], the author found the longitudinal structure function at small x by the gluon content of the nucleon target at fixed energy within the color dipole formalism. In the following we extend these results to the nuclear longitudinal structure function for comparison. In Ref. [75], the authors simulated the nuclear structure functions based on measurements of inclusive and charm reduced cross sections at an EIC. Since F_L has a larger direct contribution from gluons, information on F_L and, consequently, direct access to nuclear gluons are not currently available. However, at an EIC, the high luminosity and wide kinematic reach will enable the direct extraction of F_L , providing more information on the behavior of nuclear gluons. The potential for extracting the longitudinal proton structure function at EIC through a Rosenbluth separation method was explored in Ref. [19]. Simulated extractions of the longitudinal structure function for three different pseudodata replicas in the conservative scenario are presented, with different colors corresponding to different replicas.

III. RESULTS AND DISCUSSION

The standard representation for QCD couplings in the LO and NLO (within the $\overline{\text{MS}}$ -scheme) approximations reads

$$\alpha_s(t) = \frac{4\pi}{\beta_0 t} \quad (\text{LO}),$$

$$\alpha_s(t) = \frac{4\pi}{\beta_0 t} \left[1 - \frac{\beta_1 \ln \ln(t)}{\beta_0^2 t} \right] \quad (\text{NLO}),$$

with β_0 and β_1 as the first two coefficients of the QCD β -function,

$$\beta_0 = \frac{1}{3}(11C_A - n_f), \quad \beta_1 = \frac{1}{3}(34C_A^2 - 2n_f(5C_A + 3C_F)),$$

where $C_F = \frac{N_c^2 - 1}{2N_c}$ and $C_A = N_c$ are the Casimir operators in the fundamental and adjoint representations of the $\text{SU}(N_c)$ color group, and $t = \ln \frac{Q^2}{\Lambda^2}$ where the QCD parameter Λ for active flavors has been determined [26] based on $\alpha_s(M_Z^2) = 0.1166$,

$$\begin{aligned} \text{LO: } \Lambda(n_f = 3) &= 0.1368 \text{ GeV}, & \Lambda(n_f = 4) &= 0.1368 \text{ GeV}, & \Lambda(n_f = 5) &= 0.8080 \text{ GeV}, \\ \text{NLO: } \Lambda(n_f = 3) &= 0.3472 \text{ GeV}, & \Lambda(n_f = 4) &= 0.2840 \text{ GeV}, & \Lambda(n_f = 5) &= 0.1957 \text{ GeV}. \end{aligned} \quad (36)$$

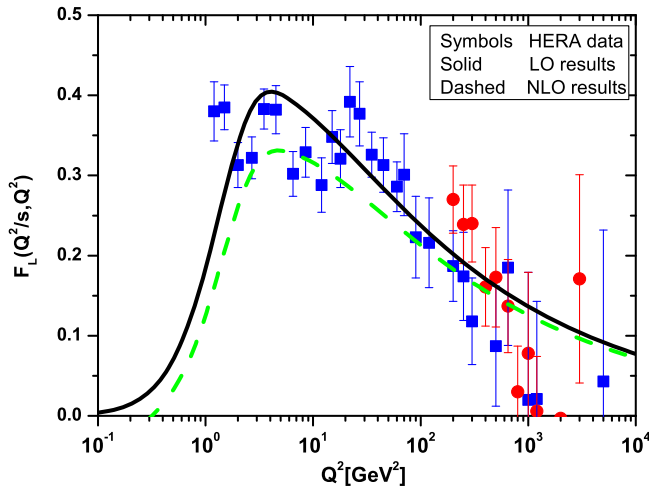


FIG. 2. The DIS longitudinal structure functions at LO (black solid curve) and NLO (green dashed curve) approximations are shown as a function of Q^2 for $\sqrt{s} = 318$ GeV. These results consider the charm effect in the rescaling of the Bjorken variable x . The coefficients are based on the results from Fit 1 in Table I. The HERA combined H1 and ZEUS reduced cross section datasets (squares-blue and circles-red) and their uncertainties are shown.

In Figs. 2 and 3, we compare the longitudinal structure function at the limit of $x = Q^2/s$ for $\sqrt{s} = 318$ and 300 GeV respectively. This comparison is made with the e^+p HERA data, as shown in Tables 1 and 2 of Ref. [76]. These tables combine the reduced cross section datasets from H1 and ZEUS, along with their respective kinematic regions. The extracted longitudinal structure functions are

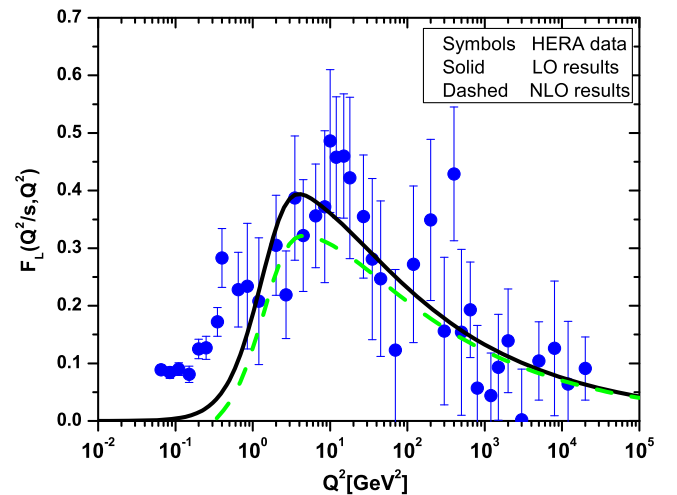


FIG. 3. The same as Fig. 1 at $\sqrt{s} = 300$ GeV.

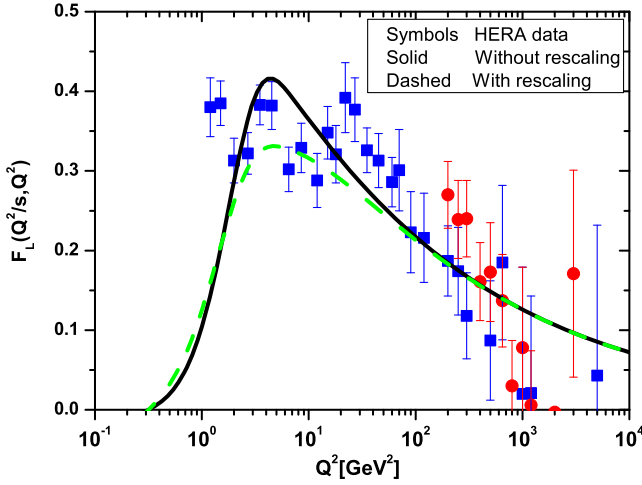


FIG. 4. The DIS longitudinal structure functions without rescaling (black solid curve) and with rescaling (green dashed curve) approximations at the NLO approximation are shown as a function of Q^2 for $\sqrt{s} = 318$ GeV. The coefficients are based on the results from Fit 1 in Table I. The HERA combined H1 and ZEUS reduced cross section datasets (squares-blue and circles-red) and their uncertainties are shown.

shown as a function of Q^2 and $x_{\min}$ in Tables VIII and IX of Ref. [20], accompanied by the uncertainties of the data points. The results at the LO and NLO approximations are obtained from the gluon distribution behavior using a dipole picture [53]. We observe that this distribution accurately represents the longitudinal structure function at both low and high Q^2 values, consistent with the HERA data. The longitudinal structure functions obtained from the dipole picture show that F_L is small at low x and low Q^2

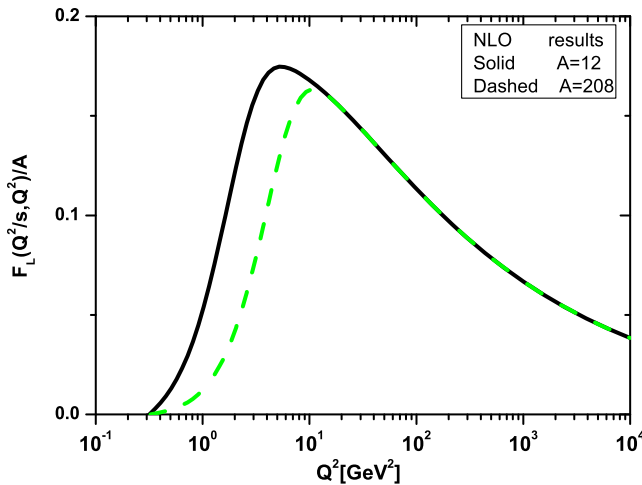


FIG. 6. The DIS longitudinal structure functions for light nuclei C-12 (black solid curve) and heavy nuclei Pb-208 (green dashed curve) at the NLO approximation are shown as a function of Q^2 for $\sqrt{s} = 89$ GeV. In these results the charm effect in the rescaling Bjorken variable x is considered. The coefficients are based on the results from fit 1 in Table I.

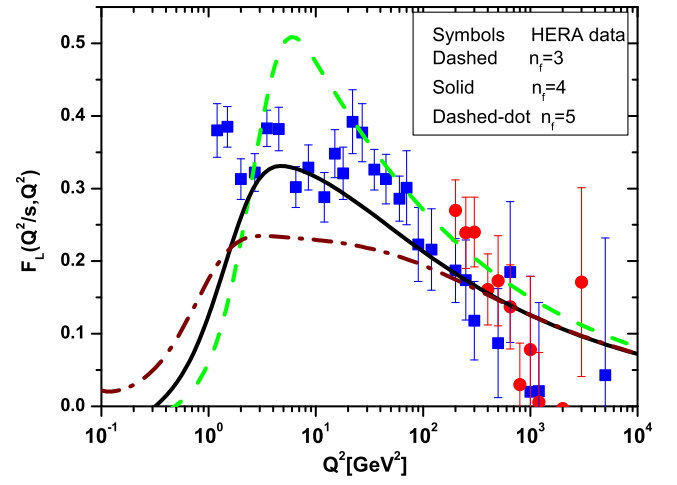


FIG. 5. The DIS longitudinal structure functions at the NLO approximation are shown by applying the active flavors ($n_f = 3$, green dashed curve), ($n_f = 4$, black solid curve) and ($n_f = 5$, brown dashed-dot curve) as a function of Q^2 for $\sqrt{s} = 318$ GeV based on the results in Table I and QCD cutoff. The HERA combined H1 and ZEUS reduced cross section datasets (squares-blue and circles-red) and their uncertainties are shown.

because the polarization of the exchanged photon is transverse at this kinematic point. The results in Figs. 2 and 3 correspond to the four active flavors (i.e., $n_f = 4$) while considering the rescaling variable, i.e., Eq. (25).

The effects of the rescaling variable in the longitudinal structure function are shown in Fig. 4. We observe that the behavior of $F_L(Q^2/s, Q^2)$ is affected by the large quark mass $m_c \approx 1.4$ GeV, whether or not the rescaling variable is used. When comparing with HERA data, the comparison without rescaling shows lower quality at moderate Q^2 . However, the comparison is significantly improved with scaling.

This rescaling strongly depends on the mass of the quarks in the $q\bar{q}$ dipole. In Fig. 5, we show the mass effects of the rescaling variable in $F_L(Q^2/s, Q^2)$. We observe that adding charm into the analysis improves the results at low and moderate Q^2 values compared to the contributions of light and bottom quarks. Nevertheless, the parameters of the bottom quark contribution can be taken into account for further analysis in order to have a full understanding of heavy quarks [59].

In Fig. 6, we show the behavior of the DIS longitudinal structure function divided by A at the NLO approximation for light nuclei C-12 and heavy nuclei Pb-208 at $x_{\min}$ in the kinematic range relevant for the EIC ($\sqrt{s} = 89$ GeV),² taking into account the rescaling variable due to the charm quark mass. We observe that the behavior of the nuclear

²The center-of-mass energies in electron-ion colliders proposed in China and the US are 15–20 GeV for EICc and 30–140 GeV for EIC, respectively.

longitudinal structure function, $F_L^A(Q^2/s, Q^2)/A$, determines the transversely polarized photon-nuclei scattering at low Q^2 values. The enhancement of $F_L^A(Q^2/s, Q^2)/A$ decreases at moderate Q^2 values as the mass number A increases, and increases as Q^2 values increase. The behavior of $F_L^A(Q^2/s, Q^2)/A$ is independent of the mass number A at large Q^2 values.

Indeed, the parameter $k = A^{1/3}$ in Eq. (35) implements the enhancement of the saturation scale based on the gluon saturation, corresponding to an increase in the density of gluons [60]. This saturation is visible in large nuclei at low Q^2 values, as shown in Fig. 6. The gluon saturation is tested by comparing the longitudinal structure functions in protons and nuclei. In Fig. 7, the behavior of the ratio $\frac{F_L^A}{AF_L^p}$ is considered at the minimum value of x given by Q^2/s at the EIC center-of-mass energy $\sqrt{s} = 89$ GeV. A clear deviation of the ratio $\frac{F_L^A}{AF_L^p}$ from the linearized description, which indicates the relevance of nonlinear terms, is visible in Fig. 7 and is significant for large nuclei. The nonlinear effects begin for light and heavy nuclei at $Q^2 < 10$ GeV². The deviation of this ratio from unity demonstrates the significance of nonlinear effects, indicating saturation physics at $x_{\min}$. The results of saturation effects in eA collisions, conducted using the color dipole approach, confirmed the findings presented in Ref. [77]. The effect of different values of k ($k = 2.289$ and $k = 5.925$) on the behavior of the ratio at $x_{\min}$ is visible in Fig. 7 and may already be observable for photonuclear reactions, where gluon densities are enhanced by $A^{1/3}$. These results are compared with the gluonic term of the P. Caucal and F. Salazar (CS) results [23] that is modified for the nuclear longitudinal structure function. This modification is based on the TMD factorization in the target

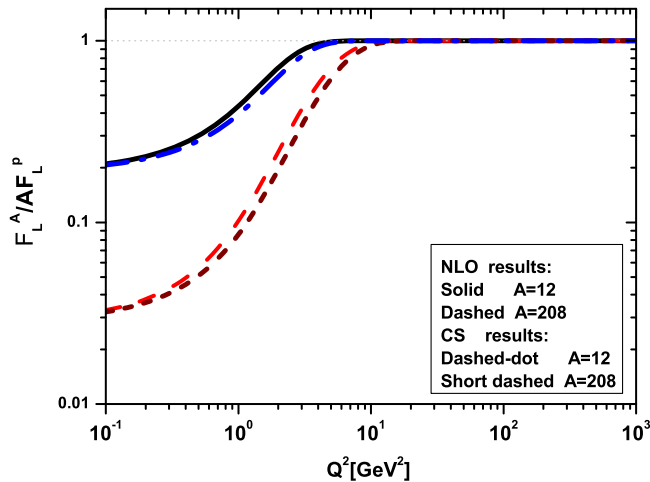


FIG. 7. The ratio of $\frac{F_L^A}{AF_L^p}(Q^2/s, Q^2)$ for $A = 12$ (black solid curve) and $A = 208$ (red dashed curve) is shown as a function of Q^2 for $\sqrt{s} = 89$ GeV and compared with the gluonic term of the CS results [23] which is modified for $A = 12$ (blue dashed-dot curve) and $A = 208$ (brown short dashed curve).

fragmentation region and consistency with the Altarelli-Martinelli relation at NLO.

Measurements of the longitudinal structure function for the deuterium at low values of Q^2 and intermediate x were presented in Ref. [78] where the experiment was conducted at Jefferson Lab (JLab). In this experiment, the separated structure functions for hydrogen and deuterium using the Rosebluth separation technique at low four-momentum transfer squared, $Q^2 < 1$ GeV², were analyzed and compared with parton distribution parametrizations and a k_T factorization approach. In Fig. 8, we compare our results with JLab data for the longitudinal structure function of deuterium as a function of x (for $x > 0.01$) for two values of $Q^2 = 0.4$ and 0.5 GeV². The longitudinal structure function for nuclei presented in this work [i.e., Eq. (35)] is based on the color dipole scattering on a large nucleus [60] where the saturation scale increases due to the nuclear “oomph factor.” Therefore, when compared to the deuterium longitudinal structure function at low Q^2 values, we expect to see differences in the results. In Fig. 8, we observe that the JLab E00-002 data are plotted between the LO and NLO results.

To further investigate the behavior of the results with Q^2 and compare them with JLab data, we present the longitudinal structure function results for deuterium versus Q^2 for $x = 0.1$ and 0.4 at the NLO approximation in Fig. 9. The data in Fig. 9 were obtained from the averaged JLab experiment (E00-002 data). In this kinematic regime, the longitudinal structure function $F_L^D(x, Q^2)$ is well described by the JLab data. We observe that, due to current conservation, the interaction of longitudinal virtual photons vanishes in the limit of real photon $Q^2 = 0$.

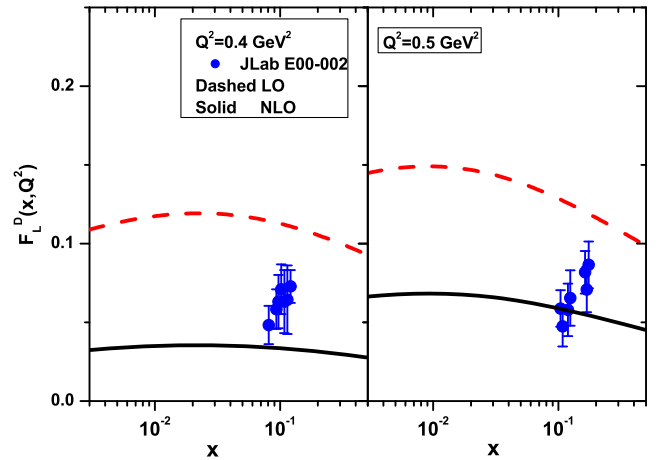


FIG. 8. Longitudinal structure function results for deuterium are shown for fixed Q^2 as a function of Bjorken x at the LO (dashed curve) and NLO (solid curve) approximations. The two panels correspond to two different Q^2 regions: $Q^2 = 0.4$ GeV² (left) and $Q^2 = 0.5$ GeV² (right). In this kinematic regime, the data are from Jefferson Lab [78] and are accompanied by total errors.

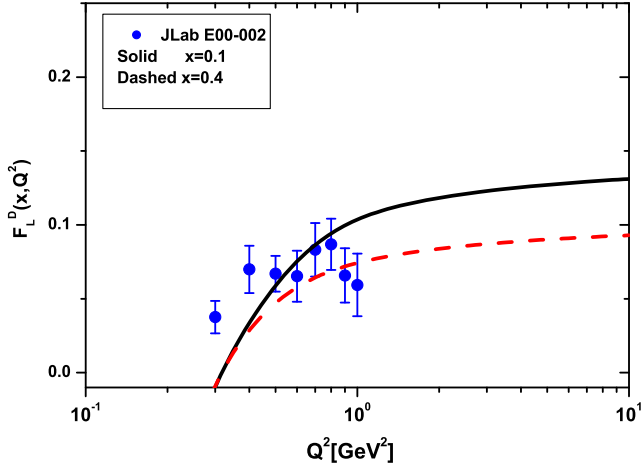


FIG. 9. The results for the deuterium longitudinal structure function are displayed for a fixed x as a function of Q^2 at the NLO approximation. Two curves are presented, representing two distinct x values: $x = 0.1$ (black solid curve) and $x = 0.4$ (red dashed curve). The averaged data for $x \geq 0.05$ are sourced from Jefferson Lab [78] and include total errors.

In conclusion, we have examined the behavior of the longitudinal structure function of protons and nuclei at $y = 1$, where $x_{\text{Bj}} = x_{\text{min}} = Q^2/s$, in both HERA and EIC kinematics. We utilized the expanding method and CDP for low x gluon distributions in this scenario. Our model provides a good description of results at moderate and large Q^2 values when compared to HERA data, and predicts nuclear longitudinal structure functions that can be measured in electron-ion collisions at x_{min} . Comparing with HERA data at moderate and large Q^2 values strongly suggests that the gluon component is the dominant factor in the longitudinal structure function within the dipole picture.

We observed that the value of F_L is small at low Q^2 values in the dipole picture. This is because the dominant gluon component is strongly suppressed and the polarization of the exchanged photon is transverse at this kinematic point. We investigated the nuclear longitudinal structure function at this limit, which reveals high gluon densities and associated nonlinear high-energy evolution. The nonlinear corrections increase with the mass number A . The ratio $\frac{F_L^A}{AF_L^p}$ can serve as an indicator for the presence of nonlinear low x dynamics in large nuclei. The longitudinal structure functions for deuterium at low four-momentum transfer squared in the LO and NLO approximations are determined and compared with the JLab E00-002 data.

ACKNOWLEDGMENTS

The author would like to thank F.E. Taylor and F. Salazar for their helpful comments and invaluable support. The author is especially grateful to Nestor Armesto, Elke Aschenauer, and Paul Newman for fruitful discussions and for allowing access to data related to the longitudinal structure function at the EIC and JLab. I am also very grateful to the Department of Physics at CERN-TH for their warm hospitality.

DATA AVAILABILITY

The data that support the findings of this article are openly available [76].

APPENDIX

The longitudinal structure function at the expanding points $z = 0$ and 0.666 is, respectively,

$$\begin{aligned}
 F_L(x, Q^2) &= \frac{2\alpha_s}{3\pi} F_2\left(\frac{4}{3}x, Q^2\right) + \frac{10\alpha_s}{27\pi} G\left(\frac{3}{2}x, Q^2\right) - \frac{28.440\alpha_s^2}{32\pi^2} F_2\left(\frac{3}{2}x, Q^2\right) - \frac{17.778\alpha_s^2}{32\pi^2} G\left(\frac{3}{2}x, Q^2\right) \\
 &= \frac{10\alpha_s}{27\pi} G\left(\frac{3}{2}x, Q^2\right) \left[1 + \frac{9}{5} \frac{F_2(\frac{4}{3}x, Q^2)}{G(\frac{3}{2}x, Q^2)}\right] - \frac{17.778\alpha_s^2}{32\pi^2} G\left(\frac{3}{2}x, Q^2\right) \left[1 + \frac{28.440}{17.778} \frac{F_2(\frac{3}{2}x, Q^2)}{G(\frac{3}{2}x, Q^2)}\right] \\
 &= \frac{10\alpha_s}{27\pi} G\left(\frac{3}{2}x, Q^2\right) [1 + \eta] - \frac{17.778\alpha_s^2}{32\pi^2} G\left(\frac{3}{2}x, Q^2\right) [1 + \xi],
 \end{aligned} \tag{A1}$$

and

$$\begin{aligned}
 F_L(x, Q^2) &= \frac{2\alpha_s}{3\pi} F_2\left(\frac{4}{2}x, Q^2\right) + \frac{10\alpha_s}{27\pi} G\left(\frac{5}{2}x, Q^2\right) - \frac{14.315\alpha_s^2}{16\pi^2} F_2\left(\frac{3}{2}x, Q^2\right) - \frac{8.948\alpha_s^2}{16\pi^2} G\left(\frac{3}{2}x, Q^2\right) \\
 &= \frac{10\alpha_s}{27\pi} G\left(\frac{5}{2}x, Q^2\right) \left[1 + \frac{9}{5} \frac{F_2(\frac{4}{2}x, Q^2)}{G(\frac{5}{2}x, Q^2)}\right] - \frac{8.948\alpha_s^2}{16\pi^2} G\left(\frac{3}{2}x, Q^2\right) \left[1 + \frac{14.315}{8.948} \frac{F_2(\frac{3}{2}x, Q^2)}{G(\frac{3}{2}x, Q^2)}\right] \\
 &= \frac{10\alpha_s}{27\pi} G\left(\frac{5}{2}x, Q^2\right) [1 + \eta] - \frac{8.948\alpha_s^2}{16\pi^2} G\left(\frac{3}{2}x, Q^2\right) [1 + \xi].
 \end{aligned} \tag{A2}$$

- [1] C. G. Callan, Jr. and D. J. Gross, *Phys. Rev. Lett.* **22**, 156 (1969).
- [2] P. Agostini *et al.* (LHeC and FCC-he Study Group Collaboration), *J. Phys. G* **48**, 110501 (2021).
- [3] EIC BNL, <https://www.bnl.gov/eic/>.
- [4] V. Andreev *et al.* (H1 Collaboration), *Eur. Phys. J. C* **74**, 2814 (2014).
- [5] H. Abramowicz *et al.* (ZEUS Collaboration), *Phys. Rev. D* **90**, 072002 (2014).
- [6] A. Abada *et al.* (FCC Collaboration), *Eur. Phys. J. C* **79**, 474 (2019).
- [7] M. Klein, *From My Vast Repertoire ...*, (World Scientific, Singapore, 2018); *Ann. Phys. (Amsterdam)* **528**, 138 (2016).
- [8] N. Armesto, P. R. Newman, W. Słomiński, and A. M. Staśto, *Phys. Rev. D* **100**, 074022 (2019).
- [9] E. Iancu, A. Leonidov, and L. McLerran, [arXiv:hep-ph/0202270](https://arxiv.org/abs/hep-ph/0202270); E. Iancu and R. Venugopalan, *QuarkGluon Plasma 3* (World Scientific, Singapore, 2004); F. Gelis, E. Iancu, J. Jalilian-Marian, and R. Venugopalan, *Annu. Rev. Nucl. Part. Sci.* **60**, 463 (2010).
- [10] A. Morreale and F. Salazar, *Universe* **7**, 312 (2021).
- [11] B. Z. Kopeliovich, L. I. Lapidus, and A. B. Zamolodchikov, *JETP Lett.* **33**, 595 (1981) [*Pis'ma Zh. Eksp. Teor. Fiz.* **33**, 612 (1981)].
- [12] N. N. Nikolaev and B. Zakharov, *Z. Phys. C* **49**, 607 (1991).
- [13] J. Bartels, K. Golec-Biernat, and H. Kowalski, *Phys. Rev. D* **66**, 014001 (2002).
- [14] E. Iancu, K. Itakura, and S. Munier, *Phys. Lett. B* **590**, 199 (2004).
- [15] J. Bartels and E. Levin, *Nucl. Phys.* **B387**, 617 (1992).
- [16] E. Iancu, K. Itakura, and L. McLerran, *Nucl. Phys.* **A708**, 327 (2002).
- [17] P. Caucal and E. Iancu, *Phys. Rev. D* **111**, 074008 (2025).
- [18] H. Mantysaari, J. Penttala, F. Salazar, and B. Schenke, *Phys. Rev. D* **111**, 054033 (2025).
- [19] J. Jimenez-Lopez, P. R. Newman, and K. Wichmann, *Phys. Rev. D* **111**, 056014 (2025).
- [20] Frank E. Taylor, *Phys. Rev. D* **111**, 052001 (2025).
- [21] L. N. Hand, *Phys. Rev.* **129**, 1834 (1963).
- [22] K. Golec-Biernat and M. Wusthoff, *Phys. Rev. D* **59**, 014017 (1998).
- [23] P. Caucal and F. Salazar, [arXiv:2502.02634](https://arxiv.org/abs/2502.02634).
- [24] D. I. Kazakov, A. V. Kotikov, G. Parente, O. A. Sampayo, J. Sánchez Guillén, *Phys. Rev. Lett.* **65**, 1535 (1990).
- [25] L. H. Orr and W. J. Stirling, *Phys. Rev. Lett.* **66**, 1673 (1991).
- [26] L. P. Kaptari, A. V. Kotikov, N. Yu. Chernikova, and Pengming Zhang, *Phys. Rev. D* **99**, 096019 (2019).
- [27] G. R. Boroun and B. Rezaei, *Phys. Lett. B* **816**, 136274 (2021).
- [28] M. Niedziela and M. Praszalowicz, *Acta Phys. Pol. B* **46**, 2019 (2015).
- [29] N. Baruah, M. K. Das, and J. K. Sarma, *Eur. Phys. J. Plus* **129**, 229 (2014).
- [30] G. R. Boroun, B. Rezaei, and J. K. Sarma, *Int. J. Mod. Phys. A* **29**, 1450189 (2014).
- [31] K. Golec-Biernat and A. M. Stasto, *Phys. Rev. D* **80**, 014006 (2009).
- [32] A. D. Martin, W. J. Stirling, and R. S. Thorne, *Phys. Lett. B* **635**, 305 (2006).
- [33] A. V. Kotikov and G. Parente, *Mod. Phys. Lett. A* **12**, 963 (1997).
- [34] M. Gluck, C. Pisano, and E. Reya, *Phys. Rev. D* **77**, 074002 (2008).
- [35] S. Moch and A. Vogt, *J. High Energy Phys.* **04** (2009) 081.
- [36] M. Modarres, M. R. Masouminia, H. Hosseinkhani, and N. Olanj, *Nucl. Phys.* **A945**, 168 (2016).
- [37] M. Mottaghizadeh and A. Mirjalili, *Phys. Lett. B* **820**, 136534 (2021).
- [38] S. Zarrin and S. Dadfar, *Int. J. Theor. Phys.* **60**, 3822 (2021).
- [39] G. Altarelli and G. Martinelli, *Phys. Lett. B* **76**, 89 (1978).
- [40] M. Gluck and E. Reya, *Nucl. Phys.* **B145**, 24 (1978).
- [41] S. Moch, J. A. M. Vermaseren, and A. Vogt, *Phys. Lett. B* **606**, 123 (2005).
- [42] G. R. Boroun, *Phys. Rev. C* **97**, 015206 (2018).
- [43] G. R. Boroun and B. Rezaei, *Eur. Phys. J. C* **72**, 2221 (2012).
- [44] M. B. Gay Ducati and P. B. Goncalves, *Phys. Lett. B* **390**, 401 (1997).
- [45] J. X. Chen, X. P. Wang *et al.*, *Chin. Phys. C* **48**, 063104 (2024).
- [46] G. R. Boroun and Y. Cai, *Chin. Phys. C* **49**, 053104 (2025).
- [47] G. R. Boroun, *Commun. Theor. Phys.* **77**, 055201 (2025).
- [48] Yuri V. Kovchegov and E. Levin, *Quantum Chromodynamics at High Energies*, Cambridge Monographs on Particle Physics, Nuclear Physics and Cosmology (Cambridge University Press, Cambridge, England, 2012).
- [49] Dmitri E. Kharzeev and E. Levin, *Phys. Rev. D* **104**, 031503 (2021).
- [50] Yu Seon Jeong, C. S. Kim, Minh Vu Luu, and Mary Hall Reno, *J. High Energy Phys.* **11** (2014) 025.
- [51] A. Donnachie and P. V. Landshoff, *Z. Phys. C* **61**, 139 (1994).
- [52] M. M. Block, L. Durand, and P. Ha, *Phys. Rev. D* **89**, 094027 (2014).
- [53] R. S. Thorne, *Phys. Rev. D* **71**, 054024 (2005).
- [54] R. S. Thorne, [arXiv:hep-ph/9805298](https://arxiv.org/abs/hep-ph/9805298).
- [55] A. D. Martin, W. J. Stirling, and R. S. Thorne, *Phys. Lett. B* **636**, 259 (2006).
- [56] G. R. Boroun, *JETP Lett.* **114**, 1 (2021).
- [57] S. Catani, M. Ciafaloni, and F. Hautmann, *Nucl. Phys.* **B366**, 135 (1991).
- [58] J. C. Collins and R. K. Ellis, *Nucl. Phys.* **B360**, 3 (1991).
- [59] K. Golec-Biernat and S. Sapeta, *J. High Energy Phys.* **03** (2018) 102.
- [60] M. A. Peredo and M. Hentschinski, *Phys. Rev. D* **109**, 014032 (2024).
- [61] F. Carvalho, F. O. Duraes, F. S. Navarra, and S. Szpigel, *Phys. Rev. C* **79**, 035211 (2009).
- [62] N. Armesto, H. Paukkunen, C. A. Salgado, and K. Tywoniuk, *Phys. Lett. B* **694**, 38 (2010).
- [63] N. Armesto, T. Lappi, H. Mäntysaari, H. Paukkunen, and M. Tevio, *Phys. Rev. D* **105**, 114017 (2022).
- [64] B. Rezaei, *Nucl. Phys.* **A1053**, 122971 (2025).
- [65] K. Kovarik *et al.*, *Phys. Rev. D* **93**, 085037 (2016).

- [66] H. Khanpour and S. Atashbar Tehrani, *Phys. Rev. D* **93**, 014026 (2016).
- [67] N. Armesto, C. A. Salgado, and U. A. Wiedemann, *Phys. Rev. Lett.* **94**, 022002 (2005).
- [68] D. A. Fagundes and M. V. T. Machado, *Phys. Rev. D* **107**, 014004 (2023).
- [69] N. N. Nikolaev, W. Schafer, B. G. Zakharov, and V. R. Zoller, *JETP Lett.* **84**, 537 (2006).
- [70] K. J. Eskola, H. Paukkunen, and C. A. Salgado, *J. High Energy Phys.* **04** (2009) 065.
- [71] D. de Florian and R. Sassot, *Phys. Rev. D* **69**, 074028 (2004).
- [72] M. Hirai, S. Kumano, and T. H. Nagai, *Phys. Rev. C* **76**, 065207 (2007).
- [73] L. Frankfurt, V. Guzey, and M. Strikman, *Phys. Rep.* **512**, 255 (2012).
- [74] M. V. T. Machado, *Eur. Phys. J. C* **47**, 365 (2006).
- [75] E. C. Aschenauer, S. Fazio, M. A. C. Lamont, H. Paukkunen, and P. Zurita, *Phys. Rev. D* **96**, 114005 (2017).
- [76] HERAhepdata, <https://www.hepdata.net/record/ins1377206>.
- [77] E. R. Cazaroto, F. Carvalho, V. P. Goncalves, and F. S. Navarra, *Phys. Lett. B* **671**, 233 (2009).
- [78] V. Tvaskis *et al.*, *Phys. Rev. C* **97**, 045204 (2018).